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Research article

A nonlocal boundary value problems for hybrid φ-Caputo fractional integro-differential equations

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Abstract: In this paper, we discuss the existence of solutions for a nonlocal boundary value problems for hybrid φ-Caputo fractional integro-differential equations. Our main result is based on a hybrid fixed point theorem due to Dhage. Finally, we give an example to illustrate our main result.

Keywords: boundary value problem; existence of solutions; hybrid φ-Caputo fractional differential equation; integro-differential equations; hybrid fixed point theorem
Mathematics Subject Classification: 34A08, 26A33

1. Introduction

Hybrid differential equations have been considered more important and served as special cases of dynamical systems. Dhage and Lakshmikantham [1] were the first to study ordinary hybrid differential equation and studied the existence of solutions for this boundary value problem. In recent years, with the wide study of fractional differential equations, the theory of hybrid fractional differential equations were also studied by several researchers, see [2–10] and the references therein.

Zhao et al. [2] studied existence and uniqueness results for the following hybrid differential equations involving Riemann-Liouville fractional derivative

$$D_{0^+}^{q}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad a.e. t \in J = [0,T]$$

$$x(0) = 0,$$

where $0 < q < 1, f \in C(J \times R \to R\backslash\{0\})$ and $g \in C(J \times R, R)$.

Zidane Baitiche et al. [11] considered the following boundary value problem of nonlinear fractional hybrid differential equations involving Caputo's derivative

$$ {}^{C}D^{\alpha}_{0^+}\Big(\frac{x(t)}{f(t,x(\mu(t)))}\Big) = g(t,x(\mu(t))), \quad t\in I=[0,1] $$

$$ a\Big[\frac{x(t)}{f(t,x(\mu(t)))}\Big]\Big|_{t=0} + b\Big[\frac{x(t)}{f(t,x(\mu(t)))}\Big]\Big|_{t=1} = c, $$

where $0<\alpha\le 1, {}^{C}D^{\alpha}_{0^+}$ is the Caputo fractional derivative. $f\in C(I\times R\to R\backslash\{0\}), g\in C(I\times R,R)$.

As we all known, the hadamard fractional differential equations are also popular in the literature, see [12–16], so some authors began to study the theory of fractional hybrid differential equation of hadamard type.

Zidane Baitiche et al. [17] studied the existence of solutions for fractional hybrid differential equation of hadamard type with dirichlet boundary conditions

$$ {}_{H}D^{\alpha}\Big(\frac{x(t)}{f(t,x(t))}\Big) = g(t,x(t)), \quad 1<t<e, \ 1<\alpha\le 2, $$

$$ x(1)=0, \quad x(e)=0, $$

where $1<\alpha\le 2$, ${}_{H}D^{\alpha}$ is the Hadamard fractional derivative, $f\in C([1,e]\times R\to R\backslash\{0\})$ and $g\in C([1,e]\times R,R)$.

In [18], M. Jamil et al. discussed the existence result for the boundary value problem of hybrid fractional integro-differential equations involving Caputo's derivative given by

$$ {}^{C}D^{\alpha}\Big(\frac{{}^{C}D^{\omega}u(t)-\sum_{i=1}^{m} I^{\beta_i} f_i(t,u(t))}{g(t,u(t))}\Big) = h(t,u(t),I^{\gamma}u(t)), \quad t\in J=[0,1], $$

$$ u(0)=0, \ D^{\omega}u(0)=0, \ u(1)=\delta u(\eta), \quad 0<\delta<1, \ 0<\eta<1, $$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order α, ${}^{C}D^{\omega}$ is the Caputo fractional derivative of order ω, $0<\alpha\le 1$, $1<\omega\le 2$.

In order to analyze fractional differential equations in a generic way, a fractional derivative with respect to another function called φ-Caputo derivative was proposed [19].

By mixing idea of the above works, we derived an existence result for the nonlocal boundary value problems of hybrid φ-Caputo fractional integro-differential equations

$$ {}^{C}D^{\alpha\,\varphi}\Big(\frac{{}^{C}D^{\beta\,\varphi}u(t)-\sum_{i=1}^{m} I^{\omega_i\,\varphi} f_i(t,u(t),I^{\mu_1\,\varphi}u(t),\cdots,I^{\mu_n\,\varphi}u(t))}{g(t,u(t),I^{\gamma_1\,\varphi}u(t),\cdots,I^{\gamma_p\,\varphi}u(t))}\Big) = h(t,u(t)), t\in J=[0,1], \tag{1.1} $$

$$ u(0)=0, \ {}^{C}D^{\beta\,\varphi}u(0)=0, \ u(1)=\sum_{j=1}^{k}\delta_j u(\xi_j), \tag{1.2} $$

where $0<\alpha\le 1$, $1<\beta\le 2$, ${}^{C}D^{\alpha\,\varphi}$ is the φ-Caputo fractional derivative of order α, ${}^{C}D^{\beta\,\varphi}$ is the φ-Caputo fractional derivative of order β, the function $\varphi:\ [0,1]\to R$ is a strictly increasing function such that $\varphi\in C^2[0,1]$ with $\varphi'(x)>0$ for all $x\in[0,1]$, $I^{\mu\,\varphi}$ denote the φ-Riemann-Liouville fractional integral of order μ, $g\in C(J\times R^{p+1},R\backslash\{0\})$, $h\in C(J\times R,R)$ and $f_i\in C(J\times R^{n+1},R)$ with

$f_i(0,\underbrace{0,\cdots,0}_{n+1}) = 0,\ w_i > 0,\ i = 1,2,\cdots,m,\ \mu_1,\cdots,\mu_n > 0$ and $\gamma_1,\cdots,\gamma_p > 0,\ 0 < \delta_j < 1,\ j = 1,2,\cdots,k,\ 0 < \xi_1 < \xi_2 < \cdots < \xi_k < 1$.

It is notable that the fractional hybrid integro-differential equation presented in this paper is the novel in the sense that the fractional derivative with respect to another function called φ-Caputo fractional derivative. Note that the hybrid fractional integro-differential equations involving Caputo's derivative in [18] is a special case of our hybrid φ-Caputo fractional integro-differential equations with $\varphi(t) = t$. Moreover, all dependent functions f_i and g in our paper are in the form of multi-term. Furthermore, our problem is more general than the work in [8], as we consider the problem with multi-point boundary conditions, while the authors in [8] only investigated two-point boundary condition.

The organization of this work is as follows. Section 2 contains some preliminary facts that we need in the sequel. In section 3, we present the solution for the hybrid fractional integro-differential equation (1.1), (1.2) and then prove our main existence results. Finally, we illustrate the obtained results by an example.

2. The preliminary lemmas

In the following and throughtout the text, $a > 0$ is a real, $x : [a,b] \to R$ an integrable function and $\varphi \in C^2[a,b]$ an increasing function such that with $\varphi'(t) \neq 0$ for all $t \in [a,b]$.

Definition 2.1 The φ-Riemann-Liouville fractional integral of x of order α is defined as follows

$$I_{a^+}^{\alpha\ \varphi} x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \varphi'(s)(\varphi(t) - \varphi(s))^{\alpha-1} x(s)ds.$$

Definition 2.2 The φ-Riemann-Liouville fractional derivative of x of order α is defined as follows

$$D_{a^+}^{\alpha\ \varphi} x(t) := \Big(\frac{1}{\varphi'(t)}\frac{d}{dt}\Big)^n I_{a^+}^{n-\alpha\ \varphi} x(t) = \frac{1}{\Gamma(n-\alpha)}\Big(\frac{1}{\varphi'(t)}\frac{d}{dt}\Big)^n \int_a^t \varphi'(s)(\varphi(t) - \varphi(s))^{n-\alpha-1} x(s)ds,$$

here $n = [\alpha] + 1$.

Remark 2.1 Let $\alpha, \beta > 0$, then the relation holds

$$I_{a^+}^{\alpha\ \varphi} I_{a^+}^{\beta\ \varphi} x(t) = I_{a^+}^{\alpha+\beta\ \varphi} x(t).$$

Definition 2.3 Let $\alpha > 0$ and $x \in C^{n-1}[a,b]$, the φ-Caputo fractional derivative of x of order α is defined as follows

$${}^C D_{a^+}^{\alpha\ \varphi} x(t) := D_{a^+}^{\alpha\ \varphi}\Big[x(t) - \sum_{k=0}^{n-1} \frac{x_\varphi^{[k]}(a)}{k!}(\varphi(t) - \varphi(a))^k\Big],\ n = [\alpha] + 1 \text{ for } \alpha \notin N,\ n = \alpha \text{ for } \alpha \in N,$$

where $x_\varphi^{[k]}(t) := \Big(\frac{1}{\varphi'(t)}\frac{d}{dt}\Big)^k x(t)$.

Theorem 2.1 [20] Let $x : [a,b] \to R$. The following results hold:

1. If $x \in C[a,b]$, then ${}^C D_{a^+}^{\alpha\ \varphi} I_{a^+}^{\alpha\ \varphi} x(t) = x(t)$;
2. If $x \in C^{n-1}[a,b]$, then

$$I_{a^+}^{\alpha\ \varphi}\ {}^C D_{a^+}^{\alpha\ \varphi} x(t) = x(t) - \sum_{k=0}^{n-1} \frac{x_\varphi^{[k]}(a)}{k!}(\varphi(t) - \varphi(a))^k.$$

Lemma 2.2 [18] Let S be a nonempty, convex, closed, and bounded set such that $S \subseteq E$, and let $A : E \to E$ and $B : S \to E$ be two operators which satisfy the following :
(H_1) A is contraction;
(H_2) B is compact and continuous, and
(H_3) $u = Au + Bv,\ \forall v \in S \Rightarrow u \in S$.

Then there exists a solution of the operator equation $u = Au + Bu$.

Let $E = C(J, R)$ be a Banach space equipped with the norm

$$\|u\| = \sup_{t\in J} |u(t)| \quad \text{and} \quad (uv)(t) = u(t)v(t), \quad \forall\ t \in J.$$

Then E is a Banach algebra with the above norm and multiplication.

3. Main results

Lemma 3.1 Suppose that $\alpha, \beta, \omega_i, i = 1, 2, \cdots, m, \gamma_i, i = 1, 2, \cdots, p, \mu_i, i = 1, 2, \cdots, n, \delta_j, \xi_j, j = 1, 2, \cdots, k$ and functions $g, h, f_i, i = 1, 2, \cdots, m$ satisfy problem (1.1), (1.2). Then the unique solution of (1.1), (1.2) is given by

$$\begin{aligned} u(t) \ &= \int_0^t \tfrac{(\varphi(t)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}\varphi'(s)g(s,u(s),I^{\gamma_1\ \varphi}u(s),\cdots,I^{\gamma_p\ \varphi}u(s)) \int_0^s \tfrac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau,u(\tau))d\tau ds \\ &+ \sum_{i=1}^m I^{\omega_i+\beta\ \varphi} f_i(t,u(t),I^{\mu_1\ \varphi}u(t),\cdots,I^{\mu_n\ \varphi}u(t)) \\ &+ \frac{\varphi(t)-\varphi(0)}{\sum_{j=1}^k \delta_j(\varphi(\xi_j)-\varphi(0)) - (\varphi(1)-\varphi(0))} \\ &\Big[\int_0^1 \tfrac{(\varphi(1)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}\varphi'(s)g(s,u(s),I^{\gamma_1\ \varphi}u(s),\cdots,I^{\gamma_p\ \varphi}u(s)) \int_0^s \tfrac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau,u(\tau))d\tau ds \\ &+ \sum_{i=1}^m I^{\omega_i+\beta\ \varphi} f_i(1,u(1),I^{\mu_1\ \varphi}u(1),\cdots,I^{\mu_n\ \varphi}u(1)) \\ &- \sum_{j=1}^k \delta_j \int_0^{\xi_j} \tfrac{(\varphi(\xi_j)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}\varphi'(s)g(s,u(s),I^{\gamma_1\ \varphi}u(s),\cdots,I^{\gamma_p\ \varphi}u(s)) \\ &\int_0^s \tfrac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau,u(\tau))d\tau ds \\ &- \sum_{j=1}^k \delta_j \sum_{i=1}^m I^{\omega_i+\beta\ \varphi} f_i(\xi_j,u(\xi_j),I^{\mu_1\ \varphi}u(\xi_j),\cdots,I^{\mu_n\ \varphi}u(\xi_j))\Big], \end{aligned} \tag{3.1}$$

where

$$I^{\omega_i+\beta\ \varphi} f_i(t, u(t), I^{\mu_1\ \varphi}u(t), \cdots, I^{\mu_n\ \varphi}u(t)) \\ = \int_0^t \frac{(\varphi(t)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s) f_i(s, u(s), I^{\mu_1\ \varphi}u(s), \cdots, I^{\mu_n\ \varphi}u(s))ds;$$

$$I^{\omega_i+\beta\ \varphi} f_i(1, u(1), I^{\mu_1\ \varphi}u(1), \cdots, I^{\mu_n\ \varphi}u(1)) \\ = \int_0^1 \frac{(\varphi(1)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s) f_i(s, u(s), I^{\mu_1\ \varphi}u(s), \cdots, I^{\mu_n\ \varphi}u(s))ds;$$

$$I^{\omega_i+\beta\ \varphi} f_i(\xi_j, u(\xi_j), I^{\mu_1\ \varphi}u(\xi_j), \cdots, I^{\mu_n\ \varphi}u(\xi_j)) \\ = \int_0^{\xi_j} \frac{(\varphi(\xi_j)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s) f_i(s, u(s), I^{\mu_1\ \varphi}u(s), \cdots, I^{\mu_n\ \varphi}u(s))ds.$$

Proof. We apply φ-Riemann-Liouville fractional integral $I^{\alpha\ \varphi}$ on both sides of (1.1), by Theorem 2.1, we have

$$\frac{{}^C D^{\beta\ \varphi}u(t) - \sum_{i=1}^m I^{\omega_i\ \varphi} f_i(t, u(t), I^{\mu_1\ \varphi}u(t), \cdots, I^{\mu_n\ \varphi}u(t))}{g(t, u(t), I^{\gamma_1\ \varphi}u(t), \cdots, I^{\gamma_p\ \varphi}u(t))} = I^{\alpha\ \varphi}h(t, u(t)) + c_0,$$

then by $u(0) = 0$, ${}^C D^{\beta\ \varphi}u(0) = 0$, $f_i(0, \underbrace{0, \cdots, 0}_{n+1}) = 0$, we get $c_0 = 0$. i.e,

$$\begin{aligned} {}^C D^{\beta\ \varphi}u(t) &= g(t, u(t), I^{\gamma_1\ \varphi}u(t), \cdots, I^{\gamma_p\ \varphi}u(t)) \int_0^t \frac{(\varphi(t)-\varphi(s))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(s)h(s, u(s))ds \\ &+ \sum_{i=1}^m I^{\omega_i\ \varphi} f_i(t, u(t), I^{\mu_1\ \varphi}u(t), \cdots, I^{\mu_n\ \varphi}u(t)). \end{aligned} \tag{3.2}$$

Apply again fractional integral $I^{\beta\ \varphi}$ on both sides of (3.2) and by Theorem 2.1, we get

$$\begin{aligned} u(t) &= \int_0^t \frac{(\varphi(t)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}\varphi'(s)g(s, u(s), I^{\gamma_1\ \varphi}u(s), \cdots, I^{\gamma_p\ \varphi}u(s)) \int_0^s \frac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau, u(\tau))d\tau ds \\ &+ \sum_{i=1}^m I^{\omega_i+\beta\ \varphi} f_i(t, u(t), I^{\mu_1\ \varphi}u(t), \cdots, I^{\mu_n\ \varphi}u(t)) + c_1 + c_2(\varphi(t) - \varphi(0)), \end{aligned} \tag{3.3}$$

$u(0) = 0$, $f_i(0, \underbrace{0, \cdots, 0}_{n+1}) = 0$ yield $c_1 = 0$, thus equation (3.3) is reduced to

$$\begin{aligned} u(t) &= \int_0^t \frac{(\varphi(t)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}\varphi'(s)g(s, u(s), I^{\gamma_1\ \varphi}u(s), \cdots, I^{\gamma_p\ \varphi}u(s)) \int_0^s \frac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau, u(\tau))d\tau ds \\ &+ \sum_{i=1}^m I^{\omega_i+\beta\ \varphi} f_i(t, u(t), I^{\mu_1\ \varphi}u(t), \cdots, I^{\mu_n\ \varphi}u(t)) + c_2(\varphi(t) - \varphi(0)), \end{aligned} \tag{3.4}$$

specially.

$$\begin{aligned} u(1) &= \int_0^1 \frac{(\varphi(1)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}\varphi'(s)g(s, u(s), I^{\gamma_1\ \varphi}u(s), \cdots, I^{\gamma_p\ \varphi}u(s)) \int_0^s \frac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau, u(\tau))d\tau ds \\ &+ \sum_{i=1}^m I^{\omega_i+\beta\ \varphi} f_i(1, u(1), I^{\mu_1\ \varphi}u(1), \cdots, I^{\mu_n\ \varphi}u(1)) + c_2(\varphi(1) - \varphi(0)), \end{aligned}$$

$$\begin{aligned} u(\xi_j) &= \int_0^{\xi_j} \frac{(\varphi(\xi_j)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}\varphi'(s)g(s, u(s), I^{\gamma_1\ \varphi}u(s), \cdots, I^{\gamma_p\ \varphi}u(s)) \int_0^s \frac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau, u(\tau))d\tau ds \\ &+ \sum_{i=1}^m I^{\omega_i+\beta\ \varphi} f_i(\xi_j, u(\xi_j), I^{\mu_1\ \varphi}u(\xi_j), \cdots, I^{\mu_n\ \varphi}u(\xi_j)) + c_2(\varphi(\xi_j) - \varphi(0)), \end{aligned}$$

from $u(1)=\sum\limits_{j=1}^{k}\delta_j u(\xi_j)$, we have

$$
\begin{aligned}
c_2 &= \frac{1}{\sum\limits_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))} \\
&\Big[\int_0^1 \tfrac{(\varphi(1)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}\varphi'(s)g(s,u(s),I^{\gamma_1}\,{}^{\varphi}u(s),\cdots,I^{\gamma_p}\,{}^{\varphi}u(s))\int_0^s \tfrac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau,u(\tau))d\tau ds \\
&+\sum_{i=1}^{m} I^{\omega_i+\beta}\,{}^{\varphi}f_i(1,u(1),I^{\mu_1}\,{}^{\varphi}u(1),\cdots,I^{\mu_n}\,{}^{\varphi}u(1)) \\
&-\sum_{j=1}^{k}\delta_j\int_0^{\xi_j}\tfrac{(\varphi(\xi_j)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}\varphi'(s)g(s,u(s),I^{\gamma_1}\,{}^{\varphi}u(s),\cdots,I^{\gamma_p}\,{}^{\varphi}u(s)) \\
&\int_0^s \tfrac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau,u(\tau))d\tau ds \\
&-\sum_{j=1}^{k}\delta_j\sum_{i=1}^{m} I^{\omega_i+\beta}\,{}^{\varphi}f_i(\xi_j,u(\xi_j),I^{\mu_1}\,{}^{\varphi}u(\xi_j),\cdots,I^{\mu_n}\,{}^{\varphi}u(\xi_j))\Big].
\end{aligned}
$$

Consequently, we can get the desired result. The proof is completed. □

Theorem 3.2 Suppose that functions $g\in C(J\times R^{p+1},R\backslash\{0\})$, $h\in C(J\times R,R)$ and $f_i\in C(J\times R^{n+1},R)$ with $f_i(0,\underbrace{0,\cdots,0}_{n+1})=0$. Furthermore, assume that

(C_1) there exist bounded mapping $\sigma:[0,1]\to R^+$, $\lambda:[0,1]\to R^+$ such that

$$|g(t,k_1,k_2,\cdots,k_{p+1})-g(t,k_1',k_2',\cdots,k_{p+1}')|\le\sigma(t)\sum_{i=1}^{p+1}|k_i-k_i'|$$

for $t\in J$ and $(k_1,k_2,\cdots,k_{p+1}),(k_1',k_2',\cdots,k_{p+1}')\in R^{p+1}$, and

$|h(t,u)-h(t,v)|\le\lambda(t)|u-v|$ for $t\in J$ and $u,v\in R$;

(C_2) there exist $\phi_i,\Omega,\chi\in C(J,R^+)$, $i=1,2,\cdots,m$ such that

$$|f_i(t,k_1,k_2,\cdots,k_{n+1})|\le\phi_i(t),\ \forall\ (t,k_1,k_2,\cdots,k_{n+1})\in J\times R^{n+1},$$

$$|h(t,u)|\le\Omega(t),\ \forall\ (t,u)\in J\times R,$$

$$|g(t,k_1,k_2,\cdots,k_{p+1})|\le\chi(t),\ \forall\ (t,k_1,k_2,\cdots,k_{p+1})\in J\times R^{p+1};$$

(C_3) there exists $r>0$ such that

$$
\begin{aligned}
&\left(1+\frac{(\varphi(1)-\varphi(0))(1+\sum\limits_{j=1}^{k}\delta_j)}{\left|\sum\limits_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))\right|}\right) \\
&\left(\chi^*\Omega^*\frac{(\varphi(1)-\varphi(0))^{\alpha}}{\Gamma(\alpha+1)}\frac{(\varphi(1)-\varphi(0))^{\beta}}{\Gamma(\beta+1)}+\sum_{i=1}^{m}\phi_i^*\frac{(\varphi(1)-\varphi(0))^{\omega_i+\beta}}{\Gamma(\omega_i+\beta+1)}\right)\le r;
\end{aligned}
\tag{3.5}
$$

$$\left(\chi^*\lambda^* + \Omega^*\sigma^* \sum_{i=1}^{p+1} \frac{(\varphi(1)-\varphi(0))^{\gamma_i}}{\Gamma(\gamma_i+1)}\right)\frac{(\varphi(1)-\varphi(0))^{\alpha}}{\Gamma(\alpha+1)}$$
$$\left(1 + \frac{(\varphi(1)-\varphi(0))(1+\sum_{j=1}^{k}\delta_j)}{\left|\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0)) - (\varphi(1)-\varphi(0))\right|}\right)\frac{(\varphi(1)-\varphi(0))^{\beta}}{\Gamma(\beta+1)} < 1, \tag{3.6}$$

where $\Omega^* = \sup_{0\le t\le 1}|\Omega(t)|$, $\phi_i^* = \sup_{0\le t\le 1}|\phi_i(t)|$, $i = 1, 2, \cdots, p$, $\chi^* = \sup_{0\le t\le 1}|\chi(t)|$, $\lambda^* = \sup_{0\le t\le 1}|\lambda(t)|$, $\sigma^* = \sup_{0\le t\le 1}|\sigma(t)|$.

Then the hybrid problem (1.1), (1.2) has at least one solution.

Proof. Define a subset S of E as

$$S = \{u \in E : \|u\| \le r\},$$

where r satisfies inequality (3.5). Clearly S is closed, convex and bounded subset of the Banach space E. Define two operators $A : E \to E$ by

$$\begin{aligned} Au(t) &= \int_0^t \frac{(\varphi(t)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}\varphi'(s)g(s,u(s),I^{\gamma_1\ \varphi}u(s),\cdots,I^{\gamma_p\ \varphi}u(s))\int_0^s \frac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau,u(\tau))d\tau ds \\ &+\frac{\varphi(t)-\varphi(0)}{\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))} \\ &\int_0^1 \frac{(\varphi(1)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}\varphi'(s)g(s,u(s),I^{\gamma_1\ \varphi}u(s),\cdots,I^{\gamma_p\ \varphi}u(s))\int_0^s \frac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau,u(\tau))d\tau ds \\ &-\frac{(\varphi(t)-\varphi(0))\sum_{j=1}^{k}\delta_j}{\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))} \\ &\int_0^{\xi_j} \frac{(\varphi(\xi_j)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}\varphi'(s)g(s,u(s),I^{\gamma_1\ \varphi}u(s),\cdots,I^{\gamma_p\ \varphi}u(s))\int_0^s \frac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau,u(\tau))d\tau ds, \end{aligned} \tag{3.7}$$

$$\begin{aligned} Bu(t) &= \sum_{i=1}^{m}\int_0^t \frac{(\varphi(t)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s)f_i(s,u(s),I^{\mu_1\ \varphi}u(s),\cdots,I^{\mu_n\ \varphi}u(s))ds \\ &+\frac{\varphi(t)-\varphi(0)}{\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))} \\ &\sum_{i=1}^{m}\int_0^1 \frac{(\varphi(1)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s)f_i(s,u(s),I^{\mu_1\ \varphi}u(s),\cdots,I^{\mu_n\ \varphi}u(s))ds \\ &-\frac{(\varphi(t)-\varphi(0))\sum_{j=1}^{k}\delta_j}{\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))} \\ &\sum_{i=1}^{m}\int_0^{\xi_j} \frac{(\varphi(\xi_j)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s)f_i(s,u(s),I^{\mu_1\ \varphi}u(s),\cdots,I^{\mu_n\ \varphi}u(s))ds. \end{aligned} \tag{3.8}$$

Then $u(t)$ is a solution of problem (1.1), (1.2) if and only if $u(t) = Au(t) + Bu(t)$. We shall show that the operators A and B satisfy all the conditions of Lemma 2.2. We split the proof into several steps.

Step 1. We first show that A is a contraction mapping. Let $u(t), v(t) \in S$, we write

$$\begin{aligned} G(s) &= g(s, u(s), I^{\gamma_1\ \varphi}u(s), \cdots, I^{\gamma_p\ \varphi}u(s)) \int_0^s \frac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau, u(\tau))d\tau \\ &-g(s, v(s), I^{\gamma_1\ \varphi}v(s), \cdots, I^{\gamma_p\ \varphi}v(s)) \int_0^s \frac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau, v(\tau))d\tau, \end{aligned}$$

then by (C_1) we have

$$\begin{aligned} |G(s)| &= \Big| g(s, u(s), I^{\gamma_1\ \varphi}u(s), \cdots, I^{\gamma_p\ \varphi}u(s)) \int_0^s \frac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau, u(\tau))d\tau \\ &-g(s, u(s), I^{\gamma_1\ \varphi}u(s), \cdots, I^{\gamma_p\ \varphi}u(s)) \int_0^s \frac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau, v(\tau))d\tau \\ &+g(s, u(s), I^{\gamma_1\ \varphi}u(s), \cdots, I^{\gamma_p\ \varphi}u(s)) \int_0^s \frac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau, v(\tau))d\tau \\ &-g(s, v(s), I^{\gamma_1\ \varphi}v(s), \cdots, I^{\gamma_p\ \varphi}v(s)) \int_0^s \frac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)h(\tau, v(\tau))d\tau \Big| \\ &\leq \Big| g(s, u(s), I^{\gamma_1\ \varphi}u(s), \cdots, I^{\gamma_p\ \varphi}u(s)) \Big| \int_0^s \frac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)\Big|h(\tau, u(\tau)) - h(\tau, v(\tau))\Big|d\tau \\ &+ \int_0^s \frac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\varphi'(\tau)|h(\tau, v(\tau))|d\tau \\ &\Big| g(s, u(s), I^{\gamma_1\ \varphi}u(s), \cdots, I^{\gamma_p\ \varphi}u(s)) - g(s, v(s), I^{\gamma_1\ \varphi}v(s), \cdots, I^{\gamma_p\ \varphi}v(s)) \Big| \\ &\leq \chi^*\lambda^* \Big\|u-v\Big\| \frac{(\varphi(s)-\varphi(0))^{\alpha}}{\Gamma(\alpha+1)} + \Omega^* \frac{(\varphi(s)-\varphi(0))^{\alpha}}{\Gamma(\alpha+1)} \sigma^* \sum_{i=1}^{p+1} \frac{(\varphi(s)-\varphi(0))^{\gamma_i}}{\Gamma(\gamma_i+1)} \|u-v\| \\ &\leq \left(\chi^*\lambda^* + \Omega^*\sigma^* \sum_{i=1}^{p+1} \frac{(\varphi(1)-\varphi(0))^{\gamma_i}}{\Gamma(\gamma_i+1)}\right) \frac{(\varphi(1)-\varphi(0))^{\alpha}}{\Gamma(\alpha+1)} \|u-v\|, \end{aligned}$$

thus we have

$$\begin{aligned} |Au(t) - Av(t)| &\leq \int_0^t \frac{(\varphi(t)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}\varphi'(s)G(s)ds + \frac{\varphi(t)-\varphi(0)}{\left|\sum\limits_{j=1}^{k} \delta_j(\varphi(\xi_j)-\varphi(0)) - (\varphi(1)-\varphi(0))\right|} \\ &\int_0^1 \frac{(\varphi(1)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}\varphi'(s)G(s)ds \\ &+ \frac{\varphi(t)-\varphi(0)}{\left|\sum\limits_{j=1}^{k} \delta_j(\varphi(\xi_j)-\varphi(0)) - (\varphi(1)-\varphi(0))\right|} \sum_{j=1}^{k} \delta_j \int_0^{\xi_j} \frac{(\varphi(\xi_j)-\varphi(s))^{\beta-1}}{\Gamma(\beta)}\varphi'(s)G(s)ds \\ &\leq \left(\chi^*\lambda^* + \Omega^*\sigma^* \sum_{i=1}^{p+1} \frac{(\varphi(1)-\varphi(0))^{\gamma_i}}{\Gamma(\gamma_i+1)}\right) \frac{(\varphi(1)-\varphi(0))^{\alpha}}{\Gamma(\alpha+1)} \\ &\left(1 + \frac{(\varphi(1)-\varphi(0))(1+\sum\limits_{j=1}^{k}\delta_j)}{\left|\sum\limits_{j=1}^{k} \delta_j(\varphi(\xi_j)-\varphi(0)) - (\varphi(1)-\varphi(0))\right|}\right) \frac{(\varphi(1)-\varphi(0))^{\beta}}{\Gamma(\beta+1)} \|u-v\|, \end{aligned}$$

which implies

$$\begin{aligned} \|Au(t) - Av(t)\| &\leq \Bigg[\left(\chi^*\lambda^* + \Omega^*\sigma^* \sum_{i=1}^{p+1} \frac{(\varphi(1)-\varphi(0))^{\gamma_i}}{\Gamma(\gamma_i+1)}\right) \frac{(\varphi(1)-\varphi(0))^{\alpha}}{\Gamma(\alpha+1)} \\ &\left(1 + \frac{(\varphi(1)-\varphi(0))(1+\sum\limits_{j=1}^{k}\delta_j)}{\left|\sum\limits_{j=1}^{k} \delta_j(\varphi(\xi_j)-\varphi(0)) - (\varphi(1)-\varphi(0))\right|}\right) \frac{(\varphi(1)-\varphi(0))^{\beta}}{\Gamma(\beta+1)}\Bigg] \|u-v\|, \end{aligned}$$

in view of (3.6), this shows that A is a contraction mapping.

Step 2. The operator B is compact and continuous on S.

First, we show that B is continuous on S. Let $\{u_n\}$ be a sequence of functions in S converging to a function $u \in S$. Then by Lebesgue dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} Bu_n(t) &= \lim_{n\to\infty}\Big[\sum_{i=1}^{m}\int_0^t \tfrac{(\varphi(t)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s) f_i(s,u_n(s),I^{\mu_1\,\varphi}u_n(s),\cdots,I^{\mu_n\,\varphi}u_n(s))ds \\
&+\frac{\varphi(t)-\varphi(0)}{\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))} \\
&\sum_{i=1}^{m}\int_0^1 \tfrac{(\varphi(1)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s) f_i(s,u_n(s),I^{\mu_1\,\varphi}u_n(s),\cdots,I^{\mu_n\,\varphi}u_n(s))ds \\
&-\frac{(\varphi(t)-\varphi(0))\sum_{j=1}^{k}\delta_j}{\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))} \\
&\sum_{i=1}^{m}\int_0^{\xi_j} \tfrac{(\varphi(\xi_j)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s) f_i(s,u_n(s),I^{\mu_1\,\varphi}u_n(s),\cdots,I^{\mu_n\,\varphi}u_n(s))ds\Big]. \\
&= \sum_{i=1}^{m}\int_0^t \tfrac{(\varphi(t)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s)\lim_{n\to\infty} f_i(s,u_n(s),I^{\mu_1\,\varphi}u_n(s),\cdots,I^{\mu_n\,\varphi}u_n(s))ds \\
&+\frac{\varphi(t)-\varphi(0)}{\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))} \\
&\sum_{i=1}^{m}\int_0^1 \tfrac{(\varphi(1)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s)\lim_{n\to\infty} f_i(s,u_n(s),I^{\mu_1\,\varphi}u_n(s),\cdots,I^{\mu_n\,\varphi}u_n(s))ds \\
&-\frac{(\varphi(t)-\varphi(0))\sum_{j=1}^{k}\delta_j}{\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))} \\
&\sum_{i=1}^{m}\int_0^{\xi_j} \tfrac{(\varphi(\xi_j)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s)\lim_{n\to\infty} f_i(s,u_n(s),I^{\mu_1\,\varphi}u_n(s),\cdots,I^{\mu_n\,\varphi}u_n(s))ds \\
&= \sum_{i=1}^{m}\int_0^t \tfrac{(\varphi(t)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s) f_i(s,u(s),I^{\mu_1\,\varphi}(s),\cdots,I^{\mu_n\,\varphi}u(s))ds \\
&+\frac{\varphi(t)-\varphi(0)}{\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))} \\
&\sum_{i=1}^{m}\int_0^1 \tfrac{(\varphi(1)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s) f_i(s,u(s),I^{\mu_1\,\varphi}u(s),\cdots,I^{\mu_n\,\varphi}u(s))ds \\
&-\frac{(\varphi(t)-\varphi(0))\sum_{j=1}^{k}\delta_j}{\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))} \\
&\sum_{i=1}^{m}\int_0^{\xi_j} \tfrac{(\varphi(\xi_j)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s) f_i(s,u(s),I^{\mu_1\,\varphi}u(s),\cdots,I^{\mu_n\,\varphi}u(s))ds = Bu(t).
\end{aligned}$$

This shows that B is continuous on S. It is sufficient to show that $B(S)$ is a uniformly bounded and equicontinuous set in E.

First, we note that

$$\begin{aligned}
|Bu(t)| &\leq \sum_{i=1}^{m}\int_0^t \tfrac{(\varphi(t)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s)|f_i(s,u(s),I^{\mu_1\ \varphi}u(s),\cdots,I^{\mu_n\ \varphi}u(s))|ds \\
&+\frac{\varphi(t)-\varphi(0)}{\left|\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))\right|} \\
&\sum_{i=1}^{m}\int_0^1 \tfrac{(\varphi(1)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s)|f_i(s,u(s),I^{\mu_1\ \varphi}u(s),\cdots,I^{\mu_n\ \varphi}u(s))|ds \\
&+\frac{(\varphi(t)-\varphi(0))\sum_{j=1}^{k}\delta_j}{\left|\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))\right|} \\
&\sum_{i=1}^{m}\int_0^{\xi_j} \tfrac{(\varphi(\xi_j)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s)|f_i(s,u(s),I^{\mu_1\ \varphi}u(s),\cdots,I^{\mu_n\ \varphi}u(s))|ds \\
&\leq \sum_{i=1}^{m}\phi_i^*\frac{(\varphi(1)-\varphi(0))^{\omega_i+\beta}}{\Gamma(\omega_i+\beta+1)}+\frac{(\varphi(1)-\varphi(0))(1+\sum_{j=1}^{k}\delta_j)}{\left|\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))\right|} \\
&\sum_{i=1}^{m}\phi_i^*\frac{(\varphi(1)-\varphi(0))^{\omega_i+\beta}}{\Gamma(\omega_i+\beta+1)} \\
&=\left(1+\frac{(\varphi(1)-\varphi(0))(1+\sum_{j=1}^{k}\delta_j)}{\left|\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))\right|}\right)\sum_{i=1}^{m}\phi_i^*\frac{(\varphi(1)-\varphi(0))^{\omega_i+\beta}}{\Gamma(\omega_i+\beta+1)}.
\end{aligned}$$

This shows that B is uniformly bounded on S.

Next, we show that B is an equicontinuous set in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $u \in S$. Then we have

$$
\begin{aligned}
|Bu(t_2)-Bu(t_1)| &= \Bigg|\sum_{i=1}^{m}\int_0^{t_2}\tfrac{(\varphi(t_2)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s)f_i(s,u(s),I^{\mu_1}{}^{\varphi}u(s),\cdots,I^{\mu_n}{}^{\varphi}u(s))ds\\
&-\sum_{i=1}^{m}\int_0^{t_1}\tfrac{(\varphi(t_1)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s)f_i(s,u(s),I^{\mu_1}{}^{\varphi}u(s),\cdots,I^{\mu_n}{}^{\varphi}u(s))ds\\
&+\frac{\varphi(t_2)-\varphi(t_1)}{\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))}\\
&\sum_{i=1}^{m}\int_0^{1}\tfrac{(\varphi(1)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s)f_i(s,u(s),I^{\mu_1}{}^{\varphi}u(s),\cdots,I^{\mu_n}{}^{\varphi}u(s))ds\\
&-\frac{(\varphi(t_2)-\varphi(t_1))\sum_{j=1}^{k}\delta_j}{\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))}\\
&\sum_{i=1}^{m}\int_0^{\xi_j}\tfrac{(\varphi(\xi_j)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)}\varphi'(s)f_i(s,u(s),I^{\mu_1}{}^{\varphi}u(s),\cdots,I^{\mu_n}{}^{\varphi}u(s))ds\Bigg|\\
&\le\sum_{i=1}^{m}\frac{\phi_i^*}{\Gamma(\omega_i+\beta)}\Bigg[\Bigg|\int_0^{t_1}[(\varphi(t_2)-\varphi(s))^{\omega_i+\beta-1}-(\varphi(t_1)-\varphi(s))^{\omega_i+\beta-1}]\varphi'(s)ds\\
&+\int_{t_1}^{t_2}[(\varphi(t_2)-\varphi(s))^{\omega_i+\beta-1}\varphi'(s)ds\Bigg|\\
&+\frac{\varphi(t_2)-\varphi(t_1)}{\left|\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))\right|}\int_0^1(\varphi(1)-\varphi(s))^{\omega_i+\beta-1}\varphi'(s)ds\\
&+\frac{(\varphi(t_2)-\varphi(t_1))\sum_{j=1}^{k}\delta_j}{\left|\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))\right|}\int_0^{\xi_j}(\varphi(\xi_j)-\varphi(s))^{\omega_i+\beta-1}\varphi'(s)ds\Bigg]\\
&\le\sum_{i=1}^{m}\frac{\phi_i^*}{\Gamma(\omega_i+\beta+1)}\Bigg[\Big|(\varphi(t_2)-\varphi(0))^{\omega_i+\beta}-(\varphi(t_1)-\varphi(0))^{\omega_i+\beta}\Big|\\
&+\frac{\varphi(t_2)-\varphi(t_1)}{\left|\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))\right|}(\varphi(1)-\varphi(0))^{\omega_i+\beta}\\
&+\frac{(\varphi(t_2)-\varphi(t_1))\sum_{j=1}^{k}\delta_j}{\left|\sum_{j=1}^{k}\delta_j(\varphi(\xi_j)-\varphi(0))-(\varphi(1)-\varphi(0))\right|}(\varphi(\xi_j)-\varphi(0))^{\omega_i+\beta}\Bigg].
\end{aligned}
$$

Let $h(t)=(\varphi(t)-\varphi(0))^{\omega_i+\beta}$. Then h is continuously differentiable function. Consequently, for all $t_1,t_2\in[0,1]$, without loss of generality, let $t_1<t_2$, then there exist positive constants M such that

$$|h(t_2)-h(t_1)|=|h'(\xi)||t_2-t_1|\le M|t_2-t_1|,\quad \xi\in(t_1,t_2).$$

On the other hand, for $\varphi\in C^{'}[0,1]$, thus there exist positive constants N such that $|\varphi(t_2)-\varphi(t_1)|=|\varphi'(\xi)||t_2-t_1|\le N|t_2-t_1|,\quad \xi\in(t_1,t_2)$, from which we deduce

$$|Bu(t_2)-Bu(t_1)|\to 0\quad\text{as}\ \ t_2-t_1\to 0.$$

Therefore, it follows from the Arzela-Ascoli theorem that B is a compact operator on S.

Step 3. Next we show that hypothesis (H_3) of Lemma 2.2 is satisfied. Let $v \in S$, then we have

$$
\begin{aligned}
|u(t)| &= |Au(t) + Bv(t)| \leq |Au(t)| + |Bv(t)| \\
&\leq \Bigg| \int_0^t \tfrac{(\varphi(t)-\varphi(s))^{\beta-1}}{\Gamma(\beta)} \varphi'(s) g(s, u(s), I^{\gamma_1\ \varphi} u(s), \cdots, I^{\gamma_p\ \varphi} u(s)) \int_0^s \tfrac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)} \varphi'(\tau) h(\tau, u(\tau)) d\tau ds \\
&+ \frac{\varphi(t) - \varphi(0)}{\sum_{j=1}^{k} \delta_j (\varphi(\xi_j) - \varphi(0)) - (\varphi(1) - \varphi(0))} \\
&\int_0^1 \tfrac{(\varphi(1)-\varphi(s))^{\beta-1}}{\Gamma(\beta)} \varphi'(s) g(s, u(s), I^{\gamma_1\ \varphi} u(s), \cdots, I^{\gamma_p\ \varphi} u(s)) \int_0^s \tfrac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)} \varphi'(\tau) h(\tau, u(\tau)) d\tau ds \\
&- \frac{(\varphi(t) - \varphi(0)) \sum_{j=1}^{k} \delta_j}{\sum_{j=1}^{k} \delta_j (\varphi(\xi_j) - \varphi(0)) - (\varphi(1) - \varphi(0))} \\
&\int_0^{\xi_j} \tfrac{(\varphi(\xi_j)-\varphi(s))^{\beta-1}}{\Gamma(\beta)} \varphi'(s) g(s, u(s), I^{\gamma_1\ \varphi} u(s), \cdots, I^{\gamma_p\ \varphi} u(s)) \int_0^s \tfrac{(\varphi(s)-\varphi(\tau))^{\alpha-1}}{\Gamma(\alpha)} \varphi'(\tau) h(\tau, u(\tau)) d\tau ds \Bigg| \\
&+ \Bigg| \sum_{i=1}^{m} \int_0^t \tfrac{(\varphi(t)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)} \varphi'(s) f_i(s, v(s), I^{\mu_1\ \varphi} v(s), \cdots, I^{\mu_n\ \varphi} v(s)) ds \\
&+ \frac{\varphi(t) - \varphi(0)}{\sum_{j=1}^{k} \delta_j (\varphi(\xi_j) - \varphi(0)) - (\varphi(1) - \varphi(0))} \\
&\sum_{i=1}^{m} \int_0^1 \tfrac{(\varphi(1)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)} \varphi'(s) f_i(s, v(s), I^{\mu_1\ \varphi} v(s), \cdots, I^{\mu_n\ \varphi} v(s)) ds \\
&- \frac{(\varphi(t) - \varphi(0)) \sum_{j=1}^{k} \delta_j}{\sum_{j=1}^{k} \delta_j (\varphi(\xi_j) - \varphi(0)) - (\varphi(1) - \varphi(0))} \\
&\sum_{i=1}^{m} \int_0^{\xi_j} \tfrac{(\varphi(\xi_j)-\varphi(s))^{\omega_i+\beta-1}}{\Gamma(\omega_i+\beta)} \varphi'(s) f_i(s, v(s), I^{\mu_1\ \varphi} v(s), \cdots, I^{\mu_n\ \varphi} v(s)) ds \Bigg| \\
&\leq \left(1 + \frac{(\varphi(1) - \varphi(0))(1 + \sum_{j=1}^{k} \delta_j)}{\left| \sum_{j=1}^{k} \delta_j (\varphi(\xi_j) - \varphi(0)) - (\varphi(1) - \varphi(0)) \right|} \right) \\
&\left(\chi^* \Omega^* \frac{(\varphi(1) - \varphi(0))^{\alpha}}{\Gamma(\alpha + 1)} \frac{(\varphi(1) - \varphi(0))^{\beta}}{\Gamma(\beta + 1)} + \sum_{i=1}^{m} \phi_i^* \frac{(\varphi(1) - \varphi(0))^{\omega_i+\beta}}{\Gamma(\omega_i + \beta + 1)} \right) \leq r,
\end{aligned}
$$

which implies $\|u\| \leq r$ and so $u \in S$.

Thus all the conditions of Lemma 2.2 are satisfied and hence the operator equation $u = Au + Bu$ has a solution in S. In consequence, the problem (1.1), (1.2) has a solution on J. This completes the proof. □

4. Example

In this section, we provide an example to illustrate our main result.

Example 4.1 Consider the following hybrid φ-Caputo fractional integro-differential equations

$$ {}^{C}D^{\frac{1}{2};\frac{t}{4}}\left(\frac{{}^{C}D^{\frac{3}{2};\frac{t}{4}}u(t)-\sum_{i=1}^{2}I^{\omega_i;\frac{t}{4}}f_i(t,u(t),I^{\frac{1}{3};\frac{t}{4}}u(t),I^{\frac{4}{3};\frac{t}{4}}u(t))}{\frac{1}{4}t^2\left(\frac{|u(t)|}{1+|u(t)|}+\frac{|I^{\frac{1}{4};\frac{t}{4}}u(t)|}{1+|I^{\frac{1}{4};\frac{t}{4}}u(t)|}+\sin I^{\frac{1}{2};\frac{t}{4}}u(t)\right)}\right)=\frac{2}{5}cos(\frac{t}{4})\left(\frac{|u(t)|}{|u(t)|+1}\right),\ \ t\in J=[0,1], \tag{4.1} $$

$$ u(0)=0,\ {}^{C}D^{\frac{3}{2};\frac{t}{4}}u(0)=0,\ u(1)=\frac{1}{3}u(\frac{1}{3}), \tag{4.2} $$

where

$$ \begin{aligned} &\sum_{i=1}^{2}I^{\omega_i;\frac{t}{4}}f_i(t,u(t),I^{\frac{1}{3};\frac{t}{4}}u(t),I^{\frac{4}{3};\frac{t}{4}}u(t))=I^{\frac{1}{3};\frac{t}{4}}\left(t\left[\frac{|u(t)|}{1+|u(t)|}+\sin(I^{\frac{1}{3};\frac{t}{4}}u(t))+\cos(I^{\frac{4}{3};\frac{t}{4}}u(t))\right]\right)\\ &+I^{\frac{2}{3};\frac{t}{4}}\left(\frac{t}{10}\left[\frac{|u(t)|}{1+|u(t)|}+\arctan(I^{\frac{1}{3};\frac{t}{4}}u(t))+\sin(I^{\frac{4}{3};\frac{t}{4}}u(t))\right]\right). \end{aligned} \tag{4.3} $$

We note that $\alpha=\frac{1}{2},\beta=\frac{3}{2},m=2,n=2,p=2,k=1,\delta=\frac{1}{3},\xi=\frac{1}{3},\omega_1=\frac{1}{3},\omega_2=\frac{2}{3},\mu_1=\frac{1}{3},\mu_2=\frac{4}{3},\gamma_1=\frac{1}{4},\gamma_2=\frac{1}{2},\varphi(t)=\frac{t}{4}$,

$$ f_1(t,u(t),I^{\frac{1}{3};\frac{t}{4}}u(t),I^{\frac{4}{3};\frac{t}{4}}u(t))=t\left[\frac{|u(t)|}{1+|u(t)|}+\sin(I^{\frac{1}{3};\frac{t}{4}}u(t))+\cos(I^{\frac{4}{3};\frac{t}{4}}u(t))\right], $$

$$ f_2(t,u(t),I^{\frac{1}{3};\frac{t}{4}}u(t),I^{\frac{4}{3};\frac{t}{4}}u(t))=\frac{t}{10}\left[\frac{|u(t)|}{1+|u(t)|}+\arctan(I^{\frac{1}{3};\frac{t}{4}}u(t))+\sin(I^{\frac{4}{3};\frac{t}{4}}u(t))\right], $$

$$ g(t,u(t),I^{\frac{1}{4};\frac{t}{4}}u(t),I^{\frac{1}{2};\frac{t}{4}}u(t))=\frac{1}{4}t^2\left(\frac{|u(t)|}{1+|u(t)|}+\frac{|I^{\frac{1}{4};\frac{t}{4}}u(t)|}{1+|I^{\frac{1}{4};\frac{t}{4}}u(t)|}+\sin I^{\frac{1}{2};\frac{t}{4}}u(t)\right), $$

$$ h(t,u(t))=\frac{2}{5}cos(\frac{t}{4})\left(\frac{|u(t)|}{|u(t)|+1}\right). $$

Thus we have

$$ \begin{aligned} &|g(t,u(t),I^{\frac{1}{4};\frac{t}{4}}u(t),I^{\frac{1}{2};\frac{t}{4}}u(t))-g(t,v(t),I^{\frac{1}{4};\frac{t}{4}}v(t),I^{\frac{1}{2};\frac{t}{4}}v(t))|\le\sigma(t)\left[1+\frac{t^{\frac{1}{4}}}{\Gamma(\frac{5}{4})}+\frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\right]|u(t)-v(t)|\\ &=\frac{t^2}{4}\left[1+\frac{t^{\frac{1}{4}}}{\Gamma(\frac{5}{4})}+\frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\right]|u(t)-v(t)|, \end{aligned} $$

$$ |h(t,u(t))-h(t,v(t))|=\frac{2}{5}\cos(\frac{t}{4})|u(t)-v(t)|. $$

Therefore,

$$\sigma^* = \sup_{0\le t\le 1} |\sigma(t)| = \sup_{0\le t\le 1} \frac{t^2}{4}\Big[1+\frac{t^{\frac{1}{4}}}{\Gamma(\frac{5}{4})}+\frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\Big] = \frac{1}{4}\Big(1+\frac{1}{\Gamma(\frac{5}{4})}+\frac{1}{\Gamma(\frac{3}{2})}\Big) = \frac{1}{4}\Big(1+\frac{1}{0.9064}+\frac{1}{0.8862}\Big) = 0.8079;$$

$$\lambda^* = \sup_{0\le t\le 1} |\lambda(t)| = \sup_{0\le t\le 1} \frac{2}{5}\cos(\frac{t}{4}) = 0.4;$$

$$\phi_1^* = \sup_{0\le t\le 1} |\phi_1(t)| = \sup_{0\le t\le 1} t(1+1+1) = 3;$$

$$\phi_2^* = \sup_{0\le t\le 1} |\phi_2(t)| = \sup_{0\le t\le 1} \frac{t}{10}(1+\frac{\pi}{2}+1) = \frac{1}{10}\times 3.57 = 0.357;$$

$$\Omega^* = \sup_{0\le t\le 1} |\Omega(t)| = \sup_{0\le t\le 1} \frac{2}{5}\cos(\frac{t}{4}) = 0.4;$$

$$\chi^* = \sup_{0\le t\le 1} |\chi(t)| = \sup_{0\le t\le 1} \frac{t^2}{4}(1+1+1) = \frac{3}{4} = 0.75.$$

Choose $r > 0.5$, then we have

$$\Big(1+\frac{\frac{1}{4}\times\frac{4}{3}}{\frac{2}{9}}\Big)\Big[0.75\times 0.4\times\frac{(\frac{1}{4})^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\times\frac{(\frac{1}{4})^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}+3\times\frac{(\frac{1}{4})^{\frac{11}{6}}}{\Gamma(\frac{17}{6})}+0.357\times\frac{(\frac{1}{4})^{\frac{13}{6}}}{\Gamma(\frac{19}{6})}\Big] = 0.4016 \le r.$$

Moreover,

$$\Big(0.75\times 0.4+0.4\times 0.8079\times\Big(\frac{(\frac{1}{4})^{\frac{1}{4}}}{\Gamma(\frac{5}{4})}+\frac{(\frac{1}{4})^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\Big)\Big)\frac{(\frac{1}{4})^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}\Big(1+\frac{\frac{1}{4}\times\frac{4}{3}}{\frac{2}{9}}\Big)\frac{(\frac{1}{4})^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} = 0.097 < 1.$$

Now, by using Theorem 3.2, it is deduced that the fractional hybrid integro-differential problem (4.1), (4.2) has a solution.

5. Conclusions

Hybrid fractional integro-differential equations have been considered more important and served as special cases of dynamical systems. In this paper, we introduced a new class of the hybrid φ-Caputo fractional integro-differential equations. By using famous hybrid fixed point theorem due to Dhage, we have developed adequate conditions for the existence of at least one solution to the hybrid problem (1.1), (1.2). The respective results have been verified by providing a suitable example.

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Conflict of interest

The authors declare no conflict of interest in this paper.

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