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Research article

Sharper bounds and new proofs of the exponential function with cotangent

Xiangyang Wu, Kang Yang, Nichang Jiang and Xiao-Diao Chen*

Key Laboratory of Complex Systems Modeling and Simulation, Hangzhou Dianzi University, Hangzhou, China

* **Correspondence:** Email: xiaodiao@hdu.edu.cn; Tel: +8657186919113; Fax: +8657186919113.

Abstract: This paper provides a Páde interpolation based method for finding improved bounds for the exponential function with cotangent. In principle, it can recover many results in prevailing methods. A new method for proving the corresponding bounds is also proposed, which can also be applied for proving more other bounding functions. Numerical experiments show that the new bounds are better than those of prevailing methods.

Keywords: bounds; inequalities; circular functions; Páde interpolation; cotangent function
Mathematics Subject Classification: 33B10, 26D05

1. Introduction

In 1978, Becker and Stark [1] proved the double inequality

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan(x)}{x} < \frac{\pi^2}{\pi^2 - 4x^2}, \tag{1.1}$$

$$\frac{-4x^2}{\pi^2} < x\cot(x) - 1 \triangleq F(x) < \frac{(\pi^2 - 8) - 4x^2}{8}. \tag{1.2}$$

holds for all $x \in (0, \pi/2)$. It is clear that the product of $x\cot(x)$ and $\frac{\tan(x)}{x}$ is equal to 1, while $F(0) = 0$. In [16], it proves that

$$1 < \frac{\sinh(x)}{x} < e^{x\coth x - 1} < \cosh x. \tag{1.3}$$

Since then, many inequalities for cotangent function were established by different ideas and methods; see e.g. [2–18], including the following inequality

$$\cos(x) < e^{x\cot x - 1} < \frac{\sin x}{x} < 1. \tag{1.4}$$

Recently, a new type of bounds for

$$G(x) = e^{x\cot(x)-1} \tag{1.5}$$

is proposed in [7], see more details in Theorems 1 and 2.

Theorem 1. [7] Let $p, q \in (-\infty, 4/\pi^2]$, $p^\star \approx 0.13484$ be the unique zero of the function $\alpha_p(\pi/2) - 1$ on $(-\infty, 4/\pi^2)$, where $\alpha_p(x) = G(x)/(1-px^2)^{1/(3p)}$ if $p \neq 0$ and $\alpha_0(x) = e^{x\cot(x)-1+x^2/3}$. Then the double inequality

$$(1-px^2)^{1/(3p)} < G(x) = e^{x\cot(x)-1} < (1-qx^2)^{1/(3q)} \tag{1.6}$$

holds for all $x \in (0, \pi/2)$ if and only if $p \geq p^\star$ and $q \leq 2/15 \approx 0.13333$.

Theorem 2. [7] For $x \in (0, \pi/2)$, one has that the double inequality

$$(1 - \frac{4}{3\pi^2}x^2)^{\pi^2/4} < G(x) < (1 - \frac{2}{15}x^2)^{5/2}. \tag{1.7}$$

Very recently, Zhu [17] proposed the following results in Theorems 3 and 4, where $h(p) = \ln(1 - \frac{\pi^2}{30} - \frac{\pi^6}{64}p) - \frac{8(45\pi^4 p + 32)}{15\pi^6 p + 32\pi^2 - 960}$,

$$\begin{aligned} p_3 &= \frac{32(30-\pi^2)}{15\pi^6} \approx 0.04467, \\ p_2 &= \frac{4}{70875} \approx 5.6437 \times 10^{-5}, \\ p_1 &= \frac{64(1-\pi^2/30-e^{-2/5})}{\pi^6} \approx 4.6143 \times 10^{-5}, \\ p_0 &\approx 3.799533 \times 10^{-5}, \end{aligned} \tag{1.8}$$

and p_0 is the unique zero of the function $h(p)$.

Theorem 3. [17] Let $0 < p < p_3$ and $x \in (0, \pi/2)$.

(i) If $p_2 \leq p < p_3$, then the function $x \to \frac{\ln(1-2x^2/15-px^6)}{x\cot x - 1} := \frac{f(x)}{g(x)}$ is strictly increasing on $(0, \pi/2)$, and therefore the double inequality

$$(1 - \frac{2x^2}{15} - px^6)^{5/2} < G(x) < (1 - \frac{2x^2}{15} - px^6)^{1/\lambda_p} \tag{1.9}$$

holds, where $\lambda_p = -\ln(1 - \frac{\pi^2}{30} - \frac{\pi^6}{64}p)$.

(ii) If $p_0 < p < p_2$, then there is an $x_0 \in (0, \pi/2)$ such that the function f/g is strictly decreasing on $(0, x_0)$ and strictly increasing on $(x_0, \pi/2)$. Consequently, the inequality

$$G(x) < (1 - \frac{2x^2}{15} - px^6)^{1/\theta_p} \tag{1.10}$$

holds, where $\theta_p = \max(2/5, \lambda_p)$. In particular, we have

$$G(x) < (1 - \frac{2x^2}{15} - px^6)^{5/2}, \ \text{for} \ p_0 < p \leq p_1, \tag{1.11}$$

$$G(x) < (1 - \frac{2x^2}{15} - px^6)^{1/\lambda_p}, \ \text{for} \ p_1 < p \leq p_2. \tag{1.12}$$

(iii) If $0 < p < p_0$, then the function f/g is strictly decreasing on $(0, \pi/2)$, and therefore the double inequality (1.9) is reversed. □

Theorem 4. [17] Let $p_2 \leq p < p_3$ and $p_0 < q \leq p_1$. Then the double inequality

$$(1 - \frac{2}{15}x^2 - px^6)^{5/2} < e^{x\cot(x)-1} < (1 - \frac{2}{15}x^2 - qx^6)^{5/2}$$

holds for all $x \in (0, \pi/2)$ with the best coefficients $p = p_2$ and $q = p_1$. In particularly, we have

$$\begin{aligned} L_{Zhu}(x) &= l_{zhu}(x)^{5/2} = (1 - \frac{2}{15}x^2 - \frac{4}{70875}x^6)^{5/2} < e^{x\cot(x)-1} \\ &< (1 - \frac{2}{15}x^2 - \frac{64(1 - \frac{1}{30}\pi^2 - e^{-2/5})x^6}{\pi^6})^{5/2} = u_{zhu}(x)^{5/2} = U_{Zhu}(x), \end{aligned} \tag{1.13}$$

for all $x \in (0, \pi/2)$. □

In this paper, we present a new method for finding new bounds of both $F(x)$ and $G(x)$, and also provide a new method for the proofs. The first thing is to find four polynomials such that $0 \leq l_1(x) \leq F(x) \leq l_2(x)$ and $0 \leq l_3(x)^\rho \leq G(x) \leq l_4(x)^\rho$, where $x \in (0, \pi/2)$ and $\rho = 8/3$ in this paper. Our method for finding the bounds is as follows. Suppose that $B(x) = \sum_{i=0}^{n} b_i x^i$ is a bounding polynomial of degree n to be found. Let $h(x) = (x\cot(x) - 1 - B(x)) \cdot \sin(x) = x\cos(x) - (1 + B(x))\sin(x)$, where $h(0) = 0$. By selecting a suitable $k \in (0, n)$, the unknown parameter of b_i can be determined by the following constraints

$$h^{(i)}(0) = 0, \ \ h^{(j)}(\pi/2) = 0, \ \ i = 1, 2, \cdots, k, \ \ j = 0, 1, \cdots, n-k, \tag{1.14}$$

which are linear in b_i. On the other hand, let $H(x) = G(x)^{1/\rho} - B(x)$. By selecting a suitable $l \in (0, n)$, the unknown parameter of b_i can be determined by the following constraints

$$H^{(i)}(0) = 0, \ \ H^{(j)}(\pi/2) = 0, \ \ i = 0, 1, \cdots, l, \ \ j = 0, 1, \cdots, n-l, \tag{1.15}$$

which are linear in b_i. Through the above way, together with the Maple software, one can find better bounds for both $F(x)$ and $G(x)$.

The main results are as follows, see also the details of Theorems 5 and 6. We also present a new method for proving the new bounds, see more details in the proofs.

Theorem 5. For all $x \in (0, \pi/2)$, we have that

$$\begin{aligned} l_1(x) &= -\frac{1}{3}x^2 - \frac{1}{45}x^4 + \frac{4}{45}\frac{\pi^4 + 60\pi^2 - 720}{\pi^6}x^6 \leq F(x) \\ &\leq -\frac{1}{3}x^2 - \frac{1}{45}x^4 + \frac{4}{45}\frac{\pi^4 + 210\pi^2 - 2160}{\pi^5}x^5 - \frac{4}{45}\frac{\pi^4 + 360\pi^2 - 3600}{\pi^6}x^6 = l_2(x). \end{aligned} \tag{1.16}$$

Theorem 6. For all $x \in (0, \pi/2)$, we have that

$$L_C(x) = l_3(x)^{8/3} \leq G(x) \leq l_4(x)^{8/3} = U_C(x). \tag{1.17}$$

where

$$\tau = \frac{\pi^4 + 960\pi^2 - 30720(1 - e^{-3/8})}{480\pi^6} \approx -7.4e - 5,$$
$$l_3(x) = 1 - \frac{1}{8}x^2 - \frac{1}{1920}x^4 - \frac{5}{64512}x^6 - \frac{19}{51609600}x^8 \text{ and } l_4(x) = 1 - \frac{1}{8}x^2 - \frac{1}{1920}x^4 + \tau x^6.$$

Remark 1. This paper uses Maple software to deduce and verify the formulae and inequalities.

2. Lemmas

We introduce Theorem 3.5.1 in Page 67, Chapter 3.5 of [19] as follows.

Theorem 7. [19] Let $w_0, w_1, \cdots, w_r$ be $r+1$ distinct points in $[a, b]$, and $n_0, \cdots, n_r$ be $r+1$ integers ≥ 0. Let $N = n_0 + \cdots + n_r + r$. Suppose that $\gamma(t)$ is a polynomial of degree N such that

$$\gamma^{(i)}(w_j) = \beta^{(i)}(w_j), \quad i = 0, \cdots, n_j, \quad j = 0, \cdots, r.$$

Then there exists $\xi_1(t) \in [a, b]$ such that

$$\beta(t) - \gamma(t) = \frac{\beta^{(N+1)}(\xi_1(t))}{(N+1)!} \prod_{i=0}^{r} (t - w_i)^{n_i+1}. \quad \square$$

Lemma 1. For all $x \in (0, \pi/2)$, we have that

$$\begin{aligned} L_{cos}(x) < \cos(x) < U_{cos}(x), \\ L_{sin}(x) < \sin(x) < U_{sin}(x), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
L_{cos}(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{-552960 + 61440\pi + 40320\pi^2 - 400\pi^4 + \pi^6}{120\pi^7}x^7 \\
&- \frac{-967680 + 115200\pi + 67200\pi^2 - 600\pi^4 + \pi^6}{60\pi^8}x^8 \\
&+ \frac{-1290240 + 161280\pi + 86400\pi^2 - 720\pi^4 + \pi^6}{90\pi^9}x^9, \\
U_{cos}(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{-138240 + 7680\pi + 13440\pi^2 - 200\pi^4 + \pi^6}{60\pi^8}x^8 \\
&- \frac{-184320 + 11520\pi + 17280\pi^2 - 240\pi^4 + \pi^6}{45\pi^9}x^9, \\
L_{sin}(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{-23040 + 8960\pi + 80\pi^2 - 200\pi^3 + \pi^5}{5\pi^7}x^7 \\
&+ \frac{8(-30240 + 11520\pi + 120\pi^2 - 240\pi^3 + \pi^5)}{15\pi^8}x^8 \\
&- \frac{2(-107520 + 40320\pi + 480\pi^2 - 800\pi^3 + 3\pi^5)}{15\pi^9}x^9, \\
U_{sin}(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{2903040 - 1290240\pi + 40320\pi^3 - 336\pi^5 + \pi^7}{1260\pi^8}x^8 \\
&- \frac{5160960 - 2257920\pi + 67200\pi^3 - 504\pi^5 + \pi^7}{1260\pi^9}x^9.
\end{aligned} \tag{2.2}$$

Proof. By using Maple software, it can be verified that

$$\begin{aligned}
&L_{cos}^{(i)}(0) = \cos^{(i)}(0),\ L_{cos}^{(j)}(\pi/2) = \cos^{(j)}(\pi/2),\ i = 0, 1, \cdots, 6, j = 0, 1, 2,\\
&U_{cos}^{(i)}(0) = \cos^{(i)}(0),\ U_{cos}^{(j)}(\pi/2) = \cos^{(j)}(\pi/2),\ i = 0, 1, \cdots, 7, j = 0, 1,\\
&L_{sin}^{(i)}(0) = \sin^{(i)}(0),\ L_{sin}^{(j)}(\pi/2) = \sin^{(j)}(\pi/2),\ i = 0, 1, \cdots, 6, j = 0, 1, 2,\\
&U_{sin}^{(i)}(0) = \sin^{(i)}(0),\ U_{sin}^{(j)}(\pi/2) = \sin^{(j)}(\pi/2),\ i = 0, 1, \cdots, 7, j = 0, 1.
\end{aligned} \tag{2.3}$$

Combining Eq (2.3) with Theorem 7, for all $x \in (0, \pi/2)$, there exists $\xi_j(x) \in (0, \pi/2)$ such that

$$\begin{aligned}
\cos(x) - L_{cos}(x) &= \frac{-\cos(\xi_2(x))}{10!}(x-0)^7(x-\pi/2)^3 > 0,\\
\cos(x) - U_{cos}(x) &= \frac{-\cos(\xi_3(x))}{10!}(x-0)^8(x-\pi/2)^2 < 0,\\
\sin(x) - L_{sin}(x) &= \frac{-\sin(\xi_4(x))}{10!}(x-0)^7(x-\pi/2)^3 > 0,\\
\sin(x) - U_{sin}(x) &= \frac{-\sin(\xi_5(x))}{10!}(x-0)^8(x-\pi/2)^2 < 0.
\end{aligned} \tag{2.4}$$

From Eq (2.4), we have completed the proof. □

Lemma 2. For all $x \in (0, \pi/2)$, we have that

$$\begin{aligned}
L_1(x) < \cos(x) < U_1(x),\\
L_2(x) < \sin(x) < U_2(x),
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
L_1(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8\\
&- \frac{113541120 - 9461760\pi - 9676800\pi^2 + 125440\pi^4 - 560\pi^6 + \pi^8}{1680\pi^{10}}x^{10}\\
&+ \frac{154828800 - 12902400\pi^2 - 13547520\pi + 161280\pi^4 - 672\pi^6 + \pi^8}{630\pi^{11}}x^{11}\\
&- \frac{567705600 - 46448640\pi^2 - 51609600\pi + 564480\pi^4 - 2240\pi^6 + 3\pi^8}{2520\pi^{12}}x^{12},\\
U_1(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 - \frac{1}{3628800}x^{10}\\
&+ \frac{-22295347200 + 928972800\pi + 2322432000\pi^2 - 38707200\pi^4 + 241920\pi^6 - 720\pi^8 + \pi^{10}}{907200\pi^{11}}x^{11}\\
&- \frac{-40874803200 + 1857945600\pi + 4180377600\pi^2 - 67737600\pi^4 + 403200\pi^6 - 1080\pi^8 + \pi^{10}}{907200\pi^{12}}x^{12},\\
L_2(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9\\
&- \frac{-743178240 + 340623360\pi - 11612160\pi^3 + 112896\pi^5 - 480\pi^7 + \pi^9}{30240\pi^{11}}x^{11}\\
&+ \frac{-1021870080 + 464486400\pi - 15482880\pi^3 + 145152\pi^5 - 576\pi^7 + \pi^9}{22680\pi^{12}}x^{12},\\
U_2(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 - \frac{1}{39916800}x^{11}\\
&+ \frac{\pi^{11} - 440\pi^9 + 126720\pi^7 - 21288960\pi^5 + 1703116800\pi^3 - 40874803200\pi + 81749606400}{19958400\pi^{12}}x^{12}.
\end{aligned}$$

Proof. By using Maple software, it can be verified that

$$\begin{aligned}
&L_1^{(i)}(0) = \cos^{(i)}(0),\ L_1^{(j)}(\pi/2) = \cos^{(j)}(\pi/2),\ i = 0, 1, \cdots, 9, j = 0, 1, 2,\\
&U_1^{(i)}(0) = \cos^{(i)}(0),\ U_1^{(j)}(\pi/2) = \cos^{(j)}(\pi/2),\ i = 0, 1, \cdots, 10, j = 0, 1,\\
&L_2^{(i)}(0) = \sin^{(i)}(0),\ L_2^{(j)}(\pi/2) = \sin^{(j)}(\pi/2),\ i = 0, 1, \cdots, 10, j = 0, 1,\\
&U_2^{(i)}(0) = \sin^{(i)}(0),\ U_2(\pi/2) = \sin(\pi/2),\ i = 0, 1, \cdots, 11,
\end{aligned} \tag{2.6}$$

Combining Eq (2.6) with Theorem 7, for all $x \in (0, \pi/2)$, there exists $\xi_j(x) \in (0, \pi/2)$ such that

$$\begin{aligned}
\cos(x) - L_1(x) &= \frac{-\sin(\xi_6(x))}{13!}(x-0)^{10}(x-\pi/2)^3 > 0,\\
\cos(x) - U_1(x) &= \frac{-\sin(\xi_7(x))}{13!}(x-0)^{11}(x-\pi/2)^2 < 0,\\
\sin(x) - L_2(x) &= \frac{\cos(\xi_8(x))}{13!}(x-0)^{11}(x-\pi/2)^2 > 0,\\
\sin(x) - U_2(x) &= \frac{\cos(\xi_9(x))}{13!}(x-0)^{12}(x-\pi/2) < 0.
\end{aligned} \tag{2.7}$$

From Eq (2.7), we have completed the proof. □

Lemma 3. For all $x \in (0, \pi/2)$, we have that

$$D_i^{(15)}(x) < 0, \quad i = 1, 2. \tag{2.8}$$

where $D_1(x) = \ln(l_3(x))$ and $D_2(x) = \ln(l_4(x))$.

Proof. By using Maple software, it can be verified that $D_i'(x) = \dfrac{l_{i+2}'(x)}{l_{i+2}(x)}$ is a rational polynomial,

$$l_i(x) > 0, \ i = 3, 4, \quad \text{for all } x \in (0, \pi/2). \tag{2.9}$$

and for all $x \in (0, \pi/2]$,

$$\begin{aligned}
&\kappa_1(x) = -\frac{6013576635102759437}{10064566985421952632540401\cdots} \cdot x^{100} \cdot (361x^4 - 1140000x^2 + 5946259200) < 0,\\
&\kappa_2(x) = \frac{980837x^{80}}{19657357393402251235430 47\cdots}(71781353358054263567x^{18}\\
&\quad + 1896452036091128015517 6x^{16} + 3337503122065455205556736 0x^{14}\\
&\quad + 1410684125562210199749128192 0x^{12} + 324894377668196709607418757120 0x^{10}\\
&\quad - 6549593947524874176492225429504 00x^8 - 7856997287656533278133322524917760 00x^6\\
&\quad - 30827118986500393498592205091614228480 0x^4 - 6819559636192018814458717714765578240 0000x^2\\
&\quad - 7305787082896788623187918918629059461120000)\\
&\quad \le \frac{980837x^{80}}{19657357393402251235430 47\cdots}(71781353358054263567(\frac{\pi}{2})^{18}\\
&\quad + 1896452036091128015517 6(\frac{\pi}{2})^{16} + 3337503122065455205556736 0(\frac{\pi}{2})^{14}\\
&\quad + 1410684125562210199749128192 0(\frac{\pi}{2})^{12} + 324894377668196709607418757120 0(\frac{\pi}{2})^{10}\\
&\quad - 7305787082896788623187918918629059461120000) < 0,
\end{aligned} \tag{2.10}$$

$$
\begin{aligned}
\kappa_3(x) &= \frac{143x^{66}}{2383768320020\cdots}(39615448364964663661602091341417x^{12} \\
&+ 25924702930522607920453711065067326x^{10} + 4152800481023073830237366768877141760x^8 \\
&+ 23163190326130302424321980054575301 5520x^6 - 25827272873676080892589660652890180684800x^4 \\
&- 5719039369320516436256950595322540776448000x^2 \\
&- 345338587141699368149962328958128651059200000) \\
&\le \frac{143x^{66}}{2383768320020\cdots}(39615448364964663661602091341417(\frac{\pi}{2})^{12} \\
&+ 25924702930522607920453711065067326(\frac{\pi}{2})^{10} + 4152800481023073830237366768877141760(\frac{\pi}{2})^8 \\
&+ 231631903261303024243219800545753015520(\frac{\pi}{2})^6 \\
&- 345338587141699368149962328958128651059200000) < 0, \\
\kappa_4(x) &= \frac{143x^{56}}{4582189212530597\cdots}(22684720993522154880577344552574181x^8 \\
&+ 5172108906700221488632662399149638400x^6 + 155423030137100025158843435168244172800x^4 \\
&- 13720914824276998512226838815949728972800x^2 \\
&- 901096344078465844454326452674876532326400) \\
&\le \frac{143x^{56}}{4582189212530597\cdots}(22684720993522154880577344552574181(\frac{\pi}{2})^8 \\
&+ 5172108906700221488632662399149638400(\frac{\pi}{2})^6 + 155423030137100025158843435168244172800(\frac{\pi}{2})^4 \\
&- 901096344078465844454326452674876532326400) < 0, \\
\kappa_5(x) &= \frac{143x^{48}}{1651517824\cdots}(5276194557590127438354017962781x^6 \\
&+ 679047775805399648766007542598188x^4 - 80513313220347669316116853785 61920x^2 \\
&- 1023064843093424843631332985958417920) \\
&\le \frac{143x^{48}}{1651517824\cdots}(5276194557590127438354017962781(\frac{\pi}{2})^6 \\
&+ 679047775805399648766007542598188(\frac{\pi}{2})^4 - 1023064843093424843631332985958417920) < 0, \\
\kappa_6(x) &= \frac{143x^{40}}{1777789\cdots}(115757114711216805713108695715x^6 + 6832160916251525607179902820304x^4 \\
&- 266942123356478045770045817856000x^2 - 38631741293537385575059485525168 00) \\
&\le \frac{143x^{40}}{1777789\cdots}(115757114711216805713108695715(\frac{\pi}{2})^6 + 6832160916251525607179902820304(\frac{\pi}{2})^4 \\
&- 3863174129353738557505948552516800) < 0, \\
\kappa_7(x) &= \frac{635462099385960525911937625063}{110229986056738920652958769673666560000000000}x^{38} - \frac{2019015\cdots}{656130\cdots}x^{36} \\
&- \frac{68975\cdots}{9112\cdots}x^{34} - \frac{372796\cdots}{34173\cdots}x^{32} - \frac{13690\cdots}{25426\cdots}x^{30} - \frac{43944\cdots}{35314\cdots}x^{28} - \frac{27072\cdots}{94593\cdots}x^{26} \\
&- \frac{40832\cdots}{60341\cdots}x^{24} - \frac{14896\cdots}{10948\cdots}x^{22} - \frac{79491\cdots}{30792\cdots}x^{20} - \frac{16220\cdots}{36657\cdots}x^{18} - \frac{12333\cdots}{18183\cdots}x^{16} \\
&- \frac{73029\cdots}{79551\cdots}x^{14} - \frac{10137\cdots}{94704\cdots}x^{12} - \frac{76318\cdots}{73987\cdots}x^{10} - \frac{37113\cdots}{47352\cdots}x^8 - \frac{94727\cdots}{23488\cdots}x^6 \\
&- \frac{78814\cdots}{73400\cdots}x^4 - \frac{40202\cdots}{3670016}x^2 - \frac{86015174817}{327680} \\
&\le \frac{635462099385960525911937625063}{110229986056738920652958769673666560000000000}(\frac{\pi}{2})^{38} - \frac{86015174817}{327680} \approx -262497 < 0,
\end{aligned}
\tag{2.11}
$$

$$
E_1(x) = D_1^{(15)}(x) \cdot \frac{(l_3(x))^{15}}{x} = \sum_{i=1}^{7} \kappa_i(x) < 0, \tag{2.12}
$$

$$
\begin{aligned}
\eta_0 &= -501275174400\tau^2 + \frac{2903105205}{4}\tau - \frac{32934084183}{163840} \approx -257631 < 0,\\
\eta_1 &= 355687428096000\tau^3 - 3436731698400\tau^2 + \frac{218585376009}{64}\tau\\
&\quad - \frac{1047210871707}{1310720} \approx -1071474 < 0,\\
\eta_2 &= 1867358997504000\tau^3 - 5984900366445\tau^2 + \frac{504250928181}{128}\tau\\
&\quad - \frac{376459750738071}{524288000} \approx -1044095 < 0,\\
\eta_3 &= 2906894736345600\tau^3 - \frac{34838803404405}{8}\tau^2 + \frac{299108147080743}{163840}\tau\\
&\quad - \frac{23823291663171}{104857600} \approx -387838 < 0,\\
\eta_4 &= -4221120552929280000\tau^4 + 2065840809832800\tau^3 - \frac{216419474310039}{128}\tau^2\\
&\quad + \frac{57764529541719}{131072}\tau - \frac{6237493525215953}{201326592000} \approx -73853 < 0,\\
\eta_5 &= -4327715629036800000\tau^4 + 824412507343425\tau^3 - \frac{204729959729295}{512}\tau^2\\
&\quad + \frac{209553428215131}{3276800}\tau - \frac{1136803695066039}{536870912000} \approx -9414 < 0,\\
\eta_6 &= -2208143432935440000\tau^4 + \frac{413697216707775}{2}\tau^3 - \frac{2012569732067895}{32768}\tau^2\\
&\quad + \frac{2396774801321901}{419430400}\tau - \frac{10786238873527117}{128849018880000} \approx -937 < 0,\\
\eta_7 &= 341942776655800320000\tau^5 - 637648788296520000\tau^4 + \frac{2231367651197685}{64}\tau^3\\
&\quad - \frac{1662186846325005}{262144}\tau^2 + \frac{20843147377511459}{67108864000}\tau - \frac{69137532971507}{32212254720000} \approx -76.36 < 0,\\
\eta_8 &= 888864844323456000000\tau^5 - \frac{25078822217774325}{2}\tau^4 + \frac{1045272345672555}{256}\tau^3\\
&\quad - \frac{46209424989618891}{104857600}\tau^2 + \frac{356707039011323}{33554432000}\tau - \frac{4764914086571639}{1236950581248000000} \approx -5.3228 < 0,\\
\eta_9 &= 325140459983539200000\tau^5 - \frac{26629649610188625}{16}\tau^4 + \frac{11051878330696203}{32768}\tau^3\\
&\quad - \frac{2112612139196559}{104857600}\tau^2 + \frac{3933405507904873}{16106127360000}\tau - \frac{503679452823547}{98956046499840000} \approx -0.32518 < 0,\\
\eta_{10} &= -649691097802306560000\tau^6 + 3640992784408980000\tau^5 - \frac{41967486562066065}{256}\tau^4\\
&\quad + \frac{639467289676215}{32768}\tau^3 - \frac{204066233468097}{335544320}\tau^2\\
&\quad + \frac{341906440205333}{8589934592000}\tau - \frac{291890139410087}{5699868278390784000} \approx -0.1752e - 1 < 0,\\
\eta_{11} &= 978842321481024000000\tau^6 + \frac{124766698802771025}{2}\tau^5 - \frac{1425514496781375}{128}\tau^4\\
&\quad + \frac{1015807525413681}{1310720}\tau^3 - \frac{10560561112101}{838860800}\tau^2\\
&\quad + \frac{131801051681299}{2748779069440000}\tau - \frac{1905905828827}{4749890231992320000} \approx -0.8660e - 3 < 0,\\
\eta_{12} &= -256287921980794704000\tau^6 + 31140314459654400\tau^5 - \frac{4752114558394203}{8192}\tau^4\\
&\quad + \frac{1090509171444693}{52428800}\tau^3 - \frac{1501709333993869}{8053063680000}\tau^2\\
&\quad + \frac{24151027789889}{54975581388800000}\tau - \frac{188828640913913}{759982437118771200000000} \approx -0.3843e - 4 < 0,\\
\eta_{13} &= 4296253386506425344000\tau^7 + 9488467274550264000\tau^6 + \frac{11426274037957779}{32}\tau^5\\
&\quad - \frac{1239412622216769}{65536}\tau^4 + \frac{9893352016248701}{25165824000}\tau^3 - \frac{87523840543139}{42949672960000}\tau^2\\
&\quad + \frac{30914886879877}{989560464998400000}\tau - \frac{1490357445577}{121597189939003392000000} \approx -1.443e - 6 < 0,
\end{aligned}
\tag{2.13}
$$

$$\eta_{14} = -5527844215510256640000\tau^7 - 285875484512337720\tau^6 + \frac{1291924461885057}{256}\tau^5 - \frac{13826812373176803}{26214400}\tau^4 + \frac{343326946955993}{67108864000}\tau^3 - \frac{704964158765867}{41231686041600000}\tau^2 + \frac{1097983117231}{63331869759897600000}\tau - \frac{472031088529}{97277751951202713600000000} \approx -7.042e-8 < 0,$$

$$\eta_{15} = 1290881951621406144400\tau^7 + \frac{47685858415039845}{4}\tau^6 + \frac{28509057433553697}{40960}\tau^5 - \frac{652812345105921}{104857600}\tau^4 + \frac{905428786841579}{16106127360000}\tau^3 - \frac{6871040565389}{65970697666560000}\tau^2 + \frac{35891718373511}{455989462271262720000000}\tau - \frac{6514498991}{437749883780412211200000000} \approx -1.38e-9 < 0,$$

$$\eta_{16} = -11748882745396532736000\tau^8 - 113767392911628456600\tau^7 - \frac{9141624158069403}{8}\tau^6 - \frac{930716732606661}{163840}\tau^5 - \frac{660154844540191}{4194304000}\tau^4 + \frac{7267075953137}{25769803776000}\tau^3 - \frac{30019058592523}{474989023199232000000}\tau^2 + \frac{417712277983}{1823957849085050880000000}\tau - \frac{443176206473}{10085757322300697346048000000000} \approx -5.275e-11 < 0,$$

$$\eta_{17} = 19881514062837947520000\tau^8 + 104693884779983499 0\tau^7 + \frac{20136540963736197}{512}\tau^6 + \frac{8747399508708861}{16384000}\tau^5 + \frac{1225653319891}{2684354560}\tau^4 + \frac{20526787243963}{4947802324992000}\tau^3 - \frac{27704045369}{37999121855938560000}\tau^2 + \frac{104443209871}{875499767560824422400000000}\tau - \frac{76607531}{537907057189370525122560000000} \approx -5.76e-12 < 0,$$

$$\eta_{18} = -29543732245204697520 0\tau^8 - \frac{224765085254554845}{4}\tau^7 - \frac{32283890093582127}{20480}\tau^6 - \frac{415883933421957}{52428800}\tau^5 - \frac{29801342399829}{1073741824000}\tau^4 - \frac{118380315053}{8246337208320000}\tau^3 - \frac{593974077697}{30399297484750848000000}\tau^2 - \frac{44361317}{23346660468288651264000000}\tau - \frac{569409841}{1290976937254489260294144000000000} \approx 1.784e-13 > 0,$$

$$\eta_{19} = 1431897161615265945600 0\tau^9 + 245605001890752504 00\tau^8 + \frac{159268306888750059}{64}\tau^7 + \frac{2251023788442429}{65536}\tau^6 + \frac{1312387773914217}{8388608000}\tau^5 + \frac{921296875499}{4294967296000}\tau^4 + \frac{2716876333171}{118747255799808000000}\tau^3 + \frac{1715236523}{270215977642229760000000}\tau^2 + \frac{11237273047}{672383821486713156403200000000}\tau + \frac{931931}{1290976937254489260294144000000000} \approx -3.839e-15 < 0,$$

$$\eta_{20} = -24471077161763232000000\tau^9 - \frac{610942806715433692 5}{4}\tau^8 - \frac{3622826148021580 5}{512}\tau^7 - \frac{3948358917784281}{6553600}\tau^6 - \frac{99744642078081}{67108864000}\tau^5 - \frac{31042151181041}{16492674416640000}\tau^4 - \frac{6336100771}{70368744177664000000}\tau^3 - \frac{9395286901}{3242591731706757120000000 0}\tau^2 - \frac{356144789}{134476764297342631280640000000000}\tau - \frac{11722711}{1239337859764309689882378240000000000 0} \approx -4.579e-16 < 0, \tag{2.14}$$

$$
\begin{aligned}
\eta_{21} &= 2729104044406740000000\tau^9 + \frac{1977607040802468075}{32}\tau^8 + \frac{11514555037676691}{8192}\tau^7 \\
&+ \frac{1316464105777821}{209715200}\tau^6 + \frac{57922641936159}{5368709120000}\tau^5 + \frac{1002435525091}{131941395333120000}\tau^4 \\
&+ \frac{1383124041299}{4559894622712627200000000}\tau^3 + \frac{506935429}{11673330234144325632000000000}\tau^2 \\
&+ \frac{427427}{1494186269970473680896000000000000}\tau + \frac{7007}{330490095937149250635300864000000000000} \\
&\approx 2.181e-17 > 0, \\
\eta_{22} &= -7883647560616988160000\tau^{10} - 1785896864433782100 0\tau^9 - \frac{857120550698457045}{512}\tau^8 \\
&- \frac{1099827119297907}{65536}\tau^7 - \frac{2176963963585411}{50331648000}\tau^6 - \frac{919968558509}{21474836480000}\tau^5 - \frac{2034106223149}{949978046398464000000}\tau^4 \\
&- \frac{974652679}{2279947311356313600000000}\tau^3 - \frac{1354796443}{336191910743356578201600000000000}\tau^2 \\
&- \frac{161161}{2581953874508978520588288000000000000}\tau - \frac{7007}{35692930361212119068612493312000000000000000} \\
&\approx -1.598e-19 < 0, \\
\eta_{23} &= 1183810760430217920000\tau^{10} + \frac{3044565517965896925}{4}\tau^9 + \frac{6859153306380375}{256}\tau^8 \\
&+ \frac{3165136713872133}{26214400}\tau^7 + \frac{5709089304919}{33554432000}\tau^6 + \frac{2966311407059}{27487790694400000}\tau^5 \\
&+ \frac{5406229829}{189995609279692800000}\tau^4 + \frac{1535577043}{437749883780412211200000000}\tau^3 \\
&+ \frac{2305303}{268953528594685262561280000000000}\tau^2 + \frac{7007}{12393378597643096898823782400000000000}\tau \\
&\approx -2.277e-20 < 0, \\
\eta_{24} &= -9724961518239567600 0\tau^{10} - \frac{155423762795676375}{8}\tau^9 - \frac{14924991873578787}{65536}\tau^8 \\
&- \frac{101999314766349}{209715200}\tau^7 - \frac{12908827798867}{32212254720000}\tau^6 - \frac{44934917027}{329853488332800000}\tau^5 \\
&- \frac{19243247023}{911978924542525440000000}\tau^4 - \frac{637637}{875499767560824422400000000}\tau^3 \\
&- \frac{49049}{645488468627244630147072000000000 0}\tau^2 \approx 6.297e-22 > 0, \\
\eta_{25} &= 1880145000711065088000\tau^{11} + 443543045444619120 0\tau^{10} + \frac{8509671157752765}{32}\tau^9 \\
&+ \frac{523781479434213}{524288}\tau^8 + \frac{1868609894151}{1677721600}\tau^7 + \frac{8344755419}{17179869184000}\tau^6 \\
&+ \frac{110311201}{1187472557998080000}\tau^5 + \frac{31244213}{7295831396340203520000000}\tau^4 \\
&+ \frac{637637}{10085757322300697346048000000000 0}\tau^3 \approx 4.511e-24 > 0, \\
\eta_{26} &= -2189584385427479040 00\tau^{11} - \frac{227015384261010345}{2}\tau^{10} - \frac{1460784428808705}{1024}\tau^9 \\
&- \frac{97426928798199}{41943040}\tau^8 - \frac{35287469217}{26843545600}\tau^7 - \frac{5099183089}{16492674416640000}\tau^6 \\
&- \frac{7014007}{379991218559385600000}\tau^5 - \frac{637637}{175099953512164884480000000 0}\tau^4 \approx -3.972e-25 < 0, \\
\eta_{27} &= 119865745825809984 00\tau^{11} + \frac{216783864176729 85}{16}\tau^{10} + \frac{236067282445995}{65536}\tau^9 \\
&+ \frac{570631798737}{209715200}\tau^8 + \frac{25509943459}{32212254720000}\tau^7 + \frac{119238119}{1979120929996800000}\tau^6 \\
&+ \frac{7014007}{455989462271262720000000 0}\tau^5 \approx 2.113e-27 > 0,
\end{aligned} \tag{2.15}
$$

$$\begin{aligned}
\eta_{28} &= -172947490910498304000\tau^{12} - 316520163620060400\tau^{11} - \frac{997201323635517}{256}\tau^{10} \\
&\quad - \frac{1123302183003}{262144}\tau^{9} - \frac{211137719793}{134217728000}\tau^{8} - \frac{13026013}{85899345920000}\tau^{7} \\
&\quad - \frac{7014007}{1424967069597696000000}\tau^{6} \approx 9.467e-29 > 0, \\
\eta_{29} &= 1322381356622726400\tau^{12} + \frac{6848484713535465}{2}\tau^{11} + \frac{2682756587511}{512}\tau^{10} + \frac{127511473089}{52428800}\tau^{9} \\
&\quad + \frac{318741423}{1073741824000}\tau^{8} + \frac{1002001}{824633720832000000}\tau^{7} \approx -8.915e-31 < 0, \\
\eta_{30} &= -388408468866818400\tau^{12} - \frac{29816278156665}{8}\tau^{11} - \frac{894298105701}{327680}\tau^{10} \\
&\quad - \frac{373900527}{838860800}\tau^{9} - \frac{6004999}{257698037760000}\tau^{8} \approx -9.443e-33 < 0, \\
\eta_{31} &= 4747592868834816000\tau^{13} + 3829002679101600\tau^{12} + \frac{12708014319}{4}\tau^{11} \\
&\quad + \frac{1589629041}{2621440}\tau^{10} + \frac{14665651}{402653184000}\tau^{9} \approx 9.574e-35 > 0, \\
\eta_{32} &= \frac{21021(-866568318536908800000\tau^{3} - 2341397200896000\tau^{2} - 6349824000\tau - 1001)\tau^{10}}{1048576000} \\
&\quad \approx 3.464e-37 > 0, \\
\eta_{33} &= -\frac{63063(7978167042048000\tau^{2} + 4690483200\tau + 611)\tau^{11}}{327680} \approx -3.151e-39 < 0, \\
\eta_{34} &= -\frac{3972969(2611740672000\tau^{2} - 34099200\tau - 13)\tau^{12}}{512} \approx -3.656e-42 < 0, \\
\eta_{35} &= \frac{42567525(-12441600\tau - 7)\tau^{13}}{4} \approx 2.017e-44 > 0, \\
\eta_{36} &= 6810804000\tau^{14} \approx 1.0457e-48 > 0, \\
\eta_{37} &= 523069747200\tau^{15} \approx -5.959e-51 < 0.
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
E_2(x) = D_2^{(15)}(x)\cdot\frac{(l_4(x))^{15}}{x} &= \sum_{i=0}^{37}\eta_i x^{2i} < x^{66}(\eta_{36}(\frac{\pi}{2})^6 + \eta_{35}(\frac{\pi}{2})^4 + \eta_{33}) \\
&+ x^{58}(\eta_{32}(\frac{\pi}{2})^6 + \eta_{31}(\frac{\pi}{2})^4 + \eta_{29}) + x^{52}(\eta_{28}(\frac{\pi}{2})^4 + \eta_{27}(\frac{\pi}{2})^2 + \eta_{26}) \\
&+ x^{44}(\eta_{25}(\frac{\pi}{2})^6 + \eta_{24}(\frac{\pi}{2})^4 + \eta_{22}) + x^{38}(\eta_{21}(\frac{\pi}{2})^4 + \eta_{19}) + x^{34}(\eta_{18}(\frac{\pi}{2})^2 + \eta_{17}) < 0,
\end{aligned} \tag{2.17}$$

Combining Eq (2.12) with (2.17), we have that

$$E_i(x) < 0 \text{ and } D_i^{(15)}(x) < 0,\ i = 1, 2,\ \forall x \in (0, \pi/2),$$

and complete the proof. □

Lemma 4. For all $x \in (0, \pi/2)$, we have that

$$\begin{aligned}
L_{D,1}(x) &= \sum_{i=2}^{14}\mu_{1,i}x^i < \ln(l_3(x)) = D_1(x) < U_{D,1}(x) = \sum_{i=2}^{14}\mu_{2,i}x^i, \\
L_{D,2}(x) &= \sum_{i=2}^{14}\mu_{3,i}x^i < \ln(l_4(x)) = D_2(x) < U_{D,2}(x) = \sum_{i=2}^{14}\mu_{4,i}x^i,
\end{aligned} \tag{2.18}$$

where

$$
\begin{aligned}
\nu_1 &= \ln(13212057600), \nu_2 = (13212057600 - 412876800\pi^2 - 430080\pi^4 - 16000\pi^6 - 19\pi^8), \nu_3 = \ln(\nu_2),\\
\eta_1 &= 3779309\pi^{24} + 6720029440\pi^{22} + 3475345121280\pi^{20} + 622471793868800\pi^{18}\\
&\quad + 187109607473152000\pi^{16} + 15736038067745587200\pi^{14} + 279828691050037248000\pi^{12}\\
&\quad + 1093032729395928760320000\pi^{10} + 7680310667432542863360000\pi^{8}\\
&\quad - 8133028049070241873920000\pi^{6} + 279776164888016320462848000000\pi^{4}\\
&\quad - 4809547467098178234561331200000\pi^{2} + 10493558110032388875406540800000 0,\\
\end{aligned}
$$

$$
\begin{aligned}
\eta_2 &= 3779309\pi^{24} + 6791004928\pi^{22} + 3602453504000\pi^{20} + 689363492536320\pi^{18}\\
&\quad + 199672880902963200\pi^{16} + 19431451283108659200\pi^{14} + 314189896793613926400 0\pi^{12}\\
&\quad + 163065427523543236608000\pi^{10} + 10634276308752751656960000\pi^{8}\\
&\quad - 112611157602511041331200 00\pi^{6} + 38738238215263798217932800000\pi^{4}\\
&\quad - 665937341598209294016184320 0000\pi^{2} + 14529541998506384596716748800 0000,\\
\eta_3 &= 3779309\pi^{24} + 6838321920\pi^{22} + 3687823319040\pi^{20} + 735100007874560\pi^{18}\\
&\quad + 208667771771289600\pi^{16} + 22021651288188518400\pi^{14} + 3405862907714469888000\pi^{12}\\
&\quad + 201890529314863054848000\pi^{10} + 13166246858455787765760000\pi^{8}\\
&\quad - 13942333798406128926720000\pi^{6} + 47961628266517083507916800000\pi^{4}\\
&\quad - 824493851502544840210513920 0000\pi^{2} + 17988956760055523786411212800 0000,\\
\eta_4 &= 3779309\pi^{24} + 6791004928\pi^{22} + 3602453504000\pi^{20} + 689363492536320\pi^{18}\\
&\quad + 199672880902963200\pi^{16} + 19431451283108659200\pi^{14} + 3141898967936139264000\pi^{12}\\
&\quad + 163065427523543236608000\pi^{10} + 10634276308752751656960000\pi^{8}\\
&\quad - 11261115760251104133120000\pi^{6} + 38738238215263798217932800000\pi^{4}\\
&\quad - 6659373415982092940161843200000\pi^{2} + 145295419985063845967167488000000,\\
\eta_5 &= 233918951325696000\pi^{8} + 19698438006374400 0000\pi^{6} + 529494013611343872 0000\pi^{4}\\
&\quad + 50831425306689011712 00000\pi^{2} - 16266056098140483747840 0000,\\
\eta_6 &= 4128377\pi^{18} + 3922088640\pi^{16} + 496550154240\pi^{14} + 124846787788800\pi^{12}\\
&\quad + 8842476362137600\pi^{10} + 408845184663552000\pi^{8} + 21520543521964032000\pi^{6}\\
&\quad + 795816895457525760000\pi^{4} + 7412916190558814208 0000\pi^{2}\\
&\quad - 50831425306689011712 00000,\\
\mu_{1,2} &= -\frac{1}{8}, \mu_{1,3} = 0, \mu_{1,4} = -\frac{1}{120}, \mu_{1,5} = 0, \mu_{1,6} = -\frac{1}{1260}, \mu_{1,7} = 0, \mu_{1,8} = -\frac{1}{12600}, \mu_{1,9} = 0,\\
\mu_{1,10} &= -\frac{10469}{1238630400}, \mu_{1,11} = 0,\\
\mu_{1,12} &= \frac{19236755865600(-\nu_1 + \nu_3)}{51609600\pi^{12}} + \frac{\eta_1}{51609600\pi^{10}(\nu_3)^2},\\
\mu_{1,13} &= \frac{26635508121600(\nu_1 - \nu_3)}{19353600\pi^{13}} - \frac{\eta_2}{19353600(\nu_2)^2\pi^{11}},\\
\mu_{1,14} &= \frac{32977295769600(\nu_3 - \nu_1)}{25804800\pi^{14}} + \frac{\eta_3}{25804800(\nu_2)^2\pi^{12}},\\
\mu_{2,2} &= -\frac{1}{8}, \mu_{2,3} = 0, \mu_{2,4} = -\frac{1}{120}, \mu_{2,5} = 0, \mu_{2,6} = -\frac{1}{1260}, \mu_{2,7} = 0, \mu_{2,8} = -\frac{1}{12600},\\
\mu_{2,9} &= 0, \mu_{2,10} = -\frac{10469}{1238630400}, \mu_{2,11} = 0, \mu_{2,12} = -\frac{217283}{231211008000},\\
\mu_{2,13} &= \frac{26635508121600\nu_2(-\nu_1 + \nu_3)}{173408256000\pi^{13}} + \frac{\eta_4}{173408256000\pi^{11}\nu_2},\\
\mu_{2,14} &= \frac{\eta_5 \cdot (\nu_1 - \nu_3)}{57802752000\pi^{14}(-\nu_2)} + \frac{\eta_6}{57802752000\pi^{12}\nu_2},
\end{aligned}
$$

$$
\begin{aligned}
\eta_7 &= 983040 - 30720\pi^2 - 32\pi^4 + 11\pi^6 - 983040e^{-3/8},\\
\eta_8 &= 589824001843200\pi^2 - 1920\pi^4 + 511\pi^6 - 58982400e^{-3/8},\\
\eta_9 &= 30474240 - 952320\pi^2 - 992\pi^4 + 211\pi^6 - 30474240e^{-3/8},\\
\eta_{10} &= -1315543449600 + 25008537600\pi^2 + 210370560\pi^4 + 1643520\pi^6 + 58344\pi^8 + 633\pi^{10}\\
&\quad + (-271790899200 + 11324620800\pi^2 - 112066560\pi^4 - 122880\pi^6 - 32\pi^8)\cdot e^{3/4},\\
&\quad + (1494849945600 - 29255270400\pi^2 - 14254080\pi^4)\cdot e^{3/8},\\
&\quad + (-422785843200 - 7077888000\pi^2 - 91422720\pi^4)e^{-3/8},\\
\eta_{11} &= (-645503385600 + 11324620800\pi^2 + 93388800\pi^4 + 983040\pi^6 + 23536\pi^8 + 211\pi^{10})\\
&\quad + (-135895449600 + 5662310400\pi^2 - 56033280\pi^4 - 61440\pi^6 - 16\pi^8)e^{3/4}\\
&\quad + (724775731200 - 14155776000\pi^2 - 6881280\pi^4)e^{3/8}\\
&\quad + (-2831155200\pi^2 - 181193932800 - 30474240\pi^4)e^{-3/8},\\
\eta_{12} &= -2514065817600 + 41523609600\pi^2 + 342097920\pi^4 + 3932160\pi^6 + 78784\pi^8 + 633\pi^{10}\\
&\quad + (-543581798400 + 22649241600\pi^2 - 224133120\pi^4 - 245760\pi^6 - 64\pi^8)e^{3/4}\\
&\quad + (2808505958400 - 54735667200\pi^2 - 26542080\pi^4)e^{3/8}\\
&\quad + (-634178764800 - 9437184000\pi^2 - 91422720\pi^4)e^{-3/8},\\
\eta_{13} &= (5435817984000 - 169869312000\pi^2 - 176947200\pi^4 + 176947200\pi^6)e^{-3/8}\\
&\quad - 2717908992000e^{-3/4} - 2717908992000 + 169869312000\pi^2 - 2477260800\pi^4\\
&\quad - 182476800\pi^6 + 5526720\pi^8 + 5760\pi^{10} - 1021\pi^{12},\\
\eta_{14} &= 6115295232000 - 679477248000\pi^2 - 10970726400\pi^4 - 66355200\pi^6\\
&\quad + 1468800\pi^8 - 70200\pi^{10} - 1021\pi^{12}\\
&\quad + (-16307453952000 + 339738624000\pi^2 + 176947200\pi^4)e^{3/8} - 2717908992000e^{-3/4}\\
&\quad + (27179089920000 + 339738624000\pi^2 + 10793779200\pi^4 + 176947200\pi^6)e^{-3/8},\\
\eta_{15} &= 18345885696000 - 1358954496000\pi^2 - 16633036800\pi^4 - 66355200\pi^6\\
&\quad - 1296000\pi^8 - 108180\pi^{10} - 1021\pi^{12}\\
&\quad + (43486543872000 + 679477248000\pi^2 + 16279142400\pi^4 + 176947200\pi^6)e^{-3/8}\\
&\quad + (-32614907904000 + 679477248000\pi^2 + 353894400\pi^4)e^{3/8} - 2717908992000e^{-3/4},\\
\mu_{3,2} &= -\frac{1}{8}, \mu_{3,3} = 0, \mu_{3,4} = -\frac{1}{120}, \mu_{3,5} = 0, \mu_{3,6} = \frac{-\eta_7}{15360\pi^6}, \mu_{3,7} = 0, \mu_{3,8} = \frac{-\eta_8}{7372800\pi^6},\\
\mu_{3,9} &= 0, \mu_{3,10} = \frac{-\eta_9}{29491200\pi^6}, \mu_{3,11} = 0, \mu_{3,12} = \frac{\eta_{10}}{3686400\pi^{12}},\\
\mu_{3,13} &= -\frac{\eta_{11}}{460800\pi^{13}}, \mu_{3,14} = \frac{\eta_{12}}{1843200\pi^{14}},\\
\mu_{4,i} &= \mu_{3,i}, \quad i = 2, 3, \cdots, 11,\\
\mu_{4,12} &= \frac{\eta_{13}}{1327104000\pi^{12}}, \mu_{4,13} = -\frac{\eta_{14}}{331776000\pi^{14}}, \mu_{4,14} = \frac{\eta_{15}}{331776000\pi^{14}}.
\end{aligned}
$$

Proof. By using Maple software, it can be verified that

$$
\begin{aligned}
&L_{D,1}^{(i)}(0) = D_1^{(i)}(0),\ L_{D,1}^{(j)}(\pi/2) = D_1^{(j)}(\pi/2),\ i = 0, 1, \cdots, 11,\ j = 0, 1, 2,\\
&U_{D,1}^{(i)}(0) = D_1^{(i)}(0),\ U_{D,1}^{(j)}(\pi/2) = D_1^{(j)}(\pi/2),\ i = 0, 1, \cdots, 12,\ j = 0, 1,\\
&L_{D,2}^{(i)}(0) = D_2^{(i)}(0),\ L_{D,2}^{(j)}(\pi/2) = D_2^{(j)}(\pi/2),\ i = 0, 1, \cdots, 11,\ j = 0, 1, 2,\\
&U_{D,2}^{(i)}(0) = D_2^{(i)}(0),\ U_{D,2}^{(j)}(\pi/2) = D_2^{(j)}(\pi/2),\ i = 0, 1, \cdots, 12,\ j = 0, 1.
\end{aligned}
\tag{2.19}
$$

Combing Eq (2.19) with Theorem 7, for all $x \in (0, \pi/2)$, there exists $\xi_i(x) \in (0, \pi/2)$ such that

$$\begin{cases} D_1(x) - L_{D,1}(x) = \dfrac{D_1^{(15)}(\xi_{10}(x))}{15!}(x-0)^{12}(x-\pi/2)^3, \\ D_1(x) - U_{D,1}(x) = \dfrac{D_1^{(15)}(\xi_{11}(x))}{15!}(x-0)^{13}(x-\pi/2)^2, \\ D_2(x) - L_{D,2}(x) = \dfrac{D_2^{(15)}(\xi_{12}(x))}{15!}(x-0)^{12}(x-\pi/2)^3, \\ D_2(x) - U_{D,2}(x) = \dfrac{D_2^{(15)}(\xi_{13}(x))}{15!}(x-0)^{13}(x-\pi/2)^2. \end{cases} \tag{2.20}$$

Combining Eq (2.20) with Lemma 3, we have that

$$\begin{cases} D_1(x) - L_{D,1}(x) > 0, \\ D_1(x) - U_{D,1}(x) < 0, \\ D_2(x) - L_{D,2}(x) > 0, \\ D_2(x) - U_{D,2}(x) < 0, \end{cases} \quad \forall x \in (0, \pi/2). \tag{2.21}$$

which is equivalent to Eq (2.18), and we complete the proof. □

3. Proof of Theorem 5

Prove that $F(x) - l_1(x) > 0$ and $F(x) - l_2(x) < 0$, for all $x \in (0, \pi/2)$.
Let $E_{i+2}(x) = (F(x) - l_i(x)) \cdot \sin(x) = x\cos(x) - (1 + l_i(x))\sin(x)$, $i = 1, 2$. It is equivalent to prove

$$E_3(x) > 0 \text{ and } E_4(x) < 0, \text{ for all } x \in (0, \pi/2). \tag{3.1}$$

For $\forall x \in (0, \pi/2)$, note that $1 + l_i(x) > 0$, $i = 1, 2$, combining with Lemma 1, we have that

$$\begin{aligned}
&\iota_0 = -480\pi^8(\pi^6 + 42\pi^4 + 2520\pi^2 - 30240) \approx 1.4e+9 > 0, \\
&\iota_1 = 30\pi^6(25\pi^7 - 24528\pi^5 + 2217600\pi^3 + 3870720\pi^2 - 26127360\pi - 17418240) \approx 9.1e+8 > 0, \\
&\iota_2 = 9\pi^5(3\pi^9 + 140\pi^7 + 100800\pi^5 - 12364800\pi^3 - 22579200\pi^2 + 187084800\pi - 12902400) \\
&\quad \approx -9.1e+7 < 0, \\
&\iota_3 = 6\pi^4(19\pi^9 - 2520\pi^7 + 403200\pi^5 - 13708800\pi^3 + 29030400\pi^2 + 19353600\pi - 38707200) \\
&\quad \approx -6.0e+7 < 0, \\
&\iota_4 = -\pi^3(\pi^{11} + 10080\pi^7 - 927360\pi^5 + 29030400\pi^3 - 38707200\pi^2 - 232243200\pi + 464486400) \\
&\quad \approx 2.4e+6 > 0, \\
&\iota_5 = 2\pi^2(\pi^{11} - 672\pi^9 + 70560\pi^7 - 1653120\pi^5 + 5806080\pi^4 - 29030400\pi^3 + 38707200\pi^2 \\
&\quad + 232243200\pi - 464486400) \approx 1.5e+6 > 0, \\
&\iota_6 = 4\pi(\pi^7 - 168\pi^5 + 13440\pi^3 - 322560\pi + 645120)(\pi^4 + 60\pi^2 - 720) \approx -38685 < 0, \\
&\iota_7 = -8(\pi^4 + 60\pi^2 - 720)(\pi^7 - 504\pi^5 + 67200\pi^3 - 2257920\pi + 5160960) \approx -23533 < 0, \\
&E_3(x) \geq xL_{cos}(x) - (1 + l_1(x))U_{sin}(x) = (\sum_{i=0}^{7} \iota_i x^i) \cdot \frac{(\pi - 2x)x^7}{226800\pi^{15}} \\
&\geq [(\iota_0 + \iota_2(\frac{\pi}{2})^2 + \iota_3(\frac{\pi}{2})^3) + x^4(\iota_4 + \iota_6(\frac{\pi}{2})^2 + \iota_7(\frac{\pi}{2})^3)] \cdot \frac{(\pi - 2x)x^7}{226800\pi^{15}} > 0,
\end{aligned} \tag{3.2}$$

$$
\begin{aligned}
&\omega_0 = -960\pi^8(\pi^4 + 210\pi^2 - 2160) \approx -9.1e + 7 < 0,\\
&\omega_1 = -5\pi^6(5\pi^7 + 144\pi^5 + 178560\pi^3 - 34560\pi^2 - 4838400\pi + 9953280) \approx -3.6e + 7 < 0,\\
&\omega_2 = 60\pi^5(\pi^7 + 480\pi^5 - 28800\pi^3 + 368640\pi - 414720) \approx 5.9e + 6 > 0,\\
&\omega_3 = 2\pi^4(\pi^9 - 180\pi^7 + 86400\pi^5 - 28800\pi^4 - 3974400\pi^3 + 8985600\pi^2\\
&\quad + 12441600\pi - 27648000) \approx 2.3e + 6 > 0,\\
&\omega_4 = -240\pi^3(7\pi^7 - 792\pi^5 + 28800\pi^3 - 46080\pi^2 - 230400\pi + 506880) \approx -155620 < 0,\\
&\omega_5 = -48\pi^2(\pi^9 - 120\pi^7 + 80\pi^6 + 120\pi^5 - 23040\pi^4 + 316800\pi^3 - 499200\pi^2\\
&\quad - 2534400\pi + 5529600) \approx -62280 < 0,\\
&\omega_6 = 64\pi(2\pi^9 + 15\pi^7 + 240\pi^6 - 94800\pi^5 - 10080\pi^4 + 6422400\pi^3 - 14227200\pi^2\\
&\quad - 53913600\pi + 140313600) \approx 2448 > 0,\\
&\omega_7 = -(32(\pi^4 + 360\pi^2 - 3600))(3\pi^5 - 800\pi^3 + 480\pi^2 + 40320\pi - 107520) \approx 866 > 0,\\
&E_4(x) \le xU_{cos}(x) - (1 + l_2(x))L_{sin}(x) = (\sum_{i=0}^{7}\omega_i x^i) \cdot \frac{(\pi - 2x)^2 x^6}{10800\pi^{15}}\\
&\le [(\omega_0 + \omega_2(\frac{\pi}{2})^2 + \omega_3(\frac{\pi}{2})^3) + x^4(\omega_4 + \omega_6(\frac{\pi}{2})^2 + \omega_7(\frac{\pi}{2})^3)] \cdot \frac{(\pi - 2x)^2 x^6}{10800\pi^{15}} < 0,
\end{aligned}
\tag{3.3}
$$

Combining with Eq (3.2) and Eq (3.3), we have that $E_3(x) > 0$ and $E_4(x) < 0$, $\forall x \in (0, \pi/2)$, and complete the proof of Eq (3.1). □

4. Proof of Theorem 6

It is equivalent to prove that for all $x \in (0, \pi/2)$

$$
\begin{cases} 8/3 \cdot D_1(x) = 8/3 \cdot \ln(l_3(x)) < x\cot(x) - 1, \\ x\cot(x) - 1 < 8/3 \cdot \ln(l_4(x)) = 8/3 \cdot D_2(x), \end{cases} \forall x \in (0, \pi/2). \tag{4.1}
$$

Let $E_{i+4}(x) = (x\cot(x) - 1 - 8/3 \cdot D_i(x)) \cdot \sin(x) = x\cos(x) - (1 + 8/3 \cdot D_i(x)) \cdot \sin(x)$, $i = 1, 2$. Eq (4.1) is equivalent to

$$
E_5(x) > 0 \text{ and } E_6(x) < 0, \quad \forall x \in (0, \pi/2). \tag{4.2}
$$

By using Maple software, it can be verified that $l_i(x) \ge e^{-3/8}$ and $1 + 8/3 \cdot \ln(l_i(x)) > 0$, $i = 3, 4$. For all $x \in (0, \pi/2)$, combining with Lemmas 2 and 4, we have that

$$
\begin{aligned}
&E_5(x) = x\cos(x) - (1 + 8/3 \cdot D_i(x)) \cdot \sin(x) > xL_1(x) - (1 + 8/3 \cdot U_{D,1}(x))U_2(x)\\
&= (\sum_{i=0}^{15}\mu_{5,i}x^i(\pi/2 - x)^{15-i}) \cdot \frac{1}{467775\pi^{26}}x^{11},\\
&E_6(x) = x\cos(x) - (1 + 8/3 \cdot D_2(x)) \cdot \sin(x) < xU_1(x) - (1 + 8/3 \cdot L_{D,2}(x)) \cdot L_2(x)\\
&= (\sum_{i=0}^{19}\mu_{6,i}x^i(\pi/2 - x)^{19-i}) \cdot x^7.
\end{aligned}
\tag{4.3}
$$

For all $\forall x \in (0, \pi/2)$, $i = 0, 1, \cdots, 15$ and $j = 0, 1, \cdots, 19$, we have that

$$
\begin{aligned}
& x^i(\pi/2 - x)^{15-i} > 0, \ \ \pi^{-12}x^{11} > 0, \ \ x^j(\pi/2 - x)^{19-j} > 0, \ \ \pi^{-11}x^7(\pi/2 - x) > 0, \\
& \mu_{5,0} = \frac{-2048\mu_{2,2}}{8505\pi^{15}} + \frac{16384\mu_{2,4}}{945\pi^{15}} - \frac{32768\mu_{2,6}}{45\pi^{15}} + \frac{131072\mu_{2,8}}{9\pi^{15}} - \frac{262144\mu_{2,10}}{3\pi^{15}} \approx 1.3e-9 > 0, \\
& \mu_{5,1} = \frac{-2048\mu_{2,2}}{567\pi^{15}} - \frac{1024\mu_{2,3}}{8505\pi^{14}} + \frac{16384\mu_{2,4}}{63\pi^{15}} + \frac{8192\mu_{2,5}}{945\pi^{14}} - \frac{32768\mu_{2,6}}{3\pi^{15}} - \frac{16384\mu_{2,7}}{45\pi^{14}} + \frac{655360\mu_{2,8}}{3\pi^{15}} \\
& \quad + \frac{65536\mu_{2,9}}{9\pi^{14}} - \frac{1310720\mu_{2,10}}{\pi^{15}} - \frac{131072\mu_{2,11}}{3\pi^{14}} \approx 1.9e-8 > 0, \\
& \mu_{5,2} = 256(\pi^2 - 46200)\frac{\mu_{2,2}}{467775\pi^{15}} - \frac{2048\mu_{2,3}}{1215\pi^{14}} - 512(\pi^2 - 30240)\frac{\mu_{2,4}}{8505\pi^{15}} + \frac{16384\mu_{2,5}}{135\pi^{14}} + 4096(\pi^2 - 17640)\frac{\mu_{2,6}}{945\pi^{15}} \\
& \quad - \frac{229376\mu_{2,7}}{45\pi^{14}} - 8192(\pi^2 - 8400)\frac{\mu_{2,8}}{45\pi^{15}} + \frac{917504\mu_{2,9}}{9\pi^{14}} + 32768(\pi^2 - 2520)\frac{\mu_{2,10}}{9\pi^{15}} - \frac{1835008\mu_{2,11}}{3\pi^{14}} - \frac{65536\mu_{2,12}}{3\pi^{13}} \\
& \quad \approx 1.3e-7 > 0, \\
& \mu_{5,3} = 1024(3\pi^{11} - 49940\pi^9 - 31680\pi^7 + 5322240\pi^5 - 425779200\pi^3 + 10218700800\pi - 20437401600)\frac{\mu_{2,2}}{467775\pi^{24}} \\
& \quad + 128(\pi^2 - 40040)\frac{\mu_{2,3}}{467775\pi^{14}} - 6656(\pi^2 - 10080)\frac{\mu_{2,4}}{8505\pi^{15}} - 256(\pi^2 - 26208)\frac{\mu_{2,5}}{8505\pi^{14}} + 53248(\pi^2 - 5880)\frac{\mu_{2,6}}{945\pi^{15}} \\
& \quad + 2048(\pi^2 - 15288)\frac{\mu_{2,7}}{945\pi^{14}} - 106496(\pi^2 - 2800)\frac{\mu_{2,8}}{45\pi^{15}} - 4096(\pi^2 - 7280)\frac{\mu_{2,9}}{45\pi^{14}} \\
& \quad + 425984(\pi^2 - 840)\frac{\mu_{2,10}}{9\pi^{15}} + 16384(\pi^2 - 2184)\frac{\mu_{2,11}}{9\pi^{14}} \\
& \quad - \frac{851968\mu_{2,12}}{3\pi^{13}} - \frac{32768\mu_{2,13}}{3\pi^{12}} \approx 6.0e-7 > 0, \\
& \mu_{5,4} = 1024(3\pi^{11} - 49940\pi^9 - 31680\pi^7 + 5322240\pi^5 - 425779200\pi^3 + 10218700800\pi - 20437401600)\frac{\mu_{2,2}}{467775\pi^{24}} \\
& \quad + 128(\pi^2 - 40040)\frac{\mu_{2,3}}{467775\pi^{14}} - 6656(\pi^2 - 10080)\frac{\mu_{2,4}}{8505\pi^{15}} - 256(\pi^2 - 26208)\frac{\mu_{2,5}}{8505\pi^{14}} + 53248(\pi^2 - 5880)\frac{\mu_{2,6}}{945\pi^{15}} \\
& \quad + 2048(\pi^2 - 15288)\frac{\mu_{2,7}}{945\pi^{14}} - 106496(\pi^2 - 2800)\frac{\mu_{2,8}}{45\pi^{15}} - 4096(\pi^2 - 7280)\frac{\mu_{2,9}}{45\pi^{14}} + 425984(\pi^2 - 840)\frac{\mu_{2,10}}{9\pi^{15}} \\
& \quad + 16384(\pi^2 - 2184)\frac{\mu_{2,11}}{9\pi^{14}} - \frac{851968\mu_{2,12}}{3\pi^{13}} \\
& \quad - \frac{32768\mu_{2,13}}{3\pi^{12}} \approx 1.8e-6 > 0, \\
& \mu_{5,5} = 1024(\pi^{11} - 5874\pi^9 - 38016\pi^7 + 6386688\pi^5 - 510935040\pi^3 + 12262440960\pi - 24524881920)\frac{\mu_{2,2}}{8505\pi^{24}} \\
& \quad + 128(\pi^{11} - 7920\pi^9 - 25344\pi^7 + 4257792\pi^5 - 340623360\pi^3 + 8174960640\pi - 16349921280)\frac{\mu_{2,3}}{8505\pi^{23}} \\
& \quad + 128(\pi^{11} - 12540\pi^9 + 38041344\pi^7 + 2128896\pi^5 - 170311680\pi^3 + 4087480320\pi - 8174960640)\frac{\mu_{2,4}}{93555\pi^{22}} \\
& \quad + 32(\pi^4 - 29040\pi^2 + 126846720)\frac{\mu_{2,5}}{467775\pi^{14}} - 1408(\pi^4 - 7488\pi^2 + 13208832)\frac{\mu_{2,6}}{8505\pi^{15}} \\
& \quad - 64(\pi^4 - 19008\pi^2 + 48432384)\frac{\mu_{2,7}}{8505\pi^{14}} + 11264(\pi^4 - 4368\pi^2 + 3669120)\frac{\mu_{2,8}}{945\pi^{15}} \\
& \quad + 512(\pi^4 - 11088\pi^2 + 13453440)\frac{\mu_{2,9}}{945\pi^{14}} - 22528(\pi^4 - 2080\pi^2 + 524160)\frac{\mu_{2,10}}{45\pi^{15}} \\
& \quad - 1024(\pi^4 - 5280\pi^2 + 1921920)\frac{\mu_{2,11}}{45\pi^{14}} + 90112(\pi^2 - 624)\frac{\mu_{2,12}}{9\pi^{13}} + 4096(\pi^2 - 1584)\frac{\mu_{2,13}}{9\pi^{12}} \\
& \quad - \frac{180224\mu_{2,14}}{3\pi^{11}} \approx 3.9e-6 > 0, \\
& \mu_{5,6} = 256(3\pi^{11} - 12760\pi^9 - 168960\pi^7 + 28385280\pi^5 - 2270822400\pi^3 + 54499737600\pi - 108999475200)\frac{\mu_{2,2}}{2835\pi^{24}} \\
& \quad + 128(\pi^{11} - 5192\pi^9 - 42240\pi^7 + 7096320\pi^5 - 567705600\pi^3 + 13624934400\pi - 27249868800)\frac{\mu_{2,3}}{2835\pi^{23}} \\
& \quad + 64(3\pi^{11} - 20680\pi^9 + 42197760\pi^7 + 14192640\pi^5 - 1135411200\pi^3 + 27249868800\pi - 54499737600)\frac{\mu_{2,4}}{31185\pi^{22}} \\
& \quad + 32(3\pi^{11} - 32120\pi^9 + 84522240\pi^7 + 7096320\pi^5 - 567705600\pi^3 + 13624934400\pi - 27249868800)\frac{\mu_{2,5}}{155925\pi^{21}} \\
& \quad + 16(\pi^6 - 24200\pi^4 + 90604800\pi^2 - 106551244800)\frac{\mu_{2,6}}{467775\pi^{15}} - 128(5\pi^4 - 31680\pi^2 + 48432384)\frac{\mu_{2,7}}{8505\pi^{14}} \\
& \quad - 32(\pi^6 - 15840\pi^4 + 34594560\pi^2 - 19372953600)\frac{\mu_{2,8}}{8505\pi^{15}} + 1024(\pi^4 - 3696\pi^2 + 2690688)\frac{\mu_{2,9}}{189\pi^{14}} \\
& \quad + 256(\pi^6 - 9240\pi^4 + 9609600\pi^2 - 1614412800)\frac{\mu_{2,10}}{945\pi^{15}} - 2048(\pi^4 - 1760\pi^2 + 384384)\frac{\mu_{2,11}}{9\pi^{14}} \\
& \quad - 512(\pi^4 - 4400\pi^2 + 1372800)\frac{\mu_{2,12}}{45\pi^{13}} + 40960(\pi^2 - 528)\frac{\mu_{2,13}}{9\pi^{12}} + 2048(\pi^2 - 1320)\frac{\mu_{2,14}}{9\pi^{11}} \approx 6.6e-6 > 0,
\end{aligned}
\tag{4.4}
$$

$$\mu_{5,7} = 2048(\pi^{11} - 3300\pi^9 - 79200\pi^7 + 13305600\pi^5 - 1064448000\pi^3 + 25546752000\pi - 51093504000)\frac{\mu_{2,2}}{4725\pi^{24}}$$
$$+ 256(\pi^{11} - 3784\pi^9 - 63360\pi^7 + 10644480\pi^5 - 851558400\pi^3 + 20437401600\pi - 40874803200)\frac{\mu_{2,3}}{2835\pi^{23}}$$
$$+ 512(\pi^{11} - 4554\pi^9 + 6747840\pi^7 + 7983360\pi^5 - 638668800\pi^3 + 15328051200\pi - 30656102400)\frac{\mu_{2,4}}{31185\pi^{22}}$$
$$+ 128(\pi^{11} - 5940\pi^9 + 10538880\pi^7 + 5322240\pi^5 - 425779200\pi^3 + 10218700800\pi - 20437401600)\frac{\mu_{2,5}}{51975\pi^{21}}$$
$$+ 128(\pi^{11} - 9020\pi^9 + 20370240\pi^7 - 17121646080\pi^5 - 212889600\pi^3 + 5109350400\pi - 10218700800)\frac{\mu_{2,6}}{467775\pi^{20}}$$
$$+ 8(\pi^6 - 19800\pi^4 + 62726400\pi^2 - 63930746880)\frac{\mu_{2,7}}{467775\pi^{14}} - 32(\pi^6 - 5280\pi^4 + 6918912\pi^2 - 2767564800)\frac{\mu_{2,8}}{945\pi^{15}}$$
$$- 16(\pi^6 - 12960\pi^4 + 23950080\pi^2 - 11623772160)\frac{\mu_{2,9}}{8505\pi^{14}} + 256(\pi^6 - 3080\pi^4 + 1921920\pi^2 - 230630400)\frac{\mu_{2,10}}{105\pi^{15}}$$
$$+ 128(\pi^6 - 7560\pi^4 + 6652800\pi^2 - 968647680)\frac{\mu_{2,11}}{945\pi^{14}} - 512(3\pi^4 - 4400\pi^2 + 823680)\frac{\mu_{2,12}}{15\pi^{13}}$$
$$- 256(\pi^4 - 3600\pi^2 + 950400)\frac{\mu_{2,13}}{45\pi^{12}} + 2048(\pi^2 - 440)\frac{\mu_{2,14}}{\pi^{11}} \approx 8.5e - 6 > 0,$$
$$\mu_{5,8} = 1024(7\pi^{11} - 18810\pi^9 - 760320\pi^7 + 127733760\pi^5 - 10218700800\pi^3 + 245248819200\pi - 490497638400)\frac{\mu_{2,2}}{14175\pi^{24}}$$
$$+ 256(7\pi^{11} - 20680\pi^9 - 633600\pi^7 + 106444800\pi^5 - 8515584000\pi^3 + 204374016000\pi - 408748032000)\frac{\mu_{2,3}}{14175\pi^{23}}$$
$$+ 128(7\pi^{11} - 23408\pi^9 + 26674560\pi^7 + 85155840\pi^5 - 6812467200\pi^3 + 163499212800\pi - 326998425600)\frac{\mu_{2,4}}{31185\pi^{22}}$$
$$+ 128(7\pi^{11} - 27720\pi^9 + 35861760\pi^7 + 63866880\pi^5 - 5109350400\pi^3 + 122624409600\pi - 245248819200)\frac{\mu_{2,5}}{155925\pi^{21}}$$
$$+ 64(\pi^{11} - 5060\pi^9 + 7729920\pi^7 - 4886576640\pi^5 - 486604800\pi^3 + 11678515200\pi - 23357030400)\frac{\mu_{2,6}}{66825\pi^{20}}$$
$$+ 8(\pi^{11} - 7480\pi^9 + 14319360\pi^7 - 10434631680\pi^5 - 243302400\pi^3 + 5839257600\pi - 11678515200)\frac{\mu_{2,7}}{66825\pi^{19}}$$
$$+ 4(\pi^8 - 15840\pi^6 + 41817600\pi^4 - 36531855360\pi^2 + 10959556608000)\frac{\mu_{2,8}}{467775\pi^{15}} - 128(\pi^6 - 4320\pi^4$$
$$+ 4790016\pi^2 - 1660538880)\frac{\mu_{2,9}}{8505\pi^{14}} - 8(\pi^8 - 10368\pi^6 + 15966720\pi^4 - 6642155520\pi^2 + 597793996800)\frac{\mu_{2,10}}{8505\pi^{15}}$$
$$+ 1024(\pi^6 - 2520\pi^4 + 1330560\pi^2 - 138378240)\frac{\mu_{2,11}}{945\pi^{14}} + 64(\pi^6 - 6048\pi^4 + 4435200\pi^2 - 553512960)\frac{\mu_{2,12}}{945\pi^{13}}$$
$$- 2048(\pi^4 - 1200\pi^2 + 190080)\frac{\mu_{2,13}}{45\pi^{12}} - \frac{128(\pi^2 - 2640)(\pi^2 - 240)\mu_{2,14}}{45\pi^{11}} \approx 8.5e - 6 > 0,$$
$$\mu_{5,9} = 2048(9\pi^{11} - 20405\pi^9 - 1330560\pi^7 + 223534080\pi^5 - 17882726400\pi^3 + 429185433600\pi - 858370867200)\frac{\mu_{2,2}}{42525\pi^{24}}$$
$$+ 256(\pi^{11} - 2420\pi^9 - 126720\pi^7 + 21288960\pi^5 - 1703116800\pi^3 + 40874803200\pi - 81749606400)\frac{\mu_{2,3}}{2025\pi^{23}}$$
$$+ 256(21\pi^{11} - 55220\pi^9 + 50635200\pi^7 + 372556800\pi^5 - 29804544000\pi^3 + 715309056000\pi - 1430618112000)$$
$$\frac{\mu_{2,4}}{155925\pi^{22}} + 64(3\pi^{11} - 8800\pi^9 + 8807040\pi^7 + 42577920\pi^5 - 3406233600\pi^3 + 81749606400\pi - 163499212800)$$
$$\frac{\mu_{2,5}}{22275\pi^{21}} + 128(7\pi^{11} - 23870\pi^9 + 26737920\pi^7 - 13244394240\pi^5 - 5960908800\pi^3 + 143061811200\pi - 286123622400)$$
$$\frac{\mu_{2,6}}{467775\pi^{20}} + 8(3\pi^{11} - 12760\pi^9 + 16600320\pi^7 - 9111674880\pi^5 - 1703116800\pi^3 + 40874803200\pi - 81749606400)$$
$$\frac{\mu_{2,7}}{66825\pi^{19}} + 8(3\pi^{11} - 18260\pi^9 + 29208960\pi^7 - 18255283200\pi^5 + 4261198233600\pi^3 + 20437401600\pi - 40874803200)$$
$$\frac{\mu_{2,8}}{467775\pi^{18}} + 2(\pi^8 - 12320\pi^6 + 26611200\pi^4 - 19670999040\pi^2 + 5114459750400)\frac{\mu_{2,9}}{467775\pi^{14}}$$
$$- 8(\pi^8 - 3456\pi^6 + 3193344\pi^4 - 948879360\pi^2 + 66421555200)\frac{\mu_{2,10}}{1215\pi^{15}} - 4(\pi^8 - 8064\pi^6 + 10160640\pi^4$$
$$- 3576545280\pi^2 + 278970531840)\frac{\mu_{2,11}}{8505\pi^{14}} + 64(\pi^6 - 2016\pi^4 + 887040\pi^2 - 79073280)\frac{\mu_{2,12}}{135\pi^{13}}$$
$$+ 32(\pi^6 - 4704\pi^4 + 2822400\pi^2 - 298045440)\frac{\mu_{2,13}}{945\pi^{12}} - 896(\pi^4 - 960\pi^2 + 126720)\frac{\mu_{2,14}}{45\pi^{11}} \approx 6.7e - 6 > 0, \tag{4.5}$$

$$
\begin{aligned}
\mu_{5,10} &= 256(3\pi^{11} - 5896\pi^9 - 608256\pi^7 + 102187008\pi^5 - 8174960640\pi^3 + 196199055360\pi - 392398110720)\frac{\mu_{2,2}}{2835\pi^{24}} \\
&+ 256(3\pi^{11} - 6160\pi^9 - 532224\pi^7 + 89413632\pi^5 - 7153090560\pi^3 + 171674173440\pi - 343348346880)\frac{\mu_{2,3}}{8505\pi^{23}} \\
&+ 128(7\pi^{11} - 15180\pi^9 + 11620224\pi^7 + 178827264\pi^5 - 14306181120\pi^3 + 343348346880\pi - 686696693760)\frac{\mu_{2,4}}{31185\pi^{22}} \\
&+ 64(7\pi^{11} - 16280\pi^9 + 13207040\pi^7 + 149022720\pi^5 - 11921817600\pi^3 + 286123622400\pi - 572247244800)\frac{\mu_{2,5}}{51975\pi^{21}} \\
&+ 32(7\pi^{11} - 17864\pi^9 + 15599232\pi^7 - 6273856512\pi^5 - 9537454080\pi^3 + 228898897920\pi - 457797795840)\frac{\mu_{2,6}}{93555\pi^{20}} \\
&+ 8(7\pi^{11} - 20328\pi^9 + 19540224\pi^7 - 8434685952\pi^5 - 7153090560\pi^3 + 171674173440\pi - 343348346880)\frac{\mu_{2,7}}{93555\pi^{19}} \\
&+ 4(\pi^{11} - 3520\pi^9 + 3852288\pi^7 - 1818077184\pi^5 + 340282736640\pi^3 + 16349921280\pi - 32699842560)\frac{\mu_{2,8}}{31185\pi^{18}} \\
&+ 2(\pi^{11} - 4840\pi^9 + 6361344\pi^7 - 3367913472\pi^5 + 681587343360\pi^3 + 8174960640\pi - 16349921280)\frac{\mu_{2,9}}{93555\pi^{17}} \\
&+ (\pi^{10} - 9240\pi^8 + 15966720\pi^6 - 9835499520\pi^4 + 2191911321600\pi^2 - 122747034009600)\frac{\mu_{2,10}}{467775\pi^{15}} \\
&- 8(\pi^8 - 2688\pi^6 + 2032128\pi^4 - 510935040\pi^2 + 30996725760)\frac{\mu_{2,11}}{2835\pi^{14}} - 2(\pi^8 - 6048\pi^6 + 6096384\pi^4 \\
&- 1788272640\pi^2 + 119558799360)\frac{\mu_{2,12}}{8505\pi^{13}} + 64(\pi^6 - 1568\pi^4 + 564480\pi^2 - 42577920)\frac{\mu_{2,13}}{315\pi^{12}} \\
&+ 16(\pi^6 - 3528\pi^4 + 1693440\pi^2 - 149022720)\frac{\mu_{2,14}}{945\pi^{11}} \approx 4.2e - 6 > 0, \\
\mu_{5,11} &= 1024(\pi^{11} - 1740\pi^9 - 285120\pi^7 + 47900160\pi^5 - 3832012800\pi^3 + 91968307200\pi - 183936614400)\frac{\mu_{2,2}}{8505\pi^{24}} \\
&+ 128(3\pi^{11} - 5368\pi^9 - 760320\pi^7 + 127733760\pi^5 - 10218700800\pi^3 + 245248819200\pi - 490497638400)\frac{\mu_{2,3}}{8505\pi^{23}} \\
&+ 512(3\pi^{11} - 5555\pi^9 + 3659040\pi^7 + 111767040\pi^5 - 8941363200\pi^3 + 214592716800\pi - 429185433600)\frac{\mu_{2,4}}{93555\pi^{22}} \\
&+ 128(7\pi^{11} - 13530\pi^9 + 9240000\pi^7 + 223534080\pi^5 - 17882726400\pi^3 + 429185433600\pi - 858370867200)\frac{\mu_{2,5}}{155925\pi^{21}} \\
&+ 128(7\pi^{11} - 14300\pi^9 + 10216800\pi^7 - 3446150400\pi^5 - 14902272000\pi^3 + 357654528000\pi - 715309056000)\frac{\mu_{2,6}}{467775\pi^{20}} \\
&+ 8(7\pi^{11} - 15400\pi^9 + 11658240\pi^7 - 4113027072\pi^5 - 11921817600\pi^3 + 286123622400\pi - 572247244800)\frac{\mu_{2,7}}{93555\pi^{19}} \\
&+ 16(\pi^{11} - 2442\pi^9 + 1995840\pi^7 - 745113600\pi^5 + 114960384000\pi^3 + 30656102400\pi - 61312204800)\frac{\mu_{2,8}}{93555\pi^{18}} \\
&+ 4(\pi^{11} - 2860\pi^9 + 2597760\pi^7 - 1043159040\pi^5 + 169630433280\pi^3 + 20437401600\pi - 40874803200)\frac{\mu_{2,9}}{93555\pi^{17}} \\
&+ 4(\pi^{11} - 3740\pi^9 + 3960000\pi^7 - 1751016960\pi^5 + 304006348800\pi^3 - 13938307891200\pi - 20437401600)\frac{\mu_{2,10}}{467775\pi^{16}} \\
&+ (\pi^{10} - 6600\pi^8 + 8870400\pi^6 - 4470681600\pi^4 + 843042816000\pi^2 - 40915678003200)\frac{\mu_{2,11}}{935550\pi^{14}} \\
&- 2(5\pi^8 - 10080\pi^6 + 6096384\pi^4 - 1277337600\pi^2 + 66421555200)\frac{\mu_{2,12}}{8505\pi^{13}} \\
&- (\pi^8 - 4320\pi^6 + 3386880\pi^4 - 812851200\pi^2 + 45984153600)\frac{\mu_{2,13}}{8505\pi^{12}} \\
&+ 16(\pi^6 - 1176\pi^4 + 338688\pi^2 - 21288960)\frac{\mu_{2,14}}{189\pi^{11}} \approx 2.0e - 6 > 0, \\
\mu_{5,12} &= 512(3\pi^{11} - 4700\pi^9 - 1267200\pi^7 + 212889600\pi^5 - 17031168000\pi^3 + 408748032000\pi - 817496064000)\frac{\mu_{2,2}}{42525\pi^{24}} \\
&+ 128(\pi^{11} - 1592\pi^9 - 380160\pi^7 + 63866880\pi^5 - 5109350400\pi^3 + 122624409600\pi - 245248819200)\frac{\mu_{2,3}}{8505\pi^{23}} \\
&+ 64(9\pi^{11} - 14608\pi^9 + 8490240\pi^7 + 510935040\pi^5 - 40874803200\pi^3 + 980995276800\pi - 1961990553600)\frac{\mu_{2,4}}{93555\pi^{22}} \\
&+ 128(9\pi^{11} - 14960\pi^9 + 8870400\pi^7 + 447068160\pi^5 - 35765452800\pi^3 + 858370867200\pi - 1716741734400)\frac{\mu_{2,5}}{467775\pi^{21}} \\
&+ 64(7\pi^{11} - 11990\pi^9 + 7286400\pi^7 - 2123573760\pi^5 - 23843635200\pi^3 + 572247244800\pi - 1144494489600)\frac{\mu_{2,6}}{467775\pi^{20}} \\
&+ 8(21\pi^{11} - 37400\pi^9 + 23443200\pi^7 - 7004067840\pi^5 - 59609088000\pi^3 + 1430618112000\pi - 2861236224000)\frac{\mu_{2,7}}{467775\pi^{19}} \\
&+ 4(3\pi^{11} - 5632\pi^9 + 3674880\pi^7 - 1132572672\pi^5 + 148171161600\pi^3 + 163499212800\pi - 326998425600)\frac{\mu_{2,8}}{93555\pi^{18}} \\
&+ 4(\pi^{11} - 2024\pi^9 + 1393920\pi^7 - 447068160\pi^5 + 60290334720\pi^3 + 40874803200\pi - 81749606400)\frac{\mu_{2,9}}{93555\pi^{17}} \\
&+ 2(3\pi^{11} - 6820\pi^9 + 5068800\pi^7 - 1713761280\pi^5 + 240139468800\pi^3 - 9217268121600\pi - 163499212800)\frac{\mu_{2,10}}{467775\pi^{16}} \\
&+ (3\pi^{11} - 8360\pi^9 + 6969600\pi^7 - 2533386240\pi^5 + 372982579200\pi^3 - 14837553561600\pi - 81749606400)\frac{\mu_{2,11}}{935550\pi^{15}}
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
&+(\pi^{10}-4400\pi^8+4435200\pi^6-1788272640\pi^4+281014272000\pi^2-11690193715200)\frac{\mu_{2,12}}{1871100\pi^{13}}\\
&\quad-4(\pi^8-1440\pi^6+677376\pi^4-116121600\pi^2+5109350400)\frac{\mu_{2,13}}{8505\pi^{12}}\\
&\quad-(\pi^8-2880\pi^6+1693440\pi^4-325140480\pi^2+15328051200)\frac{\mu_{2,14}}{17010\pi^{11}}\approx 7.1e-7>0,\\
&\mu_{5,13}=1024(\pi^{11}-1430\pi^9-696960\pi^7+117089280\pi^5-9367142400\pi^3+224811417600\pi-449622835200)\frac{\mu_{2,2}}{155925\pi^{24}}\\
&\quad+128(\pi^{11}-1440\pi^9-633600\pi^7+106444800\pi^5-8515584000\pi^3+204374016000\pi-408748032000)\frac{\mu_{2,3}}{42525\pi^{23}}\\
&\quad+128(\pi^{11}-1452\pi^9+760320\pi^7+95800320\pi^5-7664025600\pi^3+183936614400\pi-367873228800)\frac{\mu_{2,4}}{93555\pi^{22}}\\
&\quad+32(3\pi^{11}-4400\pi^9+2323200\pi^7+255467520\pi^5-20437401600\pi^3+490497638400\pi-980995276800)\frac{\mu_{2,5}}{155925\pi^{21}}\\
&\quad+128(\pi^{11}-1485\pi^9+792000\pi^7-204906240\pi^5-5960908800\pi^3+143061811200\pi-286123622400)\frac{\mu_{2,6}}{467775\pi^{20}}\\
&\quad+8(7\pi^{11}-10560\pi^9+5702400\pi^7-1490227200\pi^5-35765452800\pi^3+858370867200\pi-1716741734400)\frac{\mu_{2,7}}{467775\pi^{19}}\\
&\quad+8(\pi^{11}-1540\pi^9+844800\pi^7-223534080\pi^5+25546752000\pi^3+102187008000\pi-204374016000)\frac{\mu_{2,8}}{155925\pi^{18}}\\
&\quad+2(\pi^{11}-1584\pi^9+887040\pi^7-238436352\pi^5+27590492160\pi^3+81749606400\pi-163499212800)\frac{\mu_{2,9}}{93555\pi^{17}}\\
&\quad+4(\pi^{11}-1650\pi^9+950400\pi^7-260789760\pi^5+30656102400\pi^3-1011651379200\pi-122624409600)\frac{\mu_{2,10}}{467775\pi^{16}}\\
&\quad+(\pi^{11}-1760\pi^9+1056000\pi^7-298045440\pi^5+35765452800\pi^3-1198994227200\pi-81749606400)\frac{\mu_{2,11}}{311850\pi^{15}}\\
&\quad+(\pi^{11}-1980\pi^9+1267200\pi^7-372556800\pi^5+45984153600\pi^3-1573679923200\pi-40874803200)\frac{\mu_{2,12}}{935550\pi^{14}}\\
&\quad+\pi^{10}-2640\pi^8+1900800\pi^6-596090880\pi^4+76640256000\pi^2-2697737011200)\frac{\mu_{2,13}}{3742200\pi^{12}}\\
&\quad-(\pi^8-960\pi^6+338688\pi^4-46448640\pi^2+1703116800)\frac{\mu_{2,14}}{5670\pi^{11}}\approx 1.7e-7>0,\\
&\mu_{5,14}=256(\pi^{11}-1320\pi^9-1520640\pi^7+255467520\pi^5-20437401600\pi^3+490497638400\pi-980995276800)\frac{\mu_{2,2}}{467775\pi^{24}}\\
&\quad+128(\pi^{11}-1320\pi^9-1393920\pi^7+234178560\pi^5-18734284800\pi^3+449622835200\pi-899245670400)\frac{\mu_{2,3}}{467775\pi^{23}}\\
&\quad+64(\pi^{11}-1320\pi^9+633600\pi^7+212889600\pi^5-17031168000\pi^3+408748032000\pi-817496064000)\frac{\mu_{2,4}}{467775\pi^{22}}\\
&\quad+32(\pi^{11}-1320\pi^9+633600\pi^7+191600640\pi^5-15328051200\pi^3+367873228800\pi-735746457600)\frac{\mu_{2,5}}{467775\pi^{21}}\\
&\quad+16(\pi^{11}-1320\pi^9+633600\pi^7-149022720\pi^5-13624934400\pi^3+326998425600\pi-653996851200)\frac{\mu_{2,6}}{467775\pi^{20}}\\
&\quad+8(\pi^{11}-1320\pi^9+633600\pi^7-149022720\pi^5-11921817600\pi^3+286123622400\pi-572247244800)\frac{\mu_{2,7}}{467775\pi^{19}}\\
&\quad+4(\pi^{11}-1320\pi^9+633600\pi^7-149022720\pi^5+15328051200\pi^3+245248819200\pi-490497638400)\frac{\mu_{2,8}}{467775\pi^{18}}\\
&\quad+2(\pi^{11}-1320\pi^9+633600\pi^7-149022720\pi^5+15328051200\pi^3+204374016000\pi-408748032000)\frac{\mu_{2,9}}{467775\pi^{17}}\\
&\quad+(\pi^{11}-1320\pi^9+633600\pi^7-149022720\pi^5+15328051200\pi^3-449622835200\pi-326998425600)\frac{\mu_{2,10}}{467775\pi^{16}}\\
&\quad+(\pi^{11}-1320\pi^9+633600\pi^7-149022720\pi^5+15328051200\pi^3-449622835200\pi-245248819200)\frac{\mu_{2,11}}{935550\pi^{15}}\\
&\quad+(\pi^{11}-1320\pi^9+633600\pi^7-149022720\pi^5+15328051200\pi^3-449622835200\pi-163499212800)\frac{\mu_{2,12}}{1871100\pi^{14}}\\
&\quad+(\pi^{11}-1320\pi^9+633600\pi^7-149022720\pi^5+15328051200\pi^3-449622835200\pi-81749606400)\frac{\mu_{2,13}}{3742200\pi^{13}}\\
&\quad+(\pi^{10}-1320\pi^8+633600\pi^6-149022720\pi^4+15328051200\pi^2-449622835200)\frac{\mu_{2,14}}{7484400\pi^{11}}\approx 2.7e-8>0,\\
&\mu_{5,15}=-65536(\pi^7-168\pi^5+13440\pi^3-322560\pi+645120)\frac{\mu_{2,2}}{945\pi^{24}}-32768(\pi^7-168\pi^5+13440\pi^3-322560\pi\\
&\quad+645120)\frac{\mu_{2,3}}{945\pi^{23}}+131072(\pi^5-80\pi^3+1920\pi-3840)\frac{\mu_{2,4}}{45\pi^{22}}+65536(\pi^5-80\pi^3+1920\pi-3840)\frac{\mu_{2,5}}{45\pi^{21}}\\
&\quad-524288(\pi^3-24\pi+48)\frac{\mu_{2,6}}{9\pi^{20}}-262144(\pi^3-24\pi+48)\frac{\mu_{2,7}}{9\pi^{19}}+1048576(\pi-2)\frac{\mu_{2,8}}{3\pi^{18}}+524288(\pi-2)\frac{\mu_{2,9}}{3\pi^{17}}\\
&\quad-\frac{524288\mu_{2,10}}{3\pi^{16}}-\frac{262144\mu_{2,11}}{3\pi^{15}}-\frac{131072\mu_{2,12}}{3\pi^{14}}-\frac{65536\mu_{2,13}}{3\pi^{13}}-\frac{32768\mu_{2,14}}{3\pi^{12}}\approx 1.9e-9>0,
\end{aligned}
\tag{4.7}
$$

$$\begin{aligned}
\mu_{6,0} &= -\frac{524288\mu_{3,2}}{45\pi^{19}} + \frac{2097152\mu_{3,4}}{9\pi^{19}} - \frac{4194304\mu_{3,6}}{3\pi^{19}} \approx -7.5e-8 < 0;\\
\mu_{6,1} &= -\frac{9961472\mu_{3,2}}{45\pi^{19}} - \frac{262144\mu_{3,3}}{45\pi^{18}} + \frac{39845888\mu_{3,4}}{9\pi^{19}} + \frac{1048576\mu_{3,5}}{9\pi^{18}} - \frac{79691776\mu_{3,6}}{3\pi^{19}} - \frac{2097152\mu_{3,7}}{3\pi^{18}} \approx -1.4e-6 < 0;\\
\mu_{6,2} &= \frac{65536(\pi^2-28728)\mu_{3,2}}{945\pi^{19}} - \frac{524288\mu_{3,3}}{5\pi^{18}} - \frac{131072(\pi^2-13680)\mu_{3,4}}{45\pi^{19}} + \frac{2097152\mu_{3,5}}{\pi^{18}} + \frac{524288(\pi^2-4104)\mu_{3,6}}{9\pi^{19}}\\
&\quad - \frac{12582912\mu_{3,7}}{\pi^{18}} - \frac{1048576\mu_{3,8}}{3\pi^{17}} \approx -1.2e-5 < 0;\\
\mu_{6,3} &= \frac{1114112(\pi^2-9576)\mu_{3,2}}{945\pi^{19}} + \frac{32768(\pi^2-25704)\mu_{3,3}}{945\pi^{18}} - \frac{2228224(\pi^2-4560)\mu_{3,4}}{45\pi^{19}} - \frac{65536(\pi^2-12240)\mu_{3,5}}{45\pi^{18}}\\
&\quad + \frac{8912896(\pi^2-1368)\mu_{3,6}}{9\pi^{19}} + \frac{262144(\pi^2-3672)\mu_{3,7}}{9\pi^{18}} - \frac{17825792\mu_{3,8}}{3\pi^{17}} - \frac{524288\mu_{3,9}}{3\pi^{16}} \approx -7.3e-5 < 0;\\
\mu_{6,4} &= -\frac{2048(\pi^4-39168\pi^2+187536384)\mu_{3,2}}{8505\pi^{19}} + \frac{524288(\pi^2-8568)\mu_{3,3}}{945\pi^{18}} + \frac{16384(\pi^4-22848\pi^2+52093440)\mu_{3,4}}{945\pi^{19}}\\
&\quad - \frac{1048576(\pi^2-4080)\mu_{3,5}}{45\pi^{18}} - \frac{32768(\pi^4-10880\pi^2+7441920)\mu_{3,6}}{45\pi^{19}} + \frac{4194304(\pi^2-1224)\mu_{3,7}}{9\pi^{18}}\\
&\quad + \frac{131072(\pi^2-3264)\mu_{3,8}}{9\pi^{17}} - \frac{8388608\mu_{3,9}}{3\pi^{16}} - \frac{262144\mu_{3,10}}{3\pi^{15}} \approx -2.9e-4 < 0;\\
\mu_{6,5} &= -\frac{2048(5\pi^4-65280\pi^2+187536384)\mu_{3,2}}{2835\pi^{19}} - \frac{1024(\pi^4-34560\pi^2+148055040)\mu_{3,3}}{8505\pi^{18}} + \frac{16384\mu_{3,4}}{63\pi^{19}}\\
&\quad \cdot(\pi^4-7616\pi^2+10418688) + \frac{8192(\pi^4-20160\pi^2+41126400)\mu_{3,5}}{945\pi^{18}} - \frac{32768(3\pi^4-10880\pi^2+4465152)\mu_{3,6}}{9\pi^{19}}\\
&\quad - \frac{16384(\pi^4-9600\pi^2+5875200)\mu_{3,7}}{45\pi^{18}} + \frac{655360(\pi^2-1088)\mu_{3,8}}{3\pi^{17}} + \frac{65536(\pi^2-2880)\mu_{3,9}}{9\pi^{16}} - \frac{1310720\mu_{3,10}}{\pi^{15}}\\
&\quad - \frac{131072\mu_{3,11}}{3\pi^{14}} \approx -8.7e-4 < 0;\\
\mu_{6,6} &= -\frac{4096(17\pi^9-114000\pi^7+218736000\pi^5+5806080\pi^3-170311680\pi+371589120)\mu_{3,2}}{2835\pi^{24}}\\
&\quad - \frac{2048(\pi^4-11520\pi^2+29611008)\mu_{3,3}}{1215\pi^{18}} - \frac{512(\pi^6-30240\pi^4+115153920\pi^2-105020375040)\mu_{3,4}}{8505\pi^{19}}\\
&\quad + \frac{16384(\pi^4-6720\pi^2+8225280)\mu_{3,5}}{135\pi^{18}} + \frac{4096(\pi^6-17640\pi^4+31987200\pi^2-8751697920)\mu_{3,6}}{945\pi^{19}}\\
&\quad - \frac{229376(\pi^4-3200\pi^2+1175040)\mu_{3,7}}{45\pi^{18}} - \frac{8192(\pi^4-8400\pi^2+4569600)\mu_{3,8}}{45\pi^{17}} + \frac{917504(\pi^2-960)\mu_{3,9}}{9\pi^{16}}\\
&\quad + \frac{32768(\pi^2-2520)\mu_{3,10}}{9\pi^{15}} - \frac{1835008\mu_{3,11}}{3\pi^{14}} - \frac{65536\mu_{3,12}}{3\pi^{13}} \approx -2.0e-3 < 0;\\
\mu_{6,7} &= -\frac{4096(209\pi^9-882288\pi^7+1216930176\pi^5+210954240\pi^3-6177669120\pi+13470105600)\mu_{3,2}}{8505\pi^{24}}\\
&\quad - \frac{8192(11\pi^9-65340\pi^7+112232736\pi^5+4354560\pi^3-127733760\pi+278691840)\mu_{3,3}}{8505\pi^{23}}\\
&\quad - \frac{6656(\pi^6-10080\pi^4+23030784\pi^2-15002910720)\mu_{3,4}}{8505\pi^{19}} - \frac{256(\pi^6-26208\pi^4+88058880\pi^2-71856046080)\mu_{3,5}}{8505\pi^{18}}\\
&\quad + \frac{53248(\pi^6-5880\pi^4+6397440\pi^2-1250242560)\mu_{3,6}}{945\pi^{19}} + \frac{2048(\pi^6-15288\pi^4+24460800\pi^2-5988003840)\mu_{3,7}}{945\pi^{18}}\\
&\quad - \frac{106496(\pi^4-2800\pi^2+913920)\mu_{3,8}}{45\pi^{17}} - \frac{4096(\pi^4-7280\pi^2+3494400)\mu_{3,9}}{45\pi^{16}} + \frac{425984(\pi^2-840)\mu_{3,10}}{9\pi^{15}}\\
&\quad + \frac{16384(\pi^2-2184)\mu_{3,11}}{9\pi^{14}} - \frac{851968\mu_{3,12}}{3\pi^{13}} - \frac{32768\mu_{3,13}}{3\pi^{12}} \approx -3.7e-3 < 0;\\
\mu_{6,8} &= -\frac{2048(385\pi^9-1155264\pi^7+1211341824\pi^5+781885440\pi^3-22852730880\pi+49792942080)\mu_{3,2}}{2835\pi^{24}}\\
&\quad - \frac{2048(55\pi^9-206976\pi^7+255999744\pi^5+64512000\pi^3-1888911360\pi+4118446080)\mu_{3,3}}{2835\pi^{23}}\\
&\quad - \frac{512(25\pi^9-130560\pi^7+199487232\pi^5-97507307520\pi^3-340623360\pi+743178240)\mu_{3,4}}{2835\pi^{22}}\\
&\quad - \frac{1024(\pi^6-8736\pi^4+17611776\pi^2-10265149440)\mu_{3,5}}{2835\pi^{18}}
\end{aligned} \tag{4.8}$$

$$
\begin{aligned}
&-\frac{128(\pi^8-22464\pi^6+66044160\pi^4-47904030720\pi^2+7021362216960)\mu_{3,6}}{8505\pi^{19}}\\
&+\frac{8192(\pi^6-5096\pi^4+4892160\pi^2-855429120)\mu_{3,7}}{315\pi^{18}}+\frac{1024(\pi^6-13104\pi^4+18345600\pi^2-3992002560)\mu_{3,8}}{945\pi^{17}}\\
&-\frac{16384(3\pi^4-7280\pi^2+2096640)\mu_{3,9}}{45\pi^{16}}-\frac{2048(\pi^4-6240\pi^2+2620800)\mu_{3,10}}{45\pi^{15}}\\
&+\frac{65536(\pi^2-728)\mu_{3,11}}{3\pi^{14}}+\frac{8192(\pi^2-1872)\mu_{3,12}}{9\pi^{13}}-\frac{131072\mu_{3,13}}{\pi^{12}}-\frac{16384\mu_{3,14}}{3\pi^{11}}\approx-5.6e-3<0;\\
\mu_{6,9}=&-\frac{22528(23\pi^9-53184\pi^7+44362752\pi^5+79994880\pi^3-2332753920\pi+5078384640)\mu_{3,2}}{945\pi^{24}}\\
&-\frac{11264(25\pi^9-67392\pi^7+63576576\pi^5+59351040\pi^3-1734082560\pi+3777822720)\mu_{3,3}}{2835\pi^{23}}\\
&-\frac{512(85\pi^9-283392\pi^7+312512256\pi^5-119072378880\pi^3-3437199360\pi+7493713920)\mu_{3,4}}{2835\pi^{22}}\\
&-\frac{256(7\pi^9-31872\pi^7+43013376\pi^5-18815569920\pi^3-113541120\pi+247726080)\mu_{3,5}}{945\pi^{21}}\\
&-\frac{1408(\pi^8-7488\pi^6+13208832\pi^4-6843432960\pi^2+780151357440)\mu_{3,6}}{8505\pi^{19}}\\
&-\frac{64(\pi^8-19008\pi^6+48432384\pi^4-30996725760\pi^2+4064999178240)\mu_{3,7}}{8505\pi^{18}}\\
&+\frac{11264(\pi^6-4368\pi^4+3669120\pi^2-570286080)\mu_{3,8}}{945\pi^{17}}+\frac{512(\pi^6-11088\pi^4+13453440\pi^2-2583060480)\mu_{3,9}}{945\pi^{16}}\\
&-\frac{22528(\pi^4-2080\pi^2+524160)\mu_{3,10}}{45\pi^{15}}-\frac{1024(\pi^4-5280\pi^2+1921920)\mu_{3,11}}{45\pi^{14}}+\frac{90112(\pi^2-624)\mu_{3,12}}{9\pi^{13}}\\
&+\frac{4096(\pi^2-1584)\mu_{3,13}}{9\pi^{12}}-\frac{180224\mu_{3,14}}{3\pi^{11}}\approx-6.8e-3<0;\\
\mu_{6,10}=&-\frac{90112(25\pi^9-47160\pi^7+32510016\pi^5+137088000\pi^3-3986841600\pi+8670412800)\mu_{3,2}}{2835\pi^{24}}\\
&-\frac{45056(11\pi^9-23040\pi^7+17377920\pi^5+45158400\pi^3-1316044800\pi+2864332800)\mu_{3,3}}{2835\pi^{23}}\\
&-\frac{1024(19\pi^9-45792\pi^7+38683008\pi^5-11865434112\pi^3-1563770880\pi+3406233600)\mu_{3,4}}{567\pi^{22}}\\
&-\frac{16384(\pi^9-2934\pi^7+2866752\pi^5-978526080\pi^3-48384000\pi+105477120)\mu_{3,5}}{2835\pi^{21}}\\
&-\frac{512(13\pi^9-51120\pi^7+60455808\pi^5-23515591680\pi^3+2145160765440\pi+557383680)\mu_{3,6}}{8505\pi^{20}}\\
&-\frac{128(5\pi^8-31680\pi^6+48432384\pi^4-22140518400\pi^2+2258332876800)\mu_{3,7}}{8505\pi^{18}}\\
&-\frac{32(\pi^8-15840\pi^6+34594560\pi^4-19372953600\pi^2+2258332876800)\mu_{3,8}}{8505\pi^{17}}\\
&+\frac{1024(\pi^6-3696\pi^4+2690688\pi^2-369008640)\mu_{3,9}}{189\pi^{16}}+\frac{256(\pi^6-9240\pi^4+9609600\pi^2-1614412800)\mu_{3,10}}{945\pi^{15}}\\
&-\frac{2048(\pi^4-1760\pi^2+384384)\mu_{3,11}}{9\pi^{14}}-\frac{512(\pi^4-4400\pi^2+1372800)\mu_{3,12}}{45\pi^{13}}+\frac{40960(\pi^2-528)\mu_{3,13}}{9\pi^{12}}\\
&+\frac{2048(\pi^2-1320)\mu_{3,14}}{9\pi^{11}}\approx-6.8e-3<0;\\
\mu_{6,11}=&-\frac{8192(99\pi^9-158840\pi^7+93465792\pi^5+819624960\pi^3-23758479360\pi+51604439040)\mu_{3,2}}{945\pi^{24}}\\
&-\frac{90112(7\pi^9-12060\pi^7+7566048\pi^5+45964800\pi^3-1335398400\pi+2903040000)\mu_{3,3}}{2835\pi^{23}}\\
&-\frac{1024(49\pi^9-92640\pi^7+62966400\pi^5-16011233280\pi^3-7044710400\pi+15328051200)\mu_{3,4}}{945\pi^{22}}\\
&-\frac{512(\pi^9-2144\pi^7+1613952\pi^5-444672000\pi^3-99532800\pi+216760320)\mu_{3,5}}{45\pi^{21}}\\
&-\frac{512(5\pi^9-12816\pi^7+11021184\pi^5-3350522880\pi^3+250467655680\pi+643645440)\mu_{3,6}}{1215\pi^{20}}
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
&-\frac{128(\pi^9-3360\pi^7+3451392\pi^5-1185269760\pi^3+96761364480\pi+53084160)\mu_{3,7}}{405\pi^{19}}\\
&-\frac{32(\pi^8-5280\pi^6+6918912\pi^4-2767564800\pi^2+250925875200)\mu_{3,8}}{945\pi^{17}}\\
&-\frac{16(\pi^8-12960\pi^6+23950080\pi^4-11623772160\pi^2+1195587993600)\mu_{3,9}}{8505\pi^{16}}\\
&+\frac{256(\pi^6-3080\pi^4+1921920\pi^2-230630400)\mu_{3,10}}{105\pi^{15}}+\frac{128(\pi^6-7560\pi^4+6652800\pi^2-968647680)\mu_{3,11}}{945\pi^{14}}\\
&-\frac{512(3\pi^4-4400\pi^2+823680)\mu_{3,12}}{15\pi^{13}}-\frac{256(\pi^4-3600\pi^2+950400)\mu_{3,13}}{45\pi^{12}}+\frac{2048(\pi^2-440)\mu_{3,14}}{\pi^{11}}\approx -5.5e-3<0;\\
\mu_{6,12}={}&-\frac{2048(319\pi^9-449152\pi^7+231856128\pi^5+3917168640\pi^3-113086955520\pi+245248819200)\mu_{3,2}}{945\pi^{24}}\\
&-\frac{4096(143\pi^9-211200\pi^7+113944320\pi^5+1447649280\pi^3-41896673280\pi+90946437120)\mu_{3,3}}{2835\pi^{23}}\\
&-\frac{1024(161\pi^9-252720\pi^7+144006912\pi^5-31209615360\pi^3-37623398400\pi+81749606400)\mu_{3,4}}{2835\pi^{22}}\\
&-\frac{2048(7\pi^9-11904\pi^7+7268352\pi^5-1668280320\pi^3-1238630400\pi+2694021120)\mu_{3,5}}{945\pi^{21}}\\
&-\frac{256(17\pi^9-32256\pi^7+21534336\pi^5-5305098240\pi^3+332253757440\pi+4565237760)\mu_{3,6}}{1215\pi^{20}}\\
&-\frac{128(7\pi^9-15552\pi^7+11688192\pi^5-3145236480\pi^3+210651217920\pi+1128038400)\mu_{3,7}}{1215\pi^{19}}\\
&-\frac{32(11\pi^9-31200\pi^7+27562752\pi^5-8291082240\pi^3+601881477120\pi+743178240)\mu_{3,8}}{2835\pi^{18}}\\
&-\frac{128(\pi^8-4320\pi^6+4790016\pi^4-1660538880\pi^2+132843110400)\mu_{3,9}}{8505\pi^{16}}\\
&-\frac{8(\pi^8-10368\pi^6+15966720\pi^4-6642155520\pi^2+597793996800)\mu_{3,10}}{8505\pi^{15}}\\
&+\frac{1024(\pi^6-2520\pi^4+1330560\pi^2-138378240)\mu_{3,11}}{945\pi^{14}}+\frac{64(\pi^6-6048\pi^4+4435200\pi^2-553512960)\mu_{3,12}}{945\pi^{13}}\\
&-\frac{2048(\pi^4-1200\pi^2+190080)\mu_{3,13}}{45\pi^{12}}-\frac{128(\pi^2-2640)(\pi^2-240)\mu_{3,14}}{45\pi^{11}}\approx -3.6e-3<0;\\
\mu_{6,13}={}&-\frac{2048(1705\pi^9-2157696\pi^7+999806976\pi^5+31167037440\pi^3-895158190080\pi+1937465671680)\mu_{3,2}}{8505\pi^{24}}\\
&-\frac{1024(55\pi^9-71808\pi^7+34255872\pi^5+851558400\pi^3-24524881920\pi+53137244160)\mu_{3,3}}{405\pi^{23}}\\
&-\frac{1024(125\pi^9-169680\pi^7+83881728\pi^5-15904788480\pi^3-46169948160\pi+100143267840)\mu_{3,4}}{2835\pi^{22}}\\
&-\frac{512(11\pi^9-15696\pi^7+8112384\pi^5-1598607360\pi^3-3251404800\pi+7060193280)\mu_{3,5}}{405\pi^{21}}\\
&-\frac{256(133\pi^9-202752\pi^7+110920320\pi^5-22922403840\pi^3+1230331576320\pi+65028096000)\mu_{3,6}}{8505\pi^{20}}\\
&-\frac{256(5\pi^9-8352\pi^7+4923072\pi^5-1079930880\pi^3+60801269760\pi+1718599680)\mu_{3,7}}{1215\pi^{19}}\\
&-\frac{32(65\pi^9-124128\pi^7+80946432\pi^5-19160064000\pi^3+1143472619520\pi+13563002880)\mu_{3,8}}{8505\pi^{18}}\\
&-\frac{16(25\pi^9-59040\pi^7+44368128\pi^5-11588935680\pi^3+742899548160\pi+2229534720)\mu_{3,9}}{8505\pi^{17}}\\
&-\frac{8(\pi^8-3456\pi^6+3193344\pi^4-948879360\pi^2+66421555200)\mu_{3,10}}{1215\pi^{15}}\\
&-\frac{4(\pi^8-8064\pi^6+10160640\pi^4-3576545280\pi^2+278970531840)\mu_{3,11}}{8505\pi^{14}}\\
&+\frac{64(\pi^6-2016\pi^4+887040\pi^2-79073280)\mu_{3,12}}{135\pi^{13}}+\frac{32(\pi^6-4704\pi^4+2822400\pi^2-298045440)\mu_{3,13}}{945\pi^{12}}
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
&-\frac{896(\pi^4-960\pi^2+126720)\mu_{3,14}}{45\pi^{11}}\approx -1.9e-3<0;\\
\mu_{6,14}=&-\frac{4096(121\pi^9-140208\pi^7+59439744\pi^5+3384944640\pi^3-96566722560\pi+208461496320)\mu_{3,2}}{2835\pi^{24}}\\
&-\frac{2048(275\pi^9-324864\pi^7+140216832\pi^5+6642155520\pi^3-190067834880\pi+410791772160)\mu_{3,3}}{8505\pi^{23}}\\
&-\frac{512(15\pi^9-18144\pi^7+8003072\pi^5-1359912960\pi^3-8814919680\pi+19074908160)\mu_{3,4}}{315\pi^{22}}\\
&-\frac{8192(\pi^9-1246\pi^7+564480\pi^5-98219520\pi^3-487710720\pi+1056706560)\mu_{3,5}}{945\pi^{21}}\\
&-\frac{512(49\pi^9-63432\pi^7+29719872\pi^5-5324175360\pi^3+250869104640\pi+41617981440)\mu_{3,6}}{8505\pi^{20}}\\
&-\frac{128(7\pi^9-9536\pi^7+4666368\pi^5-867041280\pi^3+42123755520\pi+4551966720)\mu_{3,7}}{945\pi^{19}}\\
&-\frac{32(25\pi^9-36576\pi^7+18966528\pi^5-3692666880\pi^3+186320977920\pi+11519262720)\mu_{3,8}}{2835\pi^{18}}\\
&-\frac{128(5\pi^9-8136\pi^7+4572288\pi^5-946391040\pi^3+50071633920\pi+1416683520)\mu_{3,9}}{8505\pi^{17}}\\
&-\frac{16(\pi^9-1936\pi^7+1223040\pi^5-274821120\pi^3+15441592320\pi+123863040)\mu_{3,10}}{945\pi^{16}}\\
&-\frac{8(\pi^8-2688\pi^6+2032128\pi^4-510935040\pi^2+30996725760)\mu_{3,11}}{2835\pi^{14}}\\
&-\frac{2(\pi^8-6048\pi^6+6096384\pi^4-1788272640\pi^2+119558799360)\mu_{3,12}}{8505\pi^{13}}\\
&+\frac{64(\pi^6-1568\pi^4+564480\pi^2-42577920)\mu_{3,13}}{315\pi^{12}}+\frac{16(\pi^6-3528\pi^4+1693440\pi^2-149022720)\mu_{3,14}}{945\pi^{11}}\\
&\approx -8.3e-4<0;\\
\mu_{6,15}=&-\frac{4096(35\pi^9-37680\pi^7+14845824\pi^5+1596672000\pi^3-45132595200\pi+97077657600)\mu_{3,2}}{2835\pi^{24}}\\
&-\frac{16384(11\pi^9-11970\pi^7+4762800\pi^5+439084800\pi^3-12454041600\pi+26824089600)\mu_{3,3}}{8505\pi^{23}}\\
&-\frac{512(29\pi^9-31968\pi^7+12870144\pi^5-2001936384\pi^3-28426567680\pi+61312204800)\mu_{3,4}}{1701\pi^{22}}\\
&-\frac{256(13\pi^9-14560\pi^7+5945856\pi^5-936714240\pi^3-10838016000\pi+23410114560)\mu_{3,5}}{945\pi^{21}}\\
&-\frac{512(23\pi^9-26280\pi^7+10922688\pi^5-1747630080\pi^3+74085580800\pi+34464890880)\mu_{3,6}}{8505\pi^{20}}\\
&-\frac{128(35\pi^9-41040\pi^7+17442432\pi^5-2844979200\pi^3+122624409600\pi+42268262400)\mu_{3,7}}{8505\pi^{19}}\\
&-\frac{32(17\pi^9-20640\pi^7+9031680\pi^5-1509580800\pi^3+66421555200\pi+15792537600)\mu_{3,8}}{2835\pi^{18}}\\
&-\frac{16(7\pi^9-8928\pi^7+4064256\pi^5-701374464\pi^3+31677972480\pi+4644864000)\mu_{3,9}}{1701\pi^{17}}\\
&-\frac{16(11\pi^9-15120\pi^7+7281792\pi^5-1312174080\pi^3+61312204800\pi+4551966720)\mu_{3,10}}{8505\pi^{16}}\\
&-\frac{16(\pi^9-1560\pi^7+818496\pi^5-156764160\pi^3+7664025600\pi+185794560)\mu_{3,11}}{2835\pi^{15}}\\
&-\frac{2(5\pi^8-10080\pi^6+6096384\pi^4-1277337600\pi^2+66421555200)\mu_{3,12}}{8505\pi^{13}}\\
&-\frac{\pi^8-4320\pi^6+3386880\pi^4-812851200\pi^2+45984153600)\mu_{3,13}}{8505\pi^{12}}\\
&+\frac{16(\pi^6-1176\pi^4+338688\pi^2-21288960)\mu_{3,14}}{189\pi^{11}}\approx -2.7e-4<0;\\
\mu_{6,16}=&-\frac{2048(37\pi^9-37440\pi^7+13886208\pi^5+3150766080\pi^3-87880826880\pi+188024094720)\mu_{3,2}}{8505\pi^{24}}
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
&-\frac{2048(17\pi^9-17280\pi^7+6435072\pi^5+1277337600\pi^3-35765452800\pi+76640256000)\mu_{3,3}}{8505\pi^{23}}\\
&-\frac{512(31\pi^9-31680\pi^7+11854080\pi^5-1718599680\pi^3-57131827200\pi+122624409600)\mu_{3,4}}{8505\pi^{22}}\\
&-\frac{1024(7\pi^9-7200\pi^7+2709504\pi^5-394813440\pi^3-11147673600\pi+23967498240)\mu_{3,5}}{8505\pi^{21}}\\
&-\frac{128(25\pi^9-25920\pi^7+9821952\pi^5-1439907840\pi^3+56202854400\pi+72831467520)\mu_{3,6}}{8505\pi^{20}}\\
&-\frac{128(11\pi^9-11520\pi^7+4402944\pi^5-650280960\pi^3+25546752000\pi+26661519360)\mu_{3,7}}{8505\pi^{19}}\\
&-\frac{32(19\pi^9-20160\pi^7+7789824\pi^5-1161216000\pi^3+45984153600\pi+37158912000)\mu_{3,8}}{8505\pi^{18}}\\
&-\frac{256(\pi^9-1080\pi^7+423360\pi^5-63866880\pi^3+2554675200\pi+1509580800)\mu_{3,9}}{8505\pi^{17}}\\
&-\frac{8(13\pi^9-14400\pi^7+5757696\pi^5-882524160\pi^3+35765452800\pi+14120386560)\mu_{3,10}}{8505\pi^{16}}\\
&-\frac{8(5\pi^9-5760\pi^7+2370816\pi^5-371589120\pi^3+15328051200\pi+3437199360)\mu_{3,11}}{8505\pi^{15}}\\
&-\frac{2(7\pi^9-8640\pi^7+3725568\pi^5-603832320\pi^3+25546752000\pi+2229534720)\mu_{3,12}}{8505\pi^{14}}\\
&-\frac{4(\pi^8-1440\pi^6+677376\pi^4-116121600\pi^2+5109350400)\mu_{3,13}}{8505\pi^{12}}\\
&-\frac{\pi^8-2880\pi^6+1693440\pi^4-325140480\pi^2+15328051200)\mu_{3,14}}{17010\pi^{11}}\approx-6.8e-5<0;\\
\mu_{6,17}=&-\frac{2048(\pi^9-960\pi^7+338688\pi^5+224501760\pi^3-6131220480\pi+13005619200)\mu_{3,2}}{2835\pi^{24}}\\
&-\frac{1024(\pi^9-960\pi^7+338688\pi^5+198696960\pi^3-5449973760\pi+11581194240)\mu_{3,3}}{2835\pi^{23}}\\
&-\frac{512(\pi^9-960\pi^7+338688\pi^5-46448640\pi^3-4799692800\pi+10218700800)\mu_{3,4}}{2835\pi^{22}}\\
&-\frac{256(\pi^9-960\pi^7+338688\pi^5-46448640\pi^3-4180377600\pi+8918138880)\mu_{3,5}}{2835\pi^{21}}\\
&-\frac{128(\pi^9-960\pi^7+338688\pi^5-46448640\pi^3+1703116800\pi+7679508480)\mu_{3,6}}{2835\pi^{20}}\\
&-\frac{64(\pi^9-960\pi^7+338688\pi^5-46448640\pi^3+1703116800\pi+6502809600)\mu_{3,7}}{2835\pi^{19}}\\
&-\frac{32(\pi^9-960\pi^7+338688\pi^5-46448640\pi^3+1703116800\pi+5388042240)\mu_{3,8}}{2835\pi^{18}}\\
&-\frac{16(\pi^9-960\pi^7+338688\pi^5-46448640\pi^3+1703116800\pi+4335206400)\mu_{3,9}}{2835\pi^{17}}\\
&-\frac{8(\pi^9-960\pi^7+338688\pi^5-46448640\pi^3+1703116800\pi+3344302080)\mu_{3,10}}{2835\pi^{16}}\\
&-\frac{4(\pi^9-960\pi^7+338688\pi^5-46448640\pi^3+1703116800\pi+2415329280)\mu_{3,11}}{2835\pi^{15}}\\
&-\frac{2(\pi^9-960\pi^7+338688\pi^5-46448640\pi^3+1703116800\pi+1548288000)\mu_{3,12}}{2835\pi^{14}}\\
&-\frac{\pi^9-960\pi^7+338688\pi^5-46448640\pi^3+1703116800\pi+743178240)\mu_{3,13}}{2835\pi^{13}}\\
&-\frac{\pi^8-960\pi^6+338688\pi^4-46448640\pi^2+1703116800)\mu_{3,14}}{5670\pi^{11}}\approx-1.2e-5<0;\\
\mu_{6,18}=&-\frac{8388608(7\pi^3-184\pi+384)\mu_{3,2}}{3\pi^{24}}-\frac{16777216(5\pi^3-132\pi+276)\mu_{3,3}}{9\pi^{23}}+\frac{16777216(21\pi-44)\mu_{3,4}}{3\pi^{22}}\\
&+\frac{16777216(10\pi-21)\mu_{3,5}}{3\pi^{21}}-\frac{167772160\mu_{3,6}}{3\pi^{20}}-\frac{79691776\mu_{3,7}}{3\pi^{19}}-\frac{12582912\mu_{3,8}}{\pi^{18}}-\frac{17825792\mu_{3,9}}{3\pi^{17}}\\
&-\frac{8388608\mu_{3,10}}{3\pi^{16}}-\frac{1310720\mu_{3,11}}{\pi^{15}}-\frac{1835008\mu_{3,12}}{3\pi^{14}}-\frac{851968\mu_{3,13}}{3\pi^{13}}-\frac{131072\mu_{3,14}}{\pi^{12}}\approx-1.3e-6<0;
\end{aligned}
\tag{4.12}
$$

$$\begin{aligned}\mu_{6,19} =& -\frac{8388608(\pi^3-24\pi+48)\mu_{3,2}}{9\pi^{24}} - \frac{4194304(\pi^3-24\pi+48)\mu_{3,3}}{9\pi^{23}} + \frac{16777216(\pi-2)\mu_{3,4}}{3\pi^{22}} \\ &+ \frac{8388608(\pi-2)\mu_{3,5}}{3\pi^{21}} - \frac{8388608\mu_{3,6}}{3\pi^{20}} - \frac{4194304\mu_{3,7}}{3\pi^{19}} - \frac{2097152\mu_{3,8}}{3\pi^{18}} - \frac{1048576\mu_{3,9}}{3\pi^{17}} - \frac{524288\mu_{3,10}}{3\pi^{16}} \\ &- \frac{262144\mu_{3,11}}{3\pi^{15}} - \frac{131072\mu_{3,12}}{3\pi^{14}} - \frac{65536\mu_{3,13}}{3\pi^{13}} - \frac{32768\mu_{3,14}}{3\pi^{12}} \approx -6.8e-8 < 0.\end{aligned} \tag{4.13}$$

From Eqs (4.3–4.13), we have that

$$E_5(x) > 0 \text{ and } E_6(x) < 0, \quad \forall x \in (0, \pi/2),$$

and complete the proof. □

5. Discussions

It can be verified that

$$l_{zhu}^{(i)}(0) = \bar{G}^{(i)}(0), \; i = 0, 1, \cdots, 10, \tag{5.1}$$

$$u_{zhu}^{(i)}(0) = \bar{G}^{(i)}(0), \; l_{zhu}^{(j)}(\pi/2) = \bar{G}^{(j)}(\pi/2), \; i = 0, 1, \cdots, 6, \quad j = 0, 1, \tag{5.2}$$

where $\bar{G}(x) = (G(x))^{2/5}$. From the constraints in Eq (5.1) and Eq (5.2), we can recover the resulting bounds $l_{zhu}(x)$ and $u_{zhu}(x)$ in Eq (1.13). In principle, one can find much better bounds by adding other more constraints. Figure 1 shows the error plots of $L_C(x) - L_{Zhu}(x)$ and $U_C(x) - U_{Zhu}(x)$. It shows that $L_{Zhu}(x) < L_C(x) < G(x) < U_C(x) < U_{Zhu}(x)$, for all $x \in (0, \pi/2)$.

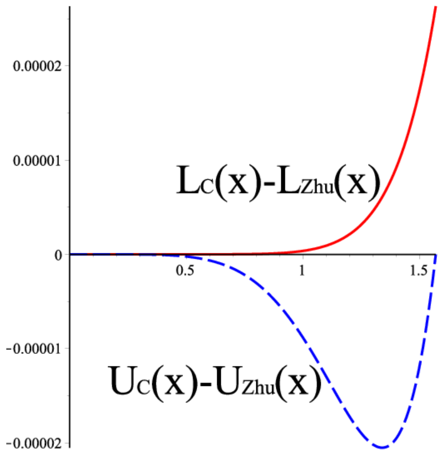

Figure 1. Error plots of $L_C(x) - L_{Zhu}(x)$ and $U_C(x) - U_{Zhu}(x)$.

6. Conclusions

This paper provides a new method to find better bounds for the exponential function with cotangent, by using the interpolation constraints at two end-points of the parametric interval. Many previous results can be recovered by using the new method in this paper. Usually, more constraints, much tighter bounds. In principle, it can be directly extended to more cases with other functions. Moreover, it also presents a new method for proving the corresponding bounds, instead of prevailing methods based on the monotonicity of some special functions.

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Conflict of interest

The authors declare that they have no conflict interests.

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