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Research article

Some New (p_1p_2, q_1q_2)-Estimates of Ostrowski-type integral inequalities via n-polynomials s-type convexity

Humaira Kalsoom[1], Muhammad Idrees[2], Artion Kashuri[3], Muhammad Uzair Awan[4] and Yu-Ming Chu[5,6,*]

[1] School of Mathematical Sciences, Zhejiang University, Hangzhou, 310027, China

[2] Zhejiang Province Key Laboratory of Quantum Technology and Device, Department of Physics, Zhejiang University, Hangzhou 310027, China

[3] Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali", Vlora, Albania

[4] Department of mathematics, Government University, Faisalabad, Pakistan

[5] Department of Mathematics, Huzhou University, Huzhou 313000, China

[6] Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science & Technology, Changsha 410114, P. R. China

* **Correspondence:** Email: chuyuming2005@126.com; Tel: +865722322189;
Fax: +865722321163.

Abstract: The purpose of this paper is to establish new generalization of Ostrowski type integral inequalities by using (p, q)-analogues which are related to the estimates of upper bound for a class of (p_1p_2, q_1q_2)-differentiable functions on co-ordinates. We first establish an integral identity for (p_1p_2, q_1q_2)-differentiable functions on co-ordinates. The result is then used to derive some estimates of upper bound for the functions whose twice partial (p_1p_2, q_1q_2)-differentiable functions are n-polynomial s-type convex functions on co-ordinates. Some new special cases from the main results are obtained and some known results are recaptured as well. At the end, an application to special means is given as well.

Keywords: quantum calculus; convex function; s-type convex functions; (p_1p_2, q_1q_2)-Ostrowski-type inequality on co-ordinates
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1. Introduction and preliminaries

Calculus is a branch of mathematics which helps us to study the derivatives and integrals. The classical derivative was convoluted with the strength regulation kind kernel and eventually, this gave upward thrust to new calculus referred to as the quantum calculus. Quantum calculus (named q-calculus) is the study of calculus without limits. In recent decades, the quantum calculus has become a powerful tool in numerous branches of mathematics and physics like q-calculus, particularly q-fractional calculus, q-integral calculus, q-transform analysis. Jackson [1] is the first researcher to define the q-analogue of derivative and integral operator as well as provided its applications. It is imperative to mention that quantum integral inequalities are more practical and informative than their classical counterparts. It has been mainly due to the fact that quantum integral inequalities can describe the hereditary properties of the processes and phenomena under investigation. Historically the subject of quantum calculus can be traced back to Euler and Jacobi, but in recent decades it has experienced a rapid development, see [2–7]. As a result, new generalizations of the classical concepts of quantum calculus have been initiated and reviewed in many literature. Tariboon and Ntouyas [8, 9] proposed the quantum calculus concepts on finite intervals and obtained several q-analogues of classical mathematical objects. This inspired other researchers to establish numerous novel results concerning quantum analogs of classical mathematical results. Noor et al. [10] provided the q-analogues of many known inequalities via the first order q-differentiable convex functions. Humaira et al. [11] obtained a new generalized q_1q_2-integral identity and established several new q_1q_2-analogues of first order q_1q_2-differentiable convex functions over finite rectangles. Wu et al. [12] gave a new corrected q-analogue of the classical Simpson inequality for preinvex function. Deng et al. [13] obtained a new generalized q-integral identity and found several new q-analogues for twice q-differentiable generalized (s, m)-preinvex functions.

The theory of post quantum calculus denoted by (p, q)-calculus is a natural generalization of the quantum calculus denoted by q-calculus, which has been studied extensively. Recently, Tunç and Göv [14] studied the concept of (p, q)-calculus on the intervals $[\chi_1, \chi_2] \subseteq \mathbb{R}$, defined the (p, q)-derivative and (p, q)-integral and established their basic properties and integral inequalities. Integral inequalities [15–27] play an important role in understanding the universe, and they can be used to find the uniqueness and existence of the linear and nonlinear differential equations. While convexity is an indispensable tool in the study of inequality theory [28–38].

It is well-known that the Hermite-Hadamard inequality [39–43] is one of the most important inequalities in the convex functions theory, which can be stated as follows.

Let $\mathcal{K} : \mathcal{I} \subset \mathbb{R} \to \mathbb{R}$ be a convex function. Then the double inequality

$$\mathcal{K}\left(\frac{\chi_1 + \chi_2}{2}\right) \leq \frac{1}{\chi_2 - \chi_1} \int_{\chi_1}^{\chi_2} \mathcal{K}(t)dt \leq \frac{\mathcal{K}(\chi_1) + \mathcal{K}(\chi_2)}{2} \tag{1.1}$$

holds for all $\chi_1, \chi_2 \in \mathcal{I}$ with $\chi_1 \neq \chi_2$.

As the generalization and refinement of the Hermite-Hadamard inequality (1.1), the Ostrowski inequality [44] can be stated in Theorem 1.1.

Theorem 1.1. *Let $\mathcal{K} : [\chi_1, \chi_2] \subseteq \mathbb{R} \to \mathbb{R}$ be continuous on $[\chi_1, \chi_2]$ and differentiable on (χ_1, χ_2) such* *that $|\mathcal{K}'(\tau)| \leq \mathcal{M}$ for all $\tau \in (\chi_1, \chi_2)$. Then the inequality*

$$\left|\mathcal{K}(\varrho) - \frac{1}{\chi_2 - \chi_1}\int_{\chi_1}^{\chi_2} \mathcal{K}(\tau)\, d\tau\right| \leq \left[\frac{1}{4} + \frac{\left(\varrho - \frac{\chi_1+\chi_2}{2}\right)^2}{(\chi_2-\chi_1)^2}\right](\chi_2 - \chi_1)\,\mathcal{M} \tag{1.2}$$

holds for all $\varrho \in [\chi_1, \chi_2]$ with the best possible constant $\frac{1}{4}$.

Inequality (1.2) can be rewritten in its equivalent form

$$\left|\mathcal{K}(\varrho) - \frac{1}{\chi_2 - \chi_1}\int_{\chi_1}^{\chi_2} \mathcal{K}(\tau)\, d\tau\right| \leq \frac{\mathcal{M}}{\chi_2 - \chi_1}\left[\frac{(\varrho-\chi_1)^2 + (\chi_2-\varrho)^2}{2}\right].$$

Recently, the generalizations, variants and applications of the Ostrowski inequality have attracted the attention of many researchers.

Now, we discuss some connections between the class of convex functions and s-type convex functions.

Definition 1.2. Let $s \in [0,1]$. Then the function $\mathcal{K} : \mathcal{I} \to \mathbb{R}$ is said to be a s-type convex function on $\mathcal{I}$ if the inequality

$$\mathcal{K}(\chi\varrho + (1-\chi)\rho) \leq [1 - s(1-\chi)]\mathcal{K}(\varrho) + [1 - s\chi]\mathcal{K}(\rho) \tag{1.3}$$

holds for all $\varrho, \rho \in \mathcal{I}$ and $\chi \in [0,1]$.

Remark 1. Definition 1.2 leads to the conclusion that
(1) If we choose $s = 1$, then we get classical convex function.
(2) If we take $s = 0$, then we have the definition of P-function [45].
(3) If $\mathcal{K}$ is s-type convex function on $\mathcal{I}$, then the range of $\mathcal{K}$ is $[0, \infty)$.

Indeed, let $\varrho \in \mathcal{I}$. Then it follows from the s-type convexity of $\mathcal{K}$ that

$$\mathcal{K}(\chi\eta_1 + (1-\chi)\varrho) \leq [1 - s(1-\chi)]\mathcal{K}(\eta_1) + [1 - s\chi]\mathcal{K}(\varrho)$$

for all $\eta_1 \in \mathcal{I}$ and $\chi \in [0,1]$. If we choose $\chi = 1$, then we get

$$\mathcal{K}(\eta_1) \leq \mathcal{K}(\eta_1) + (1-s)\mathcal{K}(\varrho)$$

$$\Rightarrow \quad (1-s)\mathcal{K}(\rho) \geq 0 \quad \Rightarrow \quad \mathcal{K}(\varrho) \geq 0$$

for all $\varrho \in \mathcal{I}$.

Proposition 1. *Every non-negative convex function is also s-type convex function.*

Proof. The proof is clearly due to

$$s(1-\chi) \leq (1-\chi), \quad \chi \geq s\chi$$

for all $\chi \in [0,1]$ and $s \in [0,1]$. □

Next, We recall the definition of n-polynomial s-type convex function.

Definition 1.3. Let $s \in [0,1]$ and $n \in \mathbb{N}$. Then the function $\mathcal{K} : \mathcal{I} \to \mathbb{R}$ is said to be a n-polynomial s-type convex function on $\mathcal{I}$ if the inequality

$$\mathcal{K}(\chi\varrho + (1-\chi)\rho) \le \frac{1}{n}\sum_{i=1}^{n}[1-(s(1-\chi))^i]\mathcal{K}(\varrho) + \frac{1}{n}\sum_{i=1}^{n}[1-(s\chi)^i]\mathcal{K}(\rho) \tag{1.4}$$

holds for $\varrho, \rho \in \mathcal{I}$ and $\chi \in [0,1]$.

Remark 2. From Definition 1.3, one has

(1) If we choose $s = 0$, then we get the P-function [45].

(2) If we take $s = 1$, then we obtain the Definition given in [46].

(3) If we choose $n = 1$, then we obtain Definition 1.2.

(4) If $\mathcal{K}$ is a n-polynomial s-type convex function, then the range of $\mathcal{K}$ is $[0,\infty)$.

Remark 3. Every non-negative n-polynomial convex function is also n-polynomial s-type convex function. Indeed

$$\frac{1}{n}\sum_{i=1}^{n}[1-(s(1-\chi))^i] \ge \frac{1}{n}\sum_{i=1}^{n}[1-(1-\chi)^i]$$

and

$$\frac{1}{n}\sum_{i=1}^{n}[1-\chi^i] \le \frac{1}{n}\sum_{i=1}^{n}[1-(s\chi)^i]$$

for all $\chi \in [0,1]$, $n \in \mathbb{N}$ and $s \in [0,1]$.

In what follows, we suppose that $\chi_1, \chi_2, \chi_3, \chi_4 \in \mathbb{R}$ with $\chi_1 < \chi_2$ and $\chi_3 < \chi_4$, $0 < q_k < p_k \le 1$ $(k = 1,2)$ are constants, $\mathcal{N} = [\chi_1,\chi_2] \times [\chi_3,\chi_4] \subseteq \mathbb{R}^2$ is a rectangle and $\mathcal{N}^\circ = (\chi_1,\chi_2)\times(\chi_3,\chi_4)$ is the interior of $\mathcal{N}$.

Definition 1.4. Let $\mathcal{K} : \mathcal{N} \to \mathbb{R}$ be a continuous function. Then the partial (p_1,q_1)-, (p_2,q_2)- and (p_1p_2, q_1q_2)-derivatives at $(z,w) \in \mathcal{N}$ are respectively defined by

$$\begin{aligned}
\frac{{}_{\chi_1}\partial_{p_1,q_1}\mathcal{K}(z,w)}{{}_{\chi_1}\partial_{p_1,q_1}z} &= \frac{\mathcal{K}(p_1z+(1-p_1)\chi_1,w) - \mathcal{K}(q_1z+(1-q_1)\chi_1,w)}{(p_1-q_1)(z-\chi_1)} \quad (z \ne \chi_1),\\
\frac{{}_{\chi_3}\partial_{p_2,q_2}\mathcal{K}(z,w)}{{}_{\chi_3}\partial_{p_2,q_2}w} &= \frac{\mathcal{K}(z,p_2w+(1-p_2)\chi_3) - \mathcal{K}(z,q_2w+(1-q_2)\chi_3)}{(p_2-q_2)(w-\chi_3)} \quad (w \ne \chi_3),\\
\frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w} &= \frac{1}{(p_1-q_1)(p_2-q_2)(z-\chi_1)(w-\chi_3)}[\mathcal{K}(q_1z+(1-q_1)\chi_1, q_2w+(1-q_2)\chi_3)\\
&\quad -\mathcal{K}(q_1z+(1-q_1)\chi_1, p_2w+(1-p_2)\chi_3) - \mathcal{K}(p_1z+(1-p_1)\chi_1, q_2w+(1-q_2)\chi_3)\\
&\quad +\mathcal{K}(p_1z+(1-p_1)\chi_1, p_2w+(1-p_2)\chi_3)].
\end{aligned}$$

Definition 1.5. Let $\mathcal{K} : \mathcal{N} \to \mathbb{R}$ be a continuous function. Then the definite (p_1p_2, q_1q_2)-integral on $\mathcal{N}$ is defined by

$$\begin{aligned}
\int_{\chi_3}^{t}\int_{\chi_1}^{s}\mathcal{K}(z,w)\,{}_{\chi_1}d_{p_1,q_1}z\,{}_{\chi_3}d_{p_2,q_2}w &= (p_1-q_1)(p_2-q_2)(s-\chi_1)(t-\chi_3)\\
\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q_1^nq_2^m}{p_1^{n+1}p_2^{m+1}}\mathcal{K}&\left(\frac{q_1^n}{p_1^{n+1}}s+\left(1-\frac{q_1^n}{p_1^{n+1}}\right)\chi_1, \frac{q_2^m}{p_2^{m+1}}t+\left(1-\frac{q_2^m}{p_2^{m+1}}\right)\chi_3\right)
\end{aligned} \tag{1.5}$$

for $(s,t) \in \mathcal{N}$.

Definition 1.6. For any real number n, the (p,q)-analogue is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p-q},$$

where $0 < q < p \leq 1$ are constants.

In the present paper, we introduce new Definitions 1.7 and 1.8 for (p_1p_2, q_1q_2)-differentiable function, and $(p,q)_{\chi_1}^{\chi_4}$, $(p,q)_{\chi_3}^{\chi_2}$ and $(p,q)^{\chi_2,\chi_4}$ integrals for two variables mappings over finite rectangles by using convex set. These new definitions will open new doors for convexity and (p,q)-calculus for two variables functions over the finite rectangles in the plane $\mathbb{R}^2$. We drive a key lemma for (p_1p_2, q_1q_2)-integral. By making use of new identity, we will obtain estimates of integral inequality whose twice partial (p_1p_2, q_1q_2)-derivatives in absolute value at certain powers are n-polynomial s-type convex function. We also discuss some new special cases of the main results. A briefly conclusion is given at the end.

Definition 1.7. Let $\mathcal{K} : \mathcal{N} \to \mathbb{R}$ be a continuous function. Then the partial (p_1,q_1)-, (p_2,q_2)- and (p_1p_2, q_1q_2)-derivatives at $(z,w) \in \mathcal{N}$ are respectively defined by

$$\frac{{}^{\chi_2}\partial_{p_1,q_1}\mathcal{K}(z,w)}{{}^{\chi_2}\partial_{p_1,q_1}z} = \frac{\mathcal{K}(p_1z+(1-p_1)\chi_2,w)-\mathcal{K}(q_1z+(1-q_1)\chi_2,w)}{(p_1-q_1)(\chi_2-z)} \quad (\chi_2 \neq z),$$

$$\frac{{}^{\chi_4}\partial_{p_2,q_2}\mathcal{K}(z,w)}{{}^{\chi_4}\partial_{p_2,q_2}w} = \frac{\mathcal{K}(z,p_2w+(1-p_2)\chi_4)-\mathcal{K}(z,q_2w+(1-q_2)\chi_4)}{(p_2-q_2)(\chi_4-w)} \quad (\chi_4 \neq w),$$

$$\begin{aligned}\frac{{}_{\chi_1}^{\chi_4}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w} &= \frac{1}{(p_1-q_1)(p_2-q_2)(z-\chi_1)(\chi_4-w)}\Big[\mathcal{K}(q_1z+(1-q_1)\chi_1, q_2w+(1-q_2)\chi_4)\\ &-\mathcal{K}(q_1z+(1-q_1)\chi_1, p_2w+(1-p_2)\chi_4)-\mathcal{K}(p_1z+(1-p_1)\chi_1, q_2w+(1-q_2)\chi_4)\\ &+\mathcal{K}(p_1z+(1-p_1)\chi_1, p_2w+(1-p_2)\chi_4)\Big] \quad (z\neq\chi_1, \quad \chi_4\neq w),\end{aligned}$$

$$\begin{aligned}\frac{{}_{\chi_3}^{\chi_2}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w} &= \frac{1}{(p_1-q_1)(p_2-q_2)(\chi_2-z)(w-\chi_3)}\Big[\mathcal{K}(q_1z+(1-q_1)\chi_2, q_2w+(1-q_2)\chi_3)\\ &-\mathcal{K}(q_1z+(1-q_1)\chi_2, p_2w+(1-p_2)\chi_3)-\mathcal{K}(p_1z+(1-p_1)\chi_2, q_2w+(1-q_2)\chi_3)\\ &+\mathcal{K}(p_1z+(1-p_1)\chi_2, p_2w+(1-p_2)\chi_3)\Big] \quad (\chi_2\neq z, \quad w\neq\chi_3),\end{aligned}$$

$$\begin{aligned}\frac{{}^{\chi_2,\chi_4}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w} &= \frac{1}{(p_1-q_1)(p_2-q_2)(\chi_2-z)(\chi_4-w)}\Big[\mathcal{K}(q_1z+(1-q_1)\chi_2, q_2w+(1-q_2)\chi_4)\\ &-\mathcal{K}(q_1z+(1-q_1)\chi_2, p_2w+(1-p_2)\chi_4)-\mathcal{K}(p_1z+(1-p_1)\chi_2, q_2w+(1-q_2)\chi_4)\end{aligned}$$

$$+\mathcal{K}(p_1 z + (1-p_1)\chi_2, p_2 w + (1-p_2)\chi_4)\Big] \quad (\chi_2 \neq z, \quad \chi_4 \neq w).$$

Definition 1.8. Let $\mathcal{K} : \mathcal{N} \to \mathbb{R}$ be a continuous function. Then the definite $(p,q)^{\chi_4}_{\chi_1}$, $(p,q)^{\chi_2}_{\chi_3}$ and $(p,q)^{\chi_2,\chi_4}$ integrals on $[\chi_1,\chi_2] \times [\chi_3,\chi_4]$ are respectively defined by

$$\int_t^{\chi_4}\int_{\chi_1}^{s} \mathcal{K}(z,w)\ {}_{\chi_1}d_{p_1,q_1}z\ {}_{\chi_3}d_{p_2,q_2}w = (p_1-q_1)(p_2-q_2)(s-\chi_1)(\chi_4-t)$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{q_1^n q_2^m}{p_1^{n+1}p_2^{m+1}} \mathcal{K}\left(\frac{q_1^n}{p_1^{n+1}}s + \left(1-\frac{q_1^n}{p_1^{n+1}}\right)\chi_1, \frac{q_2^m}{p_2^{m+1}}t + \left(1-\frac{q_2^m}{p_2^{m+1}}\right)\chi_4\right),$$

$$\int_{\chi_3}^{t}\int_{s}^{\chi_2} \mathcal{K}(z,w)\ {}^{\chi_2}d_{p_1,q_1}z\ {}_{\chi_3}d_{p_2,q_2}w\,(p_1-q_1)(p_2-q_2)(\chi_2-s)(t-\chi_3)$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{q_1^n q_2^m}{p_1^{n+1}p_2^{m+1}} \mathcal{K}\left(\frac{q_1^n}{p_1^{n+1}}s + \left(1-\frac{q_1^n}{p_1^{n+1}}\right)\chi_2, \frac{q_2^m}{p_2^{m+1}}t + \left(1-\frac{q_2^m}{p_2^{m+1}}\right)\chi_3\right)$$

and

$$\int_t^{\chi_4}\int_{s}^{\chi_2} \mathcal{K}(z,w)\ {}^{\chi_2}d_{p_1,q_1}z\ {}^{\chi_4}d_{p_2,q_2}w\,(p_1-q_1)(p_2-q_2)(\chi_2-s)(\chi_4-t)$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{q_1^n q_2^m}{p_1^{n+1}p_2^{m+1}} \mathcal{K}\left(\frac{q_1^n}{p_1^{n+1}}s + \left(1-\frac{q_1^n}{p_1^{n+1}}\right)\chi_2, \frac{q_2^m}{p_2^{m+1}}t + \left(1-\frac{q_2^m}{p_2^{m+1}}\right)\chi_4\right)$$

for $(s,t) \in [\chi_1,\chi_2] \times [\chi_3,\chi_4]$.

2. A key lemma

Lemma 2.1. *Let $\mathcal{K} : \mathcal{N} \to \mathbb{R}$ be a twice partial (p_1p_2, q_1q_2)-differentiable function on $\mathcal{N}^{\circ}$ such that the partial (p_1p_2, q_1q_2)-derivatives $\frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}_{\chi_1}\partial_{p_1,q_1}z\ {}_{\chi_3}\partial_{p_2,q_2}w}$, $\frac{{}^{\chi_4}_{\chi_1}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}_{\chi_1}\partial_{p_1,q_1}z\ {}^{\chi_4}\partial_{p_2,q_2}w}$, $\frac{{}^{\chi_2}_{\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}^{\chi_2}\partial_{p_1,q_1}z\ {}_{\chi_3}\partial_{p_2,q_2}w}$ and $\frac{{}^{\chi_2,\chi_4}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}^{\chi_2}\partial_{p_1,q_1}z\ {}^{\chi_4}\partial_{p_2,q_2}w}$ are continuous and integrable on $\mathcal{N}$. Then one has the equality*

$$\mathcal{K}(\varrho,\rho) - \frac{1}{p_1(\chi_2-\chi_1)}\left[\int_{\chi_1}^{p_1\varrho+(1-p_1)\chi_1} \mathcal{K}(u,\rho)\ {}_{\chi_1}d_{p_1,q_1}u + \int_{p_1\varrho+(1-p_1)\chi_2}^{\chi_2} \mathcal{K}(u,\rho)\ {}^{\chi_2}d_{p_1,q_1}u\right]$$
$$- \frac{1}{p_2(\chi_4-\chi_3)}\left[\int_{\chi_3}^{p_2\rho+(1-p_2)\chi_3} \mathcal{K}(\varrho,v)\ {}_{\chi_3}d_{p_2,q_2}v + \int_{p_2\rho+(1-p_2)\chi_4}^{\chi_4} \mathcal{K}(\varrho,v)\ {}^{\chi_4}d_{p_2,q_2}v\right] + \mathcal{A}$$
$$= \Delta\Big\{(\varrho-\chi_1)^2(\rho-\chi_3)^2 \int_0^1\int_0^1 zw \frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_1, w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z\ {}_{\chi_3}\partial_{p_2,q_2}w}\ {}_0d_{p_1,q_1}z\ {}_0d_{p_2,q_2}w$$

$$+ (\varrho-\chi_1)^2(\chi_4-\rho)^2 \int_0^1 \int_0^1 zw \frac{{}_{\chi_1}^{\chi_4}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_1, w\rho+(1-w)\chi_4)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w}\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w$$
$$+ (\chi_2-\varrho)^2(\rho-\chi_3)^2 \int_0^1 \int_0^1 zw \frac{{}_{\chi_3}^{\chi_2}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_2, w\rho+(1-w)\chi_3)}{{}_{\chi_3}\partial_{p_1,q_1}z\,{}^{\chi_2}\partial_{p_2,q_2}w}\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w$$
$$+ (\chi_2-\varrho)^2(\chi_4-\rho)^2 \int_0^1 \int_0^1 zw \frac{{}^{\chi_2,\chi_4}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_2, w\rho+(1-w)\chi_4)}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w}\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w\Big\}, \tag{2.1}$$

where

$$\begin{aligned}
\mathcal{A} &= \frac{1}{p_1p_2(\chi_2-\chi_1)(\chi_4-\chi_3)} \int_{\chi_1}^{p_1\varrho+(1-p_1)\chi_1} \int_{\chi_3}^{p_2\rho+(1-p_2)\chi_3} \mathcal{K}(u,v)\,{}_{\chi_1}d_{p_1,q_1}u\,{}_{\chi_3}d_{p_2,q_2}v \\
&+\frac{1}{p_1p_2(\chi_2-\chi_1)(\chi_4-\chi_3)} \int_{\chi_1}^{p_1\varrho+(1-p_1)\chi_1} \int_{p_2\rho+(1-p_2)\chi_4}^{\chi_4} \mathcal{K}(u,v)\,{}_{\chi_1}d_{p_1,q_1}u\,{}^{\chi_4}d_{p_2,q_2}v \\
&+\frac{1}{p_1p_2(\chi_2-\chi_1)(\chi_4-\chi_3)} \int_{p_1\varrho+(1-p_1)\chi_2}^{\chi_2} \int_{\chi_3}^{p_2\rho+(1-p_2)\chi_3} \mathcal{K}(u,v)\,{}^{\chi_2}d_{p_1,q_1}u\,{}_{\chi_3}d_{p_2,q_2}v \\
&+\frac{1}{p_1p_2(\chi_2-\chi_1)(\chi_4-\chi_3)} \int_{p_1\varrho+(1-p_1)\chi_2}^{\chi_2} \int_{p_2\rho+(1-p_2)\chi_4}^{\chi_4} \mathcal{K}(u,v)\,{}^{\chi_2}d_{p_1,q_1}u\,{}^{\chi_4}d_{p_2,q_2}v
\end{aligned}$$

for all $\varrho,\rho \in \mathcal{N}$ and $\Delta = \frac{q_1q_2}{(\chi_2-\chi_1)(\chi_4-\chi_3)}$.

Proof. We consider the integral

$$\int_0^1 \int_0^1 zw \frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_1, w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w$$
.

By the definition of partial (p_1p_2, q_1q_2)-derivative and definite $(p_1p_2, q_1q_2)_{\chi_1,\chi_3}$-integral, we have

$$\begin{aligned}
&\int_0^1 \int_0^1 zw \frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_1, w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w \\
&= \frac{1}{(1-q_1)(1-q_2)(\varrho-\chi_1)(\rho-\chi_3)} \\
&\times\Big[\int_0^1 \int_0^1 \mathcal{K}(zq_1\varrho+(1-zq_1)\chi_1, wq_2\rho+(1-wq_2)\chi_3)\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w \\
&-\int_0^1 \int_0^1 \mathcal{K}(zq_1\varrho+(1-zq_1)\chi_1, wp_2\rho+(1-wp_2)\chi_3)\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w \\
&-\int_0^1 \int_0^1 \mathcal{K}(zp_1\varrho+(1-zp_1)\chi_1, wq_2\rho+(1-wq_2)\chi_3)\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w \\
&+\int_0^1 \int_0^1 \mathcal{K}(zp_1\varrho+(1-zp_1)\chi_1, wp_2\rho+(1-wp_2)\chi_3)\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w\Big] \\
&= \frac{1}{(\varrho-\chi_1)(\rho-\chi_3)}
\end{aligned}$$

$$\times\Big[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n+1}p_2^{m+1}}\mathcal{K}\left(\frac{q_1^{n+1}}{p_1^{n+1}}\varrho+\left(1-\frac{q_1^{n+1}}{p_1^{n+1}}\right)\chi_1,\frac{q_2^{m+1}}{p_2^{m+1}}\rho+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)\chi_3\right)$$
$$-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n+1}p_2^{m+1}}\mathcal{K}\left(\frac{q_1^{n+1}}{p_1^{n+1}}\varrho+\left(1-\frac{q_1^{n+1}}{p_1^{n+1}}\right)\chi_1,\frac{q_2^{m}}{p_2^{m}}\rho+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)\chi_3\right)$$
$$-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n+1}p_2^{m+1}}\mathcal{K}\left(\frac{q_1^{n}}{p_1^{n+1}}\varrho+\left(1-\frac{q_1^{n}}{p_1^{n+1}}\right)\chi_1,\frac{q_2^{m+1}}{p_2^{m+1}}\rho+\left(1-\frac{q_2^{m+1}}{p_2^{m+1}}\right)\chi_3\right)$$
$$+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n+1}p_2^{m+1}}\mathcal{K}\left(\frac{q_1^{n}}{p_1^{n}}\varrho+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)\chi_1,\frac{q_2^{m}}{p_2^{m}}\rho+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)\chi_3\right)\Big]$$
$$=\frac{1}{q_1q_2(\varrho-\chi_1)(\rho-\chi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q_1^n q_2^m}{p_1^{n}p_2^{m}}\mathcal{K}\left(\frac{q_1^{n}}{p_1^{n}}\varrho+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)\chi_1,\frac{q_2^{m}}{p_2^{m}}\rho+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)\chi_3\right)$$
$$-\frac{1}{p_2q_1(\varrho-\chi_1)(\rho-\chi_3)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n}p_2^{m}}\mathcal{K}\left(\frac{q_1^{n}}{p_1^{n}}\varrho+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)\chi_1,\frac{q_2^{m}}{p_2^{m}}\rho+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)\chi_3\right)$$
$$-\frac{1}{p_1q_2(\varrho-\chi_1)(\rho-\chi_3)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{q_1^n q_2^m}{p_1^{n}p_2^{m}}\mathcal{K}\left(\frac{q_1^{n}}{p_1^{n}}\varrho+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)\chi_1,\frac{q_2^{m}}{p_2^{m}}\rho+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)\chi_3\right)$$
$$+\frac{1}{p_1p_2(\varrho-\chi_1)(\rho-\chi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n}p_2^{m}}\mathcal{K}\left(\frac{q_1^{n}}{p_1^{n}}\varrho+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)\chi_1,\frac{q_2^{m}}{p_2^{m}}\rho+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)\chi_3\right).\tag{2.2}$$

Note that

$$\frac{1}{q_1q_2(\varrho-\chi_1)(\rho-\chi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q_1^n q_2^m}{p_1^{n}p_2^{m}}\mathcal{K}\left(\frac{q_1^{n}}{p_1^{n}}\varrho+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)\chi_1,\frac{q_2^{m}}{p_2^{m}}\rho+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)\chi_3\right)$$
$$=-\frac{\mathcal{K}(\varrho,\rho)}{q_1q_2(\varrho-\chi_1)(\rho-\chi_3)}-\frac{1}{q_1q_2(\varrho-\chi_1)(\rho-\chi_3)}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^n}\mathcal{K}\left(\frac{q_1^{n}}{p_1^{n}}\varrho+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)\chi_1,\rho\right)$$
$$-\frac{1}{q_1q_2(\varrho-\chi_1)(\rho-\chi_3)}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}\mathcal{K}\left(\varrho,\frac{q_2^{m}}{p_2^{m}}\rho+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)\chi_3\right)+\frac{1}{q_1q_2(\varrho-\chi_1)(\rho-\chi_3)}$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n}p_2^{m}}\mathcal{K}\left(\frac{q_1^{n}}{p_1^{n}}\varrho+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)\chi_1,\frac{q_2^{m}}{p_2^{m}}\rho+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)\chi_3\right),\tag{2.3}$$

$$-\frac{1}{p_2q_1(\varrho-\chi_1)(\rho-\chi_3)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n}p_2^{m}}\mathcal{K}\left(\frac{q_1^{n}}{p_1^{n}}\varrho+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)\chi_1,\frac{q_2^{m}}{p_2^{m}}\rho+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)\chi_3\right)$$
$$=\frac{1}{p_2q_1(\varrho-\chi_1)(\rho-\chi_3)}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}\mathcal{K}\left(\varrho,\frac{q_2^{m}}{p_2^{m}}\rho+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)\chi_3\right)-\frac{1}{p_2q_1(\varrho-\chi_1)(\rho-\chi_3)}$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n q_2^m}{p_1^{n}p_2^{m}}\mathcal{K}\left(\frac{q_1^{n}}{p_1^{n}}\varrho+\left(1-\frac{q_1^{n}}{p_1^{n}}\right)\chi_1,\frac{q_2^{m}}{p_2^{m}}\rho+\left(1-\frac{q_2^{m}}{p_2^{m}}\right)\chi_3\right)\tag{2.4}$$

and

$$
\begin{aligned}
&-\frac{1}{p_1q_2(\varrho-\chi_1)(\rho-\chi_3)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{q_1^nq_2^m}{p_1^np_2^m}\mathcal{K}\left(\frac{q_1^n}{p_1^n}\varrho+\left(1-\frac{q_1^n}{p_1^n}\right)\chi_1,\frac{q_2^m}{p_2^m}\rho+\left(1-\frac{q_2^m}{p_2^m}\right)\chi_3\right)\\
&=\frac{1}{p_1q_2(\varrho-\chi_1)(\rho-\chi_3)}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^n}\mathcal{K}\left(\frac{q_1^n}{p_1^n}\varrho+\left(1-\frac{q_1^n}{p_1^n}\right)\chi_1,\rho\right)-\frac{1}{p_1q_2(\varrho-\chi_1)(\rho-\chi_3)}\\
&\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^nq_2^m}{p_1^np_2^m}\mathcal{K}\left(\frac{q_1^n}{p_1^n}\varrho+\left(1-\frac{q_1^n}{p_1^n}\right)\chi_1,\frac{q_2^m}{p_2^m}\rho+\left(1-\frac{q_2^m}{p_2^m}\right)\chi_3\right). \qquad (2.5)
\end{aligned}
$$

Utilizing (2.3)–(2.5) in (2.2), we get

$$
\begin{aligned}
&\int_0^1\int_0^1 zw\frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\mathcal{K}(z\varrho+(1-z)\chi_1,w\rho+(1-w)\chi_3)\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w\\
&=-\frac{\mathcal{K}(\varrho,\rho)}{q_1q_2(\varrho-\chi_1)(\rho-\chi_3)}-\frac{(p_2-q_2)(\rho-\chi_3)}{p_2q_1q_2(\varrho-\chi_1)(\rho-\chi_3)^2}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}\mathcal{K}\left(\varrho,\frac{q_2^m}{p_2^m}\rho+\left(1-\frac{q_2^m}{p_2^m}\right)\chi_3\right)\\
&-\frac{(p_1-q_1)(\varrho-\chi_1)}{p_1q_1q_2(\varrho-\chi_1)^2(\rho-\chi_3)}\sum_{n=0}^{\infty}\frac{q_1^n}{p_2^m}\mathcal{K}\left(\frac{q_1^n}{p_1^n}\varrho+\left(1-\frac{q_1^n}{p_1^n}\right)\chi_1,\rho\right)\\
&+\frac{(p_1-q_1)(p_2-q_2)(\varrho-\chi_1)(\rho-\chi_3)}{p_1p_2q_1q_2(\varrho-\chi_1)^2(\rho-\chi_3)^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^nq_2^m}{p_1^np_2^m}\mathcal{K}\left(\frac{q_1^n}{p_1^n}\varrho+\left(1-\frac{q_1^n}{p_1^n}\right)\chi_1,\frac{q_2^m}{p_2^m}\rho+\left(1-\frac{q_2^m}{p_2^m}\right)\chi_3\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1\int_0^1 zw\frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\mathcal{K}(z\varrho+(1-z)\chi_1,w\rho+(1-w)\chi_3)\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w\\
&=-\frac{\mathcal{K}(\varrho,\rho)}{q_1q_2(\varrho-\chi_1)(\rho-\chi_3)}-\frac{1}{p_2q_1q_2(\varrho-\chi_1)(\rho-\chi_3)^2}\int_{\chi_3}^{p_2\rho+(1-p_2)\chi_3}\mathcal{K}(\varrho,v)_0d_{p_2,q_2}v\\
&-\frac{1}{p_1q_1q_2(\varrho-\chi_1)^2(\rho-\chi_3)}\int_{\chi_1}^{p_1\varrho+(1-p_1)\chi_1}\mathcal{K}(u,\rho)_0d_{p_1,q_1}u\\
&+\frac{1}{p_1p_2q_1q_2(\varrho-\chi_1)^2(\rho-\chi_3)^2}\int_{\chi_1}^{p_1\varrho+(1-p_1)\chi_1}\int_{\chi_3}^{p_2\rho+(1-p_2)\chi_3}\mathcal{K}(u,v)_0d_{p_1,q_1}u_0d_{p_2,q_2}v. \qquad (2.6)
\end{aligned}
$$

Multiplying both sides of equality (2.6) by $\Delta(\varrho-\chi_1)^2(\rho-\chi_3)^2$ leads to

$$
\begin{aligned}
&\Delta(\varrho-\chi_1)^2(\rho-\chi_3)^2\int_0^1\int_0^1 zw\frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\mathcal{K}(z\varrho+(1-z)\chi_1,w\rho+(1-w)\chi_3)\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w\\
&=-\frac{(\varrho-\chi_1)(\rho-\chi_3)}{(\chi_2-\chi_1)(\chi_4-\chi_3)}\mathcal{K}(\varrho,\rho)-\frac{\varrho-\chi_1}{p_2(\chi_2-\chi_1)(\chi_4-\chi_3)}\int_{\chi_3}^{p_2\rho+(1-p_2)\chi_3}\mathcal{K}(\varrho,v)_0d_{p_2,q_2}v
\end{aligned}
$$

$$-\frac{\rho-\chi_3}{p_1(\chi_2-\chi_1)(\chi_4-\chi_3))}\int_{\chi_1}^{p_1\varrho+(1-p_1)\chi_1}\mathcal{K}(u,\rho)_0d_{p_1,q_1}u$$

$$+\frac{1}{p_1p_2(\chi_2-\chi_1)(\chi_4-\chi_3)}\int_{\chi_1}^{p_1\varrho+(1-p_1)\chi_1}\int_{\chi_3}^{p_2\rho+(1-p_2)\chi_3}\mathcal{K}(u,v)_0d_{p_1,q_1}u_0d_{p_2,q_2}v. \tag{2.7}$$

Similarly, calculating the remaining integrals and by using definition 1.8 we get

$$\Delta(\varrho-\chi_1)^2(\chi_4-\rho)^2\int_0^1\int_0^1 zw\frac{{}_{\chi_1}^{\chi_4}\partial^2_{p_1p_2,q_1q_2}}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w}\mathcal{K}(z\varrho+(1-z)\chi_1,w\rho+(1-w)\chi_4)\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w$$

$$=-\frac{(\varrho-\chi_1)(\chi_4-\rho)}{(\chi_2-\chi_1)(\chi_4-\chi_3)}\mathcal{K}(\varrho,\rho)-\frac{\varrho-\chi_1}{p_2(\chi_2-\chi_1)(\chi_4-\chi_3)}\int_{p_2\rho+(1-p_2)\chi_4}^{\chi_4}\mathcal{K}(\varrho,v)\,{}^{\chi_4}d_{p_2,q_2}v$$

$$-\frac{\chi_4-\rho}{p_1(\chi_2-\chi_1)(\chi_4-\chi_3))}\int_{\chi_1}^{p_1\varrho+(1-p_1)\chi_1}\mathcal{K}(u,\rho)\,{}_{\chi_1}d_{p_1,q_1}u$$

$$+\frac{1}{p_1p_2(\chi_2-\chi_1)(\chi_4-\chi_3)}\int_{\chi_1}^{p_1\varrho+(1-p_1)\chi_1}\int_{p_2\rho+(1-p_2)\chi_4}^{\chi_4}\mathcal{K}(u,v)\,{}_{\chi_1}d_{p_1,q_1}u\,{}^{\chi_4}d_{p_2,q_2}v, \tag{2.8}$$

$$\Delta(\chi_2-\varrho)^2(\rho-\chi_3)^2\int_0^1\int_0^1 zw\frac{{}_{\chi_3}^{\chi_2}\partial^2_{p_1p_2,q_1q_2}}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\mathcal{K}(z\varrho+(1-z)\chi_1,w\rho+(1-w)\chi_3)\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w$$

$$=-\frac{(\chi_2-\varrho)(\rho-\chi_3)}{(\chi_2-\chi_1)(\chi_4-\chi_3)}\mathcal{K}(\varrho,\rho)-\frac{\chi_2-\varrho}{p_2(\chi_2-\chi_1)(\chi_4-\chi_3)}\int_{\chi_3}^{p_2\rho+(1-p_2)\chi_3}\mathcal{K}(\varrho,v)\,{}_{\chi_3}d_{p_2,q_2}v$$

$$-\frac{\rho-\chi_3}{p_1(\chi_2-\chi_1)(\chi_4-\chi_3))}\int_{p_1\varrho+(1-p_1)\chi_2}^{\chi_2}\mathcal{K}(u,\rho)\,{}^{\chi_2}d_{p_1,q_1}u$$

$$+\frac{1}{p_1p_2(\chi_2-\chi_1)(\chi_4-\chi_3)}\int_{\chi_2}^{p_1\varrho+(1-p_1)\chi_1}\int_{p_2\rho+(1-p_2)\chi_2}^{\chi_2}\mathcal{K}(u,v)\,{}^{\chi_2}d_{p_1,q_1}u\,{}_{\chi_3}d_{p_2,q_2}v, \tag{2.9}$$

$$\Delta(\chi_2-\varrho)^2(\chi_4-\rho)^2\int_0^1\int_0^1 zw\frac{{}^{\chi_2,\chi_4}\partial^2_{p_1p_2,q_1q_2}}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w}\mathcal{K}(z\varrho+(1-z)\chi_2,w\rho+(1-w)\chi_4)\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w$$

$$=-\frac{(\chi_2-\varrho)(\chi_4-\rho)}{(\chi_2-\chi_1)(\chi_4-\chi_3)}\mathcal{K}(\varrho,\rho)-\frac{\chi_2-\varrho}{p_2(\chi_2-\chi_1)(\chi_4-\chi_3)}\int_{p_2\rho+(1-p_2)\chi_4}^{\chi_4}\mathcal{K}(\varrho,v)\,{}^{\chi_4}d_{p_2,q_2}v$$

$$-\frac{\chi_4-\rho}{p_1(\chi_2-\chi_1)(\chi_4-\chi_3))}\int_{p_1\varrho+(1-p_1)\chi_2}^{\chi_2}\mathcal{K}(u,\rho)\,{}^{\chi_2}d_{p_1,q_1}u$$

$$+\frac{1}{p_1p_2(\chi_2-\chi_1)(\chi_4-\chi_3)}\int_{p_1\varrho+(1-p_1)\chi_2}^{\chi_4}\int_{p_2\rho+(1-p_2)\chi_4}^{\chi_4}\mathcal{K}(u,v)\,{}^{\chi_2}d_{p_1,q_1}u\,{}^{\chi_4}d_{p_2,q_2}v. \tag{2.10}$$

From (2.7)–(2.10) and (2.1), we derive the desired result of Lemma 2.1. □

Remark 4. Taking $p_k = 1$ for $k = 1, 2$ we get

$$
\begin{aligned}
&\mathcal{K}(\varrho,\rho) - \frac{1}{(\chi_2-\chi_1)}\left[\int_{\chi_1}^{\varrho}\mathcal{K}(u,\rho)\,{}_{\chi_1}d_{q_1}u + \int_{\varrho}^{\chi_2}\mathcal{K}(u,\rho)\,{}^{\chi_2}d_{q_1}u\right]\\
&-\frac{1}{(\chi_4-\chi_3)}\left[\int_{\chi_3}^{\rho}\mathcal{K}(\varrho,v)\,{}_{\chi_3}d_{q_2}v + \int_{\rho}^{\chi_4}\mathcal{K}(\varrho,v)\,{}^{\chi_4}d_{q_2}v\right] + \mathcal{W}\\
&= \Delta\Big\{(\varrho-\chi_1)^2(\rho-\chi_3)^2\int_0^1\int_0^1 zw\frac{{}_{\chi_1,\chi_3}\partial^2_{q_1,q_2}\mathcal{K}(z\varrho+(1-z)\chi_1, w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{q_1}z\,{}_{\chi_3}\partial_{q_2}w}\,{}_0d_{q_1}z\,{}_0d_{q_2}w\\
&+(\varrho-\chi_1)^2(\chi_4-\rho)^2\int_0^1\int_0^1 zw\frac{{}^{\chi_4}_{\chi_1}\partial^2_{q_1,q_2}\mathcal{K}(z\varrho+(1-z)\chi_1, w\rho+(1-w)\chi_4)}{{}_{\chi_1}\partial_{q_1}z\,{}^{\chi_4}\partial_{q_2}w}\,{}_0d_{q_1}z\,{}_0d_{q_2}w\\
&+(\chi_2-\varrho)^2(\rho-\chi_3)^2\int_0^1\int_0^1 zw\frac{{}^{\chi_2}_{\chi_3}\partial^2_{q_1,q_2}\mathcal{K}(z\varrho+(1-z)\chi_2, w\rho+(1-w)\chi_3)}{{}_{\chi_3}\partial_{q_1}z\,{}^{\chi_2}\partial_{q_2}w}\,{}_0d_{q_1}z\,{}_0d_{q_2}w\\
&+(\chi_2-\varrho)^2(\chi_4-\rho)^2\int_0^1\int_0^1 zw\frac{{}^{\chi_2,\chi_4}\partial^2_{q_1,q_2}\mathcal{K}(z\varrho+(1-z)\chi_2, w\rho+(1-w)\chi_4)}{{}^{\chi_2}\partial_{q_1}z\,{}^{\chi_4}\partial_{q_2}w}\,{}_0d_{q_1}z\,{}_0d_{q_2}w\Big\}, \qquad (2.11)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{W} \quad &= \frac{1}{(\chi_2-\chi_1)(\chi_4-\chi_3)}\int_{\chi_1}^{\varrho}\int_{\chi_3}^{\rho}\mathcal{K}(u,v)\,{}_{\chi_1}d_{q_1}u\,{}_{\chi_3}d_{q_2}v\\
&+\frac{1}{(\chi_2-\chi_1)(\chi_4-\chi_3)}\int_{\chi_1}^{\varrho}\int_{\rho}^{\chi_4}\mathcal{K}(u,v)\,{}_{\chi_1}d_{q_1}u\,{}^{\chi_4}d_{q_2}v\\
&+\frac{1}{(\chi_2-\chi_1)(\chi_4-\chi_3)}\int_{\varrho}^{\chi_2}\int_{\chi_3}^{\rho}\mathcal{K}(u,v)\,{}^{\chi_2}d_{q_1}u\,{}_{\chi_3}d_{q_2}v\\
&+\frac{1}{(\chi_2-\chi_1)(\chi_4-\chi_3)}\int_{\varrho}^{\chi_2}\int_{\rho}^{\chi_4}\mathcal{K}(u,v)\,{}^{\chi_2}d_{q_1}u\,{}^{\chi_4}d_{q_2}v.
\end{aligned}
$$

3. (p_1p_2, q_1q_2)-Ostrowski type inequalities for function with two variables

In this section, we introduce (p_1p_2, q_1q_2)-Ostrowski inequalities by using n-polynomial s-type convex function on the co-ordinates.

Theorem 3.1. *Suppose that $n \in \mathbb{N}$, $s \in [0,1]$ and all the assumptions of Lemma 2.1 are true. If* $\left|\frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^{\tau_2}$, $\left|\frac{{}^{\chi_4}_{\chi_1}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w}\right|^{\tau_2}$, $\left|\frac{{}^{\chi_2}_{\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^{\tau_2}$ *and* $\left|\frac{{}^{\chi_2,\chi_4}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w}\right|^{\tau_2}$ *are n-polynomial s-type convex functions on the co-ordinates on $\mathcal{N}$ for $\tau_1, \tau_2 > 1$ with $\frac{1}{\tau_1}+\frac{1}{\tau_2} = 1$ and* $\left|\frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\varrho,\rho)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right| \le \mathcal{M}$, $\left|\frac{{}^{\chi_4}_{\chi_1}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w}\right| \le \mathcal{M}$, $\left|\frac{{}^{\chi_2}_{\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right| \le \mathcal{M}$, $\left|\frac{{}^{\chi_2,\chi_4}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w}\right| \le \mathcal{M}$, *$\varrho, \rho \in \mathcal{N}$, then the following inequality holds*

$$
\begin{aligned}
&\Bigg|\mathcal{K}(\varrho,\rho) - \frac{1}{p_1(\chi_2-\chi_1)}\left[\int_{\chi_1}^{p_1\varrho+(1-p_1)\chi_1}\mathcal{K}(u,\rho)\,{}_{\chi_1}d_{p_1,q_1}u + \int_{p_1\varrho+(1-p_1)\chi_2}^{\chi_2}\mathcal{K}(u,\rho)\,{}^{\chi_2}d_{p_1,q_1}u\right]\\
&-\frac{1}{p_2(\chi_4-\chi_3)}\left[\int_{\chi_3}^{p_2\rho+(1-p_2)\chi_3}\mathcal{K}(\varrho,v)\,{}_{\chi_3}d_{p_2,q_2}v + \int_{p_2\rho+(1-p_2)\chi_4}^{\chi_4}\mathcal{K}(\varrho,v)\,{}^{\chi_4}d_{p_2,q_2}v\right] + \mathcal{A}\Bigg|
\end{aligned}
$$

$$\leq \frac{\Delta\mathcal{M}\sqrt[\tau_2]{\left(C_{p_1,q_1}+\mathcal{D}_{p_1,q_1}\right)\left(C_{p_2,q_2}+\mathcal{D}_{p_2,q_2}\right)}\left[(\varrho-\chi_1)^2+(\chi_2-\varrho)^2\right]\left[(\rho-\chi_3)^2+(\chi_4-\rho)^2\right]}{\sqrt[\tau_1]{[1+\tau_1]_{p_1,q_1}[1+\tau_1]_{p_2,q_2}}}, \tag{3.1}$$

where

$$C_{p_k,q_k}=1-\frac{(p_k-q_k)}{n}\sum_{i=1}^{n}\sum_{e=0}^{\infty}s^i\frac{q_k^e}{p_k^{e+1}}\left(1-\frac{q_k^e}{p_k^{e+1}}\right)^i,$$

$$\mathcal{D}_{p_k,q_k}=1-\frac{1}{n}\sum_{i=1}^{n}s^i\left(\frac{p_k-q_k}{p_k^{i+1}-q_k^{i+1}}\right)$$

for $k=1,2$ *and* Δ, $\mathcal{A}$ *are defined in Lemma 2.1.*

Proof. Taking absolute value on both sides of (2.1), by applying Hölder inequality for double integrals and utilizing the fact that $\left|\frac{\partial^2_{p_1p_2,q_1q_2}\mathcal{K}}{\partial_{p_1,q_1}z\partial_{p_2,q_2}w}\right|^{\tau_2}$ is n-polynomial s-type convex on co-ordinates, we get the following inequality

$$\begin{aligned}
&\Bigg|\mathcal{K}(\varrho,\rho)-\frac{1}{p_1(\chi_2-\chi_1)}\left[\int_{\chi_1}^{p_1\varrho+(1-p_1)\chi_1}\mathcal{K}(u,\rho)\,{}_{\chi_1}d_{p_1,q_1}u+\int_{p_1\varrho+(1-p_1)\chi_2}^{\chi_2}\mathcal{K}(u,\rho)\,{}^{\chi_2}d_{p_1,q_1}u\right]\\
&-\frac{1}{p_2(\chi_4-\chi_3)}\left[\int_{\chi_3}^{p_2\rho+(1-p_2)\chi_3}\mathcal{K}(\varrho,v)\,{}_{\chi_3}d_{p_2,q_2}v+\int_{p_2\rho+(1-p_2)\chi_4}^{\chi_4}\mathcal{K}(\varrho,v)\,{}^{\chi_4}d_{p_2,q_2}v\right]+\mathcal{A}\Bigg|\\
&\leq\Delta\left(\int_0^1\int_0^1 z^{\tau_1}w^{\tau_1}\,{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w\right)^{\frac{1}{\tau_1}}\\
&\times\Bigg\{(\varrho-\chi_1)^2(\rho-\chi_3)^2\left(\int_0^1\int_0^1\left|\frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_1,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^{\tau_2}{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w\right)^{\frac{1}{\tau_2}}\\
&+(\varrho-\chi_1)^2(\chi_4-\rho)^2\left(\int_0^1\int_0^1\left|\frac{{}_{\chi_1}^{\chi_4}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_1,w\rho+(1-w)\chi_4)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w}\right|^{\tau_2}{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w\right)^{\frac{1}{\tau_2}}\\
&+(\chi_2-\varrho)^2(\rho-\chi_3)^2\left(\int_0^1\int_0^1\left|\frac{{}^{\chi_2}_{\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_2,w\rho+(1-w)\chi_3)}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^{\tau_2}{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w\right)^{\frac{1}{\tau_2}}\\
&+(\chi_2-\varrho)^2(\chi_4-\rho)^2\left(\int_0^1\int_0^1\left|\frac{{}^{\chi_2,\chi_4}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_2,w\rho+(1-w)\chi_4)}{{}^{\chi_2}\partial_{p_1,q_1}z^{\chi_4}\partial_{p_2,q_2}w}\right|^{\tau_2}{}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w\right)^{\frac{1}{\tau_2}}\Bigg\}.
\end{aligned}$$

Considering first integral

$$\int_0^1\int_0^1\left|\frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_1,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z_{\chi_3}\partial_{p_2,q_2}w}\right|^{\tau_2}{}_0d_{p_1,q_1}z_0d_{p_2,q_2}w$$

$$\leq \int_0^1 \left\{ \int_0^1 \left[\begin{array}{c} \frac{1}{n}\sum_{i=1}^n \left[1-(s(1-z))^i\right] \left| \frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\varrho,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z_{\chi_3}\partial_{p_2,q_2}w} \right|^{\tau_2} \\ +\frac{1}{n}\sum_{i=1}^n \left[1-(sz)^i\right] \left| \frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\chi_1,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z_{\chi_1}\partial_{p_2,q_2}w} \right|^{\tau_2} \end{array} \right] {}_0d_{p_1,q_1}z \right\} {}_0d_{p_2,q_2}w. \tag{3.2}$$

Computing the (p_1, q_1)-integral on the right-hand side of (3.2), we have

$$\leq \int_0^1 \left[\begin{array}{c} \frac{1}{n}\sum_{i=1}^n \left[1-(s(1-z))^i\right] \left| \frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\varrho,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z_{\chi_3}\partial_{p_2,q_2}w} \right|^{\tau_2} \\ +\frac{1}{n}\sum_{i=1}^n \left[1-(sz)^i\right] \left| \frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\chi_1,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z_{\chi_1}\partial_{p_2,q_2}w} \right|^{\tau_2} \end{array} \right] {}_0d_{p_1,q_1}z.$$

In view of the Definitions 1.4 for $k = 1, 2$, we get

$$\mathcal{C}_{p_k,q_k} = \frac{1}{n}\sum_{i=1}^n \int_0^1 \left[1-(s(1-z))^i\right] {}_0d_{p_k,q_k}z = 1 - \frac{(p_k-q_k)}{n}\sum_{i=1}^n\sum_{e=0}^\infty s^i \frac{q_k^e}{p_k^{e+1}}\left(1-\frac{q_k^e}{p_k^{e+1}}\right)^i,$$

$$\mathcal{D}_{p_k,q_k} = \frac{1}{n}\sum_{i=1}^n \int_0^1 \left[1-(sz)^i\right] {}_0d_{p_k,q_k}z = 1 - \frac{1}{n}\sum_{i=1}^n s^i\left(\frac{p_k-q_k}{p_k^{i+1}-q_k^{i+1}}\right).$$

Putting the above calculations into (3.2), we obtain

$$\leq \int_0^1 \left[\begin{array}{c} \mathcal{C}_{p_1,q_1} \left| \frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\varrho,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z_{\chi_3}\partial_{p_2,q_2}w} \right|^{\tau_2} \\ +\mathcal{D}_{p_1,q_1} \left| \frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\chi_1,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z_{\chi_3}\partial_{p_2,q_2}w} \right|^{\tau_2} \end{array} \right] {}_0d_{p_2,q_2}w. \tag{3.3}$$

Similarly, by computing the (p_2, q_2)-integral, utilizing the fact $\left| \frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\varrho,\rho)}{{}_{\chi_1}\partial_{p_1,q_1}z_{\chi_3}\partial_{p_2,q_2}w} \right| \leq \mathcal{M}, \varrho, \rho \in \mathcal{N}$ on the right-hand side of (3.3), we have

$$\begin{aligned} &\int_0^1\int_0^1 \left| \frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-w)\chi_1, w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z_{\chi_3}\partial_{p_2,q_2}w} \right|^{\tau_2} {}_0d_{p_1,q_1}z_0d_{p_2,q_2}w \\ &\leq \mathcal{M}^{\tau_2}(\mathcal{C}_{p_1,q_1}+\mathcal{D}_{p_1,q_1})(\mathcal{C}_{p_2,q_2}+\mathcal{D}_{p_2,q_2}). \end{aligned} \tag{3.4}$$

Analogously, we get

$$\begin{aligned} &\int_0^1\int_0^1 \left| \frac{{}^{\chi_4}_{\chi_1}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-w)\chi_1, \chi_4+w\rho+(1-w)\chi_4)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w} \right|^{\tau_2} {}_0d_{p_1,q_1}z_0d_{p_2,q_2}w \\ &\leq \mathcal{M}^{\tau_2}(\mathcal{C}_{p_1,q_1}+\mathcal{D}_{p_1,q_1})(\mathcal{C}_{p_2,q_2}+\mathcal{D}_{p_2,q_2}) \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\int_0^1\int_0^1 \left| \frac{{}^{\chi_2}_{\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_2, w\rho+(1-w)\chi_3)}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w} \right|^{\tau_2} {}_0d_{p_1,q_1}z_0d_{p_2,q_2}w \\ &\leq \mathcal{M}^{\tau_2}(\mathcal{C}_{p_1,q_1}+\mathcal{D}_{p_1,q_1})(\mathcal{C}_{p_2,q_2}+\mathcal{D}_{p_2,q_2}) \end{aligned} \tag{3.6}$$

and

$$\int_0^1 \int_0^1 \left| \frac{{}^{\chi_2,\chi_4}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho + (1-z)\chi_2, \chi_4 + w\rho + (1-w)\chi_4)}{{}^{\chi_2}\partial_{p_1,q_1}z^{\chi_4}\partial_{p_2,q_2}w} \right|^{\tau_2} {}_0d_{p_1,q_1}z_0d_{p_2,q_2}w$$
$$\leq \mathcal{M}^{\tau_2}(C_{p_1,q_1} + \mathcal{D}_{p_1,q_1})(C_{p_2,q_2} + \mathcal{D}_{p_2,q_2}). \tag{3.7}$$

Now by making use of the inequalities (3.4)–(3.7) and using the fact that

$$\int_0^1 \int_0^1 z^{\tau_1} w^{\tau_1} {}_0d_{p_1,q_1}z_0d_{p_2,q_2}w = \frac{1}{[1+\tau_1]_{p_1,q_1}[1+\tau_1]_{p_2,q_2}},$$

we get the desired inequality (3.1). This completes the proof. □

Corollary 1. ***I.*** *Taking $p_k = 1$ for $k = 1, 2$ in Theorem 3.1, we get*

$$\Bigg|\mathcal{K}(\varrho,\rho) - \frac{1}{(\chi_2-\chi_1)}\left[\int_{\chi_1}^{\varrho}\mathcal{K}(u,\rho)\,{}_{\chi_1}d_{q_1}u + \int_{\varrho}^{\chi_2}\mathcal{K}(u,\rho)\,{}^{\chi_2}d_{q_1}u\right]$$
$$-\frac{1}{(\chi_4-\chi_3)}\left[\int_{\chi_3}^{\rho}\mathcal{K}(\varrho,v)\,{}_{\chi_3}d_{q_2}v + \int_{\rho}^{\chi_4}\mathcal{K}(\varrho,v)\,{}^{\chi_4}d_{q_2}v\right] + \mathcal{W}\Bigg|$$
$$\leq \frac{\Delta\mathcal{M}\sqrt[\tau_2]{\left(C_{q_1}+\mathcal{D}_{q_1}\right)\left(C_{q_2}+\mathcal{D}_{q_2}\right)}\left[(\varrho-\chi_1)^2+(\chi_2-\varrho)^2\right]\left[(\rho-\chi_3)^2+(\chi_4-\rho)^2\right]}{\sqrt[\tau_1]{[1+\tau_1]_{q_1}[1+\tau_1]_{q_2}}},$$

where

$$C_{q_k} = 1 - \frac{(1-q_k)}{n}\sum_{i=1}^{n}\sum_{e=0}^{\infty} s^i q_k^e (1-q_k^e)^i,$$
$$\mathcal{D}_{q_k} = 1 - \frac{1}{n}\sum_{i=1}^{n} s^i\left(\frac{1-q_k}{1-q_k^{i+1}}\right)$$

and Δ, $\mathcal{W}$ are defined in Remark 4.

II. *Taking $q_k \to 1^-$ for $k = 1, 2$ in part **I**, we get*

$$\left|\mathcal{K}(\varrho,\rho) + \frac{1}{(\chi_2-\chi_1)(\chi_4-\chi_3)}\int_{\chi_1}^{\chi_2}\int_{\chi_3}^{\chi_4}\mathcal{K}(u,v)dvdu - Q\right|$$
$$\leq \frac{\mathcal{M}\sqrt[\tau_2]{\left(\frac{2}{n}\sum_{i=1}^{n}\left(\frac{i+1-s^i}{i+1}\right)\right)^2}}{\sqrt[\tau_1]{(1+\tau_1)^2}}\left[\frac{(\varrho-\chi_1)^2+(\chi_2-\varrho)^2}{\chi_2-\chi_1}\right]\left[\frac{(\rho-\chi_3)^2+(\chi_4-\rho)^2}{\chi_4-\chi_3}\right],$$

where

$$Q = \frac{1}{\chi_2-\chi_1}\int_{\chi_1}^{\chi_2}\mathcal{K}(u,\rho)du + \frac{1}{\chi_4-\chi_3}\int_{\chi_3}^{\chi_4}\mathcal{K}(\varrho,v)dv.$$

Remark 5. Taking $n = s = 1$ in part **II** of Corollary 1, we obtain Theorem 4 of [47].

Theorem 3.2. *Suppose that $n \in \mathbb{N}$, $s \in [0,1]$ and all the assumptions of Lemma 2.1 holds. If* $\left|\frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^\tau$, $\left|\frac{{}^{\chi_4}_{\chi_1}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w}\right|^\tau$, $\left|\frac{{}^{\chi_2}_{\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^\tau$ *and* $\left|\frac{{}^{\chi_2,\chi_4}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w}\right|^\tau$ *are n-polynomial s-type convex functions on the co-ordinates on $\mathcal{N}$ for $\tau \geq 1$, and* $\left|\frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\varrho,\rho)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right| \leq \mathcal{M}$, $\left|\frac{{}^{\chi_4}_{\chi_1}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w}\right| \leq \mathcal{M}$, $\left|\frac{{}^{\chi_2}_{\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right| \leq \mathcal{M}$, $\left|\frac{{}^{\chi_2,\chi_4}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z,w)}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w}\right| \leq \mathcal{M}$ *, $\varrho,\rho \in \mathcal{N}$, then the following inequality holds*

$$\begin{aligned}
&\Bigg|\mathcal{K}(\varrho,\rho) - \frac{1}{p_1(\chi_2-\chi_1)}\left[\int_{\chi_1}^{p_1\varrho+(1-p_1)\chi_1}\mathcal{K}(u,\rho)\,{}_{\chi_1}d_{p_1,q_1}u + \int_{p_1\varrho+(1-p_1)\chi_2}^{\chi_2}\mathcal{K}(u,\rho)\,{}^{\chi_2}d_{p_1,q_1}u\right]\\
&-\frac{1}{p_2(\chi_4-\chi_3)}\left[\int_{\chi_3}^{p_2\rho+(1-p_2)\chi_3}\mathcal{K}(\varrho,v)\,{}_{\chi_3}d_{p_2,q_2}v + \int_{p_2\rho+(1-p_2)\chi_4}^{\chi_4}\mathcal{K}(\varrho,v)\,{}^{\chi_4}d_{p_2,q_2}v\right] + \mathcal{A}\Bigg|\\
&\leq \frac{\Delta\mathcal{M}\sqrt[\tau]{\left(\mathcal{A}_{p_1,q_1}+\mathcal{B}_{p_1,q_1}\right)\left(\mathcal{A}_{p_2,q_2}+\mathcal{B}_{p_2,q_2}\right)}\left[(\varrho-\chi_1)^2+(\chi_2-\varrho)^2\right]\left[(\rho-\chi_3)^2+(\chi_4-\rho)^2\right]}{\left[(p_1+q_1)(p_2+q_2)\right]^{1-\frac{1}{\tau}}},
\end{aligned} \tag{3.8}$$

where

$$\mathcal{A}_{p_k,q_k} = \frac{1}{p_k+q_k} - \frac{(p_k-q_k)}{n}\sum_{i=1}^{n}\sum_{e=0}^{\infty} s^i \frac{q_k^{2e}}{p_k^{2e+2}}\left(1-\frac{q_k^e}{p_k^{e+1}}\right)^i,$$

$$\mathcal{B}_{p_k,q_k} = \frac{1}{p_k+q_k} - \frac{1}{n}\sum_{i=1}^{n} s^i\left(\frac{p_k-q_k}{p_k^{i+2}-q_k^{i+2}}\right)$$

for $k = 1,2$ and Δ, $\mathcal{A}$ are defined in Lemma 2.1.

Proof. Taking absolute value on both sides of (2.1), by applying power mean inequality for double integrals, we get the following inequality

$$\begin{aligned}
&\Bigg|\mathcal{K}(\varrho,\rho) - \frac{1}{p_1(\chi_2-\chi_1)}\left[\int_{\chi_1}^{p_1\varrho+(1-p_1)\chi_1}\mathcal{K}(u,\rho)_0 d_{p_1,q_1}u + \int_{\chi_2}^{p_1\varrho+(1-p_1)\chi_2}\mathcal{K}(u,\rho)_0 d_{p_1,q_1}u\right]\\
&-\frac{1}{p_2(\chi_4-\chi_3)}\left[\int_{\chi_3}^{p_2\rho+(1-p_2)\chi_3}\mathcal{K}(\varrho,v)_0 d_{p_2,q_2}v + \int_{\chi_4}^{p_2\rho+(1-p_2)\chi_4}\mathcal{K}(\varrho,v)_0 d_{p_2,q_2}v\right] + \mathcal{A}\Bigg|\\
&\leq \Delta\left(\int_0^1\int_0^1 zw_0 d_{p_1,q_1}z_0 d_{p_2,q_2}w\right)^{1-\frac{1}{\tau}}\\
&\times\Bigg\{(\varrho-\chi_1)^2(\rho-\chi_3)^2\left(\int_0^1\int_0^1 zw\left|\frac{{}_{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_1, w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^\tau {}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w\right)^{\frac{1}{\tau}}\\
&+(\varrho-\chi_1)^2(\chi_4-\rho)^2\left(\int_0^1\int_0^1 zw\left|\frac{{}^{\chi_4}_{\chi_1}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_1, w\rho+(1-w)\chi_4)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w}\right|^\tau {}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w\right)^{\frac{1}{\tau}}
\end{aligned}$$

$$+(\chi_2-\varrho)^2(\rho-\chi_3)^2\left(\int_0^1\int_0^1 zw\left|\frac{{}^{\chi_2}_{\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_2,w\rho+(1-w)\chi_3)}{{}_{\chi_2}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^\tau {}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w\right)^{\frac{1}{\tau}}$$

$$+(\chi_2-\varrho)^2(\chi_4-\rho)^2\left(\int_0^1\int_0^1 zw\left|\frac{{}^{\chi_2,\chi_4}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_2,w\rho+(1-w)\chi_4)}{{}_{\chi_2}\partial_{p_1,q_1}z^{\chi_4}\partial_{p_2,q_2}w}\right|^\tau {}_0d_{p_1,q_1}z\,{}_0d_{p_2,q_2}w\right)^{\frac{1}{\tau}}\Bigg\}.$$

Considering first integral

$$\begin{aligned}
&\int_0^1\int_0^1 zw\left|\frac{{}^{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_1,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^\tau {}_0d_{p_1,q_1}z{}_0d_{p_2,q_2}w\\
&\le\int_0^1 w\left\{\int_0^1 z\left[\begin{array}{c}\frac{1}{n}\sum_{i=1}^n\left[1-(s(1-z))^i\right]\left|\frac{{}^{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\varrho,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^\tau\\ +\frac{1}{n}\sum_{i=1}^n\left[1-(sz)^i\right]\left|\frac{{}^{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\chi_1,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^\tau\end{array}\right]{}_0d_{p_1,q_1}z\right\}{}_0d_{p_2,q_2}w.
\end{aligned}\tag{3.9}$$

Computing the (p_1,q_1)-integral on the right-hand side of (3.9), we have

$$\le\int_0^1 z\left[\begin{array}{c}\frac{1}{n}\sum_{i=1}^n\left[1-(s(1-z))^i\right]\left|\frac{{}^{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\varrho,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^\tau\\ +\frac{1}{n}\sum_{i=1}^n\left[1-(sz)^i\right]\left|\frac{{}^{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\chi_1,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^\tau\end{array}\right]{}_0d_{p_1,q_1}z.$$

In view of the Definitions 1.5 fpr $k=1,2$, we get

$$\mathcal{A}_{p_k,q_k}=\frac{1}{n}\sum_{i=1}^n\int_0^1 z\left[1-(s(1-z))^i\right]{}_0d_{p_k,q_k}z=\frac{1}{p_k+q_k}-\frac{(p_k-q_k)}{n}\sum_{i=1}^n\sum_{e=0}^\infty s^i\frac{q_k^{2e}}{p_k^{2e+2}}\left(1-\frac{q_k^e}{p_k^{e+1}}\right)^i,$$

$$\mathcal{B}_{p_k,q_k}=\frac{1}{n}\sum_{i=1}^n\int_0^1 z\left[1-(sz)^i\right]{}_0d_{p_k,q_k}z=\frac{1}{p_k+q_k}-\frac{1}{n}\sum_{i=1}^n s^i\left(\frac{p_k-q_k}{p_k^{i+2}-q_k^{i+2}}\right).$$

Putting the above calculations into (3.9), we obtain

$$\le\int_0^1 w\left[\begin{array}{c}\mathcal{A}_{p_1,q_1}\left|\frac{{}^{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\varrho,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^\tau\\ +\mathcal{B}_{p_1,q_1}\left|\frac{{}^{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\chi_1,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^\tau\end{array}\right]{}_0d_{p_2,q_2}w.\tag{3.10}$$

Similarly, by computing the (p_2,q_2)-integral, utilizing the fact $\left|\frac{{}^{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(\varrho,\rho)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|\le\mathcal{M},\varrho,\rho\in\mathcal{N}$ on the right-hand side of (3.10), we have

$$\begin{aligned}
&\int_0^1\int_0^1 zw\left|\frac{{}^{\chi_1,\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-w)\chi_1,w\rho+(1-w)\chi_3)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^\tau {}_0d_{p_1,q_1}z{}_0d_{p_2,q_2}w\\
&\le\mathcal{M}^\tau(\mathcal{A}_{p_1,q_1}+\mathcal{B}_{p_1,q_1})(\mathcal{A}_{p_2,q_2}+\mathcal{B}_{p_2,q_2}).
\end{aligned}\tag{3.11}$$

Analogously, we get

$$\int_0^1\int_0^1 zw\left|\frac{{}^{\chi_4}_{\chi_1}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-w)\chi_1,\chi_4+w\rho+(1-w)\chi_4)}{{}_{\chi_1}\partial_{p_1,q_1}z\,{}^{\chi_4}\partial_{p_2,q_2}w}\right|^{\tau}{}_0d_{p_1,q_1}z{}_0d_{p_2,q_2}w$$
$$\leq \mathcal{M}^{\tau}(\mathcal{A}_{p_1,q_1}+\mathcal{B}_{p_1,q_1})(\mathcal{A}_{p_2,q_2}+\mathcal{B}_{p_2,q_2}), \tag{3.12}$$

$$\int_0^1\int_0^1 zw\left|\frac{{}^{\chi_2}_{\chi_3}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_2,w\rho+(1-w)\chi_3)}{{}^{\chi_2}\partial_{p_1,q_1}z\,{}_{\chi_3}\partial_{p_2,q_2}w}\right|^{\tau}{}_0d_{p_1,q_1}z{}_0d_{p_2,q_2}w$$
$$\leq \mathcal{M}^{\tau}(\mathcal{A}_{p_1,q_1}+\mathcal{B}_{p_1,q_1})(\mathcal{A}_{p_2,q_2}+\mathcal{B}_{p_2,q_2}) \tag{3.13}$$

and

$$\int_0^1\int_0^1 zw\left|\frac{{}^{\chi_2,\chi_4}\partial^2_{p_1p_2,q_1q_2}\mathcal{K}(z\varrho+(1-z)\chi_2,\chi_4+w\rho+(1-w)\chi_4)}{{}^{\chi_2}\partial_{p_1,q_1}z^{\chi_4}\partial_{p_2,q_2}w}\right|^{\tau}{}_0d_{p_1,q_1}z{}_0d_{p_2,q_2}w$$
$$\leq \mathcal{M}^{\tau}(\mathcal{A}_{p_1,q_1}+\mathcal{B}_{p_1,q_1})(\mathcal{A}_{p_2,q_2}+\mathcal{B}_{p_2,q_2}). \tag{3.14}$$

Now by making use of the inequalities (3.11)–(3.14) and the fact that

$$\int_0^1\int_0^1 zw{}_0d_{p_1,q_1}z{}_0d_{p_2,q_2}w=\frac{1}{(p_1+q_1)(p_2+q_2)},$$

we get the desired inequality (3.8). This completes the proof. □

Corollary 2. ***I.*** *Taking $\tau=1$ in Theorem 3.2, we get*

$$\Bigg|\mathcal{K}(\varrho,\rho)-\frac{1}{p_1(\chi_2-\chi_1)}\left[\int_{\chi_1}^{p_1\varrho+(1-p_1)\chi_1}\mathcal{K}(u,\rho)\,{}_{\chi_1}d_{p_1,q_1}u+\int_{p_1\varrho+(1-p_1)\chi_2}^{\chi_2}\mathcal{K}(u,\rho)\,{}^{\chi_2}d_{p_1,q_1}u\right]$$
$$-\frac{1}{p_2(\chi_4-\chi_3)}\left[\int_{\chi_3}^{p_2\rho+(1-p_2)\chi_3}\mathcal{K}(\varrho,v)\,{}_{\chi_3}d_{p_2,q_2}v+\int_{p_2\rho+(1-p_2)\chi_4}^{\chi_4}\mathcal{K}(\varrho,v)\,{}^{\chi_4}d_{p_2,q_2}v\right]+\mathcal{A}\Bigg|$$
$$\leq\Delta\mathcal{M}\left(\mathcal{A}_{p_1,q_1}+\mathcal{B}_{p_1,q_1}\right)(\mathcal{A}_{p_2,q_2}+\mathcal{B}_{p_2,q_2})\left[(\varrho-\chi_1)^2+(\chi_2-\varrho)^2\right]\left[(\rho-\chi_3)^2+(\chi_4-\rho)^2\right].$$

II. *Taking $p_k=1$ for $k=1,2$ in Theorem 3.2, we obtain*

$$\Bigg|\mathcal{K}(\varrho,\rho)-\frac{1}{(\chi_2-\chi_1)}\left[\int_{\chi_1}^{\varrho}\mathcal{K}(u,\rho)\,{}_{\chi_1}d_{q_1}u+\int_{\varrho}^{\chi_2}\mathcal{K}(u,\rho)\,{}^{\chi_2}d_{q_1}u\right]$$
$$-\frac{1}{(\chi_4-\chi_3)}\left[\int_{\chi_3}^{\rho}\mathcal{K}(\varrho,v)_{\chi_3}d_{q_2}v+\int_{\rho}^{\chi_4}\mathcal{K}(\varrho,v)\,{}^{\chi_4}d_{q_2}v\right]+\mathcal{W}\Bigg|$$
$$\leq\frac{\Delta\mathcal{M}\sqrt[\tau]{\left(\mathcal{A}_{q_1}+\mathcal{B}_{q_1}\right)(\mathcal{A}_{q_2}+\mathcal{B}_{q_2})}}{\left[(1+q_1)(1+q_2)\right]^{1-\frac{1}{\tau}}}\left[(\varrho-\chi_1)^2+(\chi_2-\varrho)^2\right]\left[(\rho-\chi_3)^2+(\chi_4-\rho)^2\right],$$

where

$$\mathcal{A}_{q_k}=\frac{1}{1+q_k}-\frac{1-q_k}{n}\sum_{i=1}^{n}\sum_{e=0}^{\infty}s^iq_k^{2e}\left(1-q_k^e\right)^i,$$

$$\mathcal{B}_{q_k} = \frac{1}{1+q_k} - \frac{1}{n}\sum_{i=1}^{n} s^i\left(\frac{1-q_k}{1-q_k^{i+2}}\right),$$

and Δ, $\mathcal{W}$ are defined in Remark 4.
III. *Taking $p_k = 1$ for $k = 1, 2$, and $q_k \to 1^-$ in part **II**, we get*

$$\begin{aligned}&\left|\mathcal{K}(\varrho,\rho) + \frac{1}{(\chi_2-\chi_1)(\chi_4-\chi_3)}\int_{\chi_1}^{\chi_2}\int_{\chi_3}^{\chi_4}\mathcal{K}(u,v)dvdu - \mathcal{Q}\right|\\ &\leq \mathcal{M}4^{\frac{1}{\tau}-1}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{i^2+5i+6-6s^i}{2(i+1)(i+2)}\right)^{\frac{2}{\tau}}\left[\frac{(\varrho-\chi_1)^2+(\chi_2-\varrho)^2}{\chi_2-\chi_1}\right]\left[\frac{(\rho-\chi_3)^2+(\chi_4-\rho)^2}{\chi_4-\chi_3}\right],\end{aligned}$$

*where $\mathcal{Q}$ is defined in part **II** of Corollary 1 .*
IV. *Taking $n = 1 = s$ in part **III**, we have the following inequality*

$$\begin{aligned}&\left|\mathcal{K}(\varrho,\rho) + \frac{1}{(\chi_2-\chi_1)(\chi_4-\chi_3)}\int_{\chi_1}^{\chi_2}\int_{\chi_3}^{\chi_4}\mathcal{K}(u,v)dvdu - \mathcal{Q}\right|\\ &\leq \frac{\mathcal{M}}{4}\left[\frac{(\varrho-\chi_1)^2+(\chi_2-\varrho)^2}{\chi_2-\chi_1}\right]\left[\frac{(\rho-\chi_3)^2+(\chi_4-\rho)^2}{\chi_4-\chi_3}\right],\end{aligned}$$

*where $\mathcal{Q}$ is defined in part **II** of Corollary 1.*

4. Application to special means

Let $k \in \mathbb{R}\backslash\{-1, 0\}$, and φ_1 and φ_2 be two distinct positive real numbers. Then the generalized logarithmic mean $\mathcal{L}_k(\varphi_1, \varphi_2)$ is defined by

$$\mathcal{L}_k(\varphi_1,\varphi_2) = \left(\frac{\varphi_2^{k+1}-\varphi_1^{k+1}}{(k+1)(\varphi_2-\varphi_1)}\right)^{\frac{1}{k}}.$$

Proposition 2. *If $m, k > 1$ and $\chi_1, \chi_2, \chi_3, \chi_4$ are positive real numbers such that $\chi_1 < \chi_2$ and $\chi_3 < \chi_4$, then one has*

$$\begin{aligned}&\Bigg|\varrho^m\times\rho^k - \frac{\rho^k}{(\chi_2-\chi_1)}\left[\mathcal{L}_m^m(\varrho,\chi_1)+\mathcal{L}_m^m(\chi_2,\varrho)\right] - \frac{\varrho^k}{(\chi_4-\chi_3)}\left[\mathcal{L}_k^k(\rho,\chi_3)+\mathcal{L}_k^k(\chi_4,\rho)\right]\\ &+\frac{1}{(\chi_2-\chi_1)(\chi_4-\chi_3)}\left(\mathcal{L}_m^m(\varrho,\chi_1)+\mathcal{L}_m^m(\chi_2,\varrho)\right)\left(\mathcal{L}_k^k(\rho,\chi_3)+\mathcal{L}_k^k(\chi_4,\rho)\right)\Bigg|\\ &\leq \frac{\mathcal{M}}{(1+\tau_1)^{\frac{2}{\tau_1}}}\left[\frac{(\varrho-\chi_1)^2+(\chi_2-\varrho)^2}{\chi_2-\chi_1}\right]\left[\frac{(\rho-\chi_3)^2+(\chi_4-\rho)^2}{\chi_4-\chi_3}\right].\end{aligned}$$

Proof. Let $\mathcal{K}(\varrho, \rho) = \varrho^m \times \rho^k$ for $m, k > 1$. Then, we have

$$\int_{\chi_1}^{\varrho} u^m\times\rho^k\ {}_{\chi_1}d_{q_1}u = \rho^k\left[\frac{1-q_1}{1-q_1^{m+1}}\right]\left(\frac{\varrho^{m+1}-\chi_1^{m+1}}{\varrho-\chi_1}\right),$$

$$\int_{\varrho}^{\chi_2} u^m \times \rho^k \,{}^{\chi_2}d_{q_1}u = \rho^k \left[\frac{1-q_1}{1-q_1^{m+1}}\right]\left(\frac{\chi_2^{m+1}-\varrho^{m+1}}{\chi_2-\varrho}\right),$$

$$\int_{\chi_3}^{\rho} \varrho^m \times v^k \,{}_{\chi_3}d_{q_2}v = \varrho^m \left[\frac{1-q_2}{1-q_2^{k+1}}\right]\left(\frac{\rho^{k+1}-\chi_3^{k+1}}{\rho-\chi_3}\right),$$

$$\int_{\rho}^{\chi_4} \varrho^m \times v^k \,{}^{\chi_4}d_{q_2}v = \varrho^m \left[\frac{1-q_2}{1-q_2^{k+1}}\right]\left(\frac{\chi_4^{k+1}-\rho^{k+1}}{\chi_4-\rho}\right),$$

$$\int_{\chi_1}^{\varrho}\int_{\chi_3}^{\rho} u^m \times v^k \,{}_{\chi_1}d_{q_1}u \,{}_{\chi_3}d_{q_2}v = \left[\frac{1-q_1}{1-q_1^{m+1}}\right]\left[\frac{1-q_2}{1-q_2^{k+1}}\right]\left(\frac{\varrho^{m+1}-\chi_1^{m+1}}{\varrho-\chi_1}\right)\left(\frac{\rho^{k+1}-\chi_3^{k+1}}{\rho-\chi_3}\right),$$

$$\int_{\chi_1}^{\varrho}\int_{\rho}^{\chi_4} u^m \times v^k \,{}_{\chi_1}d_{q_1}u \,{}^{\chi_4}d_{q_2}v = \left[\frac{1-q_1}{1-q_1^{m+1}}\right]\left[\frac{1-q_2}{1-q_2^{k+1}}\right]\left(\frac{\varrho^{m+1}-\chi_1^{m+1}}{\varrho-\chi_1}\right)\left(\frac{\chi_4^{k+1}-\rho^{k+1}}{\chi_4-\rho}\right),$$

$$\int_{\varrho}^{\chi_2}\int_{\chi_3}^{\rho} u^m \times v^k \,{}^{\chi_2}d_{q_1}u \,{}_{\chi_3}d_{q_2}v = \left[\frac{1-q_1}{1-q_1^{m+1}}\right]\left[\frac{1-q_2}{1-q_2^{k+1}}\right]\left(\frac{\chi_2^{m+1}-\varrho^{m+1}}{\chi_2-\varrho}\right)\left(\frac{\rho^{k+1}-\chi_3^{k+1}}{\rho-\chi_3}\right)$$

and

$$\int_{\varrho}^{\chi_2}\int_{\rho}^{\chi_4} u^m \times v^k \,{}^{\chi_2}d_{q_1}u \,{}^{\chi_4}d_{q_2}v = \left[\frac{1-q_1}{1-q_1^{m+1}}\right]\left[\frac{1-q_2}{1-q_2^{k+1}}\right]\left(\frac{\chi_2^{m+1}-\varrho^{m+1}}{\chi_2-\varrho}\right)\left(\frac{\chi_4^{k+1}-\rho^{k+1}}{\chi_4-\rho}\right).$$

It follows from part **I** of Corollary 1 that

$$\begin{aligned}
&\Bigg|\mathcal{K}(\varrho,\rho) - \frac{\rho^k}{(\chi_2-\chi_1)}\left[\frac{1-q_1}{1-q_1^{m+1}}\right]\left[\left(\frac{\varrho^{m+1}-\chi_1^{m+1}}{\varrho-\chi_1}\right)+\left(\frac{\chi_2^{m+1}-\varrho^{m+1}}{\chi_2-\varrho}\right)\right] \\
&-\frac{\varrho^k}{(\chi_4-\chi_3)}\left[\frac{1-q_2}{1-q_2^{k+1}}\right]\left[\left(\frac{\rho^{k+1}-\chi_3^{k+1}}{\rho-\chi_3}\right)+\left(\frac{\chi_4^{k+1}-\rho^{k+1}}{\chi_4-\rho}\right)\right]+\mathcal{Z}\Bigg| \\
&\le \frac{\Delta\mathcal{M}\sqrt[\tau_2]{\left(C_{q_1}+\mathcal{D}_{q_1}\right)\left(C_{q_2}+\mathcal{D}_{q_2}\right)}\left[(\varrho-\chi_1)^2+(\chi_2-\varrho)^2\right]\left[(\rho-\chi_3)^2+(\chi_4-\rho)^2\right]}{\sqrt[\tau_1]{[1+\tau_1]_{q_1}[1+\tau_1]_{q_2}}},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{Z} = \frac{1}{(\chi_2-\chi_1)(\chi_4-\chi_3)}\left[\frac{1-q_1}{1-q_1^{m+1}}\right]\left[\frac{1-q_2}{1-q_2^{k+1}}\right]&\left[\left(\frac{\varrho^{m+1}-\chi_1^{m+1}}{\varrho-\chi_1}\right)\left(\frac{\rho^{k+1}-\chi_3^{k+1}}{\rho-\chi_3}\right)\right. \\
&+\left(\frac{\varrho^{m+1}-\chi_1^{m+1}}{\varrho-\chi_1}\right)\left(\frac{\chi_4^{k+1}-\rho^{k+1}}{\chi_4-\rho}\right)+\left(\frac{\chi_2^{m+1}-\varrho^{m+1}}{\chi_2-\varrho}\right)\left(\frac{\rho^{k+1}-\chi_3^{k+1}}{\rho-\chi_3}\right) \\
&\left.+\left(\frac{\chi_2^{m+1}-\varrho^{m+1}}{\chi_2-\varrho}\right)\left(\frac{\chi_4^{k+1}-\rho^{k+1}}{\chi_4-\rho}\right)\right],
\end{aligned}$$

$$C_{q_k} = 1-\frac{(1-q_k)}{n}\sum_{i=1}^{n}\sum_{e=0}^{\infty} s^i q_k^e\left(1-q_k^e\right)^i,$$

$$\mathcal{D}_{q_k} = 1-\frac{1}{n}\sum_{i=1}^{n} s^i\left(\frac{1-q_k}{1-q_k^{i+1}}\right).$$

□

Remark 6. Applying the same idea as in Proposition 2 and using Theorems 3.1, 3.2 and their corresponding corollaries, and choosing suitable functions, for example $\mathcal{K}(\varrho,\rho) = \varrho^m \times \rho^k$, $m,k > 1$ and $\varrho,\rho > 0$; $\mathcal{K}(\varrho,\rho) = \frac{1}{\varrho\rho}$, $\varrho,\rho > 0$; $\mathcal{K}(\varrho,\rho) = e^{\varrho+\rho}$, $\varrho,\rho \in \mathbb{R}$, and so on, we can obtain other new interesting inequalities for special means. We omit their proofs and the details are left to the interested readers.

5. Conclusions

In this paper, we have defined several new partial post quantum derivatives and integrals for the functions with two variables, provided some new generalizations in the frame of a new class of convex functions named n-polynomial s-type convex functions, found a new version (p_1p_2, q_1q_2)-Ostrowski type inequality via the class of n-polynomial s-type convex functions on co-ordinates, established a twice partial integral identitity involving (p_1p_2, q_1q_2)-differentiable functions, and generalized the Ostrowski type inequality. Our results are the generalizations of many previous known results, and our ideas and approach may lead to a lot of follow-up research.

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Conflict of interest

The authors declare that they have no competing interests.

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