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Research article

On the stability of projected dynamical system for generalized variational inequality with hesitant fuzzy relation

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Abstract: In this paper, we investigate the projected dynamical system associated with the generalized variational inequality with hesitant fuzzy relation. We establish the equivalence between the generalized variational inequality with hesitant fuzzy relation and the fuzzy fixed point problem. And we analyze the existence theorem and iterative algorithm of solutions to such problem. Furthermore, using the projection method, we propose a projection neural network for solving the generalized variational inequality with hesitant fuzzy relation and discuss the stability of the proposed projected dynamical system.

Keywords: generalized variational inequalities; hesitant fuzzy relations; level sets; projection methods; Lyapunov stability
Mathematics Subject Classification: 26E50, 49K15

1. Introduction

Many problems arising in diverse applied fields ranging from physics, economics, optimization to engineering can be formulated as variational inequalities. Actually, the variational inequality problem achieves its present-day status as a lively and fruitful area of research through the evolution of three major events [11]. First is the experience of the PIES (Project Independence Evaluation System) energy model [1] which was developed at the U.S. Department of Energy in the late 1970's provided a useful piece of practical evidence demonstrating the inability of the fixed-point methods in handling real-life applications. The second event is the publication of a paper by Smith [25] which formulated the traffic assignment problem as a variational inequality. The last event was initiated by Lars Mathiesen [21, 22] who attempted to solve the Walrasian or general equilibrium model of economic activities with some of the recent techniques developed for the nonlinear complementarity problem, which supports the benefits of the variational inequality problem approach for solving large-scale equilibrium problems. Historically, variational inequality theory, where the function is a vector-valued mapping, was introduced by Hartman and Stampacchia [12] in 1965. The most basic result on the existence of solutions to the variational inequality VI(M, F) requires the set M to be compact and convex, and the mapping f to be continuous [7], which is given by Brouwer's fixed-point theorem. Later, extended conclusions are derived by replacing the compactness of the set M by closed (which is possibly unbounded) with additional conditions on F (e.g., pseudo-monotone, strongly monotone, coercive with respect to M) [7, 8, 11].

It is well known that the theory of set-valued mappings, beside being an important mathematical theory, has become an significant tool in many practical areas, especially in economic analysis [18]. In 1982, Fang [9] extended the variational inequality to the generalized variational inequality, where the function is a set-valued mapping. The generalized variational inequality, is to find $x \in M$ and $y \in f(x)$ such that

$$y^T(x' - x) \geq 0, \ \forall x' \in M, \tag{1.1}$$

where $M \subseteq R^n$, and $f : M \to 2^{R^n}$ is a set-valued function. The most fundamental existence theorem for GVI(M, F) can be proved by Kakutani fixed-point theorem which is for set-valued function. It is worth noting that there is an equivalence between set-valued mappings and binary relations, and the more convenient discussion framework system can be chosen between the two according to the actual needs. Thus, for a given set-valued mapping f on M, the generalized variational inequality (1.1) can also be represented as follows: the generalized variational inequality, denoted by $GVI(M, \Gamma)$, is to find all solutions (x, y) such that

$$\begin{aligned} & x \in M, \\ & \langle y, x' - x \rangle \geq 0, \ \forall x' \in M, \\ & (x, y) \in \Gamma, \end{aligned} \tag{1.2}$$

where $M \subseteq R^n$, and $\Gamma \subseteq R^n \times R^n$ is a relation on R^n.

In the classical set, the nature of the element is required to be explicit, that is, it can be explicitly indicated that any element has or does not have this property. However, in the objective world, many phenomena have fuzziness, which are based on numerous fuzzy phenomena and multi-valued logic, and therefore can not be described by the classical set. For example, the linguistic interpretations such as "young" and "old", "long" and "short", are fuzzy concepts in people's concepts. In 1965, Zadeh [30] introduced the fuzzy set theory and enabled us to represent our knowledge under varied interpretations and axiomatic foundations from linguistic to computational representations. A fuzzy set u on R is a mapping $u : R \to [0, 1]$, and $u(x)$ is the degree of membership of the element x in the fuzzy set u. The fuzzy set is a generalization of the classical set whose characteristic function is valued in $\{0, 1\}$. By fuzziness, we mean a type of imprecision which is associated with fuzzy sets, that is, classes in which there is no sharp transition from membership to nonmembership. In fact, in sharp contrast to the notion of a class or a set in mathematics, most of the classes in the real world do not have crisp boundaries which separate those objects which belong to a class from those which do not. For notational purposes, it is convenient to have a device for indicating that a fuzzy set is obtained from a nonfuzzy set by fuzzifying the boundaries of the latter set. In 1970, Bellman and Zadeh [4] employed a wavy bar under a symbol which defines the nonfuzzy set.

Recently, Torra [26] introduced the concept of hesitant fuzzy set (HFS) as an extension of the FS in which the membership degree of a given element, called the hesitant fuzzy element (HFE), is defined as a set of possible values. This situation can be found in a group decision making problem. To clarify the necessity of introducing the HFS, consider a situation in which two decision makers discuss the membership degree of an element x to a set A, one wants to assign 0.2, but the other 0.4. Accordingly, the difficulty of establishing a common membership degree is not because there is a margin of error or some possibility distribution values, but because there is a set of possible values. In 2018, Alcantud and Torra investigated decomposition theorems and extension principles for the hesitant fuzzy set [2]. In 2019, Xie and Gong [27] proposed a hesitant soft fuzzy rough set model and established an approach to decision making problem based on this model.

On the other hand, the classical binary relations was also extended to the fuzzy binary relations on two ordinary sets [16]. For two given ordinary sets A and B, a fuzzy relation is a fuzzy subset of the set $A \times B$. The uncertainty environment for a variational inequality leads to certain degrees of fuzziness in the classical relation. In 2001, Hu [13] introduced the fuzzy variational inequality over a compact set by using the tolerance approach. Subsequently, Hu [14] investigated the generalized variational inequality with fuzzy relation and showed that such problems can be transformed into regular optimization problems. In 2009, Hu and Liu [15] discussed mathematical programs with fuzzy parametric variational inequalities. In 2019, Xie and Gong [28] investigated the generalized variational-like inequalities for fuzzy-vector-valued functions. In this paper, we further discuss the generalized variational inequality with hesitant fuzzy relation. In addition, real-time solutions to theses problems are always needed in engineering applications, and thus they have to be solved in real time to optimize the performance of dynamical systems. As parallel computational models, recurrent neural network possess many desirable properties for real-time information processing. In 2003, M.A. Noor [24] investigated some implicit projected dynamical systems associated with quasi variational inequalities by using the techniques of the projection and the Wiener-Hopf equations. Indeed, by means of level sets of the hesitant fuzzy relations, the generalized variational inequality with hesitant fuzzy relation can be transformed into the classical (nonfuzzy) generalized variational inequality. Thus, we further propose a projection neural network for solving such problem, which is a dynamical system, and discuss the stability of the projected dynamical system.

The rest of the paper is organized as follows. In Section 2, we recall some preliminaries related to hesitant fuzzy sets. In Section 3, we introduce the generalized variational inequality with hesitant fuzzy relation. In Section 4, the existence theorem and iterative algorithm of solutions to the generalized variational inequality with hesitant fuzzy relation are given. In Section 5, based on the projection method, we propose a projection neural network for solving the proposed problem and the stability of the projected dynamical system is investigated. Section 6 concludes this paper.

2. Preliminaries

For convenience of the reader, the basic properties of hesitant fuzzy sets are presented in this section.

Definition 2.1. *[3, 26] Let U be a fixed set, a hesitant fuzzy set (HFS) on U is in terms of a function h_E that when applied to U returns a subset of* $[0, 1]$*. To be easily understood, we express the HFS by a mathematical symbol*

$$E = \{\langle x, h_E(x)\rangle | x \in U\},$$

where $h_E(x)$ is a set of some values in $[0,1]$, representing the possible membership degrees of the element $x \in U$ to the set E. For convenience, we call $h = h_E(x)$ a hesitant fuzzy element (HFE) and $H(U)$ the set of HFSs on U. In particular, if $h_E(x)$ is a non-empty and finite subset of $[0,1]$, HFS is called a typical hesitant fuzzy set (THFS).

For each typical hesitant fuzzy set E on U, let

$$h_E(x) = \{h_E^1(x), \cdots, h_E^{l_M(x)}(x)\},$$

where $h_E^1(x) < \cdots < h_E^{l_E(x)}(x)$ and $l_E(x) = |h_E(x)|$ is the cardinality of the HFE $h_E(x)$.

Let U be the universe of discourse, $\forall F, G \in H(U)$, then [26]

(i) the complement of F is denoted by F^c such that $\forall x \in U$,

$$h_{F^c}(x) = \sim h_F(x) = \{1 - h : \forall h \in h_F(x)\};$$

(ii) the intersection of F and G is denoted by $F \cap G$ such that $\forall x \in U$,

$$h_{F\cap G}(x) = h_F(x) \,\overline{\wedge}\, h_G(x) = \{h \in h_F(x) \cup h_G(x) : h \leq \min\{h_F^+(x), h_G^+(x)\}\};$$

(iii) the union of F and G is denoted by $F \cup G$ such that $\forall x \in U$,

$$h_{F\cup G}(x) = h_F(x) \,\underline{\vee}\, h_G(x) = \{h \in h_F(x) \cup h_G(x) : h \geq \max\{h_F^-(x), h_G^-(x)\}\};$$

where $h_F^+(x)$ is the upper bound of F, i.e., $h_F^+(x) = \max\{h : h \in h_F(x)\}$, and $h_F^-(x)$ is the lower bound of F, i.e., $h_F^-(x) = \min\{h : h \in h_F(x)\}$.

(iv) We say $F \subseteq G$ if and only if $h_F(x) \leq h_G(x)$ for any $x \in U$, i.e., $h_F^-(x) \leq h_G^-(x)$ and $h_F^+(x) \leq h_G^+(x)$.

Proposition 2.2. *[29] Let F, G and H be HFSs on U, then for any $x, y, z \in U$, the following properties hold:*

(1) Idempotent:

$$h_F(x) \,\overline{\wedge}\, h_F(x) = h_F(x),\ h_F(x) \,\underline{\vee}\, h_F(x) = h_F(x).$$

(2) Commutativity:

$$h_F(x) \,\overline{\wedge}\, h_G(y) = h_G(y) \,\overline{\wedge}\, h_F(x),\ h_F(x) \,\underline{\vee}\, h_G(y) = h_G(y) \,\underline{\vee}\, h_F(x).$$

(3) Associativity:

$$h_F(x) \,\overline{\wedge}\, (h_G(y) \,\overline{\wedge}\, h_H(z)) = (h_F(x) \,\overline{\wedge}\, h_G(y)) \,\overline{\wedge}\, h_H(z),$$
$$h_F(x) \,\underline{\vee}\, (h_G(y) \,\underline{\vee}\, h_H(z)) = (h_F(x) \,\underline{\vee}\, h_G(y)) \,\underline{\vee}\, h_H(z).$$

(4) Distributivity:

$$h_F(x) \,\overline{\wedge}\, (h_G(y) \,\underline{\vee}\, h_H(z)) = (h_F(x) \,\overline{\wedge}\, h_G(y)) \,\underline{\vee}\, (h_F(x) \,\overline{\wedge}\, h_H(z)),$$
$$h_F(x) \,\underline{\vee}\, (h_G(y) \,\overline{\wedge}\, h_H(z)) = (h_F(x) \,\underline{\vee}\, h_G(y)) \,\overline{\wedge}\, (h_F(x) \,\underline{\vee}\, h_H(z)).$$

(5) De Morgan's laws:

$$\sim (h_F(x) \,\overline{\wedge}\, h_G(y)) = (\sim h_F(x)) \,\underline{\vee}\, (\sim h_G(y)),$$
$$\sim (h_F(x) \,\underline{\vee}\, h_G(y)) = (\sim h_F(x)) \,\overline{\wedge}\, (\sim h_G(y)).$$

(6) Double negation law:

$$\sim (\sim h_F(x)) = h_F(x).$$

The (α, k)-level set and strong (α, k)-level set associated with E are defined, respectively, as

$$_{\alpha,k}h_E = \{x \in U : |\{h \in h_E(x) : h \geq \alpha\}| \geq k\}$$

and

$$_{\alpha^+,k}h_E = \{x \in U : |\{h \in h_E(x) : h > \alpha\}| \geq k\}$$

for all $\alpha \in [0, 1]$ and for all $k \in \{1, 2, \cdots\}$.

Definition 2.3. *[2] Let $\mathbb{E} = \{h_{E(i)}\}_{i\in J}$ be a family of hesitant fuzzy sets on U, indexed by the set of indices J. Then the HFS associated with $\mathbb{E}$, denoted by either $h_{\mathbb{E}}$ or $\bigcup_{i\in J} h_{E(i)}$, is defined as*

$$\begin{aligned} h_{\mathbb{E}} : \quad & U \to \mathcal{P}([0,1]), \\ & x \mapsto \bigcup_{i\in J} h_{E(i)}(x), \end{aligned}$$

where $\mathcal{P}([0, 1])$ denotes the set of all subsets of $[0, 1]$.

Theorem 2.4. *[2] Let h_E be a typical hesitant fuzzy set on U. Then h_E is the HFS associated with the family of fuzzy sets $f = \{_kH\}_{k\in\mathbb{N}^+}$, i.e.,*

$$h_E = \bigcup_{k=1,2,\cdots} {}_kH,$$

where $_1H(x) = \max\{\alpha \in [0, 1] : x \in_{\alpha,1} h_E\} = h_E^{l_E(x)}(x)$ for each $x \in U$; if $_1H, \cdots, _kH$ are known, then $_{k+1}H(x) = \max\{\alpha \in [0, 1] : x \in_{\alpha,k+1} h_E\}$, if $x \in_{\alpha,k+1} h_E$ some $\alpha \in [0, 1]$, and $_{k+1}H(x) = {}_kH(x)$ otherwise.

Theorem 2.4 produces a decomposition of any THFS in terms of the simplest THFSs, which are the fuzzy sets.

Example 2.5. *[27] Let $U = \{x_1, x_2\}$, $E = \{\langle x_1, \{0.3, 0.6, 0.7\}\rangle, \langle x_2, \{0.4, 0.5\}\rangle\}$. Then*

$$_{\alpha,1}h_E = \begin{cases} \{x_1, x_2\}, & \alpha \leq 0.5, \\ \{x_1\}, & 0.5 < \alpha \leq 0.7, \\ \emptyset, & \text{otherwise}, \end{cases} \quad _{\alpha,2}h_E = \begin{cases} \{x_1, x_2\}, & \alpha \leq 0.4, \\ \{x_1\}, & 0.4 < \alpha \leq 0.6, \\ \emptyset, & \text{otherwise}, \end{cases} \quad _{\alpha,3}h_E = \begin{cases} \{x_1\}, & \alpha \leq 0.3, \\ \emptyset, & \text{otherwise}, \end{cases}$$

and $_{\alpha,4}h_E = \emptyset$ for each $\alpha \in [0, 1]$. Thus, we have

$$\begin{aligned} _1H : \quad & U \to [0,1] & _2H : \quad & U \to [0,1] & _3H : \quad & U \to [0,1] \\ & x_1 \mapsto 0.7, & & x_1 \mapsto 0.6, & & x_1 \mapsto 0.3, \\ & x_2 \mapsto 0.5, & & x_2 \mapsto 0.4, & & x_2 \mapsto 0.4, \end{aligned}$$

and $_3H = {}_4H = {}_5H = \cdots$. Therefore, $h_E = \bigcup_{k=1,2,\cdots} {}_kH = {}_1H \cup {}_2H \cup {}_3H$.

Definition 2.6. *[29] Given a universe U, a hesitant fuzzy relation on U is a hesitant fuzzy set such that $R \in H(U \times U)$, i.e., $R = \{\langle (x, y), h_R(x, y)\rangle : (x, y) \in U \times U\}$, where $h_R(x, y)$ is a set of the values in $[0, 1]$, which is used to denote the possible membership degrees of the relationships between x and y.*

R is referred to as serial if and only if $\forall x \in U$, there is a $y \in U$ such that $h_R(x, y) = 1$; R is referred to as reflexive if and only if $h_R(x, x) = 1$ holds for each $x \in U$; R is referred to as symmetric if and only if $h_R(x, y) = h_R(y, x)$ $(\forall x, y \in U)$; R is referred to as transitive if and only if $h_R(x, y) \overline{\wedge} h_R(y, z) \preceq h_R(x, z)$ $(\forall x, y, z \in U)$. If a hesitant fuzzy relation R on U is reflexive, symmetric and transitive, we say R is a hesitant fuzzy equivalent relation on U.

Definition 2.7. *Let R be a hesitant fuzzy relation on U. Then for any $\alpha \in [0,1]$ and any $k \in \{1, 2, \cdots\}$, the (α, k)-level set and strong (α, k)-level set of R, denoted by ${}_{\alpha,k}h_R$ and ${}_{\alpha^+,k}h_R$, respectively, are defined as*

$$ {}_{\alpha,k}h_R = \{(x,y) \in U \times U : |\{h \in h_R(x,y) : h \geq \alpha\}| \geq k\}, $$
$$ {}_{\alpha^+,k}h_R = \{(x,y) \in U \times U : |\{h \in h_R(x,y) : h > \alpha\}| \geq k\}. $$

For any $x \in U$, let

$$ [R(x)]_\alpha^k = \{y \in U | (x,y) \in {}_{\alpha,k}h_R\}. $$

Theorem 2.8. *Let R be a hesitant fuzzy relation on U. If R is typical, then R is a hesitant fuzzy relation associated with the family of fuzzy relations $R = \{{}_tR\}_{t\in\mathbb{N}^+}$ on U, i.e.,*

$$ h_R = \bigcup_{t=1,2,\cdots} {}_tR, $$

where ${}_1R(x,y) = \max\{\alpha \in [0,1] : (x,y) \in {}_{\alpha,1}h_R\} = h_R^{l_R(x,y)}(x,y)$ for each $(x,y) \in U \times U$. If ${}_1R, \cdots, {}_tR$ are known, then ${}_{t+1}R(x,y) = \max\{\alpha \in [0,1] : (x,y) \in {}_{\alpha,t+1}h_R\}$ for $(x,y) \in {}_{\alpha,t+1}h_R$ some $\alpha \in [0,1]$; ${}_{t+1}R(x,y) = {}_tR(x,y)$ otherwise.

Proof. Since R is a typical hesitant fuzzy relation, then according to Theorem 2.4, it is not difficult to prove the conclusion. □

Example 2.9. *Assume that Mr. X wants to buy a car. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be a set of six candidate cars. Suppose that the set of candidate cars U can be characterized by a set of parameters $V = \{v_1, v_2, v_3, v_4\}$, where v_j $(j = 1, 2, 3, 4)$ stand for "being cheap", "being beautiful", "being safe" and "being comfortable", respectively. The characteristics of six candidate choices under four parameters are represented by a hesitant fuzzy relation matrix $R(u_i, v_j)_{6\times 4}$, which describes the attractiveness of the cars which Mr. X is going to buy, as follows:*

$$ R = \begin{pmatrix} 0.6, 0.7 & 0.4, 0.5, 0.6, 0.7 & 0.5, 0.6, 0.7 & 0.4, 0.5, 0.6, 0.7 \\ 0.4, 0.5, 0.6, 0.7 & 0.6, 0.7, 0.8 & 0.6, 0.7, 0.8 & 0.5, 0.7, 0.8 \\ 0.5, 0.6, 0.7 & 0.6, 0.7, 0.8 & 0.8, 0.9 & 0.6, 0.7, 0.8 \\ 0.4, 0.5, 0.6, 0.7 & 0.5, 0.7, 0.8 & 0.6, 0.7, 0.8 & 0.7, 0.9 \\ 0.5, 0.6 & 0.4, 0.5, 0.6 & 0.5, 0.6, 0.7 & 0.5, 0.6 \\ 0.3, 0.4, 0.5, 0.6 & 0.6, 0.7 & 0.7 & 0.6, 0.7 \end{pmatrix}. $$

By Theorem 2.8, we have $h_R = \bigcup_{t=1}^{4} {}_tR$, where

$$ {}_1R = \begin{pmatrix} 0.7 & 0.7 & 0.7 & 0.7 \\ 0.7 & 0.8 & 0.8 & 0.8 \\ 0.7 & 0.8 & 0.9 & 0.8 \\ 0.7 & 0.8 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 & 0.6 \\ 0.6 & 0.7 & 0.7 & 0.7 \end{pmatrix}, \quad {}_2R = \begin{pmatrix} 0.6 & 0.6 & 0.6 & 0.6 \\ 0.6 & 0.7 & 0.7 & 0.7 \\ 0.6 & 0.7 & 0.8 & 0.7 \\ 0.6 & 0.7 & 0.7 & 0.7 \\ 0.5 & 0.5 & 0.6 & 0.5 \\ 0.5 & 0.6 & 0.7 & 0.6 \end{pmatrix}, $$

$$
{}_3R = \begin{pmatrix} 0.6 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.6 & 0.6 & 0.5 \\ 0.5 & 0.6 & 0.8 & 0.6 \\ 0.5 & 0.5 & 0.6 & 0.7 \\ 0.5 & 0.4 & 0.5 & 0.5 \\ 0.4 & 0.6 & 0.7 & 0.6 \end{pmatrix}, \quad {}_4R = \begin{pmatrix} 0.6 & 0.4 & 0.5 & 0.4 \\ 0.4 & 0.5 & 0.6 & 0.5 \\ 0.5 & 0.5 & 0.8 & 0.6 \\ 0.4 & 0.5 & 0.5 & 0.7 \\ 0.5 & 0.4 & 0.5 & 0.5 \\ 0.3 & 0.6 & 0.7 & 0.6 \end{pmatrix}.
$$

3. The generalized variational inequality with hesitant fuzzy relation

Definition 3.1. *Let $M \subseteq R^n$, $f : M \to 2^{R^n}$ be a set-valued mapping, and R is a hesitant fuzzy relation on $M \times R^n$. Then the generalized variational inequality with hesitant fuzzy relation, denoted by $GVI(M, R)$, is defined as*

$$
\begin{array}{ll} \text{find} & (x, y) \\ \text{s.t.} & x \in M, \\ & \langle y, x' - x \rangle \geq 0, \ \forall x' \in M, \\ & \langle (x, y), h_R(x, y) \rangle \in R, \end{array} \tag{3.1}
$$

where $R = \{\langle (x, y), h_R(x, y) \rangle : y \underset{\sim}{=} f(x)\} \subseteq H(R^m \times R^m)$, here the wavy bar under a symbol plays the role of a fuzzifier, that is, a transformation which takes a nonfuzzy set into a fuzzy set which is approximately equal to it. In other words, $y \underset{\sim}{=} f(x)$ is a fuzzy equality and "$\underset{\sim}{=}$" denotes the fuzzified version of "$=$" with the linguistic interpretation "approximately equal to".

Remark 3.2. *For $y, f(x) \in R^n$, since $y \underset{\sim}{=} f(x)$, then $y_j \underset{\sim}{=} f_j(x)$, $j = 1, 2, \cdots, n$, which actually determines a hesitant fuzzy set, whose membership function denoted by h_{R_j}, $j = 1, 2, \cdots, n$. The membership grade $h_{R_j}(x, y)$ can be interpreted as the degree to which the regular equality $y_j = f_j(x)$, $j = 1, 2, \cdots, n$, is satisfied. It is commonly assumed that $h_{R_j}(x, y)$ should be 0 if the regular equality $y_j = f_j(x)$ is strongly violated, and 1 if it is satisfied. In this sense, for $j = 1, 2, \cdots, n$, we can obtain a membership function h_{R_j} in the following forms*

$$
h_{R_j}(x, y) = \begin{cases} 1, & y_j - f_j(x) = 0, \\ h_{l_j}(y_j - f_j(x)), & -c_j \leq y_j - f_j(x) < 0, \\ h_{r_j}(y_j - f_j(x)), & 0 < y_j - f_j(x) \leq d_j, \\ 0, & \text{otherwise}, \end{cases}
$$

where $c_j, d_j \geq 0$, are the tolerance levels which a decision maker can tolerate in the accomplishment of the fuzzy equality $y_j \underset{\sim}{=} f_j(x)$.

Remark 3.3. *If R is a fuzzy relation on $M \times R^n$, then (3.1) reduces to the generalized variational inequality with fuzzy relation proposed by Hu [14].*

Remark 3.4. *Since all the components of $y \underset{\sim}{=} f(x)$ have to be satisfied, for the hesitant fuzzy relation R, we define its membership function as*

$$
h_R(x, y) = \bigcup_{j=1,2,\cdots,n} \{h_{R_j}(x, y)\}.
$$

Definition 3.5. *We say (x, y) is a (α, k)-level solution to the problem $GVI(M, R)$ if (x, y) solves the problem, denoted by $GVI(M, {}_{\alpha,k}h_R)$,*

$$\begin{array}{ll} \text{find} & (x, y) \\ \text{s.t.} & x \in M, \\ & \langle y, x' - x\rangle \geq 0,\ \forall x' \in M, \\ & \langle (x, y), h_R(x, y)\rangle \in {}_{\alpha,k}h_R, \end{array} \tag{3.2}$$

where $\alpha \in [0, 1]$, $k \in \mathbb{N}^+$, ${}_{\alpha,k}h_R = \{(x, y) \in M \times R^m : |\{h \in h_R(x, y) : h \geq \alpha\}| \geq k\}$.

4. The existence theorem and iterative algorithm of solutions to the generalized variational inequality with hesitant fuzzy relation

Definition 4.1. *Let $M \subseteq R^n$, and R be a hesitant fuzzy relation on $M \times R^n$. For all $x_1, x_2 \in M$, R is said to be*

(1) monotone, if for all $y_1 \in [R(x_1)]^k_\alpha$, $y_2 \in [R(x_2)]^k_\alpha$,

$$\langle y_1 - y_2, x_1 - x_2\rangle \geq 0.$$

(2) strongly monotone, if there exists a constant $\delta \in (0, 1)$ such that

$$\langle y_1 - y_2, x_1 - x_2\rangle \geq \delta\|x_1 - x_2\|$$

for all $y_1 \in [R(x_1)]^k_\alpha$, $y_2 \in [R(x_2)]^k_\alpha$, where $\|\cdot\|$ and $\langle\cdot\rangle$ denote norm and inner product on R^n, respectively.

(3) pseudo-monotone, if for all $y_1 \in [R(x_1)]^k_\alpha$, $y_2 \in [R(x_2)]^k_\alpha$,

$$\langle y_1, x_2 - x_1\rangle \geq 0 \Rightarrow \langle y_2, x_2 - x_1\rangle \geq 0.$$

(4) Lipschitz continuous, if there exists a constant $L \in (0, 1)$ such that

$$D([R(x_1)]^k_\alpha, [R(x_2)]^k_\alpha) \leq L\|x_1 - x_2\|,$$

where D is the Hausdorff metric on R^n.

Definition 4.2. *[5] The distance of a point $x_0 \in R^n$ to a closed set $C \subseteq R^n$, in the norm $\|\cdot\|$, is defined as*

$$dist(x_0, C) = \inf\{\|x_0 - x\| : x \in C\}.$$

The infimum here is always achieved. We refer to any point $z \in C$ which is closest to x_0, i.e., satisfies $\|z - x_0\| = \mathrm{dist}(x_0, C)$, as a projection of x_0 on C, denoted by $P_C(x_0)$.

In other words, $P_C : R^n \to C$, and $P_C(x_0) = \mathrm{argmin}\{\|x_0 - x\| : x \in C\}$, we refer to P_C as projection on C.

Lemma 4.3. *[17] Let $M \subseteq R^n$ be a closed and convex set. Then*

$$(x - P_M(x))^T(y - P_M(x)) \leq 0,\ \forall x \in R^n, \forall y \in M, \tag{4.1}$$

$$\|P_M(x) - P_M(y)\| \leq \|x - y\|,\ \forall x, y \in R^n. \tag{4.2}$$

Theorem 4.4. *Let $M \subseteq R^n$, R be a hesitant fuzzy relation on $M \times R^n$. If R is Lipschitz continuous. Then there exists a point $x \in M$ such that $x \in [R(x)]^k_\alpha$, where $\alpha \in [0,1], k \in \mathbb{N}^+$, that is, x is a fixed point of R.*

Proof. Let $x_0 \in M$ and $x_1 \in [R(x_0)]^k_\alpha$. Then there exists $x_2 \in [R(x_1)]^k_\alpha$ and

$$||x_2 - x_1|| \leq L||x_1 - x_0||,$$

where $L \in (0,1)$. Since R and $x_2 \in [R(x_1)]^k_\alpha$, there is a point $x_3 \in [R(x_2)]^k_\alpha$ such that

$$||x_3 - x_2|| \leq L||x_2 - x_1|| \leq L^2||x_1 - x_0||.$$

Then we can obtain a sequence $\{x_n\}$ of points of M satifying $x_{n+1} \in [R(x_n)]^k_\alpha$ and

$$||x_{n+1} - x_n|| \leq L||x_n - x_{n-1}|| \leq L^n||x_1 - x_0||$$

for all $n \geq 1$. Therefore, we have

$$\begin{aligned} ||x_{n+m} - x_n|| \leq & ||x_{n+m} - x_{n+m-1}|| + ||x_{n+m-1} - x_{n+m-2}|| + \cdots + ||x_{n+1} - x_n|| \\ \leq & (L^{n+m-1} + \cdots + L^n)||x_1 - x_0|| \\ \leq & \frac{L^n}{1-L}||x_1 - x_0|| \end{aligned}$$

for all $n, m \geq 1$, thus, the sequence $\{x_{n+1}\}$ is a Cauchy sequence, which implies that $x_n \to x \in R^n$. Therefore, the sequence $[R(x_n)]^k_\alpha$ converges to $x_{n+1} \in [R(x)]^k_\alpha$ weakly, and since $x_{n+1} \in [R(x_n)]^k_\alpha$ for all n, then $x \in [R(x)]^k_\alpha$, therefore, x is a fixed point of R. □

Theorem 4.5. *Let $M \subseteq R^n$ be a closed and convex set, R be a hesitant fuzzy relation on $M \times R^n$. Then (x, y) is a solution of $GVI(M, R)$ if and only if*

$$x = P_M[x - \rho y], \tag{4.3}$$

where $y \in [R(x)]^k_\alpha$ for $\alpha \in [0,1], k \in \mathbb{N}^+$, $\rho > 0$ is a constant, and P_M is the projection of R^n on to M.

Proof. If (x, y) is a solution to $GVI(M, R)$, then $x \in M, y \in [R(x)]^k_\alpha$, and

$$\langle y, x' - x\rangle \geq 0, \ \forall x' \in M.$$

Thus, for a constant $\rho > 0$, we have $\langle \rho y, x' - x\rangle \geq 0, \ \forall x' \in M$. Then for all $v \in M$,

$$\begin{aligned} ||v - (x - \rho y)||^2 = & ||v - x||^2 + 2\langle v - x, \rho y\rangle + ||\rho y||^2 \\ \geq & ||\rho y||^2 \\ = & ||x - (x - \rho y)||^2. \end{aligned}$$

Therefore, $x = \min\limits_{x \in M} \frac{1}{2}||v - (x - \rho y)||^2$, that is, $x = P_M[x - \rho y]$, where $\rho > 0$.

Conversely, if $x = P_M[x - \rho y]$, and $y \in [R(x)]^k_\alpha$, where $\rho > 0$, then $x \in M$. By (4.1) of Lemma 4.3, we obtain

$$\langle P_M[x - \rho y] - (x - \rho y), v - P_M[x - \rho y]\rangle \geq 0, \ \forall v \in M,$$

that is,

$$\langle x - (x - \rho y), v - x\rangle \geq 0, \ \forall v \in M,$$

thus, we have $\langle \rho y, v - x)\rangle \geq 0, \ \forall v \in M$. Since $\rho > 0$ is a constant, then $\langle y, v - x)\rangle \geq 0, \ \forall v \in M$, where $y \in [R(x)]^k_\alpha$. Therefore, (x, y) is a solution of $GVI(M, R)$. □

Theorem 4.5 indicates that $GVI(M,R)$ is equivalent to the following fuzzy fixed point problem

$$H(x) = P_M[x - \rho y], \tag{4.4}$$

where $y \in [R(x)]_\alpha^k$. Accordingly, we can give the following iterative algorithm.
Algorithm 1 For a given $x_0 \in M$ such that $y_0 \in [R(x_0)]_\alpha^k$, where $\alpha \in [0,1], k \in \mathbb{N}^+$.

Step 1. Let

$$x_1 = P_M[x_0 - \rho y_0],$$

where $\rho > 0$ is a constant.

Step 2. Since $y_0 \in [R(x_0)]_\alpha^k$, there exists $y_0 \in [R(x_0)]_\alpha^k$ such that $\|y_0 - y_1\| \leq D([R(x_0)]_\alpha^k, [R(x_1)]_\alpha^k)$. Let

$$x_2 = P_M[x_1 - \rho y_1].$$

Step 3. Find x_n and y_n by the following iterative methods

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq D([R(x_{n+1})]_\alpha^k, [R(x_n)]_\alpha^k), \\ x_{n+1} &= P_M[x_n - \rho y_n], \ n = 1, 2, \cdots . \end{aligned} \tag{4.5}$$

Remark 4.6. *Let R be a fuzzy relation on $M \times R^n$, (x_n, y_n) and (x, y) be the solutions to (4.5) and (3.1), respectively. If R is strongly monotone and Lipschitz continuous, then $x_n \to x$ strongly, and $y_n \to y$ strongly (see Theorem 3.1 in [23]).*

Remark 4.7. *Let R be a hesitant fuzzy relation on $M \times R^n$. If R is Lipschitz continuous, then according to Theorem 3.1 proved by L.W. Liu and Y.Q. Li in [20], the set-valued operator R cannot be monotone.*

5. Stability of the dynamical system for generalized variational inequality with hesitant fuzzy relation

Let $M \subseteq R^n$ be a closed and convex set, R be a hesitant fuzzy relation on $M \times R^n$. Consider the following projected neural network associated with the generalized variational inequality with hesitant fuzzy relation (3.1):

$$\frac{dx(t)}{dt} = \lambda\{P_M[x - \rho y] - x\}, \ x(t_0) = x_0, \tag{5.1}$$

where $\rho > 0$, λ are constants, and $y \in [R(x)]_\alpha^k$, $\alpha \in [0,1]$, $k \in \mathbb{N}^+$. $x(t) = (x_1(t), x_2(t), \cdots, x_m(t))^T$ denotes the state vector of neurons, m is the number of neurons, and the initial value x_0 is given randomly. It is a dynamical system.

Without loss of generality, consider the following nonlinear dynamical system [6]

$$\begin{cases} \frac{dx}{dt} = f(t, x), \\ x(t_0) = x_0, \end{cases} \tag{5.2}$$

where $t \in R$, $x \in M \subseteq R^n$, x_0 is the initial state. If there exists a state $x^\star$ in the state space satisfying

$$f(t, x^\star) = 0, \ \forall t \geq t_0,$$

then we say $x^\star$ is an equilibrium state or an equilibrium point of the system (5.2). The equilibrium point $x^\star$ is said to be stable in the sense of Lyapunov, if for any $\varepsilon > 0$, there exists $\delta > 0$, when $\|x(t_0) - x^\star\| < \delta$, we have $\|x(t_0) - x^\star\| < \varepsilon(t \geq t_0)$; $x^\star$ is said to be asymptotically stable, if $x^\star$ is stable and satisfies $x(t) \to x^\star(t \to \infty)$; $x^\star$ is said to be globally asymptotically stable, if for any initial point, $x^\star$ is asymptotically stable; $x^\star$ is said to be globally exponentially stable, if for any solution of the system $x(t)$, there exist $k > 0, \eta > 0$, such that

$$\|x(t) - x^\star\| \leq k\|x(t_0) - x^\star\| exp(-\eta(t - t_0)),\ \forall t \geq t_0.$$

The system (5.2) is said to globally converges to the set $M' \subseteq R^n$, if for any initial point, the solution of the system $x(t)$ satisfies

$$\lim_{t\to\infty} dist(x(t), M') = 0,$$

where $dist(x(t), M') = \inf_{y\in M'} \|x - y\|$.

Lemma 5.1. *(LaSalle's invariance principle) [19] Let $f(t, x)$ be continuous in the system (5.2). If there exists a continuously differentiable function $V : R^n \to R^1$ satisfying the following conditions*

(i) there exists a constant $r > 0$, such that the set $M_r = \{x \in R^n : V(x) \leq r\}$ is bounded,

(ii) for all $x \in M_r$, $\frac{dV(x)}{dt} \leq 0$,

then for all $x_0 \in M_r$, when $t \to \infty$, $x(t)$ converges to the largest invariant subset of the set $\{x \in R^n : \frac{dV(x)}{dt} \leq 0\}$.

Lemma 5.2. *(Gronwall's inequality) [10] Let $x(x), y(t)$ be real-valued nonnegative continuous functions with domain $\{t : t \geq t_0\}$, and let $a(t) = a_0(|t - t_0|)$, where a_0 is a monotone increasing function. If for $t \geq t_0$,*

$$x(t) \leq a(t) + \int_{t_0}^{t} x(s)y(s)ds,$$

then

$$x(t) \leq a(t)\exp\Big(\int_{\mathrm{t_0}}^{\mathrm{t}} \mathrm{y(s)ds}\Big).$$

Theorem 5.3. *Let $M \subseteq R^n$ be a closed and convex set, R be a hesitant fuzzy relation on $M\times R^n$. $(x^\star, y^\star)$ is a solution of $GVI(M, R)$ if and only if $x^\star$ is an equilibrium point of the dynamical system (5.1).*

Proof. According to Theorem 4.5, $(x^\star, y^\star)$ is a solution of $GVI(M, R)$ if and only if

$$x^\star = P_M[x^\star - \rho y^\star],$$

where $y^\star \in [R(x^\star)]^k_\alpha, \rho > 0$ is a constant, that is,

$$P_M[x^\star - \rho y^\star] - x^\star = 0,$$

namely, $x^\star$ is an equilibrium point of the dynamical system (5.1). □

Theorem 5.4. *Let $M \subseteq R^n$ be a closed and convex set, R be a hesitant fuzzy relation on $M \times R^n$. If R is Lipschitz continuous, then for any $x_0 \in R^n$, there exists a unique continuous solution $x(t)$ of dynamical system (5.1) with $x(t_0) = x_0$, where $t \in [t_0, \infty)$.*

Proof. Let

$$G(x) = \lambda\{P_M[x - \rho y] - x\},\ y \in [R(x)]^k_\alpha,\ \alpha \in [0, 1], k \in \mathbb{N}^+.$$

Then for any $x_1, x_2 \in R^n$, since R is Lipschitz continuous, and by (4.2), we have

$$\begin{aligned}
\|G(x_1) - G(x_2)\| \leq& \lambda\{\|P_M[x_1 - \rho y_1] - P_M[x_2 - \rho y_2]\| + \|x_1 - x_2\|\} \\
\leq& \lambda\{\|x_1 - x_2\| + \|(x_1 - \rho y_1) - (x_2 - \rho y_2)\|\} \\
\leq& \lambda\{2 + \rho L\}\|x_1 - x_2\|,
\end{aligned}$$

where $y_1 \in [R(x_1)]^k_\alpha,\ y_2 \in [R(x_2)]^k_\alpha,\ \rho > 0,\ L > 0$. Thus, $G(x)$ is Lipschitz continuous. Then by the existence and uniqueness theorem of solutions for an ordinary differential equation, for any $x_0 \in R^n$, there exists a unique continuous solution $x(t)$ of dynamical system (5.1) with $x(t_0) = x_0$ over $[t_0, T]$.

On the other hand, since for any $x \in R^n$,

$$\begin{aligned}
\|G(x)\| =& \lambda\{\|P_M[x - \rho y] - x\|\} \\
\leq& \lambda\{\|P_M[x - \rho y] - P_M[x]\| + \|P_M(x) - P_M[x^\star]\| + \|P_M[x^\star] - x\|\} \\
\leq& \lambda\rho\|y\| + \lambda\|x - x^\star\| + \lambda\|P_M[x^\star]\| + \lambda\|x\| \\
\leq& \lambda(2 + \rho L)\|x\| + \lambda\{\|x^\star\| + \|P_M[x^\star]\|\},
\end{aligned}$$

then

$$\|x(t)\| \leq \|x_0\| + \int_{t_0}^{t} \|G(x(s))\|ds \leq (\|x_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|x(s)\|ds,$$

where $k_1 = \lambda\{\|x^\star\| + \|P_M[x^\star]\|\}$, $k_2 = \lambda(2 + \rho L)$. Therefore, by Lemma 5.2, we have

$$\|x(t)\| \leq \{\|x_0\| + k_1(t - t_0)\}\mathrm{exp(k_2(t - t_0)),\ t \in [t_0, T)}.$$

It implies that $x(t)$ is bounded on $[t_0, T)$, then by the extension theorem of solutions for an ordinary differential equation, we have $T = \infty$. □

Theorem 5.5. *Let $M \subseteq R^n$ be a closed and convex set, R be a hesitant fuzzy relation on $M \times R^n$. If R is pseudo-monotone and Lipschitz continuous, then the dynamical system (5.1) is stable in the sense of Lyapunov and globally converges to the solution set S of $GVI(M, R)$.*

Proof. Since R is Lipschitz continuous, by Theorem 5.4, the dynamical system (5.1) has a unique continuous solution $x(t)$. Suppose that $x^\star \in M$ is an equilibrium point of the dynamical system (5.1), then $x^\star$ is a solution of $GVI(M, R)$, it follows that $(y^\star)^T(x - x^\star) \geq 0,\ \forall x \in M$, where $y^\star \in [R(x^\star)]^k_\alpha$, and since R is pseudo-monotone, then we have $y^T(x - x^\star) \geq 0,\ \forall x \in M$, where $y \in [R(x)]^k_\alpha$. Setting $x = P_M[x - \rho y]$, then

$$\langle y, P_M[x - \rho y] - x^\star\rangle \geq 0.$$

On the other hand, for $x^\star \in M$, by (4.1) of Lemma 4.3, we have

$$\langle P_M[x - \rho y] - (x - \rho y), x^\star - P_M[x - \rho y]\rangle \geq 0,$$

that is,

$$\langle P_M[x - \rho y] - x, x^\star - P_M[x - \rho y]\rangle + \langle \rho y, x^\star - P_M[x - \rho y]\rangle \geq 0,$$

therefore, we obtain

$$\langle P_M[x-\rho y]-x, x^\star - x + (x - P_M[x-\rho y])\rangle \geq 0,$$

thus, we have

$$\langle x-x^\star, x-P_M[x-\rho y]\rangle \geq \|x-P_M[x-\rho y]\|^2.$$

Hence, for the following Lyapunov function

$$V(x)=\lambda\|x-x^\star\|^2,\ x\in R^n,$$

we have

$$\frac{dV(x)}{dt}=\frac{dV}{dx}\frac{dx}{dt}=2\lambda\langle x-x^\star, P_M[x-\rho y]-x\rangle \leq 0,$$

where $x\in M_0=\{x\in M : V(x)\leq V(x_0)\}$. Therefore, the dynamical system (5.1) is stable in the sense of Lyapunov.

Furthermore, since $V(x)$ is continuously differentiable on the bounded set M_0, by LaSalle's invariance principle, $x(t)$ converges to the largest invariant subset of the set $\{x\in M : \frac{dV}{dt}=0\}$. Since $\frac{dV}{dt}=0 \Leftrightarrow \frac{dx}{dt}=0$, then $\{x\in M : \frac{dV}{dt}=0\}=\{x\in M : \frac{dx}{dt}=0\}=M_0\cap S$, therefore, $\lim\limits_{t\to\infty} dist(x(t),S)=0$, that is, the dynamical system (5.1) globally converges to the solution set S of $GVI(M,R)$. □

Theorem 5.6. *Let $M\subseteq R^n$ be a closed and convex set, R be a hesitant fuzzy relation on $M\times R^n$. If R is Lipschitz continuous, then for $\lambda<0$, the dynamical system (5.1) globally exponentially converges to the solution of $GVI(M,R)$.*

Proof. Since R is Lipschitz continuous, by Theorem 5.4, the dynamical system (5.1) has a unique continuous solution $x(t)$. Let $x^\star\in M$ is an equilibrium point of the dynamical system (5.1), and consider the following Lyapunov function

$$V(x)=\lambda\|x-x^\star\|^2,\ x\in R^n,$$

we have

$$\begin{aligned}\frac{dV}{dt}&=2\lambda\langle x(t)-x^\star, P_M[x(t)-\rho y]-x(t)\rangle\\ &=-2\lambda\|x(t)-x^\star\|^2+2\lambda\langle x(t)-x^\star, P_M[x(t)-\rho y]-x^\star\rangle.\end{aligned}$$

On the other hand, for the equilibrium point $x^\star\in M$, by Theorem 5.3, we have $x^\star$ is a solution of $GVI(M,R)$), that is, $x^\star=P_M[x^\star-\rho y^\star]$, thus, by (4.1) of Lemma 4.3 and R is Lipschitz continuous, we obtain

$$\begin{aligned}\|P_M[x(t)-\rho y]-x^\star\| =&\|P_M[x(t)-\rho y]-P_M[x^\star-\rho y^\star]\|\\ \leq&\|x-x^\star-\rho(y-y^\star)\|\\ \leq&\|x-x^\star\|+\rho L\|x-x^\star\|\\ \leq&(1+\rho L)\|x-x^\star\|,\end{aligned}$$

where $\rho>0, L>0, y\in[R(x)]^k_\alpha, y^\star\in[R(x^\star)]^k_\alpha, \alpha\in[0,1], k\in\mathbb{N}^+$. Therefore, we have

$$\frac{dV}{dt}=\frac{d}{dt}(\lambda\|x(t)-x^\star\|^2)\leq 2\alpha\lambda\|x(t)-x^\star\|^2,$$

where $\alpha=\rho L$. Setting $\lambda_1=-\lambda$, then $\lambda_1>0$, and we have

$$\|x(t)-x^\star\|\leq\|x(t_0)-x^\star\| exp(-\alpha\lambda_1(t-t_0)),$$

that is, the dynamical system (5.1) globally exponentially converges to the solution of $GVI(M,R)$. □

6. Conclusion

As a generalization of fuzzy relation, hesitant fuzzy relation is a very useful tool in situations where there are some difficulties in determining the membership of an element to a set caused by a doubt between a few different values. In this paper, we obtained the existence theorem and iterative algorithm of solutions to the generalized inequality with hesitant fuzzy relation. Furthermore, we proposed a projected neural network model for solving this type variational inequality by using the projection method. Compared with classical optimization approaches, the prominent advantage of neural computing is that it can converge to the equilibrium point (optimal solution) rapidly, and this advantage motivates us to propose an efficient algorithm, which is based on the neural network approach, for the variational inequality problem. The proposed projected dynamical system is shown to be stable in the sense of Lyapunov, globally convergent and globally exponentially convergent under various conditions.

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Conflict of Interest

The authors declare that they have no conflict of interest in this paper.

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