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Research article

The Faber polynomial expansion method and the Taylor-Maclaurin coefficient estimates of Bi-Close-to-Convex functions connected with the q-convolution

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Abstract: In this paper, we introduce a new class of analytic and bi-close-to-convex functions connected with q-convolution, which are defined in the open unit disk. We find estimates for the general Taylor-Maclaurin coefficients of the functions in this subclass by using the Faber polynomial expansion method. Several corollaries and consequences of our main results are also briefly indicated.

Keywords: analytic functions; univalent functions; Bieberbach conjecture (de Branges theorem); Carathéodory Lemma; Faber polynomial expansion; Bi-Close-to-Convex functions; convolution of analytic functions; q-derivative (or q-difference) operator; q-convolution; Poisson operator and Pascal distribution operator
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1. Introduction, definitions and preliminaries

In his survey-cum-expository review article, Srivastava [35] included also a brief overview of the classical q-analysis versus the so-called (p, q)-analysis with an obviously redundant additional parameter p (see, for details, [35, p. 340]). The present sequel to Srivastava's widely-cited review article [35], we apply the concept of q-convolution in order to introduce and study the general Taylor-Maclaurin coefficient estimates for functions belonging to a new class of normalized analytic and bi-close-to-convex functions in the open unit disk, which we have defined here.

Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \Delta), \tag{1.1}$$

where Δ denotes the open unit disk in the complex z-plane given by

$$\Delta := \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| < 1\}.$$

Also let $\mathcal{S} \subset \mathcal{A}$ consist of functions which are also univalent in Δ.

If the function f is given by (1.1) and the function $\Upsilon \in \mathcal{A}$ is given by

$$\Upsilon(z) = z + \sum_{n=2}^{\infty} \psi_n z^n \qquad (z \in \Delta), \tag{1.2}$$

then the Hadamard product (or convolution) of the functions f and Υ is defined by defined by

$$(f * \Upsilon)(z) := z + \sum_{n=2}^{\infty} a_n \psi_n z^n =: (\Upsilon * f)(z) \qquad (z \in \Delta).$$

For $0 \leqq \alpha < 1$, we let $S^*(\alpha)$ denote the class of functions $g \in \mathcal{S}$ which are starlike of order α in Δ such that

$$\Re\left(\frac{zg'(z)}{g(z)}\right) > \alpha \qquad (z \in \Delta).$$

We denote by $C(\alpha)$ the class of functions $f \in \mathcal{S}$ which are close-to-convex of order α in Δ such that (see [10, 24])

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha \qquad (z \in \Delta),$$

where

$$g \in S^*(0) =: S^*.$$

We note that

$$S^*(\alpha) \subset C(\alpha) \subset \mathcal{S} \qquad \text{and} \qquad |a_n| < n \qquad (\forall\ f \in \mathcal{S};\ n \in \mathbb{N} \setminus \{1\})$$

by the Bieberbach conjecture or the De Branges Theorem (see [3, 10]), $\mathbb{N}$ being the set of natural numbers (or the positive integers).

In the above-cited review article, Srivastava [35] made use of various operators of q-calculus and fractional q-calculus. We begin by recalling the definitions and notations as follows (see also [33] and [45, pp. 350–351]).

The q-shifted factorial is defined, for $\lambda, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, by

$$(\lambda; q)_n = \begin{cases} 1 & (n = 0) \\ (1-\lambda)(1-\lambda q)\cdots\left(1-\lambda q^{n-1}\right) & (n \in \mathbb{N}). \end{cases}$$

By using the q-gamma function $\Gamma_q(z)$, we get

$$\left(q^{\lambda}; q\right)_n = \frac{(1-q)^n \ \Gamma_q(\lambda+n)}{\Gamma_q(\lambda)} \qquad (n \in \mathbb{N}_0),$$

where (see [19, 33])

$$\Gamma_q(z) = (1-q)^{1-z} \ \frac{(q;q)_\infty}{(q^z;q)_\infty} \qquad (|q| < 1).$$

We note also that

$$(\lambda; q)_\infty = \prod_{n=0}^{\infty} (1-\lambda q^n) \qquad (|q| < 1),$$

and that the q-gamma function $\Gamma_q(z)$ satisfies the following recurrence relation:

$$\Gamma_q(z+1) = [z]_q \ \Gamma_q(z),$$

where $[\lambda]_q$ denotes the basic (or q-) number defined as follows:

$$[\lambda]_q := \begin{cases} \dfrac{1-q^{\lambda}}{1-q} & (\lambda \in \mathbb{C}) \\ 1 + \sum\limits_{j=1}^{\ell-1} q^j & (\lambda = \ell \in \mathbb{N}). \end{cases} \tag{1.3}$$

Using the definition in (1.3), we have the following consequences:

(i) For any non-negative integer $n \in \mathbb{N}_0$, the *q-shifted factorial* is given by

$$[n]_q! := \begin{cases} 1 & (n = 0) \\ \prod\limits_{k=1}^{n} [k]_q & (n \in \mathbb{N}). \end{cases}$$

(ii) For any positive number r, the generalized *q-Pochhammer symbol* is defined by

$$[r]_{q,n} := \begin{cases} 1 & (n = 0) \\ \prod\limits_{k=r}^{r+n-1} [k]_q & (n \in \mathbb{N}). \end{cases}$$

In terms of the classical (Euler's) gamma function $\Gamma(z)$, it is easily seen that

$$\lim_{q \to 1-} \{\Gamma_q(z)\} = \Gamma(z).$$

We also observe that

$$\lim_{q\to 1-}\left\{\frac{\left(q^{\lambda};q\right)_n}{(1-q)^n}\right\}=(\lambda)_n\,,$$

where $(\lambda)_n$ is the familiar Pochhammer symbol defined by

$$(\lambda)_n=\begin{cases}1 & (n=0)\\ \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N}).\end{cases}$$

For $0<q<1$, the *q-derivative operator* (or, equivalently, the *q-difference operator*) D_q is defined by (see [22]; see also [14, 16, 21])

$$\begin{aligned}D_q\,(f*\Upsilon)\,(z)\; &= D_q\left(z+\sum_{n=2}^{\infty}a_n\,\psi_n\,z^n\right)\\ &:=\frac{(f*\Upsilon)\,(z)-(f*\Upsilon)\,(qz)}{z(1-q)}=1+\sum_{n=2}^{\infty}[n]_q\,a_n\,\psi_n\,z^{n-1}\qquad(z\in\Delta),\end{aligned}$$

where, as in the definition (1.3),

$$[n]_q=\begin{cases}\dfrac{1-q^n}{1-q}=1+\sum\limits_{j=1}^{n-1}q^j & (n\in\mathbb{N})\\ \\ 0 & (n=0).\end{cases}\tag{1.4}$$

Remark 1. Whereas a q-extension of the class of starlike functions was introduced in 1990 in [20] by means of the q-derivative operator D_q, a firm footing of the usage of the q-calculus in the context of Geometric Function Theory was actually provided and the generalized basic (or q-) hypergeometric functions were first used in Geometric Function Theory in an earlier book chapter published in 1989 by Srivastava (see, for details, [34]; see also the recent works [25, 27, 32, 36, 37, 39, 40, 46, 51–53, 55–57]).

For $\lambda>-1$ and $0<q<1$, El-Deeb *et al.* [14] defined the linear operator $\mathcal{H}_{\Upsilon}^{\lambda,q}:\mathcal{A}\to\mathcal{A}$ by

$$\mathcal{H}_{\Upsilon}^{\lambda,q}f(z)*\mathcal{M}_{q,\lambda+1}(z)=z\,D_q\,(f*\Upsilon)\,(z)\qquad(z\in\Delta),$$

where the function $\mathcal{M}_{q,\lambda}(z)$ is given by

$$\mathcal{M}_{q,\lambda}(z)=z+\sum_{n=2}^{\infty}\frac{[\lambda]_{q,n-1}}{[n-1]_q!}\,z^n\qquad(z\in\Delta).$$

A simple computation shows that

$$\mathcal{H}_{\Upsilon}^{\lambda,q}f(z)=z+\sum_{n=2}^{\infty}\frac{[n]_q!}{[\lambda+1]_{q,n-1}}\,a_n\psi_n\,z^n\qquad(\lambda>-1;\;0<q<1;\;z\in\Delta).\tag{1.5}$$

From the defining relation (1.5), we can easily verify that the following relations hold true for all $f\in\mathcal{A}$:

$$\text{(i)}\quad [\lambda+1]_q\mathcal{H}_{\Upsilon}^{\lambda,q}f(z)=[\lambda]_q\mathcal{H}_{\Upsilon}^{\lambda+1,q}f(z)+q^{\lambda}\,z\,D_q\left(\mathcal{H}_{\Upsilon}^{\lambda+1,q}f(z)\right)\qquad(z\in\Delta);$$

$$\text{(ii)}\quad \mathcal{I}_{\Upsilon}^{\lambda}f(z):=\lim_{q\to 1-}\mathcal{H}_{\Upsilon}^{\lambda,q}f(z)=z+\sum_{n=2}^{\infty}\frac{n!}{(\lambda+1)_{n-1}}\,a_n\psi_n z^m\qquad(z\in\Delta).\tag{1.6}$$

Remark 2. If we take different particular cases for the coefficients ψ_n, we obtain the following special cases for the operator $\mathcal{H}_h^{\lambda,q}$:

(i) For $\psi_n = 1$, we obtain the operator $\mathfrak{I}_q^{\lambda}$ defined by Arif *et al.* [2] as follows (see also Srivastava [47]):

$$\mathfrak{I}_q^{\lambda} f(z) := z + \sum_{n=2}^{\infty} \frac{[n]_q!}{[\lambda+1]_{q,n-1}}\, a_n\, z^n \qquad (z \in \Delta); \tag{1.7}$$

(ii) For

$$\psi_n = \frac{(-1)^{n-1}\,\Gamma(\upsilon+1)}{4^{n-1}\,(n-1)!\,\Gamma(n+\upsilon)} \qquad \text{and} \qquad \upsilon > 0,$$

we obtain the operator $\mathcal{N}_{\upsilon,q}^{\lambda}$ defined by El-Deeb and Bulboacă [12] and El-Deeb [11] as follows (see also [16]):

$$\begin{aligned}\mathcal{N}_{\upsilon,q}^{\lambda} f(z) &:= z + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\Gamma(\upsilon+1)}{4^{n-1}\,(n-1)!\,\Gamma(n+\upsilon)} \cdot \frac{[n]_q!}{[\lambda+1]_{q,n-1}}\, a_n\, z^n \\ &= z + \sum_{n=2}^{\infty} \frac{[n]_q!}{[\lambda+1]_{q,n-1}}\, \phi_n\, a_n\, z^n \end{aligned} \tag{1.8}$$

$$(\upsilon > 0;\ \lambda > -1;\ 0 < q < 1;\ z \in \Delta),$$

where

$$\phi_n := \frac{(-1)^{n-1}\,\Gamma(\upsilon+1)}{4^{n-1}\,(n-1)!\,\Gamma(n+\upsilon)} \qquad (n \in \mathbb{N} \setminus \{1\}); \tag{1.9}$$

(iii) For

$$\psi_n = \left(\frac{n+1}{m+n}\right)^{\alpha}, \quad \alpha > 0 \qquad \text{and} \qquad n \in \mathbb{N}_0,$$

we obtain the operator $\mathcal{M}_{m,q}^{\lambda,\alpha}$ defined by El-Deeb and Bulboacă (see [13, 43]) as follows:

$$\mathcal{M}_{m,q}^{\lambda,\alpha} f(z) := z + \sum_{n=2}^{\infty} \left(\frac{m+1}{m+n}\right)^{\alpha} \cdot \frac{[n]_q!}{[\lambda+1]_{q,n-1}}\, a_n\, z^n \qquad (z \in \Delta); \tag{1.10}$$

(iv) For

$$\psi_n = \frac{\rho^{n-1}}{(n-1)!}\, e^{-\rho} \qquad \text{and} \qquad \rho > 0,$$

we obtain a q-analogue of the Poisson operator defined in [30] by

$$\mathcal{I}_q^{\lambda,\rho} f(z) := z + \sum_{n=2}^{\infty} \frac{\rho^{n-1}}{(n-1)!}\, e^{-\rho} \cdot \frac{[n]_q!}{[\lambda+1]_{q,n-1}}\, a_n\, z^n \qquad (z \in \Delta); \tag{1.11}$$

(v) For

$$\psi_n = \binom{m+n-2}{n-1}\, \theta^{n-1}\, (1-\theta)^m \qquad (m \in \mathbb{N};\ 0 \leqq \theta \leqq 1),$$

we get a q-analogue $\Psi_{q,\theta}^{\lambda,m}$ of the Pascal distribution operator as follows (see [15]):

$$\Psi_{q,\theta}^{\lambda,m} f(z) := z + \sum_{n=2}^{\infty} \binom{m+n-2}{n-1} \theta^{n-1} (1-\theta)^m \cdot \frac{[n]_q!}{[\lambda+1]_{q,n-1}} \, a_n \, z^n \tag{1.12}$$

$$(z \in \Delta).$$

If f and F are analytic functions in Δ, we say that the function f is subordinate to the function F, written as $f(z) \prec F(z)$, if there exists a Schwarz function s, which is analytic in Δ with $s(0) = 0$ and $|s(z)| < 1$ for all $z \in \Delta$, such that

$$f(z) = F\big(s(z)\big) \qquad (z \in \Delta).$$

Furthermore, if the function F is univalent in Δ, then we have the following equivalence (see, for example, [7, 28])

$$f(z) \prec F(z) \iff f(0) = F(0) \qquad \text{and} \qquad f(\Delta) \subset F(\Delta).$$

The Koebe one-quarter theorem (see [10]) asserts that the image of Δ under every univalent function $f \in \mathcal{S}$ contains the disk of radius $\frac{1}{4}$. Therefore, every function $f \in \mathcal{S}$ has an inverse f^{-1} which satisfies the following inequality:

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f)\,;\; r_0(f) \geqq \frac{1}{4}\right),$$

where

$$\begin{aligned} g(w) = f^{-1}(w) &= w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \\ &= w + \sum_{n=2}^{\infty} A_n \, w^n. \end{aligned}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ denote the class of normalized analytic and bi-univalent functions in Δ given by (1.1). The class Σ of analytic and bi-univalent functions was introduced by Lewin [26], where it was shown that

$$f \in \Sigma \implies |a_2| < 1.51.$$

Brannan and Clunie [4] improved Lewin's result to the following form:

$$f \in \Sigma \implies |a_2| < \sqrt{2}$$

and, subsequently, Netanyahu [29] proved that

$$f \in \Sigma \implies |a_2| < \frac{4}{3}.$$

It should be noted that the following functions:

$$f_1(z) = \frac{z}{1-z}, \quad f_2(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right) \qquad \text{and} \qquad f_3(z) = -\log(1-z),$$

together with their corresponding inverses given by

$$f_1^{-1}(w) = \frac{w}{1+w}, \quad f_2^{-1}(w) = \frac{e^{2w}-1}{e^{2w}+1} \qquad \text{and} \qquad f_3^{-1}(w) = \frac{e^w - 1}{e^w},$$

are elements of the analytic and bi-univalent function class Σ (see [14, 48]). A brief history and interesting examples of the analytic and bi-univalent function class Σ can be found in (for example) [5, 48].

Brannan and Taha [6] (see also [48]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^*(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order α $(0 \leqq \alpha < 1)$, respectively (see [5]). Indeed, following Brannan and Taha [6], a function $f \in \mathcal{A}$ is said to be in the class $S_\Sigma^*(\alpha)$ of bi-starlike functions of order α $(0 < \alpha \leqq 1)$, if each of the following conditions is satisfied:

$$f \in \Sigma \qquad \text{and} \qquad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \Delta)$$

and

$$\left|\arg\left(\frac{z\mathcal{F}'(w)}{\mathcal{F}(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \Delta),$$

where the function $\mathcal{F}$ is the analytic extension of f^{-1} to Δ, given by

$$\mathcal{F}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \qquad (w \in \Delta). \tag{1.13}$$

A function $f \in A$ is said to be in the class $K_\Sigma^*(\alpha)$ of bi-convex functions of order α $(0 < \alpha \leqq 1)$, if each of the following conditions is satisfied:

$$f \in \Sigma, \quad \text{with} \quad \left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (z \in \Delta)$$

and

$$\left|\arg\left(1 + \frac{zg''(w)}{g'(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (w \in \Delta).$$

The classes $S_\Sigma^*(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order α in Δ and bi-convex functions of order α $(0 < \alpha \leqq 1)$ in Δ, corresponding to the function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S_\Sigma^*(\alpha)$ and $K_\Sigma(\alpha)$, non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ are known (see [6, 35, 48]). In fact, this pioneering work by Srivastava *et al.* [48] happens to be one of the most important studies of the bi-univalent function class Σ. It not only revived the study of the bi-univalent function class Σ in recent years, but it has also inspired remarkably many investigations in this area including the present paper. Some of these many recent papers dealing with problems involving the analytic and bi-univalent functions such as those considered in this article include [1, 9, 17, 23, 48], and indeed also many other works (see, for example, [38, 44, 54]).

Sakar and Güney [31] introduced and studied the following class:

$$\mathcal{T}_\Sigma(\lambda, \beta) \quad (0 \leqq \lambda \leqq 1;\ 0 \leqq \beta < 1).$$

In the same way, we define the following subclass of bi-close-to-convex functions $\mathcal{H}_\Sigma^{q,\lambda}(\eta, \beta, \Upsilon)$ as follows.

Definition 1. For $0 \leqq \eta < 1$ and $0 \leqq \beta \leqq 1$, a function $f \in \Sigma$ has the form (1.1) and the function Υ given by (1.2), the function f is said to be in the class $\mathcal{H}_{\Sigma}^{q,\lambda}(\eta,\beta,\Upsilon)$ if there exists a function $g \in \mathcal{S}^{*}$ such that

$$\Re\left(\frac{z\left(\mathcal{H}_{\Upsilon}^{\lambda,q}f(z)\right)' + \beta z^2\left(\mathcal{H}_{\Upsilon}^{\lambda,q}f(z)\right)''}{(1-\beta)\,\mathcal{H}_{\Upsilon}^{\lambda,q}g(z) + \beta z\left(\mathcal{H}_{\Upsilon}^{\lambda,q}g(z)\right)'}\right) > \eta \qquad (z \in \Delta) \tag{1.14}$$

and

$$\Re\left(\frac{z\left(\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{F}(w)\right)' + \beta z^2\left(\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{F}(w)\right)''}{(1-\beta)\,\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{G}(w) + \beta z\left(\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{G}(w)\right)'}\right) > \eta \qquad (w \in \Delta), \tag{1.15}$$

where the function $\mathcal{F}$ is the analytic extension of f^{-1} to Δ, and is given by (1.13), and $\mathcal{G}$ is the analytic extension of g^{-1} to Δ as follows:

$$\mathcal{G}(w) = w - b_2w^2 + \left(2b_2^2 - b_3\right)w^3 - \left(5b_2^3 - 5b_2b_3 + b_4\right)w^4 + \cdots \qquad (w \in \Delta). \tag{1.16}$$

We note that, if $b_n = a_n \quad (n \in \mathbb{N} \setminus \{1\})$, $\mathcal{S}_{\Sigma}^{q,\lambda}(\eta,\beta,\Upsilon)$ becomes the class of bi-starlike functions satisfying the following inequalities:

$$\Re\left(\frac{z\left(\mathcal{H}_{\Upsilon}^{\lambda,q}f(z)\right)' + \beta z^2\left(\mathcal{H}_{\Upsilon}^{\lambda,q}f(z)\right)''}{(1-\beta)\,\mathcal{H}_{\Upsilon}^{\lambda,q}f(z) + \beta z\left(\mathcal{H}_{\Upsilon}^{\lambda,q}f(z)\right)'}\right) > \eta \qquad (z \in \Delta). \tag{1.17}$$

and

$$\Re\left(\frac{z\left(\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{F}(w)\right)' + \beta z^2\left(\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{F}(w)\right)''}{(1-\beta)\,\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{F}(w) + \beta z\left(\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{F}(w)\right)'}\right) > \eta \qquad (w \in \Delta). \tag{1.18}$$

Remark 3. Each of the following limit cases when $q \to 1-$ is worthy of note.

(i) Putting $q \to 1-$, we obtain

$$\lim_{q\to 1-} \mathcal{H}_{\Sigma}^{q,\lambda}(\eta,\beta,h) =: \mathcal{P}_{\Sigma}^{\lambda}(\eta,\beta,h),$$

where $\mathcal{P}_{\Sigma}^{\lambda}(\eta,\beta,\Upsilon)$ represents the functions $f \in \Sigma$ that satisfy (1.14) and (1.15) with $\mathcal{H}_{\Upsilon}^{\lambda,q}$ replaced by $\mathcal{I}_{\Upsilon}^{\lambda}$ as in (1.6).

(ii) Putting

$$\psi_n = \frac{(-1)^{n-1}\Gamma(\upsilon+1)}{4^{n-1}\,(n-1)!\,\Gamma(m+\upsilon)} \qquad (\upsilon > 0),$$

we obtain the class $\mathcal{B}_{\Sigma}^{q,\lambda}(\eta,\beta,\upsilon)$ representing the functions $f \in \Sigma$ that satisfy (1.14) and (1.15) with $\mathcal{H}_{\Upsilon}^{\lambda,q}$ replaced by $\mathcal{N}_{\upsilon,q}^{\lambda}$ as in (1.8).

(iii) Putting

$$\psi_n = \left(\frac{n+1}{m+n}\right)^{\alpha} \qquad (\alpha > 0;\ m \geqq \mathbb{N}_0),$$

we obtain the class $\mathcal{L}_{\Sigma}^{\lambda,q}(\eta,\beta,m,\alpha)$ consisting of the functions $f \in \Sigma$ that satisfy (1.14) and (1.15) with $\mathcal{H}_{\Upsilon}^{\lambda,q}$ replaced by $\mathcal{M}_{m,q}^{\lambda,\alpha}$ as in (1.10).

(iv) Putting

$$\psi_n = \frac{\rho^{n-1}}{(n-1)!}\, e^{-\rho} \qquad (\rho > 0),$$

we obtain the class $\mathcal{M}_{\Sigma}^{q,\lambda}(\eta,\beta,\rho)$ representing the functions $f \in \Sigma$ which satisfy the inequalities in (1.14) and (1.15) with $\mathcal{H}_{\Upsilon}^{\lambda,q}$ replaced by $\mathcal{I}_{q}^{\lambda,\rho}$ as in (1.11).

(v) Putting

$$\psi_n = \binom{m+n-2}{n-1} \theta^{n-1}(1-\theta)^m \qquad (m \in \mathbb{N};\ 0 \leqq \theta \leqq 1),$$

we get the class $\mathcal{W}_{\Sigma}^{q,\lambda}(\eta,\beta,m,\theta)$ of the functions $f \in \Sigma$ which satisfy the inequalities in (1.14) and (1.15) with $\mathcal{H}_{\Upsilon}^{\lambda,q}$ replaced by $\Psi_{q,\theta}^{\lambda,m}$ occurring in (1.12).

Using the Faber polynomial expansion of functions $f \in \mathcal{A}$ which have the normalized form (1.1), the coefficients of its inverse map may be expressed as follows (see [18]):

$$\mathcal{F}(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots)\, w^n = w + \sum_{n=2}^{\infty} A_n\, w^n, \tag{1.19}$$

where

$$\begin{aligned}\mathcal{K}_{n-1}^{-n}(a_2,a_3,\cdots) = {} & \frac{(-n)!}{(-2n+1)!\,(n-1)!} a_2^{n-1} \\ & + \frac{(-n)!}{(2(-n+1))!\,(n-3)!}\, a_2^{n-3}\, a_3 \\ & + \frac{(-n)!}{(-2n+3)!\,(n-4)!}\, a_2^{n-4}\, a_4 \\ & + \frac{(-n)!}{(2(-n+2))!\,(n-5)!}\, a_2^{n-5} \left[a_5 + (-n+2)\, a_3^2\right] \\ & + \frac{(-n)!}{(-2n+5)!\,(n-6)!}\, a_2^{n-6} \left[a_6 + (-2n+5)\, a_3\, a_4\right] \\ & + \sum_{j \geqq 7} a_2^{n-j} U_j \end{aligned} \tag{1.20}$$

such that U_j with $7 \leqq j \leqq n$ is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$. Here such expressions as (for example) $(-n)!$ are to be interpreted *symbolically* by

$$(-n)! \equiv \Gamma(1-n) := (-n)(-n-1)(-n-2)\cdots \qquad (n \in \mathbb{N}_0).$$

In particular, the first three terms of $\mathcal{K}_{n-1}^{-n}$ are given by

$$\mathcal{K}_1^{-2} = -2a_2,$$

$$\mathcal{K}_2^{-3} = 3\left(2a_2^2 - a_3\right)$$

and

$$\mathcal{K}_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right).$$

In general, an expansion of $\mathcal{K}_m^{-n}$ $(n \in \mathbb{N})$ is given by (see [1, 8, 41, 42, 47, 49, 50])

$$\mathcal{K}_m^{-n} = na_m + \frac{n(n-1)}{2}\,\mathcal{D}_m^2 + \frac{n!}{3!\ (n-3)!}\,\mathcal{D}_m^3 + \cdots + \frac{n!}{m!\,(n-m)!}\,\mathcal{D}_m^m,$$

where

$$\mathcal{D}_m^n = \mathcal{D}_m^n(a_2, a_3, a_4, \cdots)$$

and, alternatively,

$$\mathcal{D}_m^n(a_2, a_3, \cdots, a_{m+1}) = \sum_{i_1,\cdots,i_m}\left(\frac{n!}{i_1!\ \cdots\ i_m!}\right) a_2^{i_1}\ \cdots\ a_{m+1}^{i_m},$$

where $a_1 = 1$ and the sum is taken over all non-negative integers $i_1, \cdots, i_m$ satisfying the following constraints:

$$i_1 + i_2 + \cdots + i_m = n$$

and

$$i_1 + 2i_2 + \cdots + mi_m = m.$$

Evidently, we have

$$\mathcal{D}_m^m(a_2, a_3, \cdots, a_{m+1}) = a_2^m.$$

The following Lemma will be needed to prove our results.

The Carathéodory Lemma. (see [10]) *If $\phi \in \mathfrak{P}$ and*

$$\phi(z) = 1 + \sum_{n=1}^{\infty} c_n\, z^n,$$

then

$$|c_n| \leqq 2 \qquad (n \in \mathbb{N}).$$

This inequality is sharp for all positive integers n. Here $\mathfrak{P}$ is the family of all functions ϕ, which analytic and have positive real part in Δ, with $\phi(0) = 1$.

2. A set of main results

In this section, we apply the above-described Faber polynomial expansion method, we derive bounds for the general Taylor-Maclaurin coefficients of functions in $\mathcal{H}_\Sigma^{q,\lambda}(\eta, \beta, \Upsilon)$.

Theorem 1. *Let the function f given by* (1.1) *belong to the class $\mathcal{H}_\Sigma^{q,\lambda}(\eta, \beta, \Upsilon)$. Suppose also that*

$$0 \leqq \eta < 1, \quad 0 \leqq \beta \leqq 1, \quad \lambda > -1 \qquad \text{and} \qquad 0 < q < 1.$$

If $a_k = 0$ for $2 \leqq k \leqq n-1$, then

$$|a_n| \leqq \frac{2\,(1-\eta)\,[\lambda+1]_{q,n-1}}{n\,[1+(n-1)\beta]\ [n]_q!\,\psi_n} + 1.$$

Proof. If $f \in \mathcal{H}_{\Sigma}^{q,\lambda}(\eta,\beta,\Upsilon)$, then there exists a function $g(z)$, given by

$$g(z) := z + \sum_{n=2}^{\infty} b_n\, z^n \in S^*,$$

such that

$$\Re\left(\frac{z\left(\mathcal{H}_{\Upsilon}^{\lambda,q} f(z)\right)' + \beta z^2 \left(\mathcal{H}_{\Upsilon}^{\lambda,q} f(z)\right)''}{(1-\beta)\,\mathcal{H}_{\Upsilon}^{\lambda,q} g(z) + \beta z \left(\mathcal{H}_{\Upsilon}^{\lambda,q} g(z)\right)'}\right) > \eta \qquad (z \in \Delta).$$

Moreover, by using the Faber polynomial expansion, we have

$$\begin{aligned}
&\frac{z\left(\mathcal{H}_{\Upsilon}^{\lambda,q} f(z)\right)' + \beta z^2 \left(\mathcal{H}_{\Upsilon}^{\lambda,q} f(z)\right)''}{(1-\beta)\,\mathcal{H}_{\Upsilon}^{\lambda,q} g(z) + \beta z \left(\mathcal{H}_{\Upsilon}^{\lambda,q} g(z)\right)'} \\
&\quad = 1 + \sum_{n=2}^{\infty}\Bigg([1+\beta(n-1)]\ \frac{[n]_q!}{[\lambda+1]_{q,n-1}}\,\psi_n\,(na_n - b_n) \\
&\qquad + \sum_{t=1}^{n-2} \frac{[n,q]!}{[\lambda+1,q]_{n-1}}\,\psi_n\,[1+(n-t-1)\beta] \\
&\qquad\quad \cdot K_t^{-1}\,[(1+\beta)\,b_2, (1+2\beta)\,b_3, \cdots, (1+t\beta)\,b_{t+1}] \\
&\qquad\quad \cdot [(n-t)\ a_{n-t} - b_{n-t}]\Bigg) z^{n-1} \qquad (z \in \Delta).
\end{aligned} \tag{2.1}$$

Also, for the inverse map $\mathcal{F} = f^{-1}$, there exists a function $\mathcal{G}(w)$, given by

$$\mathcal{G}(w) = w + \sum_{n=2}^{\infty} B_n\, w^n \in S^*,$$

such that

$$\Re\left(\frac{z\left(\mathcal{H}_{\Upsilon}^{\lambda,q} \mathcal{F}(w)\right)' + \beta z^2 \left(\mathcal{H}_{\Upsilon}^{\lambda,q} \mathcal{F}(w)\right)''}{(1-\beta)\,\mathcal{H}_{\Upsilon}^{\lambda,q} \mathcal{G}(w) + \beta z \left(\mathcal{H}_{\Upsilon}^{\lambda,q} \mathcal{G}(w)\right)'}\right) > \eta \qquad (w \in \Delta),$$

the Faber polynomial expansion of the inverse map $\mathcal{F} = f^{-1}$ is given by

$$\mathcal{F}(w) = w + \sum_{n=2}^{\infty} A_n\, w^n,$$

so we have

$$\frac{z\left(\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{F}(w)\right)' + \beta z^2\left(\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{F}(w)\right)''}{(1-\beta)\,\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{G}(w) + \beta z\left(\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{G}(w)\right)'}$$
$$= 1 + \sum_{n=2}^{\infty}\Bigg([1+\beta(n-1)]\ \frac{[n]_q!}{[\lambda+1]_{q,n-1}}\,\psi_n\,(nA_n - B_n)$$
$$+ \sum_{t=1}^{n-2}\frac{[n]_q!}{[\lambda+1]_{q,n-1}}\psi_n\,[1+(n-t-1)\beta]$$
$$\cdot K_t^{-1}\,[(1+\beta)\,B_2, (1+2\beta)\,B_3, \cdots, (1+t\beta)\,B_{t+1}]$$
$$\cdot [(n-t)\,A_{n-t} - B_{n-t}]\Bigg)\,w^{n-1} \qquad (w \in \Delta). \tag{2.2}$$

Now, since

$$f \in \mathcal{H}_{\Sigma}^{q,\lambda}(\eta,\beta,\Upsilon) \qquad \text{and} \qquad \mathcal{F} = f^{-1} \in \mathcal{H}_{\Sigma}^{q,\lambda}(\eta,\beta,\Upsilon),$$

there are the following two positive real part functions:

$$U(z) = 1 + \sum_{n=1}^{\infty} c_n\, z^n$$

and

$$V(w) = 1 + \sum_{n=1}^{\infty} d_n\, w^n,$$

for which

$$\Re(U(z)) > 0 \qquad \text{and} \qquad \Re(V(w)) > 0 \qquad (z, w \in \Delta),$$

so that

$$\frac{z\left(\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{F}(w)\right)' + \beta z^2\left(\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{F}(w)\right)''}{(1-\beta)\,\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{G}(w) + \beta z\left(\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{G}(w)\right)'} = \eta + (1-\eta)\ U(z)$$
$$= 1 + (1-\eta)\sum_{n=1}^{\infty} c_n\, z^n \tag{2.3}$$

and

$$\frac{z\left(\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{F}(w)\right)' + \beta z^2\left(\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{F}(w)\right)''}{(1-\beta)\,\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{G}(w) + \beta z\left(\mathcal{H}_{\Upsilon}^{\lambda,q}\mathcal{G}(w)\right)'} = \eta + (1-\eta)\ V(w)$$
$$= 1 + (1-\eta)\sum_{n=1}^{\infty} d_n\, w^n. \tag{2.4}$$

Now, under the assumption that $a_k = 0$ for $0 \leqq k \leqq n-1$, we obtain $A_n = -a_n$. Then, by using (2.1) and comparing the corresponding coefficients in (2.3), we obtain

$$\left[1+\beta(n-1)\right] \frac{[n]_q!}{[\lambda+1]_{q,n-1}} \psi_n \left(na_n - b_n\right) = (1-\eta)\ c_{n-1}. \tag{2.5}$$

Similarly, by using (2.2) in the Eq (2.4), we find that

$$\left[1+\beta(n-1)\right] \frac{[n]_q!}{[\lambda+1]_{q,n-1}} \psi_n \left(nA_n - B_n\right) = (1-\eta)\ d_{n-1}, \tag{2.6}$$

$$\left[1+\beta(n-1)\right] \frac{[n]_q!}{[\lambda+1]_{q,n-1}} \psi_n \left(na_n - b_n\right) = (1-\eta)\ c_{n-1} \tag{2.7}$$

and

$$-\left[1+\beta(n-1)\right] \frac{[n]_q!}{[\lambda+1]_{q,n-1}} \psi_n \left(-na_n - B_n\right) = (1-\eta)\ d_{n-1}. \tag{2.8}$$

Taking the moduli of both members of (2.7) and (2.8) for

$$|b_n| \leqq n \qquad \text{and} \qquad |B_n| \leqq n,$$

and applying the Carathéodory Lemma, we conclude that

$$|a_n| \leqq \frac{2(1-\eta)[\lambda+1]_{q,n-1}}{n\left[1+(n-1)\beta\right]\ [n]_q!\,\psi_n} + 1,$$

which completes the proof of Theorem 1. □

If we set

$$\psi_n = \frac{(-1)^{n-1}\Gamma(\upsilon+1)}{4^{n-1}\ (n-1)!\ \Gamma(n+\upsilon)} \qquad (\upsilon > 0)$$

in Theorem 1, we obtain the following special case.

Corollary 1. *Let the function f given by* (1.1) *belong to the class $\mathcal{B}_{\Sigma}^{q,\lambda}(\eta,\beta,\upsilon)$. Suppose also that*

$$0 \leqq \eta < 1, \quad 0 \leqq \beta \leqq 1, \quad \lambda > -1, \quad \upsilon > 0 \qquad \text{and} \qquad 0 < q < 1.$$

If $a_k = 0$ for $2 \leqq k \leqq n-1$, then

$$|a_n| \leqq \frac{2(1-\eta)[\lambda+1]_{q,n-1}}{n\left[1+(n-1)\beta\right]\ [n]_q!\ \phi_n} + 1,$$

where ϕ_n is given by (1.9).

Upon putting

$$\psi_n = \left(\frac{n+1}{m+n}\right)^{\alpha} \qquad (\alpha > 0;\ m \in \mathbb{N}_0)$$

in Theorem 1, we obtain the following result.

Corollary 2. *Let the function f given by* (1.1) *belong to the class* $\mathcal{L}_{\Sigma}^{q,\lambda}(\eta,\beta,m,\alpha)$. *Suppose also that*

$$0 \leqq \eta < 1, \quad 0 \leqq \beta \leqq 1, \quad \lambda > -1, \quad \alpha > 0, \quad m \in \mathbb{N}_0 \qquad \text{and} \qquad 0 < q < 1.$$

If $a_k = 0$ *for* $2 \leqq k \leqq n-1$, *then*

$$|a_n| \leqq \frac{2(1-\eta)(m+n)^{\alpha}\ [\lambda+1]_{q,n-1}}{n[1+(n-1)\beta]\ [n]_q!\ (n+1)^{\alpha}} + 1.$$

If we take

$$\psi_n = \frac{\rho^{n-1}}{(n-1)!}\, e^{-\rho} \qquad (\rho > 0)$$

in Theorem 1, we obtain the following special case.

Corollary 3. *Let the function f given by* (1.1) *belong to the class* $\mathcal{M}_{\Sigma}^{q,\lambda}(\eta,\beta,\rho)$. *Suppose also that*

$$0 \leqq \eta < 1, \quad 0 \leqq \beta \leqq 1, \quad \lambda > -1, \quad \rho > 0 \qquad \text{and} \qquad 0 < q < 1.$$

If $a_k = 0$ *for* $2 \leqq k \leqq n-1$, *then*

$$|a_n| \leqq \frac{2(1-\eta)\ (n-1)!\ [\lambda+1]_{q,n-1}}{n[1+(n-1)\beta]\ [n]_q!\ \rho^{n-1}\ e^{-\rho}} + 1.$$

Upon setting

$$\psi_n = \binom{m+n-2}{n-1}\theta^{n-1}\ (1-\theta)^m \qquad (m \in \mathbb{N};\ 0 \leqq \theta \leqq 1)$$

in Theorem 1, we are led to the following result for the above-defined class $\mathcal{W}_{\Sigma}^{q,\lambda}(\eta,\beta,m,\theta)$.

Corollary 4. *Let the function f given by* (1.1) *belong to the following class*:

$$\mathcal{W}_{\Sigma}^{q,\lambda}(\eta,\beta,m,\theta)$$

$$(0 \leqq \eta < 1;\ 0 \leqq \beta \leqq 1;\ \lambda > -1;\ 0 < q < 1;\ m \in \mathbb{N};\ 0 \leqq \theta \leqq 1).$$

If $a_k = 0$ *for* $2 \leqq k \leqq n-1$, *then*

$$|a_n| \leqq \frac{2(1-\eta)[\lambda+1]_{q,n-1}}{n[1+(n-1)\beta]\ [n]_q!\ \binom{m+n-2}{n-1}\theta^{n-1}\ (1-\theta)^m} + 1.$$

In particular, if we let $g(z) = f(z)$, we obtain the class $\mathcal{S}_{\Sigma}^{q,\lambda}(\eta,\beta,\Upsilon)$, which is a subclass of $\mathcal{H}_{\Sigma}^{q,\lambda}(\eta,\beta,\Upsilon)$. We then give the next theorem, which involves the coefficients of this subclass of the analytic and bi-starlike functions in Δ.

Theorem 2. *Let the function f given by* (1.1) *belong to the class* $\mathcal{S}_{\Sigma}^{q,\lambda}(\eta,\beta,\Upsilon)$. *Suppose also that*

$$\gamma \geqq 1, \quad \eta \geqq 0, \quad \lambda > -1, \quad 0 \leqq \beta < 1 \qquad \text{and} \qquad 0 < q < 1.$$

Then

$$|a_2| \leqq \begin{cases} \frac{2(1-\eta)[\lambda+1]_q}{(1+\beta)\ [2]_q!\,\psi_2} & \left(0 \leqq \eta < 1 - \frac{(1+\beta)^2 \left([2]_q!\right)^2 [\lambda+2]_q\ \psi_2^2}{2\left(1+2\beta-\beta^2\right)\ [3]_q!\,[\lambda+1]_q\ \psi_3}\right) \\ \\ \sqrt{\frac{2(1-\eta)[\lambda+1]_{q,2}}{\left(1+2\beta-\beta^2\right)\ [3]_q!\,\psi_3}} & \left(1 - \frac{(1+\beta)^2 \left([2]_q!\right)^2 [\lambda+2]_q\ \psi_2^2}{2\left(1+2\beta-\beta^2\right)\ [3]_q!\,[\lambda+1]_q\psi_3} \leqq \eta < 1\right) \end{cases} \tag{2.9}$$

and

$$|a_3| \leqq \begin{cases} \frac{2(1-\eta)[\lambda+1]_{q,2}}{\left(1+2\beta-\beta^2\right)\ [3]_q!\,\psi_3} & \left(0 \leqq \eta < 1 - \frac{(1+\beta)^2 \left([2]_q!\right)^2 [\lambda+2]_q\ \psi_2^2}{2\left(1+2\beta-\beta^2\right)\ [3]_q!\,[\lambda+1]_q\ \psi_3}\right) \\ \\ \frac{(1-\eta)}{(1+2\beta)}\left(\frac{[\lambda+1]_{q,2}}{[3]_q!\,\psi_3} + \frac{2(1-\eta)[\lambda+1]_q^2}{\left([2]_q!\right)^2\ \psi_2^2}\right) & \left(1 - \frac{(1+\beta)^2 \left([2]_q!\right)^2 [\lambda+2]_q\ \psi_2^2}{2\left(1+2\beta-\beta^2\right)\ [3]_q![\lambda+1]_q\ \psi_3} \leqq \eta < 1\right). \end{cases} \tag{2.10}$$

Proof. Putting $n = 2$ and $n = 3$ in (2.5) and (2.6), we have

$$(1+\beta)\ \frac{[2]_q!}{[\lambda+1]_q}\,\psi_2 a_2 = (1-\eta)\,c_1, \tag{2.11}$$

$$\left[2\,(1+2\beta)\ a_3 - (1+\beta)^2\,a_2^2\right]\frac{[3]_q!}{[\lambda+1]_{q,2}}\,\psi_3 = (1-\eta)\,c_2, \tag{2.12}$$

$$-\,(1+\beta)\ \frac{[2]_q!}{[\lambda+1]_q}\,\psi_2 a_2 = (1-\eta)\,d_1 \tag{2.13}$$

and

$$\left[-2\,(1+2\beta)\ a_3 + \left(3+6\beta-\beta^2\right)a_2^2\right]\frac{[3]_q!}{[\lambda+1]_{q,2}}\,\psi_3 = (1-\eta)\,d_2. \tag{2.14}$$

From (2.11) and (2.13), by using the Carathéodory Lemma, we obtain

$$\begin{aligned} |a_2| &= \frac{(1-\eta)\,[\lambda+1]_q\,|c_1|}{(1+\beta)\,[2]_q!\psi_2} = \frac{(1-\beta)\,[\lambda+1]_q\,|d_1|}{(1+\gamma+2\eta)\,[2]_q!\psi_2} \\ &\leq \frac{2\,(1-\eta)\,[\lambda+1]_q}{(1+\beta)\,[2]_q!\psi_2}. \end{aligned} \tag{2.15}$$

Also, from (2.12) and (2.14), we have

$$2\left(1+2\beta-\beta^2\right)\ \frac{[3]_q!}{[\lambda+1]_{q,2}}\,\psi_3 a_2^2 = (1-\beta)\,(c_2+d_2)\,.$$

Thus, by using the Carathéodory Lemma, we obtain

$$|a_2| \leqq \sqrt{\frac{2\,(1-\beta)\,[\lambda+1]_{q,2}}{(1+2\beta-\beta^2)\ [3]_q!\,\psi_3}}. \tag{2.16}$$

From (2.15) and (2.16), we obtain the desired estimate on the coefficient $|a_2|$ as asserted in (2.9). In order to find the bound on the coefficient $|a_3|$, we subtract (2.14) from (2.12), so that

$$4\,(1+2\beta)\ \frac{[3]_q!}{[\lambda+1]_{q,2}}\,\psi_3\left(a_3 - a_2^2\right) = (1-\eta)\,(c_2-d_2)\,,$$

that is,

$$a_3 = a_2^2 + \frac{(1-\eta)\,(c_2 - d_2)\,[\lambda+1]_{q,2}}{4\,(1+2\beta)\ \ [3]_q!\ \psi_3}. \tag{2.17}$$

Now, upon substituting the value of a_2^2 from (2.16) into (2.17) and using the Carathéodory Lemma, we find that

$$|a_3| \leqq \frac{2\,(1-\beta)\,[\lambda+1]_{q,2}}{(1+2\beta-\beta^2)\ \ [3]_q!\ \psi_3}. \tag{2.18}$$

Moreover, upon substituting the value of a_2^2 from (2.11) into (2.12), we have

$$a_3 = \frac{(1-\eta)}{2\,(1+2\beta)} \left(\frac{[\lambda+1]_{q,2}\ c_2}{[3]_q!\psi_3} + \frac{(1-\eta)\,[\lambda+1]_q^2 c_1^2}{\left([2]_q!\right)^2 \psi_2^2} \right).$$

Applying the Carathéodory Lemma, we obtain

$$|a_3| \leqq \frac{(1-\eta)}{(1+2\beta)} \left(\frac{[\lambda+1]_{q,2}}{[3]_q!\psi_3} + \frac{2\,(1-\eta)\,[\lambda+1]_q^2}{\left([2]_q!\right)^2 \psi_2^2} \right). \tag{2.19}$$

Finally, by combining (2.18) and (2.19), we have the desired estimate on the coefficient $|a_3|$ as asserted in (2.10). The proof of Theorem 2 is thus completed. □

3. Conclusions

In our present investigation, we have made use of the concept of q-convolution with a view to introducing a new class of analytic and bi-close-to-convex functions in the open unit disk. For functions belonging to this analytic and bi-univalent function class, we have derived estimates for the general coefficients in their Taylor-Maclaurin series expansions in the open unit disk. Our methodology is based essentially upon the Faber polynomial expansion method. We have also presented a number of corollaries and consequences of our main results.

In his recently-published review-cum-expository review article, in addition to applying the q-analysis to Geometric Function Theory of Complex Analysis, Srivastava [35] pointed out the fact that the results for the q-analogues can easily (and possibly trivially) be translated into the corresponding results for the (p, q)-analogues (with $0 < q < p \leqq 1$) by applying some obvious parametric and argument variations, the additional parameter p being *redundant*. Of course, this exposition and observation of Srivastava [35, p. 340] would apply also to the results which we have considered in our present investigation for $0 < q < 1$.

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Conflict of interest

The authors declare that there is no conflict of interest in respect of this article.

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