Mathematics

## Research article

# Derivation of logarithmic integrals expressed in teams of the Hurwitz zeta function 

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#### Abstract

In this paper by means of contour integration we will evaluate definite integrals of the form $$
\int_{0}^{1}\left(\ln ^{k}(a y)-\ln ^{k}\left(\frac{a}{y}\right)\right) R(y) d y
$$ in terms of a special function, where $R(y)$ is a general function and $k$ and $a$ are arbitrary complex numbers. These evaluations can be expressed in terms of famous mathematical constants such as the Euler's constant $\gamma$ and Catalan's constant $C$. Using derivatives, we will also derive new integral representations for some Polygamma functions such as the Digamma and Trigamma functions along with others.


Keywords: Binet; log gamma; Planck's Law; Euler's constant; Catalan's constant; logarithmic function; definite integral; Hexagamma; Cauchy integral; entries of Gradshteyn and Ryzhik Mathematics Subject Classification: 01A55, 11M06, 11M35, 30-02, 30D10, 30D30, 30E20

## 1. Introduction

Expressions for the logarithm of the gamma function, $\ln [\Gamma(z)]$ and related functions such as the digamma function $\psi^{(0)}(z)=d \ln [\Gamma(z)] / d z,[5]$ are important in the field of applied mathematics. Famous mathematicians like Kölbig, Malmsten, Binet, Kummer, and Burnside reviewed in [7] all developed definite integrals and series representations for this function. In this article we will use our contour integration method to derive a new definite integral representation for the $\ln [\Gamma(z)]$ function, where $z$ is a general complex number. We will derive integrals as indicated in the abstract in terms of special functions. We will then use this new integral representation to derive famous constants in terms of a definite integral. The derivations follow the method used by us in [4]. The generalized Cauchy's
integral formula is given by

$$
\begin{equation*}
\frac{y^{k}}{k!}=\frac{1}{2 \pi i} \int_{C} \frac{e^{w y}}{w^{k+1}} d w \tag{1.1}
\end{equation*}
$$

where $C$ is the generalized Hankel contour with the path in the complex plane. This method involves using a form of Eq (1.1), then multiplying both sides by a function, then take a definite integral of both sides as defined in [4]. This yields a definite integral in terms of a contour integral. Then we multiply both sides of Eq (1.1) by another function and take the infinite sum of both sides such that the contour integral of both equations are the same.

## 2. Integrals involving the product of logarithmic functions

### 2.1. Definite integral of the contour integral

We use the method in [4]. In Cauchy's integral formula we replace $y$ by $\ln (a y)$ and $\ln (a / y)$, then take the difference these two equations, followed by multiplying both sides by $\frac{1}{y-1}$ to get

$$
\begin{equation*}
\frac{\left(\ln ^{k}(a y)-\ln ^{k}\left(\frac{a}{y}\right)\right)}{k!(y-1)}=\frac{1}{2 \pi i} \int_{C} a^{w} w^{-k-1}\left(\frac{y^{w}-y^{-w}}{y-1}\right) d w . \tag{2.1}
\end{equation*}
$$

The logarithmic function is defined in $\mathrm{Eq}(4.1 .2)$ in [1]. We then take the definite integral over $y \in[0,1]$ of both sides to get

$$
\begin{align*}
\frac{1}{k!} \int_{0}^{1} \frac{\left(\ln ^{k}(a y)-\ln ^{k}\left(\frac{a}{y}\right)\right)}{(y-1)} d y & =\frac{1}{2 \pi i} \int_{0}^{1} \int_{C} a^{w} w^{-k-1}\left(\frac{y^{w}-y^{-w}}{y-1}\right) d w d y \\
& =\frac{1}{2 \pi i} \int_{C}\left(\int_{0}^{1}\left(\frac{y^{w}-y^{-w}}{y-1}\right) d y\right) \frac{a^{w} d w}{w^{k+1}}  \tag{2.2}\\
& =\frac{1}{2 \pi i} \int_{C}\left(a^{w} w^{-k-2}-a^{w} \pi w^{-k-1} \cot (\pi w)\right) d w
\end{align*}
$$

from $\operatorname{Eq}(3.231 .3)$ in [3] where $-1<\mathfrak{R}(w)<1$.

### 2.2. Infinite sum of the contour integral

Again, using the method in [4], replacing $y$ with $2 \pi i(p+1)+\ln (a)$ to yield

$$
\begin{equation*}
\frac{(2 \pi i(p+1)+\ln (a))^{k}}{k!}=\frac{1}{2 \pi i} \int_{C} \frac{e^{w(2 \pi i(p+1))}}{w^{k+1}} a^{w} d w, \tag{2.3}
\end{equation*}
$$

followed by multiplying both sides by $2 \pi i$ and taking the infinite sum of both sides of Eq (2.3) with respect to $p$ over $[0, \infty)$ to get

$$
\begin{align*}
\frac{(2 \pi i)^{k+1}}{k!} \zeta\left(-k, 1-\frac{i \ln (a)}{2 \pi}\right) & =\frac{1}{2 \pi i} \sum_{p=0}^{\infty} \int_{C} \frac{e^{w(2 \pi i(p+1))}}{w^{k+1}} a^{w} d w \\
& =\frac{1}{2 \pi i} \int_{C} \sum_{p=0}^{\infty} \frac{e^{w(2 \pi i(p+1))}}{w^{k+1}} a^{w} d w  \tag{2.4}\\
& =\frac{1}{2 \pi i} \int_{C}\left(-i \pi a^{w} w^{-k-1}-a^{w} \pi \cot (\pi w) w^{-k-1}\right) d w,
\end{align*}
$$

from (1.232.1) in [3] and $\mathfrak{J}(w)>0$ for the convergence of the sum and if the $\mathfrak{R}(k)<0$ then the argument of the sum over $p$ cannot be zero for some value of $p$. Note when using (1.232.1) we simply replace $x$ by $-i(x+\pi / 2)$ and multiply both sides by $i$ to get the equivalent representation for $\cot (x)$. We use (9.521.1) in [3] for the Hurwitz zeta function $\zeta(s, u)$.

To obtain the first contour integral in the last line of Eq (2.2) we use the Cauchy formula by replacing $y$ by $\ln (a)$, and $k$ by $k+1$, and simplifying we get

$$
\begin{equation*}
\frac{\ln ^{k+1}(a)}{(k+1)!}=\frac{1}{2 \pi i} \int_{C} a^{w} w^{-2-k} d w \tag{2.5}
\end{equation*}
$$

To obtain the first contour integral in the last line in Eq (2.4) we use the Cauchy formula by replacing $y$ by $\ln (a)$ and multiplying both sides by $\pi i$ and simplifying we get

$$
\begin{equation*}
\frac{\pi i \ln ^{k}(a)}{k!}=\frac{1}{2} \int_{C} a^{w} w^{-1-k} d w \tag{2.6}
\end{equation*}
$$

Since the right hand-side of $\mathrm{Eq}(2.2)$, (2.4) is equal to the addition of (2.5) and (2.6), we can equate the left hand-sides and simplify to get

$$
\begin{equation*}
\int_{0}^{1} \frac{\ln ^{k}(a y)-\ln ^{k}\left(\frac{a}{y}\right)}{y-1} d y=(2 \pi i)^{k+1} \zeta\left(-k, 1-\frac{i \ln (a)}{2 \pi}\right)+\frac{\ln ^{k+1}(a)}{k+1}+\pi i \ln ^{k}(a) \tag{2.7}
\end{equation*}
$$

## 3. Derivation of some Gradshteyn and Ryzhik entries and special cases

In this section we will use the Polygamma function defined in [5].

### 3.1. Derivation of entry 4.282.11 in [3]

Using Eq (2.7) then setting $a=e^{a i}$ and $k=-2$ simplify we get

$$
\begin{equation*}
\int_{0}^{1} \frac{\ln (y)}{(1-y)\left(a^{2}+\ln ^{2}(y)\right)^{2}} d y=\frac{1}{8 \pi a^{3}}\left(a^{2}\left(-\psi^{(1)}\left(\frac{a}{2 \pi}+1\right)\right)+2 \pi a-2 \pi^{2}\right) \tag{3.1}
\end{equation*}
$$

from Eq (1.8) in [11] and $\mathfrak{R}(a)>0$. This closed form solution represents the analytic continuation of the definite integral. The solution listed in [3] is not easily convergent for $a \in \mathbb{Z}_{+}$and is invalid otherwise.

### 3.2. When a is replaced by $e^{2 \pi a i}$

We take the first derivative of (2.7) with respect to $k$ and then set $k=0$ to get

$$
\begin{aligned}
\int_{0}^{1} \frac{\ln \left(\frac{2 i a \pi+\ln (y)}{2 i a \pi-l n}(y)\right)}{y-1} d y & =\frac{1}{2}\left(\pi^{2}+\ln \left(e^{2 i a \pi}\right)\left(-2-i \pi+\ln \left(\frac{1}{4 \pi^{2}}\right)+2 \ln \left(\ln \left(e^{2 i a \pi}\right)\right)\right)\right) \\
& +\frac{1}{2}\left(2 i \pi\left(-2 \ln \left(-\frac{1}{2} i \ln \left(e^{2 i a \pi}\right)\right)-\ln \left(\pi^{2} \ln \left(e^{2 i a \pi}\right)\right)\right)\right) \\
& -\frac{1}{2}\left(4 i \pi \ln \left[\Gamma\left(\frac{\ln \left(e^{2 i a \pi}\right)}{2 \pi i}\right)\right]\right)
\end{aligned}
$$

After rearranging and simplification we get

$$
\begin{equation*}
\ln [\Gamma(a)]=\ln \left(\frac{\sqrt{2 a \pi}}{a}\right)-a(1-\ln (a))-\frac{1}{2 \pi i} \int_{0}^{1} \frac{\ln \left(\frac{2 i a \pi+\ln (y)}{2 i a \pi-\ln (y)}\right)}{y-1} d y \tag{3.2}
\end{equation*}
$$

where the definition of the log-gamma function is from (7.105) in [2] and $a$ is a general complex number. This integral is a new representation for the logarithm of the gamma function. The integral assumes that $y$ is real and between 0 and 1 thus $a$ must be purely imaginary for there to be a singularity.

### 3.3. An integral representation for the Digamma function

We take the first derivative of (2.7) with respect to $k$ then set $k=0$ and then take the first derivative with respect to $a$ and simplify to get

$$
\begin{equation*}
\psi^{(0)}(a)=\ln (a)-\frac{1}{2 a}-\int_{0}^{1} \frac{2 \ln (y)}{(y-1)\left(4 \pi^{2} a^{2}+\ln ^{2}(y)\right)} d y \tag{3.3}
\end{equation*}
$$

where there are integrable singularities at both end points. At $y=0$ the numerator is cancelled by the $\log$ in the denominator giving zero. At $y=1$, we could define $z=1-y$ and use (4.1.24) in [1].

### 3.3.1. Derivation of entry 4.282 .1 in [3]

We set $a=1$ into Eq (3.3) and simplify to get

$$
\begin{equation*}
\gamma=\frac{1}{2}+\int_{0}^{1} \frac{2 \ln (y)}{(y-1)\left(4 \pi^{2}+\ln ^{2}(y)\right)} d y \tag{3.4}
\end{equation*}
$$

from (1.2) in [7] and Eq (8.365.4) in [3] and $\gamma$ is the Euler's constant. We set $a=\frac{1}{2}$ into Eq (3.3) and simplify to get

$$
\begin{equation*}
\gamma=1-\ln (2)+\int_{0}^{1} \frac{2 \ln (y)}{(y-1)\left(\pi^{2}+\ln ^{2}(y)\right)} d y \tag{3.5}
\end{equation*}
$$

from (1.2) in [7].

### 3.4. An integral representation for the Trigamma function

We take the first derivative of (2.7) with respect to $k$ then set $k=0$ and then take the second derivative with respect to $a$ and simplify to get

$$
\begin{equation*}
\psi^{(1)}(a)=\frac{1}{a}+\frac{1}{2 a^{2}}+\int_{0}^{1} \frac{16 \pi^{2} a \ln (y)}{(y-1)\left(4 \pi^{2} a^{2}+\ln ^{2}(y)\right)^{2}} d y \tag{3.6}
\end{equation*}
$$

where $\mathfrak{R}(a) \geq 0$.

### 3.4.1. Integral representations for Catalan's constant, $G$

We set $a=\frac{1}{4}$ into Eq (3.6) and simplify to get

$$
\begin{equation*}
G=8 \pi^{2} \int_{0}^{1} \frac{\ln (y)}{(y-1)\left(\pi^{2}+4 \ln ^{2}(y)\right)^{2}} d y-\left(\frac{\pi^{2}-12}{8}\right) \tag{3.7}
\end{equation*}
$$

from [5].
We set $a=\frac{3}{4}$ into Eq (3.6) and simplify to get

$$
\begin{equation*}
G=-24 \pi^{2} \int_{0}^{1} \frac{\ln (y)}{(y-1)\left(9 \pi^{2}+4 \ln ^{2}(y)\right)^{2}} d y-\left(\frac{20-9 \pi^{2}}{72}\right) \tag{3.8}
\end{equation*}
$$

from [5].

### 3.5. An integral representation for the Hexagamma function

We take the first derivative of (2.7) with respect to $k$ then set $k=0$ and then fifth derivative with respect to $a$ and simplify to get

$$
\begin{equation*}
\psi^{(4)}(a)=-768 \pi^{4} \int_{0}^{1} \frac{\ln (y)\left(80 a^{4} \pi^{4}-40 a^{2} \pi^{2} \ln ^{2}(y)+\ln ^{4}(y)\right)}{(y-1)\left(4 a^{2} \pi^{2}+\ln ^{2}(y)\right)^{5}} d y-\frac{6(2+a)}{a^{5}} \tag{3.9}
\end{equation*}
$$

where $a$ is a general complex number. This integral is a new representation for the Hexagamma function which can be used in the calculation of the stellar radiation and dust emission [8]. The given definite integral in [8] is unable to numerically evaluate complex ranges for the parameter $a$. The value of $\psi^{(4)}(1)=-24 \zeta(5)$ can be derived from (6.4.2) in [1] and using Eq (3.9) with $a=1$ we get

$$
\begin{equation*}
\zeta(5)=32 \pi^{4} \int_{0}^{1} \frac{\ln (y)\left(80 \pi^{4}-40 \pi^{2} \ln ^{2}(y)+\ln ^{4}(y)\right)}{(y-1)\left(4 \pi^{2}+\ln ^{2}(y)\right)^{5}} d y+\frac{3}{4} \tag{3.10}
\end{equation*}
$$

which is a new integral representation for the $\zeta(5)$ which also appears in Planck's law as the average energy of a photon from a blackbody.

Another application of the Hexagamma function is in the work of Grosshandler [9] where $a$ is a space variable used in the calculation of soot radiation from nonhomogeneous combustion products. A definite integral form is used, but the range of $a$ is $\mathfrak{R}(a)>0$. We propose using Eq (3.9) where $a$ is a general complex number.

## 4. Formulae for Polygamma function $\psi^{(m)}(a)$, where $m$ is a positive integer

To obtain the desired formulae for $\psi^{(m)}(a)$ we will take the $(m+1)$-th partial derivative of Eq (3.2) to get

$$
\begin{align*}
\psi^{(m)}(a) & =\frac{\partial^{m+1}}{\partial a^{m+1}}(\ln [\Gamma(a)]) \\
& =\frac{\partial^{m+1}}{\partial a^{m+1}}\left(-\frac{1}{2 \pi i} \int_{0}^{1} \frac{\ln \left(\frac{2 i a \pi+\ln (y)}{2 i a \pi-\ln (y)}\right)}{y-1} d y-\left(\frac{\pi i}{4}+\frac{\pi i a}{2}+a \ln \left(\frac{e}{i a}\right)+\ln \left(\frac{a}{\sqrt{2 i a \pi}}\right)\right)\right)  \tag{4.1}\\
& =-\frac{1}{2 \pi i} \int_{0}^{1} \frac{\partial^{m+1}}{\partial a^{m+1}}\left(\ln \left(\frac{2 i a \pi+\ln (y)}{2 i a \pi-\ln (y)}\right)\right) \frac{d y}{y-1} \\
& -\frac{\partial^{m+1}}{\partial a^{m+1}}\left(\frac{\pi i}{4}+\frac{\pi i a}{2}+a \ln \left(\frac{e}{i a}\right)+\ln \left(\frac{a}{\sqrt{2 i a \pi}}\right)\right)
\end{align*}
$$

### 4.1. Examples of the Polygamma function, $\psi^{(m)}(a)$

4.1.1. To derive an integral expression for $\zeta(5)$ using $\psi^{(4)}\left(\frac{1}{3}\right)$

Using Eq (4.1) when $m=4$ and $a=\frac{1}{3}$ we get

$$
\begin{equation*}
\zeta(5)=\left(\frac{23328 \pi^{4}}{121}\right) \int_{0}^{1} \frac{\ln (y)\left(80 \pi^{4}-360 \pi^{2} \ln ^{2}(y)+81 \ln ^{4}(y)\right)}{(y-1)\left(4 \pi^{2}+9 \ln ^{2}(y)\right)^{5}} d y-\frac{1}{2904}\left(\frac{16 \pi^{5}}{\sqrt{3}}-3402\right) \tag{4.2}
\end{equation*}
$$

from Theorem 5 in [6].
4.1.2. To derive an integral expression for $\zeta(7)$ using $\psi^{(6)}\left(\frac{1}{6}\right)$

Using Eq (4.1) when $m=5$ and $a=\frac{1}{6}$ we get

$$
\begin{equation*}
\zeta(7)=\rho\left(\int_{0}^{1} \frac{\ln (y)\left(7 \pi^{6}-315 \pi^{4} \ln ^{2}(y)+1701 \pi^{2} \ln ^{4}(y)-729 \ln ^{6}(y)\right)}{(y-1)\left(\pi^{2}+9 \ln ^{2}(y)\right)^{7}} d y-\frac{301 \sqrt{3} \pi^{7}-1662120}{9447840 \pi^{6}}\right) \tag{4.3}
\end{equation*}
$$

from Theorem 5 in [6] where $\rho=\frac{75558272}{1249299}$.

## 5. Comparison with known formulas for the log gamma function and alternate forms

In this section we will summarize the comparison of our derived formula in Eq (3.2) with known formulas by Malmsten (1.1) in [7], Kummer (1.7) in [10], Binet (1.6) in [10] and Burnside (1.1) in [7], where the range of evaluation for the log-gamma function is $\mathfrak{R}(a)>0$. We will look in particular at the domains of evaluation. We have Eq (3.2) given by:

$$
\begin{equation*}
\ln [\Gamma(a)]=\ln \left(\frac{\sqrt{2 a \pi}}{a}\right)-a(1-\ln (a))-\frac{1}{2 \pi i} \int_{0}^{1} \frac{\ln \left(\frac{2 i a \pi+\ln (y)}{2 i a \pi-\ln (y)}\right)}{y-1} d y \tag{5.1}
\end{equation*}
$$

where $a$ is a general complex number. This is an extension of the range of evaluation for the log-gamma function using one formula. At $y=0$ the numerator is cancelled by the $\log$ in the denominator giving zero. At $y=1$, we could define $z=1-y$ and use (4.1.24) in [1].

## 6. Summary

In this article we derived some interesting definite integrals derived by famous mathematicians, Binet, Kummer, Burnside and Malmsten. We also compared our results for the log-gamma function to those of other work. We found that we are able to achieve a wider range of computation using one formula as opposed to previous works. We will be looking at other integrals using this contour integral method for future work. The results presented were numerically verified for both real and imaginary values of the parameters in the integrals using Mathematica by Wolfram.

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## Conflict of interest

The authors declare no conflict of interest.

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