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Research article

Existence of infinitely many small solutions for fractional Schrödinger-Poisson systems with sign-changing potential and local nonlinearity

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Abstract: In this paper, we prove the existence of infinitely many small solutions for the following fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = f(x,u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $(-\Delta)^\alpha$ denotes the fractional Laplacian of order $\alpha \in (0,1)$ and V is allowed to be sign-changing. We obtain infinitely many small solutions via a dual method. Our main tool is a critical point theorem which was established by Kajikiya.

Keywords: fractional Schrödinger-Poisson system; infinitely many small solutions; dual methods
Mathematics Subject Classification: 35J60, 35J20

1. Introduction and preliminaries

This article deals mainly with the following fractional Schrödinger-Poisson systems

$$\begin{cases} (-\Delta)^s u + V(x)u + \phi u = f(x,u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $(-\Delta)^\alpha$ denotes the fractional Laplacian of order $\alpha \in (0,1)$ and V is allowed to be sign-changing. In (1.1), the first equation is a nonlinear fractional Schrödinger equation in which the potential ϕ satisfies a nonlinear fractional Poisson equation. For this reason, system (1.1) is called a fractional Schrödinger-Poisson system, also known as the fractional Schrödinger-Maxwell system, which is not only a physically relevant generalization of the classical NLS but also an important model in the study of fractional quantum mechanics. For more details about the physical background, we refer the reader to [3, 4] and the references therein.

It is well known that the fractional Schrödinger-Poisson system was first introduced by Giammetta in [8] and the diffusion is fractional only in the Poisson equation. Afterwards, in [14], the authors proved the existence of radial ground state solutions of (1.1) when $V(x) \equiv 0$ and nonlinearity $f(x,u)$ is of subcritical or critical growth. Very recently, in [15], the author proved infinitely many solutions via Fountain Theorem for (1.1) and $V(x)$ is positive. However, to the best of our knowledge, for the sign-changing potential case, there are not many results for problem (1.1).

In recent years, the following potential function was discussed: (V_1) $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} V(x) > -\infty$, which is called sign-changing potential. It is well known that the Schrödinger equation has already attracted a great deal of interest in the recent years with the above sign-changing potential. Many researchers studied infinitely many solutions for Schrödinger equation with the above sign-changing potential and some different growth conditions on f, see [11, 13]. On the basis of the previous work, many authors considered infinitely many solutions for different equations with sign-changing potential and some different growth conditions on f, see [1, 5–7, 10, 11, 13, 18–22] and the references therein. In particular, Bao [7] and Zhou [13] studied infinitely many small solutions for Schrödinger-Poisson equation with sign-changing potential. However, we know that there are few papers which deal with infinitely many small solutions without any growth conditions via dual methods.

Inspired by [7, 13], we will study infinitely many small solutions for the problem (1.1) under the following assumptions on V and f:

(V_2) There exists a constant $d_0 > 0$ such that

$$\lim_{|y|\to\infty} meas\left(\left\{x \in \mathbb{R}^3 : |x-y| \le d_0, V(x) \le M\right\}\right) = 0, \ \forall\ M > 0.$$

(f_1) There exists constant $\delta_1 > 0$ and $1 < r_1 < 2$ such that $f \in C(\mathbb{R}^3 \times [-\delta_1, \delta_1], \mathbb{R})$ and

$$|f(x,t)| \le a(x)|t|^{r_1-1}, \quad |t| \le \delta_1, \quad \forall x \in \mathbb{R}^3,$$

where $a(x) \in L^{\frac{2}{2-r_1}}(\mathbb{R}^3)$ is a positive continuous function.

(f_2) $\lim\limits_{t\to 0} \frac{f(x,t)}{t} = +\infty$ uniformly for $x \in \mathbb{R}^3$.

(f_3) There exists a constant $\delta_2 > 0$ such that $f(x,-t) = -f(x,t)$ for any $|t| \le \delta_2$ and all $x \in \mathbb{R}^3$.

Note that condition (V_2) is usually applied to meet the compact embedding.

Next, we are ready to state the main result of this paper.

Theorem 1.1 *Suppose that* (V_1), (V_2) and (f_1)–(f_3) *hold. Then when* $s \in (\frac{3}{4}, 1)$, $t \in (0,1)$ *satisfying* $4s + 2t \ge 3$, *problem* (1.1) *has infinitely many solutions* $\{u_k\}$ *such that*

$$\frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u_k|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x) u_k^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_k}^t u_k^2 dx - \int_{\mathbb{R}^3} F(x,u_k) dx \le 0$$

and $u_k \to 0$ as $k \to \infty$.

Throughout this paper, $C > 0$ denote various positive constants which are not essential to our problem and may change from line to line.

2. Variational framework and main results

Before stating this section, we first notice the following fact: by (V_1), we can conclude that there exists a constant V_0 such that $\widetilde{V}(x) := V(x) + V_0 > 0$ for all $x \in \mathbb{R}^3$. Let $\widetilde{f}(x,u) = f(x,u) + V_0 u$ and consider the following new equation

$$\begin{cases} (-\Delta)^s u + \widetilde{V}(x)u + \phi u = \widetilde{f}(x,u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{2.1}$$

It is easy to check that the hypotheses (V_1), (V_2) and (f_1)–(f_3) still hold for $\widetilde{V}$ and $\widetilde{f}$ provided that those hold for V and f. In what follows, we just need to study the equivalent Eq (2.1). Therefore, throughout this section, we make the following assumption instead of (V_1)

$(\widetilde{V}_1)$ $\widetilde{V}(x) \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} \widetilde{V}(x) > 0$.

To this end, we define the Gagliardo seminorm by

$$[u]_{\alpha,p} = \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^p}{|x-y|^{N+\alpha p}} dxdy \right)^{\frac{1}{p}},$$

where $u : \mathbb{R}^3 \to \mathbb{R}$ is a measurable function.

On the one hand, we define fractional Sobolev space by

$$W^{\alpha,p}(\mathbb{R}^3) = \left\{ u \in L^p(\mathbb{R}^3) : u \text{ is measurable and } [u]_{\alpha,p} < \infty \right\}$$

endowed with the norm

$$\|u\|_{\alpha,p} = \left([u]^p_{\alpha,p} + \|u\|^p_p \right)^{\frac{1}{p}}, \tag{2.2}$$

where

$$\|u\|_p = \left(\int_{\mathbb{R}^3} |u(x)|^p dx \right)^{\frac{1}{p}}.$$

If $p = 2$, the space $W^{\alpha,2}(\mathbb{R}^3)$ is an equivalent definition of the fractional Sobolev spaces based on the Fourier analysis, that is,

$$H^\alpha(\mathbb{R}^3) := W^{\alpha,2}(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (1 + |\xi|^{2\alpha})|\widetilde{u}|^2 d\xi < \infty \right\},$$

endowed with the norm

$$\|u\|_{H^\alpha} = \left(\int_{\mathbb{R}^3} |\xi|^{2\alpha} |\widetilde{u}|^2 d\xi + \int_{\mathbb{R}^3} |u|^2 d\xi \right)^{\frac{1}{2}},$$

where $\widetilde{u}$ denotes the usual Fourier transform of u. Furthermore, we know that $\|\cdot\|_{H^\alpha}$ is equivalent to the norm

$$\|u\|_{H^\alpha} = \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx + \int_{\mathbb{R}^3} u^2 dx \right)^{\frac{1}{2}}.$$

Let $\Omega \subseteq \mathbb{R}^3$ and $L^p(\Omega)$, $1 \le p \le +\infty$ be a Lebesgue space, the norm in $L^p(\Omega)$ is denoted by $|\cdot|_{p,\Omega}$. Let $H^\alpha_0(\Omega)$, $\Omega \subset \mathbb{R}^3$, and $H^\alpha(\mathbb{R}^3)$ denote the usual fractional Sobolev spaces (see [9]). Under the assumption $(\widetilde{V}_1)$, our working space is defined by

$$E = \left\{ u \in H^\alpha(\mathbb{R}^3) : \int_{\mathbb{R}^3} \widetilde{V}(x) u^2 dx < \infty \right\} \tag{2.3}$$

and

$$E(\Omega) = \{u \in H_0^\alpha(\Omega) : \int_\Omega \widetilde{V}(x)u^2dx < \infty\}.$$

Thus, E is a Hilbert space with the inner product

$$(u,v)_{E_V} = \int_{\mathbb{R}^3} \left(|\xi|^{2\alpha}\widetilde{u}(\xi)\widetilde{v}(\xi) + \widetilde{u}(\xi)\widetilde{v}(\xi)\right) d\xi + \int_{\mathbb{R}^3} \widetilde{V}(x)u(x)v(x)dx,$$

$$(u,v)_{E,\Omega} = \int_\Omega \left(|\xi|^{2\alpha}\widetilde{u}(\xi)\widetilde{v}(\xi) + \widetilde{u}(\xi)\widetilde{v}(\xi)\right) d\xi + \int_\Omega \widetilde{V}(x)u(x)v(x)dx,$$

and the norm

$$\|u\|_{E_V} = \left(\int_{\mathbb{R}^3} \left(|\xi|^{2\alpha}|\widetilde{u}(\xi)|^2 + |\widetilde{u}(\xi)|^2\right) d\xi + \int_{\mathbb{R}^3} \widetilde{V}(x)u^2(x)dx\right)^{\frac{1}{2}},$$

$$\|u\|_{E,\Omega} = \left(\int_\Omega \left(|\xi|^{2\alpha}|\widetilde{u}(\xi)|^2 + |\widetilde{u}(\xi)|^2\right) d\xi + \int_\Omega \widetilde{V}(x)u^2(x)dx\right)^{\frac{1}{2}},$$

Moreover, $\|\cdot\|_{E_V}$ and $\|u\|_{E,\Omega}$ are equivalent to the following norms

$$\|u\| := \|u\|_E = \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}}u|^2dx + \int_{\mathbb{R}^3} \widetilde{V}(x)u^2dx\right)^{\frac{1}{2}},$$

and

$$\|u\|_{E,\Omega} = \left(\int_\Omega |(-\Delta)^{\frac{\alpha}{2}}u|^2dx + \int_\Omega \widetilde{V}(x)u^2dx\right)^{\frac{1}{2}},$$

where the corresponding inner product are

$$(u,v)_E = \int_{\mathbb{R}^3} \left((-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}v + \widetilde{V}(x)uv\right) dx.$$

and

$$(u,v)_{E,\Omega} = \int_\Omega \left((-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}v + \widetilde{V}(x)uv\right) dx.$$

The homogeneous Sobolev space $D^{\alpha,2}(\mathbb{R}^3)$ is defined by

$$D^{\alpha,2}(\mathbb{R}^3) = \left\{u \in L^{2^*_\alpha}(\mathbb{R}^3) : |\xi|^\alpha\widetilde{u}(\xi) \in L^2(\mathbb{R}^3)\right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{D^{\alpha,2}} = \left(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}}u|^2dx\right)^{\frac{1}{2}} = \left(\int_{\mathbb{R}^3} |\xi|^{2\alpha}|\widetilde{u}(\xi)|^2d\xi\right)^{\frac{1}{2}},$$

endowed with the inner product

$$(u,v)_{D^{\alpha,2}} = \int_{\mathbb{R}^3} (-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}vdx.$$

Then $D^{\alpha,2}(\mathbb{R}^3) \hookrightarrow L^{2^*_\alpha}(\mathbb{R}^3)$, that is, there exists a constant $C_0 > 0$ such that

$$\|u\|_{2^*_\alpha} \leq C_0 \|u\|_{D^{\alpha,2}}. \tag{2.4}$$

Next, we give the following lemmas which discuss the continuous and compact embedding for $E \hookrightarrow L^p(\mathbb{R}^3)$ for all $p \in [2, 2^*_\alpha]$. In the rest of paper, we use the norm $\|\cdot\|$ in E. Motivated by Lemma 3.4 in [16], we can prove the following Lemma 2.1 in the same way. Here we omit it.

Lemma 2.1 *E is continuously embedded into $L^p(\mathbb{R}^3)$ for $2 \leq p \leq 2^*_\alpha := \frac{6}{3-2\alpha}$ and compactly embedded into $L^p(\mathbb{R}^3)$ for all $s \in [2, 2^*_\alpha)$.*

Lemma 2.2 ([[9], *Theorem* 6.5]) *For any $\alpha \in (0,1)$, $D^{\alpha,2}(\mathbb{R}^3)$ is continuously embedded into $L^{2^*_\alpha}(\mathbb{R}^3)$, that is, there exists $S_\alpha > 0$ such that*

$$\left(\int_{\mathbb{R}^3} |u|^{2^*_\alpha} dx\right)^{\frac{2}{2^*_\alpha}} \leq S_\alpha \int_{\mathbb{R}^3} |(-\Delta)^{\frac{\alpha}{2}} u|^2 dx \quad \forall\, u \in D^{\alpha,2}(\mathbb{R}^3).$$

Next, let $\alpha = s \in (0,1)$. Using Hölder's inequality, it follows from Lemma 2.1 and Lemma 2.2 that for every $u \in E$ and $s, t \in (0,1)$, we have

$$\int_{\mathbb{R}^3} u^2 v dx \leq \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2t}} dx\right)^{\frac{3+2t}{6}} \left(\int_{\mathbb{R}^3} |v|^{2^*_t} dx\right)^{\frac{1}{2^*_t}} \leq \gamma_{\frac{12}{3+2t}} S_t^{\frac{1}{2}} \|u\|^2 \|v\|_{D^{t,2}}, \tag{2.5}$$

where we used the following embedding

$$E \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3) \text{ if } 2t + 4s \geq 3.$$

By the Lax-milgram theorem, there exists a unique $\phi_u^t \in D^{t,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} v(-\Delta)^t \phi_u^t dx = \int_{\mathbb{R}^3} (-\Delta)^{\frac{t}{2}} \phi_u^t (-\Delta)^{\frac{t}{2}} v dx = \int_{\mathbb{R}^3} u^2 v dx, \quad v \in D^{t,2}(\mathbb{R}^3). \tag{2.6}$$

Hence, ϕ_u^t satisfies the Poisson equation

$$(-\Delta)^t \phi_u^t = u^2, \quad x \in \mathbb{R}^3.$$

Moreover, ϕ_u^t has the following integral expression

$$\phi_u^t(x) = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} dy, \quad x \in \mathbb{R}^3,$$

which is called t-Riesz potential, where

$$c_t = \pi^{-\frac{3}{2}} 2^{-2t} \frac{\Gamma(\frac{3}{2} - 2t)}{\Gamma(t)}.$$

Thus $\phi_u^t(x) \geq 0$ for all $x \in \mathbb{R}^3$, from (2.5) and (2.6), we have

$$\|\phi_u^t\|_{D^{t,2}} \leq S_t^{\frac{1}{2}} \|u\|^2_{L^{\frac{12}{3+2t}}} \leq C_1 \|u\|^2 \quad \text{if } 2t + 4s \geq 3. \tag{2.7}$$

Therefore, by Hölder's inequality and Lemma 2.1 and Lemma 2.2, there exist $\widetilde{C}_1 > 0$, $\widetilde{C}_2 > 0$ such that

$$\int_{\mathbb{R}^3} \phi_u^t u^2 dx \leq \left(\int_{\mathbb{R}^3} |\phi_u^t|^{2_t^*} dx \right)^{\frac{1}{2_t^*}} \left(\int_{\mathbb{R}^3} |u|^{\frac{12}{3+2t}} dx \right)^{\frac{3+2t}{6}} \leq \widetilde{C}_1 \|\phi_u^t\|_{D^{t,2}} \|u\|^2 \leq \widetilde{C}_2 \|u\|^4.$$

Now, we define a cut-off function $h \in C(\mathbb{R}, \mathbb{R})$ such that $0 \leq h(t) \leq 1$, $h(-t) = h(t)$ for all $t \in \mathbb{R}$, $h(t) \equiv 1$ for all $|t| \leq d$, $h(t) \equiv 0$ for all $|t| \geq 2d$ and h is decreasing in $[d, 2d]$, where $0 < d \leq \frac{1}{2} \min\{\delta_1, \delta_2, 1\}$. Let

$$f_h(x,u) = f(x,u)h(u), \quad \forall\, (x,u) \in \mathbb{R}^3 \times \mathbb{R}, \tag{2.8}$$

and

$$F_h(x,u) = \int_0^u f_h(x,t)dt, \quad \forall\, (x,u) \in \mathbb{R}^3 \times \mathbb{R}. \tag{2.9}$$

Consider the following modified fractional Schrödinger-Poisson system

$$\begin{cases} (-\Delta)^s u + \widetilde{V}(x)u + \phi u = f_h(x,u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \phi = u^2, & x \in \mathbb{R}^3. \end{cases} \tag{2.10}$$

and define the cut-off functional by

$$\mathcal{J}_h(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|(-\Delta)^{\frac{s}{2}} u|^2 + \widetilde{V}(x)|u|^2 \right) dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u^t u^2 dx - \int_{\mathbb{R}^3} F_h(x,u) dx.$$

Moreover, the derivative of J is

$$\langle \mathcal{J}_h'(u), v \rangle = \int_{\mathbb{R}^3} \left((-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} v + \widetilde{V}(x)uv + \phi_u^t uv - f(x,u)v \right) dx, \quad \forall\, u, v \in E. \tag{2.11}$$

Then $u \in E$, satisfies $|u| \leq l$, is a critical point of the functional $\mathcal{J}_h$, u is a weak solution of (1.1). Since the embedding $E(\Omega) \hookrightarrow L^r(\Omega)$ is continuous, where $r \in [2, 2_s^*]$ and $\Omega \subset \mathbb{R}^3$, then there exists a constant ϱ_r such that $|u|_{r,\Omega} \leq \varrho_r \|u\|_{E,\Omega}$. By Lemma 2.1, we know that $E(\Omega) \hookrightarrow L^p(\Omega)$ is compact for all $p \in [2, 2_s^*)$. Similar to [7], the energy functional $\mathcal{J}_h : E \to \mathbb{R}$ is well defined and of class $C^1(E, \mathbb{R})$. Obviously, it can be proved that if u is a critical point of $\mathcal{J}_h$, then the pair (u, ϕ_u^t) is a solution of system (1.1).

Let Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and the genus $\gamma(A) \geq k$. For more details on genus, we refer the readers to [23]. To prove the existence of infinitely many solutions, we mainly apply the following critical point theorem established in [2].

Lemma 2.3 [2] *Let E be an infinite dimensional Banach space and $\mathcal{J}_h \in C^1(E, \mathbb{R})$ an even functional with $\mathcal{J}_h(0) = 0$. Suppose that $\mathcal{J}_h$ satisfies*

$(\mathcal{J}_1)$ $\mathcal{J}_h$ is bounded from below and satisfies (PS) condition.

$(\mathcal{J}_2)$ For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} \mathcal{J}_h(u) < 0$.

Then there exists a critical point sequence $\{u_k\}$ such that $\mathcal{J}_h(u_k) \leq 0$ and $\lim_{k\to\infty} u_k = 0$.

In order to prove our main result by Lemma 2.3, we need the following lemmas.

Lemma 2.4 *Assume that a sequence $\{u_n\} \subset E$, $u_n \rightharpoonup u$ in E as $n \to \infty$ and $\{\|u_n\|\}$ be a bounded sequence. Then, as $n \to \infty$, we have*

$$\int_{\mathbb{R}^3} (\phi_{u_n}^t u_n - \phi_u^t u)(u_n - u) dx \to 0. \tag{2.12}$$

Proof. Take a sequence $\{u_n\} \subset E$ such that $u_n \rightharpoonup u$ in E as $n \to \infty$ and $\{\|u_n\|\}$ is a bounded sequence. By Lemma 2.1, we have $u_n \to u$ in $L^p(\mathbb{R}^3)$ where $2 \leq p < 2_s^*$, and $u_n \to u$ a.e. on $\mathbb{R}^3$. Hence $\sup_{n\in\mathbb{N}} \|u_n\| < \infty$ and $\|u\|$ is finite. Since $s \in (\frac{3}{4}, 1)$, then we know that $E \hookrightarrow L^{\frac{6}{2s}}(\mathbb{R}^3)$ holds. Hence by (2.4) and (2.7), we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} (\phi_{u_n}^t u_n - \phi_u^t u)(u_n - u)dx\right| &\leq \left(\int_{\mathbb{R}^3} (\phi_{u_n}^t u_n - \phi_u^t u)^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} (u_n - u)^2 dx\right)^{\frac{1}{2}} \\
&\leq \sqrt{2}\left[\int_{\mathbb{R}^3} (|\phi_{u_n}^t u_n|^2 + |\phi_u^t u|^2)\right]^{\frac{1}{2}} \|u_n - u\|_2 \\
&\leq C_3(\|\phi_{u_n}^t\|_{2_s^*}^2 \|u_n\|_{\frac{6}{2s}}^2 + \|\phi_u^t\|_{2_s^*}^2 \|u\|_{\frac{6}{2s}}^2)^{\frac{1}{2}} \|u_n - u\|_2 \\
&\leq C_3(\|u_n\|^4 + \|u\|^4)^{\frac{1}{2}} \|u_n - u\|_2 \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

This completes the proof of this lemma. $\square$

Lemma 2.5 *Suppose that* (V_1), (V_2) *and* (f_1), (f_2) *hold. Then* $\mathcal{J}_h$ *is bounded from below and satisfies the (PS) condition on* E.

Proof. By (V_1), (V_2), (f_1), $f_2)$ and the definition of h, we can get

$$|F_h(x, v)| \leq \frac{a(x)}{r_1}|v|^{r_1} + \frac{V_0}{2}v^2, \quad \forall\, (x, v) \in (\mathbb{R}^3, \mathbb{R}).$$

For any given $v \in E$, let $\Omega = \{x \in \mathbb{R}^3 : |v| \leq 1\}$. By Hölder's inequality and the definition of $\mathcal{J}_h$, one has

$$\begin{aligned}
\mathcal{J}_h(v) &= \frac{1}{2}\|v\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_v^t v^2 dx - \int_{\mathbb{R}^3} F_h(x, v)dx \\
&\geq \frac{C}{2}\|v\|_{E,\Omega}^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_v^t v^2 dx - \int_{\Omega} F_h(x, v)dx \\
&\geq \frac{C}{2}\|v\|_{E,\Omega}^2 - \int_{\Omega}\left(\frac{a(x)}{r_1}|v|^{r_1} + \frac{V_0}{2}v^2\right)dx \\
&\geq \frac{C}{2}\|v\|_{E,\Omega}^2 - \int_{\Omega}\left(\frac{a(x)}{r_1}|v|^{r_1} + \frac{V_0}{2}v^{r_1}\right)dx \\
&\geq \frac{C}{2}\|v\|_{E,\Omega}^2 - \frac{1}{r_1}|a(x)|_{\frac{2}{2-r_1},\Omega}\|v\|_{2,\Omega}^{r_1} - \frac{V_0}{2}\|v\|_{r_1,\Omega}^{r_1} \\
&\geq \frac{C}{2}\|v\|_{E,\Omega}^2 - \frac{\varrho_2^{r_1}}{r_1}|a(x)|_{\frac{2}{2-r_1},\mathbb{R}^3}\|v\|_{E,\Omega}^{r_1} - \frac{V_0\varrho_{r_1}^{r_1}}{2}\|v\|_{E,\Omega}^{r_1},
\end{aligned} \tag{2.13}$$

which implies that $\mathcal{J}_h$ is bounded from below by $r_1 \in (1, 2)$. Next we prove $\mathcal{J}_h$ satisfies the (PS) condition. Let $\{v_n\} \subset E$ be any (PS) sequence of $\mathcal{J}_h$, that is, $\{\mathcal{J}_h(v_n)\}$ is bounded and $\mathcal{J}_h'(v_n) \to 0$. For each $n \in \mathbb{N}$, set $\Omega_n = \{x \in \mathbb{R}^3 : |v_n| \leq 1\}$. Then by (2.13), we have

$$C \geq \mathcal{J}_h(v_n) \geq \frac{C}{2}\|v_n\|_{E,\Omega_n}^2 - \frac{\varrho_2^{r_1}}{r_1}|a(x)|_{\frac{2}{2-r_1},\mathbb{R}^3}\|v_n\|_{E,\Omega_n}^{r_1} - \frac{V_0\varrho_{r_1}^{r_1}}{2}\|v_n\|_{E,\Omega_n}^{r_1},$$

which implies that $\|v_n\|_{E,\Omega_n} \le C$ and C is independent of n. Thus

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_n} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx + \frac{1}{2}\int_{\Omega_n} \widetilde{V}(x) v_n^2 dx + \frac{1}{4}\int_{\Omega_n} \phi_{v_n}^t v_n^2 dx \\
&= \mathcal{J}_h(v_n) + \int_{\Omega_n} F_h(x, v_n) dx \\
&\le C + \frac{\varrho_2^{r_1}}{r_1} |a(x)|_{\frac{2}{2-r_1},\mathbb{R}^3} \|v_n\|_{E,\Omega_n}^{r_1} + \frac{V_0 \varrho_{r_1}^{r_1}}{2} \|v_n\|_{E,\Omega_n}^{r_1} \le C,
\end{aligned} \tag{2.14}$$

where C is independent of n. Similarly

$$\begin{aligned}
\mathcal{J}_h(v_n) &= \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \widetilde{V}(x) v_n^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{v_n}^t v_n^2 dx - \int_{\mathbb{R}^3} F_h(x, v_n) dx \\
&\ge \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega_n} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega_n} \widetilde{V}(x) v_n^2 dx + \frac{1}{4}\int_{\mathbb{R}^3\setminus\Omega_n} \phi_{v_n}^t v_n^2 dx - \int_{\Omega_n} F_h(x, v_n) dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega_n} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3\setminus\Omega_n} \widetilde{V}(x) v_n^2 dx + \frac{1}{4}\int_{\mathbb{R}^3\setminus\Omega_n} \phi_{v_n}^t v_n^2 dx \\
&\le \mathcal{J}_h(v_n) + \int_{\Omega_n} F_h(x, v_n) dx \\
&\le C + \frac{\varrho_2^{r_1}}{r_1} |a(x)|_{\frac{2}{2-r_1},\mathbb{R}^3} \|v_n\|_{E,\Omega_n}^{r_1} + \frac{V_0 \varrho_{r_1}^{r_1}}{2} \|v_n\|_{E,\Omega_n}^{r_1} \\
&\le C,
\end{aligned} \tag{2.15}$$

where C is independent of n. Combining (2.14) with (2.15), we have

$$S_n^2 := \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \widetilde{V}(x) v_n^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{v_n}^t v_n^2 dx$$

is bounded independent of n. Hence, as in the proof of Lemma 3.1 in [12], we have

$$C\|v_n\| \le \frac{1}{2}\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \widetilde{V}(x) v_n^2 dx \le S_n^2 \le C,$$

which implies that $\{v_n\}$ is bounded in E. Going if necessary to a subsequence, we can assume $v_n \rightharpoonup v$ in E. Since the embedding $E \hookrightarrow L^p(\mathbb{R}^3)$ is compact, then $v_n \to v$ in $L^p(\mathbb{R}^3)$ for all $2 \le p < 2_s^*$ and $v_n \to v$ a.e. on $\mathbb{R}^3$.

By (f_2) and Hölder's inequality, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} \left(f_h(x, v_n) - f_h(x, v)\right)(v_n - v) dx\right| \\
&\le \int_{\mathbb{R}^3} \left(|a(x)||v_n|^{r_1} + V_0|v_n| + |a(x)||v|^{r_1} + V_0|v|\right)|v_n - v| dx \\
&\le C_2 \left(|a(x)|_{\frac{2}{2-r_1},\mathbb{R}^3} \|v_n\|_{2,\mathbb{R}^3}^{r_1} + V_0\|v_n\|_{2,\mathbb{R}^3} + |a(x)|_{\frac{2}{2-r_1},\mathbb{R}^3} \|v\|_{2,\mathbb{R}^3}^{r_1} + V_0\|v\|_{2,\mathbb{R}^3}\right) \|v_n - v\|_{2,\mathbb{R}^3} \\
&= o_n(1).
\end{aligned} \tag{2.16}$$

On the other hand, by Lemma 2.4, we get that

$$\int_{\mathbb{R}^3} (\phi_{v_n}^t v_n - \phi_v^t v)(v_n - v)dx \to 0, \quad \text{as } n \to \infty. \tag{2.17}$$

Hence together with (2.16) and (2.17), we get

$$\begin{aligned} o_n(1) &= \langle \mathcal{J}_h'(v_n) - \mathcal{J}_h'(v), v_n - v \rangle \\ &= \|v_n - v\|^2 + \int_{\mathbb{R}^3} (\phi_{v_n}^t v_n - \phi_v^t v)(v_n - v)dx \\ &\qquad - \int_{\mathbb{R}^3} (f_h(x, v_n) - f_h(x, v))(v_n - v)dx \\ &\geq C_3 \|v_n - v\|^2 + o_n(1). \end{aligned}$$

This implies $v_n \to v$ in E and this completes the proof. □

Similar to the proof of Lemma 3.2 in [7] and Lemma 3.2 in [17], we can get the following lemma.

Lemma 2.6. *For any $k \in \mathbb{N}$, there exists a closed symmetric subsets $A_k \subset E$ such that the genus $\gamma(A_k) \geq k$ and $\sup_{v \in A_k} \mathcal{J}(v) < 0$.*

Proof. Let E_n be any n-dimensional subspace of E. Since all norms are equivalent in a finite dimensional space, there is a constant $\beta = \beta(E_n)$ such that

$$\|v\| \leq \beta \|v\|_2$$

for all $v \in E_n$, where $\|\cdot\|_2$ is the usual norm of $L^2(\mathbb{R}^3)$.

Next, we claim that there exists a constant $M > 0$ such that

$$\frac{1}{2} \int_{\mathbb{R}^3} |v|^2 dx \geq \int_{|v|>l} |v|^2 dx \tag{2.18}$$

for all $v \in E_n$ and $\|v\| \leq M$. In fact, if (2.18) is false, then exists a sequence $\{v_k\} \subset E_n \backslash \{0\}$ such that $v_k \to 0$ in E and

$$\frac{1}{2} \int_{\mathbb{R}^3} |v_k|^2 dx < \int_{|v_k|>l} |v_k|^2 dx$$

for all $k \in \mathbb{N}$. Let $u_k = \frac{v_k}{\|v_k\|_{2,\mathbb{R}^3}}$. Then

$$\frac{1}{2} < \int_{|v_k|>l} |u_k|^2 dx, \quad \text{for all } k \in \mathbb{N}. \tag{2.19}$$

On the other hand, we can assume that $u_k \to u$ in E since E_n is finite dimensional. Hence $u_k \to u$ in $L^2(\mathbb{R}^3)$. Moreover, it can be deduced from $v_k \to 0$ in E that

$$meas\{x \in \mathbb{R}^3 : |v_k| > l\} \to 0, \quad k \to \infty.$$

Therefore,

$$\int_{|v_k|>l} |u_k|^2 dx \leq 2 \int_{\mathbb{R}^3} |u_k - u|^2 + \int_{|v_k|>l} u^2 dx \to 0, \quad k \to \infty$$

which contradicts (2.19) and hence (2.18) holds. By (f_1), we can choose a l small enough such that

$$f(x,v) \geq \frac{1}{8}\left(\frac{1}{2}+\frac{1}{4}S_t\varrho^4_{\frac{12}{3+2t}}\right)\beta v^2.$$

for all $x \in \mathbb{R}^3$ and $0 \leq v \leq 2l$. This inequality implies that

$$F_h(x,v) = F(x,v) \leq 4\left(\frac{1}{2}+\frac{1}{4}S_t\varrho^4_{\frac{12}{3+2t}}\right)\beta v^2. \tag{2.20}$$

The assumption (f_3) implies $F_h(x,v)$ is even in v. Thus, by (2.20), we have

$$\begin{aligned}
\mathcal{J}_h(v) &= \frac{1}{2}\int_\Omega |(-\Delta)^{\frac{s}{2}}v|^2dx + \frac{1}{2}\int_\Omega \widetilde{V}(x)v^2dx + \frac{1}{4}\int_\Omega \phi_v^t v^2dx - \int_\Omega F_h(x,v)dx \\
&\leq \frac{1}{2}\|v\|^2 + \frac{1}{4}S_t\varrho^4_{\frac{12}{3+2t}}\|v\|^4 - \int_{|v|\leq l} F_h(x,|v|)dx \\
&\leq \frac{1}{2}\|v\|^2 + \frac{1}{4}S_t\varrho^4_{\frac{12}{3+2t}}\|v\|^2 - 4\left(\frac{1}{2}+\frac{1}{4}S_t\varrho^4_{\frac{12}{3+2t}}\right)\beta^2\int_{|v|\leq l}|v|^2dx \\
&= \left(\frac{1}{2}+\frac{1}{4}S_t\varrho^4_{\frac{12}{3+2t}}\right)\|v\|^2 - 4\left(\frac{1}{2}+\frac{1}{4}S_t\varrho^4_{\frac{12}{3+2t}}\right)\beta^2\left(\int_{\mathbb{R}^3}|v|^2dx - \int_{|v|>l}|v|^2dx\right) \\
&\leq -\left(\frac{1}{2}+\frac{1}{4}\tau_*^2\tau^4_{\frac{12}{5}}\right)\|v\|^2
\end{aligned}$$

for all $v \in E_n$ with $\|v\| \leq \min\{M,1\}$. Let $0 < \rho \leq \min\{M,1\}$ and $A_n = \{v \in E_n : \|v\| = \rho\}$. We conclude that $\gamma(A_n) \geq n$ and

$$\sup_{v\in A_n}\mathcal{J}_h(v) \leq -\left(\frac{1}{2}+\frac{1}{4}S_t\varrho^4_{\frac{12}{3+2t}}\right)\rho^2 < 0.$$

This completes the proof.

Proof of Theorem 1.1 . By (f_1)-(f_3), we know that $\mathcal{J}_h$ is even and $\mathcal{J}_h(0) = 0$. Furthermore, Lemmas 2.5 and 2.6 imply that $\mathcal{J}_h$ has a critical sequence $\{v_n\}$ such that $\mathcal{J}_h(v_n) \leq 0$ and $v_n \to 0$ as $n \to \infty$. Thus, we get by Lemma 2.3 that problem (1.1) has infinitely many small solutions. This completes the proof.

3. Conclusions

We consider a class of fractional Schrödinger-Poisson systems with sign-changing potential. According to the assumptions, we construct an equivalent new system. By dual method and the critical point theorem, we proved the existence of infinitely many small solutions.

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Conflict of interest

The authors declare that they have no conflict interests.

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