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Research article

Hermite-Hadamard and Jensen's type inequalities for modified (p, h)-convex functions

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Abstract: In this study, we will derive the conception of modified (p, h)-convex functions which will unify p-convexity with modified h-convexity. We will investigate the fundamental properties of modified (p, h)-convexity. Furthermore, we will derive the Hermite-Hadamard, Fejér and Jensen's type inequalities for this generalization.

Keywords: p-convex function; modified h-convex function; Hermite-Hadamard inequality; Fejér type inequalities; Jensen's type inequalities
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1. Introduction

In non-linear programming and optimization theory, convexity plays an important role. Three important areas of non-linear analysis are monotone operator theory, convex analysis and theory of non-expansive mapping. In early 1960 these theories get emerged. Theses areas have got the attention of many researcher and many connections have been identified between them over the past few years. The notion of convexity has been expanded and generalization in numerous ways utilizing novel and modern methods in recent years. Convexity also plays vital part in fields outside mathematics such as chemistry, biology, physics and other sciences. It is always interesting to generalize the definition of convexity from different aspects because the wide range of applications. Recently the object of numerous studies have been the convexity of functions and sets. For further studies on generalization of convexity one can see [1–9] and references therein.

In this paper, we introduce the concept of modified (p, h)-convex functions as generalization of convex functions. Some basic results under various conditions for the modified (p, h)-convex functions are investigated. We investigate the Jensen and Hermite-Hadamard type inequalities related to modified (p, h)-convex functions. For more on Hermite-Hadamard inequalities, we will refere to the reader [10–20].

Let us see some basic definitions and generalizations of convex functions [21, 22].

Definition 1.1. **(Similarly ordered function)** [17] Two functions f and g are called similarly ordered (f is g-monotone) on $I \subset \mathbb{R}$, if

$$\langle f(r) - f(s), g(r) - g(s)\rangle \geq 0,$$

for all $r, s \in I$.

Definition 1.2. **(Super (Sub) multiplicative function)** [4] A function $h : J \to \mathbb{R}$ is said to be supermultiplicative if

$$h(xy) \geq h(x)h(y), \tag{1.1}$$

for all $x, y \in J$, where $J \subset \mathbb{R}$. If inequality (1.1) is reversed, then h is said to be a submultiplicative function.

Definition 1.3. [19] Let $f \in L_1[a, b]$, the Riemann-Liouville Integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ are defined by

$$J^{\alpha}_{a^+}(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

and

$$J^{\alpha}_{b^-}(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-\alpha} x^{\alpha-1} dx,$$

is the Gamma function.

Definition 1.4. **(Convex function)** Let $I \subseteq \mathbb{R}$ be an interval, then a function $f : I \to \mathbb{R}$ is called convex if the following inequality holds:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad \forall x, y \in I \text{ and } t \in [0, 1].$$

Definition 1.5. **(p-convex set)** [18] The interval I is called p-convex set if

$$[tr^p + (1-t)s^p]^{\frac{1}{p}} \in I,$$

for $t \in [0,1]$ and for all $r, s \in I$, whereas p = $\frac{n}{m}$ or p = 2k+1, m = 2t+1, n = 2u+1, and t, u, k$\in N$.

Definition 1.6. **(p-convex function)** [18] A function f from X to $\mathbb{R}$ is known as p-convex function whereas I is a p-convex set, if

$$f\left([tr^p + (1-t)s^p]^{\frac{1}{p}}\right) \le tf(r) + (1-t)f(s),$$

for $t \in [0,1]$ and $\forall\, r, s \in I$.

Definition 1.7. **(Modified h-convex function)** [8] Let $f, h : J \subset \mathbb{R} \to \mathbb{R}$ be non-negative functions. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called modified h-convex function if

$$f(tr + (1-t)s) \le h(t)f(r) + (1-h(t))f(s),$$

for $t \in [0,1]$ and $\forall\, r, s \in J$.

Definition 1.8. **((p,h)-convex function)** [9] Assume $h : J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative and non-zero function. A function $f : I \to \mathbb{R}$, where I is $p-$convex set in $\mathbb{R}$ is called (p,h)-convex function, if f is non-negative and

$$f([tr^p + (1-t)s^p]^{\frac{1}{p}}) \le h(t)f(r) + h(1-t)f(s),$$

for $t \in (0,1)$ and $\forall\, r, s \in I$, where $p > 0$.

Definition 1.9. **(Modified (p,h)-convex function)** Assume $h : J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative and non-zero function. A function $f : I \to \mathbb{R}$, where I is $p-$convex set in $\mathbb{R}$ is called modified (p,h)-convex function, if f is non-negative and

$$f([tr^p + (1-t)s^p]^{\frac{1}{p}}) \le h(t)f(r) + (1-h(t))f(s), \tag{1.2}$$

for $t \in (0,1)$ and $\forall\, r, s \in I$, where $p > 0$.

Likewise, if the inequality sign in (1.2) is inverted, then f is known as a modified (p,h)-concave function.

Of course, if we put in (1.2)

1. $p = 1$ then we get modified h-convex function;
2. $p = 1$ and $h(t) = t$ then we get classical convex function.

The paper is organized as follows: In next section, we will derive some basic properties of this generalization. However, the third, fourth and fifth sections are devoted to develop Hermite-Hadamard inequality, Jensen type inequality and Fejér type inequalities for modified (p,h)-convex functions.

2. Basic results

In this section, we will verify our basic properties.

Proposition 1. *Assume $f_i : I \subset \mathbb{R} \to \mathbb{R}$ be modified (p,h)-convex function, suppose $\mu_i, \cdots, \mu_n$ be positive scalers. Consider a function g from $\mathbb{R}$ to $\mathbb{R}$ so that*

ADD then g is modified (p,h)-convex function.

Proof. We know that $f_i : I \subset \mathbb{R} \to \mathbb{R}$ be modified (p,h)-convex functions. Then $t \in [0,1]$ and $\forall$, $r, s \in I$, we have

$$\begin{aligned} g(tr^p + (1-t)s^p)^{\frac{1}{p}} &= \sum_{i=1}^{n} \mu_i f_i(tr^p + (1-t)s^p)^{\frac{1}{p}}, \\ &\leq \sum_{i=1}^{n} \mu_i (h(t) f_i(r) + (1-h(t)) f_i(s)), \\ &= h(t) \sum_{i=1}^{n} \mu_i f_i(r) + (1-h(t)) \sum_{i=1}^{n} \mu_i f_i(s), \\ &= h(t) g(r) + (1-h(t)) g(s). \end{aligned}$$

The proof is completed. □

Proposition 2. *Let $h : J \subset \mathbb{R} \to [0,1]$. If $g : I \to \mathbb{R}$ is modified (p,h)-convex function, and $f : I \to \mathbb{R}$ is convex and increasing, then $f \circ g$ is also modified (p,h)-convex function.*

Proof. Since g is modified (p,h)-convex function on I, we obtained

$$f \circ g([tx^p + (1-t)y^p]^{\frac{1}{p}}) = f\left(g\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right)\right) \leq f(h(t)g(x) + (1-h(t))g(y)).$$

Then by using the convexity of f, we obtain

$$\begin{aligned} f(h(t)g(x) + (1-h(t))g(y)) &\leq h(t) f(g(x)) + (1-h(t)) f(g(y)) \\ &= h(t)(f \circ g)(x) + (1-h(t)) f \circ g)(y), \end{aligned}$$

which implies that $f \circ g$ is modified (p,h)-convex function. □

Proposition 3. *Let $h : J \subset \mathbb{R} \to [0,1]$. Further let $\left\{f_j : I \to \mathbb{R},\ j \in \mathbb{N}\right\}$ is non empty collection of modified (p,h)-convex functions such that for each $x \in I$, $\max_{j \in J} f_j(x)$ exists in $\mathbb{R}$, then the function $f : I \to \mathbb{R}$ defined by $f(x) = \max_{j \in J} f_j(x)$ for each $x \in I$ is modified (p,h)-convex.*

Proof. For any $x, y \in I$ and $t \in [0,1]$, we have

$$\begin{aligned} f([tx^p + (1-t)y^p]^{\frac{1}{p}}) &= \max_{j \in J} f_j([tx^p + (1-t)y^p]^{\frac{1}{p}}) \\ &\leq \max_{j \in J} \left\{h(t) f_j(x) + (1-h(t)) f_j(y)\right\} \\ &\leq h(t) \max_{j \in J} f_j(x) + (1-h(t)) \max_{j \in J} f_j(y)), \\ &= h(t) f(x) + (1-h(t)) f(y), \end{aligned}$$

which is required. □

Proposition 4. *Let $h : J \subset \mathbb{R} \to [0,1]$. Further let g and f are two modified (p,h)-convex functions. Then the product of f and g will be a modified (p,h)-convex function if g and f are similarly ordered.*

Proof. We know that g and f are modified (p,h)-convex function. Then

$$\begin{aligned}
& f((1-t)a_1^p + ta_2^p)^{\frac{1}{p}} g((1-t)a_1^p + ta_2^p)^{\frac{1}{p}} \\
& \leq [(1-h(t))f(r) + h(t)f(s)][(1-h(t))g(r) + h(t)g(s)] \\
& = [1-h(t)]^2 f(r)g(r) + h(t)(1-h(t))f(r)g(s) + h(t)(1-h(t))f(s)g(r) \\
& + [h(t)]^2 f(s)g(s) \\
& = (1-h(t))f(r)g(r) + h(t)f(s)g(s) - (1-h(t))f(r)g(r) - h(t)f(s)g(s) \\
& + [1-h(t)]^2 f(r)g(r) \\
& + h(t)(1-h(t))f(r)g(s) + h(t)(1-h(t))f(s)g(r) + [h(t)]^2 f(s)g(s) \\
& = (1-h(t))f(r)g(r) + h(t)f(s)g(s) - h(t)(1-h(t)) \\
& \times [f(r)g(r) + f(s)g(s) - f(s)g(r) - f(r)g(s)] \\
& \leq (1-h(t))f(r)g(r) + h(t)f(s)g(s).
\end{aligned}$$

That's the required result. □

3. Hermite-Hadamard type inequalities

Theorem 3.1. *Assume f from I to $\mathbb{R}$ be modified (p,h)-convex function on the interval $[a_1, a_2]$ with $a_1 < a_2$ then we have*

$$\begin{aligned}
\int_0^1 f\left(\frac{a_1^p + a_2^p}{2}\right)^{\frac{1}{p}} dt &\leq \left(\frac{p}{a_2^p - a_1^p}\right) \int_{a_1}^{a_2} r^{p-1} f(r) dr \\
&\leq f(a_1) + \{f(a_2) - f(a_1)\} \int_0^1 h(t) dt. \qquad (3.1)
\end{aligned}$$

Proof. Let $u^p = ta_1^p + (1-t)a_2^p$ and $v^p = (1-t)a_1^p + ta_2^p$, then we get

$$\begin{aligned}
f\left(\frac{a_1^p + a_2^p}{2}\right)^{\frac{1}{p}} &= f\left(\frac{u^p + v^p}{2}\right)^{\frac{1}{p}} = f\left(\frac{(ta_1^p + (1-t)a_2^p) + ((1-t)a_1^p + ta_2^p)}{2}\right)^{\frac{1}{p}} \\
&\leq h\left(\frac{1}{2}\right) f[(ta_1^p + (1-t)a_2^p)^{\frac{1}{p}}] + [1-h\left(\frac{1}{2}\right)] f[((1-t)a_1^p + ta_2^p)^{\frac{1}{p}}].
\end{aligned}$$

Integrating inequality over $t \in [0,1]$, we get

$$\begin{aligned}
\int_0^1 f\left(\frac{a_1^p + a_2^p}{2}\right)^{\frac{1}{p}} dt &\leq h\left(\frac{1}{2}\right)\int_0^1 f\left[(ta_1^p + (1-t)a_2^p)^{\frac{1}{p}}\right] dt + \left[1 - h\left(\frac{1}{2}\right)\right] \\
&\times \int_0^1 f\left[((1-t)a_1^p + ta_2^p)^{\frac{1}{p}}\right] dt \\
&\leq h\left(\frac{1}{2}\right)\frac{p}{a_2^p - a_1^p}\int_{a_1}^{a_2} r^{p-1} f(r)dr + \frac{p}{a_2^p - a_1^p}\left[1 - h\left(\frac{1}{2}\right)\right] \\
&\times \int_{a_1}^{a_2} r^{p-1} f(r)dr \\
&= \frac{p}{a_2^p - a_1^p}\int_{a_1}^{a_2} r^{p-1} f(r)dr. \qquad (3.2)
\end{aligned}$$

Now, we know that

$$\begin{aligned}
\int_{a_1}^{a_2} r^{p-1} f(r)dr &= \frac{a_2^p - a_1^p}{p}\int_0^1 f\left[(ta_2^p + (1-t)a_1^p)^{\frac{1}{p}}\right] dt \\
&\leq \frac{a_2^p - a_1^p}{p}\int_0^1 [h(t)f(a_2) + (1 - h(t))f(a_1)]dt \\
&= \frac{a_2^p - a_1^p}{p}\left[f(a_1) + \{f(a_2) - f(a_1)\}\int_0^1 h(t)dt\right].
\end{aligned}$$

Thus,

$$\frac{p}{a_2^p - a_1^p}\int_{a_1}^{a_2} r^{p-1} f(r)dr \leq f(a_1) + \{f(a_2) - f(a_1)\}\int_0^1 h(t)dt. \qquad (3.3)$$

Combining (3.2) and (3.3), we obtain the required result. □

Remark 1. If we put $p = 1$ in (3.1) then we gain Hermite-Hadamard type inequality for modified h-convexity, (see [8]).

Remark 2. If we put $p = 1$ and $h(t) = t$ in (3.1) then we attain classical Hermite-Hadamard type inequality.

Theorem 3.2. *Assume f be modified (p,h)-convex function and $f \in L_1[a_1,a_2]$, with $a_1 < a_2$. Then*

$$\frac{1}{\alpha} f\left(\frac{a_1^p + a_2^p}{2}\right)^{\frac{1}{p}} \leq \frac{\Gamma(\alpha)}{(a_2^p - a_1^p)^\alpha}\left[\left(1 - h\left(\frac{1}{2}\right)\right) J^\alpha_{a_1^+} f(a_2) + h(\frac{1}{2}) J^\alpha_{a_2^-} f(a_1)\right], \tag{3.4}$$

and

$$\frac{\Gamma(\alpha)}{(a_2^p - a_1^p)^\alpha}\left[J^\alpha_{a_1^+} f(a_2) + J^\alpha_{a_2^-} f(a_1)\right] \leq \frac{f(a_1) + f(a_2)}{\alpha}. \tag{3.5}$$

Proof. We know that f is modified (p,h)-convex function, so we have

$$f\left(\frac{r^p + s^p}{2}\right)^{\frac{1}{p}} \leq \left\{1 - h\left(\frac{1}{2}\right)\right\} f(r) + h\left(\frac{1}{2}\right) f(s).$$

Let $r^p = (ta_1^p + (1-t)a_2^p)$, and $s^p = ((1-t)a_1^p + ta_2^p)$, then

$$f\left(\frac{a_1^p + a_2^p}{2}\right)^{\frac{1}{p}} \leq \left\{1 - h(\frac{1}{2})\right\} f(ta_1^p + (1-t)a_2^p)^{\frac{1}{p}} + h(\frac{1}{2}) f((1-t)a_1^p + ta_2^p)^{\frac{1}{p}}.$$

Multiplying above inequality by $t^{\alpha-1}$, then integrating over $t \in [0,1]$, we get

$$\begin{aligned}
&\frac{1}{\alpha} f\left(\frac{a_1^p + a_2^p}{2}\right)^{\frac{1}{p}} = f\left(\frac{a_1^p + a_2^p}{2}\right)^{\frac{1}{p}} \int_0^1 t^{\alpha-1} dt \\
&\leq \left\{1 - h\left(\frac{1}{2}\right)\right\} \int_0^1 t^{\alpha-1} f(ta_1^p + (1-t)a_2^p)^{\frac{1}{p}} + h\left(\frac{1}{2}\right) \int_0^1 t^{\alpha-1} f((1-t)a_1^p + ta_2^p)^{\frac{1}{p}} dt \\
&= \left\{1 - h\left(\frac{1}{2}\right)\right\} \int_{a_1}^{a_2} \left(\frac{a_2^p - u^p}{a_2^p - a_1^p}\right)^{\alpha-1} f(u) u^{p-1} \frac{p}{a_2^p - a_1^p} du \\
&+ h\left(\frac{1}{2}\right) \int_{a_1}^{a_2} \left(\frac{v^p - a_1^p}{a_2^p - a_1^p}\right)^{\alpha-1} f(v) v^{p-1} \frac{p}{a_2^p - a_1^p} dv \\
&= \frac{\Gamma(\alpha)}{\left(a_2^p - a_1^p\right)^\alpha}\left[\left(1 - h\left(\frac{1}{2}\right)\right) J^\alpha_{a_1^+} f(a_2) + h\left(\frac{1}{2}\right) J^\alpha_{a_2^-} f(a_1)\right],
\end{aligned}$$

which is (3.4).

We know that f is a modified (p,h)-convex function, then

$$\begin{aligned}
&f(ta_1^p + (1-t)a_2^p)^{\frac{1}{p}} + f((1-t)a_1^p + ta_2^p)^{\frac{1}{p}} \\
&\leq h(t) f(a_1) + (1 - h(t)) f(a_2) + (1 - h(t)) f(a_1) + h(t) f(a_2) \\
&= f(a_1) + f(a_2).
\end{aligned}$$

Multiplying above inequality by $t^{\alpha-1}$ and then integrating over $t \in [0,1]$, we get

$$\int_0^1 t^{\alpha-1} f(ta_1^p + (1-t)a_2^p)^{\frac{1}{p}} dt + \int_0^1 t^{\alpha-1} f((1-t)a_1^p + ta_2^p)^{\frac{1}{p}} dt$$
$$\leq [f(a_1) + f(a_2)] \int_0^1 t^{\alpha-1} dt,$$

from which we have

$$\frac{\Gamma(\alpha)}{(a_2^p - a_1^p)^\alpha} \left[J^\alpha_{a_1^+} f(a_2) + J^\alpha_{a_2^-} f(a_1) \right] \leq \frac{f(a_1) + f(a_2)}{\alpha}.$$

The result is completed. □

Remark 3. If $p = 1$ in (3.4) and (3.5) then we will get the result for modified h-convex function, (see [8]).

4. Jensen type inequality

The following expression is useful to prove Jensen type inequality for modified (p, h)-convex functions.

Assume f from I to $\mathbb{R}$ be an modified (p, h)-convex function. For $r_1, r_2 \in I$ and $\alpha_1 + \alpha_2 = 1$, we have $f(\alpha_1 r_1^p + \alpha_2 r_2^p)^{\frac{1}{p}} \leq h(\alpha_1) f(r_1) + (1 - h(\alpha_1)) f(r_2)$.

Also when $m > 2$ for $r_1, r_2, ..., r_m \in I$, $\sum_{i=1}^m \alpha_i = 1$ and $T_i = \sum_{j=1}^i \alpha_j$, we have

$$\begin{aligned} f\left(\sum_{i=1}^m \alpha_i r_i^p\right)^{\frac{1}{p}} &= f\left(T_{m-1} \sum_{i=1}^{m-1} \frac{\alpha_i}{T_{m-1}} r_i^p + \alpha_m r_m^p\right)^{\frac{1}{p}} \\ &\leq h(T_{m-1}) f\left(\sum_{i=1}^{m-1} \frac{\alpha_i}{T_{m-1}} r_i^p\right)^{\frac{1}{p}} + (1 - h(T_{m-1})) f(r_m). \end{aligned} \tag{4.1}$$

Theorem 4.1. *Assume f from I to $\mathbb{R}$ be an modified (p, h)-convex function and h be non-negative super-multiplicative function. If $T_i = \sum_{j=1}^i \alpha_j$ for $i = 1, ..., m$, so that $T_m = 1$ where $m \in \mathbb{N}$, then*

$$f\left(\sum_{i=1}^m \alpha_i r_i^p\right)^{\frac{1}{p}} \leq f(r_m) + \sum_{i=1}^{m-1} h(T_i) \{f(r_i) - f(r_{i+1})\}. \tag{4.2}$$

Proof. By using (4.1) it follows that:

$$
\begin{aligned}
f\left(\sum_{i=1}^{m}\alpha_i r_i^p\right)^{\frac{1}{p}} &\leq h(T_{m-1})f\left(\sum_{i=1}^{m-1}\frac{\alpha_i}{T_{m-1}}r_i^p\right)^{\frac{1}{p}} + (1-h(T_{m-1}))f(r_m)\\
&= (1-h(T_{m-1}))f(r_m) + h(T_{m-1})f\left[\frac{T_{m-2}}{T_{m-1}}\left(\left[\sum_{i=1}^{m-2}\frac{\alpha_i}{T_{m-2}}r_i^p\right]^{\frac{1}{p}}\right)^p + \frac{\alpha_{m-1}}{T_{m-1}}r_{m-1}^p\right]^{\frac{1}{p}},\\
&\leq (1-h(T_{m-1}))f(r_m) + h(T_{m-1})\Bigg[h\left(\frac{T_{m-2}}{T_{m-1}}\right)f\left(\sum_{i=1}^{m-2}\frac{\alpha_i}{T_{m-2}}r_i^p\right)^{\frac{1}{p}}\\
&+\left(1-h\left(\frac{T_{m-2}}{T_{m-1}}\right)\right)f(r_{m-1})\Bigg],
\end{aligned}
$$

using the fact that h is supermultiplicative function, we have

$$
\begin{aligned}
&\leq (1-h(T_{m-1}))f(r_m) + h(T_{m-2})f\left(\sum_{i=1}^{m-2}\frac{\alpha_i}{T_{m-2}}r_i^p\right)^{\frac{1}{p}}\\
&+ h(T_{m-1})\left(1-h\left(\frac{T_{m-2}}{T_{m-1}}\right)\right)f(r_{m-1})\\
&= f(r_m) + h(T_{m-1})f(r_{m-1}) - h(T_{m-1})f(r_m) - h(T_{m-2})f(r_{m-1})\\
&+ h(T_{m-2})f\left(\sum_{i=1}^{m-2}\frac{\alpha_i}{T_{m-2}}r_i^p\right)^{\frac{1}{p}},\\
&= f(r_m) + h(T_{m-1})[f(r_{m-1}) - f(r_m)] - h(T_{m-2})f(r_{m-1})\\
&+ h(T_{m-2})f\left(\left[\frac{T_{m-3}}{T_{m-2}}\left(\left[\sum_{i=1}^{m-3}\frac{\alpha_i}{T_{m-3}}r_i^p\right]^{\frac{1}{p}}\right)^p + \frac{\alpha_{m-2}}{T_{m-2}}r_{m-2}^p\right]^{\frac{1}{p}}\right),
\end{aligned}
$$

from (4.1), we get

$$
\begin{aligned}
&\leq f(r_m) + h(T_{m-1})[f(r_{m-1}) - f(r_m)] - h(T_{m-2})f(r_{m-1})\\
&+ h(T_{m-2})\left[h\left(\frac{T_{m-3}}{T_{m-2}}\right)f\left(\sum_{i=1}^{m-3}\frac{\alpha_i}{T_{m-3}}r_i^p\right)^{\frac{1}{p}} + \left(1-h\left(\frac{T_{m-3}}{T_{m-2}}\right)\right)f(r_{m-2})\right],
\end{aligned}
$$

using the fact that h is supermultiplicative function,

$$
\begin{aligned}
&\leq f(r_m) + h(T_{m-1})[f(r_{m-1}) - f(r_m)] - h(T_{m-2})f(r_{m-1}) \\
&+ h(T_{m-3})f\left(\sum_{i=1}^{m-3} \frac{\alpha_i}{T_{m-3}} r_i^p\right)^{\frac{1}{p}} + h(T_{m-2})f(r_{m-2}) - h(T_{m-3})f(r_{m-2}) \\
&= f(r_m) + h(T_{m-1})[f(r_{m-1}) - f(r_m)] + h(T_{m-2})[f(r_{m-2}) - f(r_{m-1})] \\
&- h(T_{m-3})f(r_{m-2}) + h(T_{m-3})f\left(\sum_{i=1}^{m-3} \frac{\alpha_i}{T_{m-3}} r_i^p\right)^{\frac{1}{p}} \\
&\leq \ldots \\
&\leq f(r_m) + h(T_{m-1})[f(r_{m-1}) - f(r_m)] + h(T_{m-2})[f(r_{m-2}) - f(r_{m-1})] \\
&+ h(T_{m-3})[f(r_{m-3}) - f(r_{m-2})] + \ldots \\
&+ h(T_2)[f(r_2) - f(r_3)] + h(T_1)[f(r_1) - f(r_2)] \\
&= f(r_m) + \sum_{i=1}^{m-1} h(T_i)[f(r_i) - f(r_{i+1})].
\end{aligned}
$$

The result is completed. □

Remark 4. If we choose $p = 1$ and $h(t) = t$ in (4.2) then we have classical Jensen type inequality.

5. Fejér type inequality

Theorem 5.1. *Assume that g and f are functions such that f is modified (p, h_1)-convex function and g is modified (p, h_2)-convex function, $fg \in L_1([v_1, v_2])$ and $h_1 h_2 \in L_1([0, 1])$ along $v_1, v_2 \in I$ and $v_1 < v_2$. Then we get*

$$
\begin{aligned}
\frac{p}{v_2^p - v_1^p} \int_{v_1}^{v_2} r^{p-1} f(r)g(r)dr &\leq f(v_2)g(v_2) + l(v_1, v_2) \int_0^1 h_1(t)h_2(t)dt \\
&+ m(v_1, v_2) \int_0^1 h_2(t)dt + n(v_1, v_2) \int_0^1 h_1(t)dt, \qquad (5.1)
\end{aligned}
$$

where

$$
\begin{aligned}
l(v_1, v_2) &= [f(v_1)g(v_1) - f(v_2)g(v_1) - f(v_1)g(v_2) + f(v_2)g(v_2)]; \\
m(v_1, v_2) &= [f(v_2)g(v_1) - f(v_2)g(v_2)]; \\
n(v_1, v_2) &= [f(v_1)g(v_2) - f(v_2)g(v_2)].
\end{aligned}
$$

Proof. We know that f be modified (p, h_1)-convex function and g be modified (p, h_2)-convex function, we have

$$f([tv_1^p+(1-t)v_2^p]^{\frac{1}{p}}) \leq h_1(t)f(v_1)+(1-h_1(t))f(v_2);$$
$$g([tv_1^p+(1-t)v_2^p]^{\frac{1}{p}}) \leq h_2(t)g(v_1)+(1-h_2(t))g(v_2),$$

for each $t \in [0,1]$. Because f and g are non-negative, we attain the inequality

$$\begin{aligned}
&f([tv_1^p+(1-t)v_2^p]^{\frac{1}{p}})g([tv_1^p+(1-t)v_2^p]^{\frac{1}{p}})\\
&\leq h_1(t)h_2(t)f(v_1)g(v_1)+h_1(t)(1-h_2(t))f(v_1)g(v_2)\\
&+(1-h_1(t))h_2(t)f(v_2)g(v_1)+(1-h_1(t))(1-h_2(t))f(v_2)g(v_2),
\end{aligned}$$

integrating above inequality over (0,1), we attain the inequality

$$\begin{aligned}
&\int_0^1 f([tv_1^p+(1-t)v_2^p]^{\frac{1}{p}})g([tv_1^p+(1-t)v_2^p]^{\frac{1}{p}})dt\\
&\leq f(v_1)g(v_1)\int_0^1 h_1(t)h_2(t)dt+f(v_1)g(v_2)\int_0^1 h_1(t)(1-h_2(t))dt\\
&+f(v_2)g(v_1)\int_0^1 (1-h_1(t))h_2(t))dt\\
&+f(v_2)g(v_2)\int_0^1 (1-h_1(t))(1-h_2(t))dt.
\end{aligned}$$

By setting $r=[tv_1^p+(1-t)v_2^p]^{\frac{1}{p}}$, we get

$$\begin{aligned}
&\frac{p}{v_2^p-v_1^p}\int_{v_1}^{v_2} r^{p-1}f(r)g(r)dr \leq f(v_2)g(v_2)\\
&+[f(v_1)g(v_1)-f(v_2)g(v_1)-f(v_1)g(v_2)+f(v_2)g(v_2)]\int_0^1 h_1(t)h_2(t)dt\\
&+[f(v_2)g(v_1)-f(v_2)g(v_2)]\int_0^1 h_2(t)dt+[f(v_1)g(v_2)-f(v_2)g(v_2)]\int_0^1 h_1(t)dt\\
&\leq f(v_2)g(v_2)+l(v_1,v_2)\int_0^1 h_1(t)h_2(t)dt+m(v_1,v_2)\int_0^1 h_2(t)dt+n(v_1,v_2)\int_0^1 h_1(t)dt.
\end{aligned}$$

□

Remark 5. If we take $p=1$ and $h(t)=t$ in (5.1) then we will get the result for convex function.

6. Conclusions

Convexity play an important rule in applied sciences and mathematics. In this paper, we introduced modified (p, h)-convex functions which unify p-convexity with modified h-convexity. We investigated the fundamental properties of modified (p, h)-convexity and gave the Hermite-Hadamard, Fejér and Jensen's type inequalities.

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Conflict of interest

The authors declare that no competing interests exist.

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