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Research article

New Hermite-Hadamard type inequalities for exponentially convex functions and applications

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Abstract: The investigation of the proposed techniques is effective and convenient for solving the integrodifferential and difference equations. The present investigation depends on two highlights; the novel Hermite-Hadamard type inequalities for $\mathcal{K}$-conformable fractional integral operator in terms of a new parameter $\mathcal{K} > 0$ and weighted version of Hermite-Hadamard type inequalities for exponentially convex functions in the classical sense. By using an integral identity together with the Hölder-İşcan and improved power-mean inequality we establish several new inequalities for differentiable exponentially convex functions. This generalizes the Hadamard fractional integrals and Riemann-Liouville into a single form. Our contribution expands some innovative studies in this line. Moreover, two suitable examples are presented to demonstrate the novelty of the results established, the first one about the contributions of the modified Bessel functions and the other is about σ-digamma function. Finally, various applications for some special means as arithmetic, geometric and logarithmic are given.

Keywords: convex function; exponentially convex functions; Hermite-Hadamard inequality; $\mathcal{K}$-conformable fractional integral; weighted inequality
Mathematics Subject Classification: 26A33, 26A51, 26D10

1. Introduction and preliminaries

Convexity has played a crucial role in the advancement of pure and applied mathematics [1–10]. Due to its robustness, convex functions and convex sets have been generalized and extended in many mathematical areas [11–20]. In particular, many inequalities can be found in the literature [21–30] via convexity theory.

Integral inequalities [31-34] have numerous applications in number theory, combinatorics, orthogonal polynomials, hypergeometric functions, quantum theory, linear programming, optimization theory, mechanics and in the theory of relativity. This subject has received considerable attention from researchers [35–38] and hence it is assumed as an incorporative the subject between mathematics, statistics, economics, and physics [39-41].

To the best of our knowledge, the Hermite-Hadamard inequality is a well-known, paramount and extensively useful inequality in the applied literature [42–45]. This inequality is of pivotal significance because of other classical inequalities such as the Hardy, Opial, Lynger, Ostrowski, Minkowski, Hölder, Ky-Fan, Beckenbach-Dresher, Levinson, arithmetic-geometric, Young, Olsen and Gagliardo-Nirenberg inequalities, which are closely related to the classical Hermite-Hadamard inequality [46]. It can be stated as follows:

Let $\varphi : \mathcal{I} \subseteq \mathbb{R} \mapsto \mathbb{R}$ be a convex function and $c, d \in \mathcal{I}$ with $c < d$. Then

$$\varphi\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d \varphi(z)dz \leq \frac{\varphi(c)+\varphi(d)}{2}. \tag{1.1}$$

In [47], Fejér contemplated the important generalizations that is the weighted generalization of the Hermite-Hadamard inequality.

Let $\mathcal{I} \subseteq \mathbb{R}$ and a function $\varphi : \mathcal{I} \mapsto \mathbb{R}$ be a convex function. Then the inequalities

$$\varphi\left(\frac{c+d}{2}\right)\int_c^d w(z)dz \leq \int_c^d \varphi(z)w(z)dz \leq \frac{\varphi(c)+\varphi(d)}{2}\int_c^d w(z)dz. \tag{1.2}$$

hold, where $w : \mathcal{I} \mapsto \mathbb{R}$ is non-negative, integrable and symmetric with respect to $\frac{c+d}{2}$. If we choose $w(z) = 1$, then (1.2) reduces to (1.1). Several classical inequalities can be obtained with the help of inequality (1.1) by considering the use of peculiar convex function φ. Moreover, these inequalities for convex functions have a very important role in both applied and pure mathematics.

In recent years, integral inequalities have been derived via fractional analysis, which has emerged as another interesting technique. Due to advancement in inequalities, the comprehensive investigation of exponentially convex functions as the $\mathcal{K}$-conformable fractional integral in the present paper is new.

The class of exponentially-convex functions were introduced by Dragomir and Gomm [48]. Bernstein [49] and Antczak [50] introduced these exponentially convex functions implicitly and discuss their role in mathematical programming. The proliferating research on big data analysis and deep learning has recently intensified the interest in information theory involving exponentially-convex functions. The smoothness of exponentially-convex functions is exploited for statistical learning, sequential prediction, and stochastic optimization.

Now we recall the concept of exponentially convex functions, which is mainly due to M. A. Noor and K. I. Noor [51, 52].

Definition 1.1. *(See [51, 52]) A real-valued function $\varphi : \mathcal{M} \subseteq \mathbb{R} \mapsto \mathbb{R}$ is said to be an exponentially convex on $\mathcal{M}$ if the inequality*

$$e^{\varphi(\xi c+(1-\xi)d)} \leq \xi e^{\varphi(c)} + (1-\xi)e^{\varphi(d)} \tag{1.3}$$

holds for all $c, d \in \mathcal{M}$ and $\xi \in [0, 1]$.

It is well-known that a function $\varphi : \mathcal{I} = [c, d] \subseteq \mathbb{R} \mapsto \mathbb{R}$ is an exponentially convex function if and only if it satisfies the inequality

$$e^{\varphi(\frac{c+d}{2})} \leq \frac{1}{d-c}\int_c^d e^{\varphi(z)}dz \leq \frac{e^{\varphi(c)} + e^{\varphi(d)}}{2} \tag{1.4}$$

for all $c, d \in \mathcal{I}$ with $c < d$. Inequality (1.4) provide the upper and lower estimates for the exponential integral, is called the Hermite-Hadamard inequality.

Recently, the fractional calculus has attracted the consideration of several researchers [53, 54]. The impact and inspiration of the fractional calculus in both theoretical, applied science and engineering arose out substantially. Fractional integral operators are sometimes the gateway to physical problems that cannot be expressed by classical integral, sometimes for the solution of problems expressed in fractional order. In recent years, a lot of new operator have been defined. Some of these operators are very close to classical operators in terms of their characteristics and definitions. Various studies in the literature, on distinct fractional operators such as the classical Riemann-Liouville, Caputo, Katugampola, Hadamard, and Marchaud versions have shown versatility in modeling and control applications across various disciplines. However, such forms of fractional derivatives may not be able to explain the dynamic performance accurately, hence many authors are found to be sorting out new fractional differentiations and integrations which have a kernel depending upon a function and this makes the range of definition expanded [55, 56].

Now, we recall the basic definitions and new notations of conformable fractional operators.

Definition 1.2. ([3]). *Let $\varphi \in L_1([c, d])$. Then the Riemann-Liouville integrals $\mathcal{J}^{\delta}_{c^+}\varphi$ and $\mathcal{J}^{\delta}_{d^-}\varphi$ of order $\delta > 0$ are defined by*

$$(\mathcal{J}^{\delta}_{c^+})\varphi(z) = \frac{1}{\Gamma(\delta)}\int_c^z (z-\xi)^{\delta-1}\varphi(\xi)\, d\xi \qquad (z > c) \tag{1.5}$$

and

$$(\mathcal{J}^{\delta}_{d^-})\varphi(z) = \frac{1}{\Gamma(\delta)}\int_z^d (\xi-z)^{\delta-1}\varphi(\xi)\, d\xi \qquad (z < d), \tag{1.6}$$

respectively, where $\Gamma(\cdot)$ is the Euler Gamma function defined by $\Gamma(\delta) = \int_0^{\infty} e^{-\xi}\xi^{\delta-1}d\xi$.

In [57], Jarad et al. defined a new fractional integral operator that has several special cases among many other features as follows:

$${}^{\gamma}\mathcal{J}^{\delta}_{c^+}\varphi(z) = \frac{1}{\Gamma(\gamma)}\int_c^z \left(\frac{(z-c)^{\delta} - (\xi-c)^{\delta}}{\delta}\right)^{\gamma-1} \frac{\varphi(\xi)}{(\xi-c)^{1-\delta}}d\xi \tag{1.7}$$

and

$${}^{\gamma}\mathcal{J}^{\delta}_{d^-}\varphi(z) = \frac{1}{\Gamma(\gamma)}\int_z^d \left(\frac{(d-z)^{\delta} - (d-\xi)^{\delta}}{\delta}\right)^{\gamma-1} \frac{\varphi(\xi)}{(d-\xi)^{1-\delta}}d\xi. \tag{1.8}$$

Remark 1.1. *It is easy to see the following connections:*

(1) Let $c = 0$ *and* $\delta = 1$. *Then* (1.7) *reduces to the Riemann-Liouville operator that is given in* (1.5) *and alike the other.*

(2) If we set $c = 0$ *and* $\delta \mapsto 0$, *then the new conformable fractional integral coincides with the generalized fractional integral [58].*

(3) Furthermore, (1.8) *reduces to the Riemann-Liouville operator if we set* $d = 0$ *and* $\delta = 1$. *It also corresponds the Hadamard fractional integral [58] once* $d = 0$ *and* $\delta \mapsto 0$ *with the generalized fractional integral.*

The generalized $\mathcal{K}$-conformable fractional integrals are defined by

$$ {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{c^+}\varphi(z) = \frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\gamma)}\int_c^z \left(\frac{(z-c)^{\delta}-(\xi-c)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}-1}\frac{\varphi(\xi)}{(\xi-c)^{1-\delta}}d\xi \tag{1.9} $$

and

$$ {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{d^-}\varphi(z) = \frac{1}{\mathcal{K}\Gamma_{\mathcal{K}}(\gamma)}\int_z^d \left(\frac{(d-z)^{\delta}-(d-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}-1}\frac{\varphi(\xi)}{(d-\xi)^{1-\delta}}d\xi. \tag{1.10} $$

Definition 1.3. *Let* $\mathcal{K} > 0$. *Then the* $\mathcal{K}$*-Gamma function* $\Gamma_{\mathcal{K}}$ *is defined by*

$$ \Gamma_{\mathcal{K}}(\delta) = \lim_{y\to\infty}\frac{y!\,\mathcal{K}^y\,(y\mathcal{K})^{\frac{\delta}{\mathcal{K}}-1}}{(\delta)_{y,\mathcal{K}}}. \tag{1.11} $$

If $Re(\delta) > 0$, *then the* $\mathcal{K}$*-Gamma function in integral form is defined by*

$$ \Gamma_{\mathcal{K}}(\delta) = \int_0^{\infty} e^{-\frac{\xi^{\mathcal{K}}}{\mathcal{K}}}\xi^{\delta-1}d\xi \tag{1.12} $$

with $\delta\Gamma_{\mathcal{K}}(\delta) = \Gamma_{\mathcal{K}}(\delta+\mathcal{K})$, *where* $\Gamma_{\mathcal{K}}(\cdot)$ *stands for the* $\mathcal{K}$*-gamma function.*

This paper is aimed at establishing some new integral inequalities for exponentially convexity via $\mathcal{K}$-conformable fractional integrals linked with inequality (1.1). We present some inequalities for the class of mappings whose derivatives in absolute values are exponentially convex. In addition, we obtain some new inequalities linked with (1.2) and exponentially convexity via classical integrals. Moreover, we apply the novel approach of Hölder-İşcan and improved power-mean inequality are better than the Hölder and power-mean inequality. Moreover, we illustrate two examples to show the applicability and supremacy of the proposed technique. As an application, the inequalities for special means are derived.

2. Certain Hermite-Hadamard type inequalities

In this section, we demonstrate the Hermite-Hadamard type inequalities for exponentially convex function via $\mathcal{K}$-conformable fractional integral operator.

Theorem 2.1. *Let $\mathcal{K} > 0$, $\delta > 0$, $\gamma > 0$ and $\varphi : \mathcal{I} \mapsto \mathbb{R}$ be an exponentially convex function such that $c, d \in \mathcal{I}$ with $c < d$ and $e^{\varphi} \in L_1([c, d])$. Then the following inequality holds for $\mathcal{K}$-conformable fractional integrals:*

$$e^{\varphi(\frac{c+d}{2})} \leq \frac{\delta^{\frac{\gamma}{\mathcal{K}}} 2^{\frac{\delta\gamma}{\mathcal{K}}-1}\Gamma_{\mathcal{K}}(\gamma+\mathcal{K})}{(d-c)^{\frac{\delta\gamma}{\mathcal{K}}}} \left\{ {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^-} e^{\varphi(c)} + \mathcal{K}\Gamma_{\mathcal{K}}(\gamma) {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^+} e^{\varphi(d)} \right\} \leq \frac{e^{\varphi(c)} + e^{\varphi(d)}}{2}. \tag{2.1}$$

Proof. Since φ is exponentially convex on $\mathcal{I}$, for $\xi = \frac{1}{2}$, we have

$$e^{\varphi(\frac{z_1+z_2}{2})} \leq \frac{e^{\varphi(z_1)} + e^{\varphi(z_2)}}{2}$$

for all $z_1, z_2 \in \mathcal{I}$. Thus, if we choose $z_1 = \frac{\xi}{2}c + \frac{2-\xi}{2}d$ and $z_2 = \frac{2-\xi}{2}c + \frac{\xi}{2}d$, for $c, d \in \mathcal{I}$ and $\xi \in [0, 1]$, we have

$$2e^{\varphi(\frac{c+d}{2})} \leq e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)} + e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)}. \tag{2.2}$$

Moreover, multiplying both sides of (2.2) by $\left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}-1}(1-\xi)^{\delta-1}$ and then making use of integration with respect to ξ over $[0, 1]$, we can combine the resulting inequality with the definition of integral operator as follows

$$\begin{aligned}
&2e^{\varphi(\frac{c+d}{2})} \int_0^1 \left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}-1} (1-\xi)^{\delta-1} d\xi \\
&\leq \int_0^1 \left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}-1} (1-\xi)^{\delta-1} e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)} d\xi + \int_0^1 \left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}-1} (1-\xi)^{\delta-1} e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)} d\xi \\
&\leq \left(\frac{2}{d-c}\right)^{\frac{\delta\gamma}{\mathcal{K}}} \left\{ \int_c^{\frac{c+d}{2}} \left(\frac{\left(\frac{d-c}{2}\right)^{\delta} - \left(\frac{c+d}{2}-u\right)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}-1} \left(\frac{c+d}{2}-u\right)^{\delta-1} e^{\varphi(u)} du \right. \\
&\qquad \left. + \int_{\frac{c+d}{2}}^d \left(\frac{\left(\frac{d-c}{2}\right)^{\delta} - \left(u-\frac{c+d}{2}\right)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}-1} \left(u-\frac{c+d}{2}\right)^{\delta-1} e^{\varphi(u)} du \right\} \\
&= \left(\frac{2}{d-c}\right)^{\frac{\delta\gamma}{\mathcal{K}}} \left\{ \mathcal{K}\Gamma_{\mathcal{K}}(\gamma) {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^-} e^{\varphi(c)} + \mathcal{K}\Gamma_{\mathcal{K}}(\gamma) {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^+} e^{\varphi(d)} \right\}.
\end{aligned}$$

It is clear to see that

$$\int_0^1 \left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}-1} (1-\xi)^{\delta-1} d\xi = \frac{\mathcal{K}}{\gamma\delta^{\frac{\gamma}{\mathcal{K}}}}.$$

Consequently, we get

$$\frac{2\mathcal{K}}{\gamma\delta^{\frac{\gamma}{\mathcal{K}}}} e^{\varphi(\frac{c+d}{2})} \leq \left(\frac{2}{d-c}\right)^{\frac{\delta\gamma}{\mathcal{K}}} \left\{ \mathcal{K}\Gamma_{\mathcal{K}}(\gamma) {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^-} e^{\varphi(c)} + \mathcal{K}\Gamma_{\mathcal{K}}(\gamma) {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^+} e^{\varphi(d)} \right\}. \tag{2.3}$$

This completes the proof of the first inequality (2.1).

To prove the second part of the inequality (2.1), by a similar discussion, we will start with the exponentially convexity of φ, then for $\xi \in [0,1]$, we have

$$e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}+e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)} \leq e^{\varphi(c)}+e^{\varphi(d)}. \tag{2.4}$$

By multiplying (2.4) by $\left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}-1}(1-\xi)^{\delta-1}$ and then integrating the above estimate with respect to ξ over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 \left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}-1}(1-\xi)^{\delta-1}e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}d\xi+\int_0^1 \left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}-1}(1-\xi)^{\delta-1}e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)}d\xi\\
&\leq \left[e^{\varphi(c)}+e^{\varphi(d)}\right]\int_0^1 \left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}-1}(1-\xi)^{\delta-1}d\xi.
\end{aligned}$$

After simplification, we get

$$\left(\frac{2}{d-c}\right)^{\frac{\delta\gamma}{\mathcal{K}}}\left\{\mathcal{K}\Gamma_{\mathcal{K}}(\gamma)\,{}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^-}e^{\varphi(c)}+\mathcal{K}\Gamma_{\mathcal{K}}(\gamma)\,{}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^+}e^{\varphi(d)}\right\} \leq \frac{\mathcal{K}\left[e^{\varphi(c)}+e^{\varphi(d)}\right]}{\gamma\delta^{\frac{\gamma}{\mathcal{K}}}}.$$

The proof is completed. □

Throughout this investigation, we use $\mathcal{I}^{\circ}$ to denote the interior of the interval $\mathcal{I} \subseteq \mathbb{R}$. In order to establish our main results, we need a lemma which we present in this section.

Lemma 2.2. *Let $\mathcal{K} > 0$, δ, $\gamma > 0$ and $\varphi : \mathcal{I} \mapsto \mathbb{R}$ be a differentiable and exponentially convex function on $\mathcal{I}^{\circ}$ such that $c, d \in \mathcal{I}^{\circ}$ with $c < d$ and $(e^{\varphi})' \in L_1([c,d])$. Then the following equality holds for $\mathcal{K}$-conformable fractional integrals:*

$$\begin{aligned}
&\frac{\delta^{\frac{\gamma}{\mathcal{K}}}2^{\frac{\delta\gamma}{\mathcal{K}}-1}\Gamma_{\mathcal{K}}(\gamma+\mathcal{K})}{(d-c)^{\frac{\delta\gamma}{\mathcal{K}}}}\left[{}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^-}e^{\varphi(c)}+{}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^+}e^{\varphi(d)}\right]-e^{\varphi\left(\frac{c+d}{2}\right)} \qquad (2.5)\\
&=\frac{\delta^{\frac{\gamma}{\mathcal{K}}}(d-c)}{4}\int_0^1 \left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}}\left\{e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}\varphi'\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)-e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)}\varphi'\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)\right\}d\xi.
\end{aligned}$$

Proof. Integrating by parts and changing variable of definite integral yield

$$\begin{aligned}
&\int_0^1 \left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}}e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}\varphi'\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)d\xi\\
&=\frac{2}{d-c}\left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}}e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}\Bigg|_0^1-\frac{2\delta^{\frac{\gamma}{\mathcal{K}}}}{\delta^{\frac{\gamma}{\mathcal{K}}}(d-c)}\int_0^1 \left(1-(1-\xi)^{\delta}\right)^{\frac{\gamma}{\mathcal{K}}}(1-\xi)^{\delta-1}e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}d\xi\\
&=\frac{2}{\delta^{\frac{\gamma}{\mathcal{K}}}(d-c)}e^{\varphi\left(\frac{c+d}{2}\right)}-\frac{2^{\frac{\delta\gamma}{\mathcal{K}}}\delta^{\frac{\gamma}{\mathcal{K}}}}{\delta^{\frac{\gamma}{\mathcal{K}}-1}(d-c)^{\frac{\delta\gamma}{\mathcal{K}}+1}}\int_c^{\frac{c+d}{2}}\left(\frac{\left(\frac{d-c}{2}\right)^{\delta}-\left(\frac{c+d}{2}-u\right)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}-1}\left(\frac{c+d}{2}-u\right)^{\delta-1}e^{\varphi(u)}du
\end{aligned}$$

$$= \frac{2}{\delta^{\frac{\gamma}{\mathcal{K}}}(d-c)} e^{\varphi(\frac{c+d}{2})} - \frac{2^{\frac{\delta\gamma}{\mathcal{K}}+1}\Gamma_{\mathcal{K}}(\gamma+\mathcal{K})}{\delta^{\frac{\gamma}{\mathcal{K}}-1}(d-c)^{\frac{\delta\gamma}{\mathcal{K}}+1}} {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^-} e^{\varphi(c)}. \tag{2.6}$$

By similar argument, we have

$$\int_0^1 \left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}} e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)} \varphi'\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right) d\xi = -\frac{2}{\delta^{\frac{\gamma}{\mathcal{K}}}(d-c)} e^{\varphi(\frac{c+d}{2})} + \frac{2^{\frac{\delta\gamma}{\mathcal{K}}+1}\Gamma_{\mathcal{K}}(\gamma+\mathcal{K})}{\delta^{\frac{\gamma}{\mathcal{K}}-1}(d-c)^{\frac{\delta\gamma}{\mathcal{K}}+1}} {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^+} e^{\varphi(d)}. \tag{2.7}$$

Subtracting these two equations leads to Lemma 2.2. □

Now we are in a position to establish some new integral inequalities of Hermite-Hadamard type for differentiable convex functions. The first main result is Theorem 2.3.

Theorem 2.3. *Let $\mathcal{K} > 0$, δ, $\gamma > 0$ and $\varphi : \mathcal{I} \mapsto \mathbb{R}$ be a differentiable and exponentially convex function on $\mathcal{I}^\circ$ such that $c, d \in \mathcal{I}^\circ$ with $c < d$ and $(e^{\varphi})' \in L_1([c,d])$. If $|(e^{\varphi})'|^q$ is convex on $\mathcal{I}$ for some fixed $p, q > 1$, $q^{-1} + p^{-1} = 1$. Then the following inequality holds for $\mathcal{K}$-conformable fractional integrals:*

$$\begin{aligned}
&\left|\frac{\delta^{\frac{\gamma}{\mathcal{K}}}2^{\frac{\delta\gamma}{\mathcal{K}}-1}\Gamma_{\mathcal{K}}(\gamma+\mathcal{K})}{(d-c)^{\frac{\delta\gamma}{\mathcal{K}}}}\left[{}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^-} e^{\varphi(c)} + {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^+} e^{\varphi(d)}\right] - e^{\varphi(\frac{c+d}{2})}\right| \\
&\leq \frac{\delta^{\frac{1}{q}-1}(d-c)}{4^{1+\frac{1}{q}}}\left[\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\right)\right]^{1-\frac{1}{q}}\left[\left\{\left[\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\right)+2\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\right)+\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\right)\right]|e^{\varphi(c)}\varphi'(c)|^q\right.\right. \\
&\quad + \left[\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\right)-2\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\right)+\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\right)\right]\left|e^{\varphi(d)}\varphi'(d)\right|^q \\
&\qquad \left.\left. + \Upsilon_1(c,d)\left[\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\right)-\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\right)\right]\right\}^{\frac{1}{q}}\right. \\
&\qquad + \left\{\left[\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\right)-2\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\right)+\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\right)\right]\left|e^{\varphi(c)}\varphi'(c)\right|^q\right. \\
&\qquad\quad + \left[\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\right)+2\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\right)+\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\right)\right]\left|e^{\varphi(d)}\varphi'(d)\right|^q \\
&\qquad\quad \left.\left.\left. + \Upsilon_1(c,d)\left[\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\right)-\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\right)\right]\right\}^{\frac{1}{q}}\right],
\end{aligned} \tag{2.8}$$

where

$$\Upsilon_1(c,d) = \left|e^{\varphi(c)}\varphi'(d)\right|^q + \left|e^{\varphi(d)}\varphi'(c)\right|^q. \tag{2.9}$$

Proof. It follows from Lemma 2.2 and the power-mean inequality that

$$\left|\frac{\delta^{\frac{\gamma}{\mathcal{K}}}2^{\frac{\delta\gamma}{\mathcal{K}}-1}\Gamma_{\mathcal{K}}(\gamma+\mathcal{K})}{(d-c)^{\frac{\delta\gamma}{\mathcal{K}}}}\left[{}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^-} e^{\varphi(c)} + {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^+} e^{\varphi(d)}\right] - e^{\varphi(\frac{c+d}{2})}\right|$$

$$\leq \frac{\delta^{\frac{\gamma}{\mathcal{K}}}(d-c)}{4}\left\{\left(\int_0^1\left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}}d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}}\left|e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}\varphi'\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)\right|^q d\xi\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}}d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1\left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}}\left|e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)}\varphi'\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)\right|^q d\xi\right)^{\frac{1}{q}}\right\}. \tag{2.10}$$

Utilizing the convexity of $|(e^{\varphi})'|^q$ on $\mathcal{I}$, we have

$$\begin{aligned}
&\int_0^1\left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}}\left|e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}\varphi'\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)\right|^q d\xi\\
&\leq \int_0^1\left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}}\left[\frac{2-\xi}{2}|e^{\varphi(c)}|^q+\frac{\xi}{2}|e^{\varphi(d)}|^q\right]\left[\frac{2-\xi}{2}|\varphi'(c)|^q+\frac{\xi}{2}|\varphi'(d)|^q\right]d\xi\\
&= \int_0^1\left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}}\left[\left(\frac{2-\xi}{2}\right)^2\left|e^{\varphi(c)}\varphi'(c)\right|^q+\left(\frac{\xi}{2}\right)^2|e^{\varphi(d)}\varphi'(d)|^q\right.\\
&\qquad\left.+\frac{\xi(2-\xi)}{4}\left[|e^{\varphi(c)}\varphi'(d)|^q+|e^{\varphi(d)}\varphi'(c)|^q\right]\right]d\xi\\
&= \frac{1}{4\delta^{\frac{\gamma}{\mathcal{K}}}}\left\{\left[\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\right)+2\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\right)+\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\right)\right]\left|e^{\varphi(c)}\varphi'(c)\right|^q\right.\\
&\quad+\left[\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\right)-2\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\right)+\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\right)\right]\left|e^{\varphi(d)}\varphi'(d)\right|^q\\
&\qquad\left.+\Upsilon_1(c,d)\left[\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\right)-\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\right)\right]\right\}.
\end{aligned}\tag{2.11}$$

Analogously, we have

$$\begin{aligned}
&\int_0^1\left(\frac{1-(1-\xi)^{\delta}}{\delta}\right)^{\frac{\gamma}{\mathcal{K}}}\left|e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)}\varphi'\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)\right|^q d\xi\\
&\leq \frac{1}{4\delta^{\frac{\gamma}{\mathcal{K}}}}\left\{\left[\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\right)-2\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\right)+\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\right)\right]|e^{\varphi(c)}\varphi'(c)|^q\right.\\
&\quad+\left[\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\right)+2\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\right)+\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\right)\right]|e^{\varphi(d)}\varphi'(d)|^q\\
&\qquad\left.+\Upsilon_1(c,d)\left[\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\right)-\mathbb{B}\left(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\right)\right]\right\}.
\end{aligned}\tag{2.12}$$

Substituting the above two inequalities into the inequality (2.10), we get the required inequality (2.8). This completes the proof. □

Theorem 2.4. *Let $\mathcal{K} > 0, \delta, \gamma > 0$ and $\varphi : \mathcal{I} \mapsto \mathbb{R}$ be a differentiable and exponentially convex function on $\mathcal{I}^\circ$ such that $c, d \in \mathcal{I}^\circ$ with $c < d$ and $(e^\varphi)' \in L_1([c, d])$. If $|(e^\varphi)'|^q$ is convex on $\mathcal{I}$ for some fixed $q > 1$, $q^{-1} + p^{-1} = 1$. Then the following inequality holds for $\mathcal{K}$-conformable fractional integrals:*

$$\begin{aligned}
&\left|\frac{\delta^{\frac{\gamma}{\mathcal{K}}} 2^{\frac{\delta\gamma}{\mathcal{K}}-1}\Gamma_{\mathcal{K}}(\gamma+\mathcal{K})}{(d-c)^{\frac{\delta\gamma}{\mathcal{K}}}}\Big[{}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^-}e^{\varphi(c)} + {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^+}e^{\varphi(d)}\Big] - e^{\varphi(\frac{c+d}{2})}\right| \\
&\leq \frac{\delta^{\frac{\gamma}{\mathcal{K}}}(d-c)}{4}\left[\frac{1}{\delta^{\frac{p\delta}{\mathcal{K}}+1}}\mathbb{B}\left(\frac{p\gamma}{\mathcal{K}}+1, \frac{1}{\delta}\right)\right]^{\frac{1}{p}}\left[\left\{\frac{7|e^{\varphi(c)}\varphi'(c)|^q + |e^{\varphi(d)}\varphi'(d)|^q + 2\Upsilon_1(c,d)}{12}\right\}^{\frac{1}{q}}\right. \\
&\qquad\left. + \left\{\frac{|e^{\varphi(c)}\varphi'(c)|^q + 7|e^{\varphi(d)}\varphi'(d)|^q + 2\Upsilon_1(c,d)}{12}\right\}^{\frac{1}{q}}\right], \qquad (2.13)
\end{aligned}$$

where $\Upsilon_1(c, d)$ is given in (2.9).

Proof. It follows from Lemma 2.2 and the noted Hölder's integral inequality that

$$\begin{aligned}
&\left|\frac{\delta^{\frac{\gamma}{\mathcal{K}}} 2^{\frac{\delta\gamma}{\mathcal{K}}-1}\Gamma_{\mathcal{K}}(\gamma+\mathcal{K})}{(d-c)^{\frac{\delta\gamma}{\mathcal{K}}}}\Big[{}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^-}e^{\varphi(c)} + {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^+}e^{\varphi(d)}\Big] - e^{\varphi(\frac{c+d}{2})}\right| \\
&\leq \frac{\delta^{\frac{\gamma}{\mathcal{K}}}(d-c)}{4}\left\{\left(\int_0^1\left(\frac{1-(1-\xi)^\delta}{\delta}\right)^{\frac{p\gamma}{\mathcal{K}}}d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left|e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}\varphi'\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)\right|^q d\xi\right)^{\frac{1}{q}}\right. \\
&\quad\left. + \left(\int_0^1\left(\frac{1-(1-\xi)^\delta}{\delta}\right)^{\frac{p\gamma}{\mathcal{K}}}d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left|e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)}\varphi'\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)\right|^q d\xi\right)^{\frac{1}{q}}\right\}. \qquad (2.14)
\end{aligned}$$

Utilizing the convexity of $|(e^\varphi)'|^q$ on $\mathcal{I}$, we have

$$\begin{aligned}
&\int_0^1\left|e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}\varphi'\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)\right|^q d\xi \\
&\leq \int_0^1\Big[\frac{2-\xi}{2}|e^{\varphi(c)}|^q + \frac{\xi}{2}|e^{\varphi(d)}|^q\Big]\Big[\frac{2-\xi}{2}|\varphi'(c)|^q + \frac{\xi}{2}|\varphi'(d)|^q\Big]d\xi \\
&= \int_0^1\left[\left(\frac{2-\xi}{2}\right)^2|e^{\varphi(c)}\varphi'(c)|^q + \left(\frac{\xi}{2}\right)^2|e^{\varphi(d)}\varphi'(d)|^q + \frac{\xi(2-\xi)}{4}\Big[|e^{\varphi(c)}\varphi'(d)|^q + |e^{\varphi(d)}\varphi'(c)|^q\Big]\right]d\xi \\
&= \frac{7}{12}|e^{\varphi(c)}\varphi'(c)|^q + \frac{1}{12}|e^{\varphi(d)}\varphi'(d)|^q + \frac{2}{12}\Upsilon_1(c,d). \qquad (2.15)
\end{aligned}$$

Analogously, we have

$$\int_0^1\left|e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)}\varphi'\Big(\frac{\xi}{2}c+\frac{2-\xi}{2}d\Big)\right|^q d\xi \leq \frac{1}{12}|e^{\varphi(c)}\varphi'(c)|^q + \frac{7}{12}|e^{\varphi(d)}\varphi'(d)|^q + \frac{2}{12}\Upsilon_1(c,d). \qquad (2.16)$$

Substituting the above two inequalities into the inequality (2.14), we get the required inequality (2.13). This completes the proof. □

3. Some better approaches of Hermite-Hadamard type inequalities

In this section, we will derive the new generalizations by employing the Hölder İşcan [59] and improved power-mean [60] inequalities.

Theorem 3.1. *Let $\mathcal{K}, \delta, \gamma > 0$ and $\varphi : \mathcal{I} \mapsto \mathbb{R}$ be a differentiable and exponentially convex function on $\mathcal{I}^\circ$ such that $c, d \in \mathcal{I}^\circ$ with $c < d$ and $(e^\varphi)' \in L_1([c, d])$. If $|(e^\varphi)'|^q$ is convex on $\mathcal{I}$ for some fixed $q > 1$, $q^{-1} + p^{-1} = 1$. Then the following inequality holds for $\mathcal{K}$-conformable fractional integrals:*

$$\begin{aligned}
&\left|\frac{\delta^{\frac{\gamma}{\mathcal{K}}}2^{\frac{\delta\gamma}{\mathcal{K}}-1}\Gamma_{\mathcal{K}}(\gamma+\mathcal{K})}{(d-c)^{\frac{\delta\gamma}{\mathcal{K}}}}\Big[{}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^-}e^{\varphi(c)} + {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^+}e^{\varphi(d)}\Big] - e^{\varphi(\frac{c+d}{2})}\right| \\
&\leq \frac{\delta^{\frac{\gamma}{\mathcal{K}}}(d-c)}{2}\Bigg[\left(\frac{\mathbb{B}(1+\frac{p\gamma}{\mathcal{K}},\frac{2}{\delta})}{\delta^{\frac{p\gamma}{\mathcal{K}}+1}}\right)^{\frac{1}{p}}\left\{\frac{17|e^{\varphi(c)}\varphi'(c)|^q + |e^{\varphi(d)}\varphi'(d)|^q + 3\Upsilon_1(c,d)}{48}\right\}^{\frac{1}{q}} \\
&\quad\times\left[\frac{\mathbb{B}(1+\frac{p\gamma}{\mathcal{K}},\frac{1}{\delta}) - \mathbb{B}(1+\frac{p\gamma}{\mathcal{K}},\frac{2}{\delta})}{\delta^{\frac{p\gamma}{\mathcal{K}}+1}}\right]^{\frac{1}{p}}\left\{\frac{11|e^{\varphi(c)}\varphi'(c)|^q + 3|e^{\varphi(d)}\varphi'(d)|^q + 5\Upsilon_1(c,d)}{48}\right\}^{\frac{1}{q}} \\
&\quad+\left[\frac{\mathbb{B}(1+\frac{p\gamma}{\mathcal{K}},\frac{2}{\delta})}{\delta^{\frac{p\gamma}{\mathcal{K}}+1}}\right]^{\frac{1}{p}}\left\{\frac{|e^{\varphi(c)}\varphi'(c)|^q + 17|e^{\varphi(d)}\varphi'(d)|^q + 3\Upsilon_1(c,d)}{48}\right\}^{\frac{1}{q}} \\
&\quad\times\left[\frac{\mathbb{B}(1+\frac{p\gamma}{\mathcal{K}},\frac{1}{\delta}) - \mathbb{B}(1+\frac{p\gamma}{\mathcal{K}},\frac{2}{\delta})}{\delta^{\frac{p\gamma}{\mathcal{K}}+1}}\right]^{\frac{1}{p}}\left\{\frac{3|e^{\varphi(c)}\varphi'(c)|^q + 11|e^{\varphi(d)}\varphi'(d)|^q + 5\Upsilon_1(c,d)}{48}\right\}^{\frac{1}{q}}\Bigg], \qquad (3.1)
\end{aligned}$$

where $\Upsilon_1(c, d)$ is given in (2.9).

Proof. It follows from Lemma 2.2 and the Hölder-İşcan inequality [59] that

$$\begin{aligned}
&\left|\frac{\delta^{\frac{\gamma}{\mathcal{K}}}2^{\frac{\delta\gamma}{\mathcal{K}}-1}\Gamma_{\mathcal{K}}(\gamma+\mathcal{K})}{(d-c)^{\frac{\delta\gamma}{\mathcal{K}}}}\Big[{}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^-}e^{\varphi(c)} + {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^+}e^{\varphi(d)}\Big] - e^{\varphi(\frac{c+d}{2})}\right| \\
&\leq \frac{\delta^{\frac{\gamma}{\mathcal{K}}}(d-c)}{4}\Bigg[\left\{\left(\int_0^1(1-\xi)\left(\frac{1-(1-\xi)^\delta}{\delta}\right)^{\frac{p\gamma}{\mathcal{K}}}d\xi\right)^{\frac{1}{p}}\left(\int_0^1(1-\xi)\left(\left|e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}\varphi'(\frac{2-\xi}{2}c+\frac{\xi}{2}d)\right|^q d\xi\right)^{\frac{1}{q}}\right\} \\
&\quad\times\left\{\left(\int_0^1\xi\left(\frac{1-(1-\xi)^\delta}{\delta}\right)^{\frac{p\gamma}{\mathcal{K}}}d\xi\right)^{\frac{1}{p}}\left(\int_0^1\xi\left(\left|e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}\varphi'(\frac{2-\xi}{2}c+\frac{\xi}{2}d)\right|^q d\xi\right)^{\frac{1}{q}}\right\} \\
&\quad+\left\{\left(\int_0^1(1-\xi)\left(\frac{1-(1-\xi)^\delta}{\delta}\right)^{\frac{p\gamma}{\mathcal{K}}}d\xi\right)^{\frac{1}{p}}\left(\int_0^1(1-\xi)\left(\left|e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)}\varphi'(\frac{\xi}{2}c+\frac{2-\xi}{2}d)\right|^q d\xi\right)^{\frac{1}{q}}\right. \\
&\quad\times\left.\left\{\left(\int_0^1\xi\left(\frac{1-(1-\xi)^\delta}{\delta}\right)^{\frac{p\gamma}{\mathcal{K}}}d\xi\right)^{\frac{1}{p}}\left(\int_0^1\xi\left(\left|e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)}\varphi'(\frac{\xi}{2}c+\frac{2-\xi}{2}d)\right|^q d\xi\right)^{\frac{1}{q}}\right\}\right]. \qquad (3.2)
\end{aligned}$$

Utilizing the convexity of $|(e^\varphi)'|^q$ on $\mathcal{I}$, we get

$$\int_0^1(1-\xi)\left|e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}\varphi'(\frac{2-\xi}{2}c+\frac{\xi}{2}d)\right|^q d\xi$$

$$\leq \int_0^1 (1-\xi)\Big[\frac{2-\xi}{2}|e^{\varphi(c)}|^q + \frac{\xi}{2}|e^{\varphi(d)}|^q\Big]\Big[\frac{2-\xi}{2}|\varphi'(c)|^q + \frac{\xi}{2}|\varphi'(d)|^q\Big]d\xi$$

$$= \int_0^1 (1-\xi)\Big[\Big(\frac{2-\xi}{2}\Big)^2|e^{\varphi(c)}\varphi'(c)|^q + \Big(\frac{\xi}{2}\Big)^2|e^{\varphi(d)}\varphi'(d)|^q + \frac{\xi(2-\xi)}{4}\Big[|e^{\varphi(c)}\varphi'(d)|^q + |e^{\varphi(d)}\varphi'(c)|^q\Big]\Big]d\xi$$

$$= \frac{17}{48}|e^{\varphi(c)}\varphi'(c)|^q + \frac{1}{48}|e^{\varphi(d)}\varphi'(d)|^q + \frac{3}{48}\Upsilon_1(c,d), \tag{3.3}$$

$$\int_0^1 \xi\Big|e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}\varphi'\Big(\frac{2-\xi}{2}c + \frac{\xi}{2}d\Big)\Big|^q d\xi \leq \frac{11}{48}|e^{\varphi(c)}\varphi'(c)|^q + \frac{3}{48}|e^{\varphi(d)}\varphi'(d)|^q + \frac{5}{48}\Upsilon_1(c,d), \tag{3.4}$$

$$\int_0^1 (1-\xi)\Big|e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)}\varphi'\Big(\frac{\xi}{2}c + \frac{2-\xi}{2}d\Big)\Big|^q d\xi \leq \frac{1}{48}|e^{\varphi(c)}\varphi'(c)|^q + \frac{17}{48}|e^{\varphi(d)}\varphi'(d)|^q + \frac{3}{48}\Upsilon_1(c,d) \tag{3.5}$$

and

$$\int_0^1 \xi\Big|e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)}\varphi'\Big(\frac{\xi}{2}c + \frac{2-\xi}{2}d\Big)\Big|^q d\xi \leq \frac{3}{48}|e^{\varphi(c)}\varphi'(c)|^q + \frac{11}{48}|e^{\varphi(d)}\varphi'(d)|^q + \frac{5}{48}\Upsilon_1(c,d), \tag{3.6}$$

where we have used the indeitites

$$\int_0^1 (1-\xi)\Big(\frac{1-(1-\xi)^\delta}{\delta}\Big)^{\frac{p\gamma}{\mathcal{K}}}d\xi = \frac{1}{\delta^{\frac{p\gamma}{\mathcal{K}}+1}}\mathbb{B}\Big(1+\frac{p\gamma}{\mathcal{K}},\frac{2}{\delta}\Big),$$

$$\int_0^1 \xi\Big(\frac{1-(1-\xi)^\delta}{\delta}\Big)^{\frac{p\gamma}{\mathcal{K}}}d\xi = \frac{1}{\delta^{\frac{p\gamma}{\mathcal{K}}+1}}\Big[\mathbb{B}\Big(1+\frac{p\gamma}{\mathcal{K}},\frac{1}{\delta}\Big) - \mathbb{B}\Big(1+\frac{p\gamma}{\mathcal{K}},\frac{2}{\delta}\Big)\Big]. \tag{3.7}$$

Combining (3.2)–(3.7) leads to the required inequality (4.4). This completes the proof. □

Theorem 3.2. *Let $\mathcal{K},\delta,\gamma > 0$ and $\varphi : \mathcal{I} \mapsto \mathbb{R}$ be a differentiable and exponentially convex function on $\mathcal{I}^\circ$ such that $c,d \in \mathcal{I}^\circ$ with $c < d$ and $(e^\varphi)' \in L_1([c,d])$. If $|(e^\varphi)'|^q$ is convex on $\mathcal{I}$ for some fixed $p,q > 1$, $q^{-1}+p^{-1} = 1$. Then the following inequality holds for $\mathcal{K}$-conformable fractional integrals:*

$$\Big|\frac{\delta^{\frac{\gamma}{\mathcal{K}}}2^{\frac{\delta\gamma}{\mathcal{K}}-1}\Gamma_{\mathcal{K}}(\gamma+\mathcal{K})}{(d-c)^{\frac{\delta\gamma}{\mathcal{K}}}}\Big[{}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^-}e^{\varphi(c)} + {}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^+}e^{\varphi(d)}\Big] - e^{\varphi(\frac{c+d}{2})}\Big|$$

$$\leq \frac{2^{2/q}(d-c)}{2\delta}$$

$$\times\Big[\Big(\mathbb{B}\Big(1+\frac{\gamma}{\mathcal{K}},\frac{2}{\delta}\Big)\Big)^{1-\frac{1}{q}}\Big[\Big\{\Omega_1(\delta,\gamma;\mathcal{K})|e^{\varphi(c)}\varphi'(c)|^q + \Omega_2(\delta,\gamma;\mathcal{K})|e^{\varphi(d)}\varphi'(d)|^q + \Upsilon_1(c,d)\Omega_3(\delta,\gamma;\mathcal{K})\Big\}^{\frac{1}{q}}$$

$$\times\Big(\mathbb{B}(1+\frac{\gamma}{\mathcal{K}},\frac{1}{\delta}) - \mathbb{B}(1+\frac{\gamma}{\mathcal{K}},\frac{2}{\delta})\Big)^{1-\frac{1}{q}}$$

$$\times\left\{\Omega_4(\delta,\gamma;\mathcal{K})|e^{\varphi(c)}\varphi'(c)|^q+\Omega_5(\delta,\gamma;\mathcal{K})|e^{\varphi(d)}\varphi'(d)|^q+\Upsilon_1(c,d)\Omega_6(\delta,\gamma;\mathcal{K})\right\}^{\frac{1}{q}}$$
$$+\left(\mathbb{B}\Big(1+\frac{\gamma}{\mathcal{K}},\frac{2}{\delta}\Big)\right)^{1-\frac{1}{q}}\left\{\Omega_2(\delta,\gamma;\mathcal{K})|e^{\varphi(c)}\varphi'(c)|^q+\Omega_1(\delta,\gamma;\mathcal{K})|e^{\varphi(d)}\varphi'(d)|^q+\Upsilon_1(c,d)\Omega_3(\delta,\gamma;\mathcal{K})\right\}^{\frac{1}{q}}$$
$$\times\left(\mathbb{B}(1+\frac{\gamma}{\mathcal{K}},\frac{1}{\delta})-\mathbb{B}(1+\frac{\gamma}{\mathcal{K}},\frac{2}{\delta})\right)^{1-\frac{1}{q}}$$
$$\times\left\{\Omega_5(\delta,\gamma;\mathcal{K})|e^{\varphi(c)}\varphi'(c)|^q+\Omega_4(\delta,\gamma;\mathcal{K})|e^{\varphi(d)}\varphi'(d)|^q+\Upsilon_1(c,d)\Omega_6(\delta,\gamma;\mathcal{K})\right\}^{\frac{1}{q}}\Bigg], \tag{3.8}$$

where

$$\begin{aligned}
\Omega_1(\delta,\gamma;\mathcal{K})&=\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big)+2\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\Big)+\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big),\\
\Omega_2(\delta,\gamma;\mathcal{K})&=\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big)-2\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\Big)+\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big),\\
\Omega_3(\delta,\gamma;\mathcal{K})&=\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big),\\
\Omega_4(\delta,\gamma;\mathcal{K})&=\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\Big)+\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big),\\
\Omega_5(\delta,\gamma;\mathcal{K})&=\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\Big)-3\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)+3\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big),\\
\Omega_6(\delta,\gamma;\mathcal{K})&=\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)+\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\Big),
\end{aligned} \tag{3.9}$$

and $\Upsilon_1(c,d)$ *is given in* (2.9).

Proof. Making use of Lemma 2.2 and the improved power-mean inequality [60], we have

$$\begin{aligned}
&\left|\frac{\delta^{\frac{\gamma}{\mathcal{K}}}2^{\frac{\delta\gamma}{\mathcal{K}}-1}\Gamma_{\mathcal{K}}(\gamma+\mathcal{K})}{(d-c)^{\frac{\delta\gamma}{\mathcal{K}}}}\Big[{}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^-}e^{\varphi(c)}+{}^{\gamma}_{\mathcal{K}}\mathcal{J}^{\delta}_{(\frac{c+d}{2})^+}e^{\varphi(d)}\Big]-e^{\varphi(\frac{c+d}{2})}\right|\\
&\le\frac{\delta^{\frac{\gamma}{\mathcal{K}}}(d-c)}{4}\Bigg[\Bigg\{\Big(\int_0^1(1-\xi)\Big(\frac{1-(1-\xi)^{\delta}}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}d\xi\Big)^{1-\frac{1}{q}}\\
&\times\Big(\int_0^1(1-\xi)\Big(\frac{1-(1-\xi)^{\delta}}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}\Big|e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}\varphi'\Big(\frac{2-\xi}{2}c+\frac{\xi}{2}d\Big)\Big|^q d\xi\Big)^{\frac{1}{q}}\Bigg\}\\
&\times\Bigg\{\Big(\int_0^1\xi\Big(\frac{1-(1-\xi)^{\delta}}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}d\xi\Big)^{1-\frac{1}{q}}\Big(\int_0^1\xi\Big(\frac{1-(1-\xi)^{\delta}}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}\Big|e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}\varphi'\Big(\frac{2-\xi}{2}c+\frac{\xi}{2}d\Big)\Big|^q d\xi\Big)^{\frac{1}{q}}\Bigg\}\\
&+\Bigg\{\Big(\int_0^1(1-\xi)\Big(\frac{1-(1-\xi)^{\delta}}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}d\xi\Big)^{1-\frac{1}{q}}\Big(\int_0^1(1-\xi)\Big(\frac{1-(1-\xi)^{\delta}}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}\Big|e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)}\varphi'\Big(\frac{\xi}{2}c+\frac{2-\xi}{2}d\Big)\Big|^q d\xi\Big)^{\frac{1}{q}}\\
&\times\Bigg\{\Big(\int_0^1\xi\Big(\frac{1-(1-\xi)^{\delta}}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}d\xi\Big)^{1-\frac{1}{q}}\Big(\int_0^1\xi\Big(\frac{1-(1-\xi)^{\delta}}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}\Big|e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)}\varphi'\Big(\frac{\xi}{2}c+\frac{2-\xi}{2}d\Big)\Big|^q d\xi\Big)^{\frac{1}{q}}\Bigg\}\Bigg].
\end{aligned} \tag{3.10}$$

Utilizing the convexity of $|(e^{\varphi})'|^q$ on $\mathcal{I}$, we obtain

$$\begin{aligned}
&\int_0^1 (1-\xi)\Big(\frac{1-(1-\xi)^{\delta}}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}\Big|e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}\varphi'\Big(\frac{2-\xi}{2}c+\frac{\xi}{2}d\Big)\Big|^q d\xi\\
&\leq \int_0^1 (1-\xi)\Big(\frac{1-(1-\xi)^{\delta}}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}\Big[\frac{2-\xi}{2}|e^{\varphi(c)}|^q+\frac{\xi}{2}|e^{\varphi(d)}|^q\Big]\Big[\frac{2-\xi}{2}|\varphi'(c)|^q+\frac{\xi}{2}|\varphi'(d)|^q\Big]d\xi\\
&= \int_0^1 (1-\xi)\Big(\frac{1-(1-\xi)^{\delta}}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}\Big[\Big(\frac{2-\xi}{2}\Big)^2|e^{\varphi(c)}\varphi'(c)|^q+\Big(\frac{\xi}{2}\Big)^2|e^{\varphi(d)}\varphi'(d)|^q\\
&\qquad\qquad +\frac{\xi(2-\xi)}{4}\Big[|e^{\varphi(c)}\varphi'(d)|^q+|e^{\varphi(d)}\varphi'(c)|^q\Big]\Big]d\xi\\
&= \frac{1}{4\delta^{\frac{\gamma}{\mathcal{K}}+1}}\Big[\Big\{\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big)+2\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\Big)+\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)\Big\}|e^{\varphi(c)}\varphi'(c)|^q\\
&\qquad +\Big\{\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big)-2\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\Big)+\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)\Big\}|e^{\varphi(d)}\varphi'(d)|^q\\
&\qquad +\Upsilon_1(c,d)\Big\{\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big)\Big\}\Big]\\
&= \frac{1}{4\delta^{\frac{\gamma}{\mathcal{K}}+1}}\Big[\Omega_1(\delta,\gamma;\mathcal{K})|e^{\varphi(c)}\varphi'(c)|^q+\Omega_2(\delta,\gamma;\mathcal{K})|e^{\varphi(d)}\varphi'(d)|^q+\Upsilon_1(c,d)\Omega_3(\delta,\gamma;\mathcal{K})\Big], \qquad (3.11)
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \xi\Big(\frac{1-(1-\xi)^{\delta}}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}\Big|e^{\varphi\left(\frac{2-\xi}{2}c+\frac{\xi}{2}d\right)}\varphi'\Big(\frac{2-\xi}{2}c+\frac{\xi}{2}d\Big)\Big|^q d\xi\\
&\leq \frac{1}{4\delta^{\frac{\gamma}{\mathcal{K}}+1}}\Big[\Big\{\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\Big)+\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big)\Big\}|e^{\varphi(c)}\varphi'(c)|^q\\
&\qquad +\Big\{\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\Big)-3\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)+3\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big)\Big\}|e^{\varphi(d)}\varphi'(d)|^q\\
&\qquad +\Upsilon_1(c,d)\Big\{\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)+\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\Big)\Big\}\Big]\\
&= \frac{1}{4\delta^{\frac{\gamma}{\mathcal{K}}+1}}\Big[\Omega_4(\delta,\gamma;\mathcal{K})|e^{\varphi(c)}\varphi'(c)|^q+\Omega_5(\delta,\gamma;\mathcal{K})|e^{\varphi(d)}\varphi'(d)|^q+\Upsilon_1(c,d)\Omega_6(\delta,\gamma;\mathcal{K})\Big], \qquad (3.12)
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 (1-\xi)\Big(\frac{1-(1-\xi)^{\delta}}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}\Big|e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)}\varphi'\Big(\frac{\xi}{2}c+\frac{2-\xi}{2}d\Big)\Big|^q d\xi\\
&\leq \frac{1}{4\delta^{\frac{\gamma}{\mathcal{K}}+1}}\Big[\Big\{\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big)-2\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\Big)+\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)\Big\}|e^{\varphi(c)}\varphi'(c)|^q\\
&\qquad +\Big\{\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big)+2\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\Big)+\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)\Big\}|e^{\varphi(d)}\varphi'(d)|^q
\end{aligned}$$

$$+ \Upsilon_1(c,d)\Big\{\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big)\Big\}\Big]$$

$$= \frac{1}{4\delta^{\frac{\gamma}{\mathcal{K}}+1}}\Big[\Omega_2(\delta,\gamma;\mathcal{K})|e^{\varphi(c)}\varphi'(c)|^q + \Omega_1(\delta,\gamma;\mathcal{K})|e^{\varphi(d)}\varphi'(d)|^q + \Upsilon_1(c,d)\Omega_3(\delta,\gamma;\mathcal{K})\Big] \tag{3.13}$$

and

$$\begin{aligned}
&\int_0^1 \xi\Big(\frac{1-(1-\xi)^\delta}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}\Big|e^{\varphi\left(\frac{\xi}{2}c+\frac{2-\xi}{2}d\right)}\varphi'\Big(\frac{\xi}{2}c+\frac{2-\xi}{2}d\Big)\Big|^q d\xi\\
&\le \frac{1}{4\delta^{\frac{\gamma}{\mathcal{K}}+1}}\Big[\Big\{\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\Big)-3\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)+3\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big)\Big\}|e^{\varphi(c)}\varphi'(c)|^q\\
&\quad+\Big\{\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\Big)+\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big)\Big\}|e^{\varphi(d)}\varphi'(d)|^q\\
&\quad+\Upsilon_1(c,d)\Big\{\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{4}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{3}{\delta}\Big)-\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{2}{\delta}\Big)+\mathbb{B}\Big(\frac{\gamma}{\mathcal{K}}+1,\frac{1}{\delta}\Big)\Big\}\Big]\\
&= \frac{1}{4\delta^{\frac{\gamma}{\mathcal{K}}+1}}\Big[\Omega_5(\delta,\gamma;\mathcal{K})|e^{\varphi(c)}\varphi'(c)|^q + \Omega_4(\delta,\gamma;\mathcal{K})|e^{\varphi(d)}\varphi'(d)|^q + \Upsilon_1(c,d)\Omega_6(\delta,\gamma;\mathcal{K})\Big],
\end{aligned} \tag{3.14}$$

where we have used the facts that

$$\begin{aligned}
&\int_0^1 (1-\xi)\Big(\frac{1-(1-\xi)^\delta}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}d\xi = \frac{1}{\delta^{\frac{\gamma}{\mathcal{K}}+1}}\mathbb{B}\Big(1+\frac{\gamma}{\mathcal{K}},\frac{2}{\delta}\Big),\\
&\int_0^1 \xi\Big(\frac{1-(1-\xi)^\delta}{\delta}\Big)^{\frac{\gamma}{\mathcal{K}}}d\xi = \frac{1}{\delta^{\frac{\gamma}{\mathcal{K}}+1}}\Big[\mathbb{B}\Big(1+\frac{\gamma}{\mathcal{K}},\frac{1}{\delta}\Big)-\mathbb{B}\Big(1+\frac{\gamma}{\mathcal{K}},\frac{2}{\delta}\Big)\Big].
\end{aligned} \tag{3.15}$$

Combining (3.10)–(3.15) gives the required inequality (3.8). This completes the proof. □

4. Examples

In this section, we present some examples to demonstrate the applications of our proposed results on modified Bessel functions and σ-digamma functions.

Example 4.1. *Let $z \in \mathbb{R}$ and $\Im_\rho : \mathbb{R} \mapsto [1,\infty)$ be the modified Bessel function of the first kind defined by*

$$\Im_\rho(z) = \sum_{n\ge 0}\frac{z^{\rho+2n}}{2^{\rho+2n}n!\Gamma(\rho+n+1)}.$$

The the first order derivative formula of $\Im_\rho$ is given by

$$\Im'(z) = \frac{z}{2(\rho+1)}\Im_{\rho+1}(z). \tag{4.1}$$

By use of Theorem 2.4 and identity (4.1)*, we get*

$$\left|\frac{1}{d-c}\int_c^d e^{\Im_\rho(z)}dz - e^{\Im_\rho\left(\frac{c+d}{2}\right)}\right| \le \frac{d-c}{8(\rho+1)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\times \quad \left[\left\{\frac{7c|e^{\Im_\rho(c)}\Im_{\rho+1}(c)|^q + d|e^{\Im_\rho(d)}\Im_{\rho+1}(d)|^q + 2\{c|e^{\Im_\rho(d)}\Im_{\rho+1}(c)|^q + d|e^{\Im_\rho(c)}\Im_{\rho+1}(d)|^q\}}{12}\right\}^{\frac{1}{q}}\right.$$
$$\left. + \left\{\frac{c|e^{\Im_\rho(c)}\Im_{\rho+1}(c)|^q + 7d|e^{\Im_\rho(d)}\Im_{\rho+1}(d)|^q + 2\{c|e^{\Im_\rho(d)}\Im_{\rho+1}(c)|^q + d|e^{\Im_\rho(c)}\Im_{\rho+1}(d)|^q\}}{12}\right\}^{\frac{1}{q}}\right],$$

where $\rho > -1$, $c, d \in \mathbb{R}$ *satisfy the assumptions that* $\delta = \gamma = 1$ *and* $0 < c < d$, $\mathcal{K} = 1$. *Specifically, for* $\Im_{-1/2}(z) = \cosh z$ *and* $\Im_{1/2}(z) = \frac{\sinh z}{z}$, *one has*

$$\left|\frac{1}{d-c}\int_c^d e^{\cosh(z)}dz - e^{\cosh(\frac{c+d}{2})}\right| \leq \frac{d-c}{8(\rho+1)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\times \quad \left[\left\{\frac{7|e^{\sinh(c)}\sinh(c)|^q + |e^{\sinh(d)}\sinh(d)|^q + 2\{|e^{\sinh(d)}\sinh(c)|^q + |e^{\sinh(c)}\sinh(d)|^q\}}{12}\right\}^{\frac{1}{q}}\right.$$
$$\left. + \left\{\frac{|e^{\sinh(c)}\sinh(c)|^q + 7|e^{\sinh(d)}\sinh(d)|^q + 2\{|e^{\sinh(d)}\sinh(c)|^q + |e^{\sinh(c)}\sinh(d)|^q\}}{12}\right\}^{\frac{1}{q}}\right].$$

Example 4.2. *Consider the* σ*-analogue of the digamma function* ϕ_σ *given by*

$$\phi_\sigma(z) = -\ln(1-\sigma) + \ln\sigma\sum_{\kappa=0}^{\infty}\frac{\sigma^{\kappa+z}}{1-\sigma^{\kappa+z}} = -\ln(1-\sigma) + \ln\sigma\sum_{\kappa=0}^{\infty}\frac{\sigma^{-\kappa z}}{1-\sigma^{-z\kappa}}. \tag{4.2}$$

For $\sigma > 1$ *and* $z > 0$, *the* σ*-digamma function* ϕ_σ *can be expressed by*

$$\begin{aligned}\phi_\sigma(z) &= -\ln(1-\sigma) + \ln\sigma\left[z - \frac{1}{2} - \sum_{\kappa=0}^{\infty}\frac{\sigma^{-(\kappa+z)}}{1-\sigma^{-(\kappa+z)}}\right]\\ &= -\ln(1-\sigma) + \ln\sigma\left[z - \frac{1}{2} - \sum_{\kappa=0}^{\infty}\frac{\sigma^{-\kappa z}}{1-\sigma^{-\kappa z}}\right].\end{aligned} \tag{4.3}$$

It follows from $\lim_{\sigma\to 1^+}\phi_\sigma(z) = \lim_{\sigma\to 1^-}\phi_\sigma(z) = \phi(z)$ *that* $z \mapsto \phi'_\sigma(z)$ *is a completely monotonic function on an interval* $(0,\infty)$ *for all* $\sigma > 0$, *and consequently,* $z \mapsto \phi'_\sigma(z)$ *is convex on the same interval. Let* $\varphi_\sigma(z) = \phi'_\sigma(z)$ *with* $q > 0$. *Then* $\varphi'_\sigma(z) = \phi''_\sigma(z)$ *is completely monotonic on the interval* $(0,\infty)$, *and from Theorem 3.1 we have*

$$\left|e^{\phi_\sigma(\frac{c+d}{2})} - \left(\frac{e^{\phi_\sigma(d)} - e^{\phi_\sigma(c)}}{d-c}\right)\right|$$
$$\leq \frac{(d-c)}{2}\left[\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left\{\frac{17|e^{\phi_\sigma(c)}\phi''_\sigma(c)|^q + |e^{\phi_\sigma(d)}\phi''_\sigma(d)|^q + 3\Upsilon_1(c,d)}{48}\right\}^{\frac{1}{q}}\right.$$
$$\times\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left\{\frac{11|e^{\phi_\sigma(c)}\phi''_\sigma(c)|^q + 3|e^{\phi_\sigma(d)}\phi''_\sigma(d)|^q + 5\Upsilon_1(c,d)}{48}\right\}^{\frac{1}{q}}$$
$$+\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left\{\frac{|e^{\phi_\sigma(c)}\phi''_\sigma(c)|^q + 17|e^{\phi_\sigma(d)}\phi''_\sigma(d)|^q + 3\Upsilon_1(c,d)}{48}\right\}^{\frac{1}{q}}$$
$$\left.\times\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\frac{3|e^{\phi_\sigma(c)}\phi''_\sigma(c)|^q + 11|e^{\phi_\sigma(d)}\phi''_\sigma(d)|^q + 5\Upsilon_1(c,d)}{48}\right\}^{\frac{1}{q}}\right], \tag{4.4}$$

for all $\sigma \in (0,1)$, $\gamma = 1 = \delta$, $d > c > 0$, $\mathcal{K} = 1$ *and* $\Upsilon_1(c,d)$ *is given in* (2.9).

5. Weighted Hermite-Hadamard type inequalities for differentiable functions

In order to prove our main results in this section, we need the following lemma.

Lemma 5.1. *Let $\varphi : \mathcal{I} \mapsto \mathbb{R}$ be a differentiable and exponentially convex functions on $\mathcal{I}^\circ$ such that $c, d \in \mathcal{I}^\circ$ with $c < d$ and $(e^\varphi)' \in L_1([c,d])$. Also, let $w : [c,d] \mapsto \mathbb{R}$ be a differentiable mapping such that w is symmetric with respect to $(c+d)/2$. Then the identity*

$$\frac{1}{d-c}\int_c^d e^{\varphi(z)}e^{w(z)}dz - \frac{1}{d-c}e^{\varphi\left(\frac{c+d}{2}\right)}\int_c^d e^{w(z)}dz = (d-c)\int_0^1 \phi(\xi)e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)d\xi \quad (5.1)$$

holds for all $z \in [c,d]$, where

$$\phi(\xi) = \begin{cases} \displaystyle\int_0^\xi e^{w(cs+(1-s)d)}ds, & \xi \in [0,1/2), \\ \displaystyle -\int_\xi^1 e^{w(cs+(1-s)d)}ds, & \xi \in [1/2,1]. \end{cases}$$

Proof. Note that

$$\begin{aligned}\theta &= \int_0^1 \phi(\xi)e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)d\xi \\ &= \int_0^{\frac{1}{2}}\Big(\int_0^\xi e^{w(cs+(1-s)d)}ds\Big)e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)d\xi \\ &\qquad + \int_{\frac{1}{2}}^1\Big(-\int_\xi^1 e^{w(cs+(1-s)d)}ds\Big)e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)d\xi \\ &= \theta_1+\theta_2.\end{aligned}$$

Making use of integration by parts, we get

$$\begin{aligned}\theta_1 &= \Big(\int_0^\xi e^{w(cs+(1-s)d)}ds\Big)\frac{e^{\varphi(\xi c+(1-\xi)d)}}{c-d}\Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} e^{w(c\xi+(1-\xi)d)}\frac{e^{\varphi(\xi c+(1-\xi)d)}}{c-d}d\xi \\ &= \Big(\int_0^{\frac{1}{2}} e^{w(cs+(1-s)d)}ds\Big)\frac{e^{\varphi\left(\frac{c+d}{2}\right)}}{c-d} - \int_0^{\frac{1}{2}} e^{w(c\xi+(1-\xi)d)}\frac{e^{\varphi(\xi c+(1-\xi)d)}}{c-d}d\xi.\end{aligned}$$

Similarly, one has

$$\theta_2 = \Big(\int_{\frac{1}{2}}^1 e^{w(cs+(1-s)d)}ds\Big)\frac{e^{\varphi\left(\frac{c+d}{2}\right)}}{c-d} - \int_{\frac{1}{2}}^1 e^{w(c\xi+(1-\xi)d)}\frac{e^{\varphi(\xi c+(1-\xi)d)}}{c-d}d\xi.$$

Therefore,

$$\theta = \theta_1 + \theta_2 = \left(\int_0^1 e^{w(cs+(1-s)d)} ds \right) \frac{e^{\varphi\left(\frac{c+d}{2}\right)}}{c-d} - \int_0^1 e^{w(c\xi+(1-\xi)d)} \frac{e^{\varphi(\xi c+(1-\xi)d)}}{c-d} d\xi,$$

and identity (5.1) can be obtained by the change of variable technique and multiplying the both sided by $(d-c)$ in the above formula. □

Theorem 5.2. *Let $\varphi : \mathcal{I} \mapsto \mathbb{R}$ be a differentiable and exponentially convex function on $\mathcal{I}^\circ$ such that $c, d \in \mathcal{I}^\circ$ with $c < d$. Also, let $w : [c,d] \mapsto \mathbb{R}$ be a differentiable mapping and symmetric with respect to $(c+d)/2$. If $\left|(e^\varphi)'\right|$ is convex on $\mathcal{I}$. Then the inequality*

$$\begin{aligned}
&\left| \frac{1}{d-c} \int_c^d e^{\varphi(z)} e^{w(z)} dz - \frac{1}{d-c} e^{\varphi\left(\frac{c+d}{2}\right)} \int_c^d e^{w(z)} dz \right| \\
&\leq \left[\frac{1}{6(d-c)^3} \int_{\frac{c+d}{2}}^d e^{w(z)} dz \right] \left\{ 8\left[(z-c)^3 - (d-z)^3\right] \left[\frac{\left|e^{\varphi(c)}\varphi'(c)\right| + \left|e^{\varphi(d)}\varphi'(d)\right|}{2} \right] \right. \\
&\qquad \left. + \Upsilon(c,d)\left[(d-c)^3 - 2(d-z)^2(3d-c-2z)\right] \right\} \qquad (5.2)
\end{aligned}$$

holds for all $z \in [c,d]$, where

$$\Upsilon(c,d) = \left\{ \left|e^{\varphi(c)}\varphi'(d)\right| + \left|e^{\varphi(d)}\varphi'(c)\right| \right\}.$$

Proof. It follows from Lemma 5.1 and the hypothesis given in Theorem 5.2 that

$$\begin{aligned}
&\left| \frac{1}{d-c} \int_c^d e^{\varphi(z)} e^{w(z)} dz - \frac{1}{d-c} e^{\varphi\left(\frac{c+d}{2}\right)} \int_c^d e^{w(z)} dz \right| \\
&\leq (d-c) \left[\left\{ \int_0^{\frac{1}{2}} \left(\int_0^\xi e^{w(cs+(1-s)d)} ds \right) \left[e^{\varphi(\xi c+(1-\xi)d)} \varphi'(\xi c + (1-\xi)d) \right] d\xi \right\} \right. \\
&\qquad \left. + \left\{ \int_{\frac{1}{2}}^1 \left(\int_\xi^1 e^{w(cs+(1-s)d)} ds \right) \left[e^{\varphi(\xi c+(1-\xi)d)} \varphi'(\xi c + (1-\xi)d) \right] d\xi \right\} \right] \\
&\leq (d-c) \left[\left\{ \int_0^{\frac{1}{2}} \left(\int_0^\xi e^{w(cs+(1-s)d)} ds \right) \left[\xi^2 \left|e^{\varphi(c)}\varphi'(c)\right| + (1-\xi)^2 \left|e^{\varphi(d)}\varphi'(d)\right| + \xi(1-\xi)\Upsilon(c,d) \right] d\xi \right\} \right. \\
&\qquad \left. + \left\{ \int_{\frac{1}{2}}^1 \left(\int_\xi^1 e^{w(cs+(1-s)d)} ds \right) \left[\xi^2 |e^{\varphi(c)}\varphi'(c)| + (1-\xi)^2 \left|e^{\varphi(d)}\varphi'(d)\right| + \xi(1-\xi)\Upsilon(c,d) \right] d\xi \right\} \right] \\
&= \rho_1 + \rho_2. \qquad (5.3)
\end{aligned}$$

Exchanging the integration order gives

$$\begin{aligned}
\rho_1 &= \int_0^{\frac{1}{2}} \int_0^{\xi} e^{w(cs+(1-s)d)} \left[\xi^2 \left|e^{\varphi(c)}\varphi'(c)\right| + (1-\xi)^2 \left|e^{\varphi(d)}\varphi'(d)\right| + \xi(1-\xi)\Upsilon(c,d)\right] ds d\xi \\
&= \int_0^{\frac{1}{2}} \int_s^{\frac{1}{2}} e^{w(cs+(1-s)d)} \left[\xi^2 \left|e^{\varphi(c)}\varphi'(c)\right| + (1-\xi)^2 \left|e^{\varphi(d)}\varphi'(d)\right| + \xi(1-\xi)\Upsilon(c,d)\right] d\xi ds \\
&\leq \int_0^{\frac{1}{2}} e^{w(cs+(1-s)d)} \Bigg[\left\{\left(\frac{1}{24} - \frac{s^3}{3}\right)\left|e^{\varphi(c)}\varphi'(c)\right|\right\} + \left\{\left(-\frac{1}{24} + \frac{s^3}{3}\right)\left|e^{\varphi(d)}\varphi'(d)\right|\right\} \\
&\qquad + \Upsilon(c,d)\left\{-\frac{s^2(3-2s)}{6} + \frac{1}{12}\right\}\Bigg] \\
&= \int_{\frac{c+d}{2}}^{d} e^{w(z)} \Bigg[\left\{\left(\frac{1}{24} - \frac{(d-z)^3}{3(d-c)^3}\right)\left|e^{\varphi(c)}\varphi'(c)\right|\right\} + \left\{\left(-\frac{1}{24} + \frac{(z-c)^3}{3(d-c)^3}\right)\left|e^{\varphi(d)}\varphi'(d)\right|\right\} \\
&\qquad + 2\Upsilon(c,d)\left\{(d-c)^3 - 2(d-z)^2(d-3c+2z)\right\}\Bigg]. \qquad (5.4)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\rho_2 &= \int_{\frac{1}{2}}^{1} \int_{\xi}^{1} e^{w(cs+(1-s)d)} \left[\xi^2 \left|e^{\varphi(c)}\varphi'(c)\right| + (1-\xi)^2 \left|e^{\varphi(d)}\varphi'(d)\right| + \xi(1-\xi)\Upsilon(c,d)\right] ds d\xi \\
&\leq \int_c^{\frac{c+d}{2}} e^{w(z)} \Bigg[\left\{\left(-\frac{1}{24} + \frac{(z-c)^3}{3(d-c)^3}\right)\left|e^{\varphi(c)}\varphi'(c)\right|\right\} + \left\{\left(\frac{1}{24} - \frac{(d-z)^3}{3(d-c)^3}\right)\left|e^{\varphi(d)}\varphi'(d)\right|\right\} \\
&\qquad + 2\Upsilon(c,d)\left\{(d-c)^3 - 2(d-z)^2(d-3c+2z)\right\}\Bigg]. \qquad (5.5)
\end{aligned}$$

Since $w(z)$ is symmetric with respect to $z = \frac{c+d}{2}$, for $w(z) = w(c+d-z)$, we get

$$\rho_1 = \rho_2.$$

combining (5.3)–(5.5) leads to (5.2). This completes the proof □

Theorem 5.3. *Let $\varphi : \mathcal{I} \mapsto \mathbb{R}$ be a differentiable and exponentially convex function on $\mathcal{I}^\circ$ such that $c, d \in \mathcal{I}^\circ$ with $c < d$. Also, let $w : [c,d] \mapsto \mathbb{R}$ be a differentiable mapping and symmetric with respect to $(c+d)/2$. If $\left|(e^\varphi)'\right|^q$ is convex on $\mathcal{I}$ for some $q > 1$ with $p^{-1} + q^{-1} = 1$. Then the inequality*

$$\left|\frac{1}{d-c}\int_c^d e^{\varphi(z)}e^{w(z)}dz - \frac{1}{d-c}e^{\varphi\left(\frac{c+d}{2}\right)}\int_c^d e^{w(z)}dz\right|$$

$$\leq (d-c)\left[\frac{1}{(d-c)^2}\int_d^{\frac{c+d}{2}} e^{pw(z)}\left(z-\frac{c+d}{2}\right)dz\right]^{1/p}\cdot\left[\left(\frac{3\left|e^{\varphi(c)}\varphi'(c)\right|^q+11\left|e^{\varphi(d)}\varphi'(d)\right|^q+6\Upsilon_1(c,d)}{192}\right)^{1/q}\right.$$
$$\left.+\left(\frac{11\left|e^{\varphi(c)}\varphi'(c)\right|^q+3\left|e^{\varphi(d)}\varphi'(d)\right|^q+6\Upsilon_1(c,d)}{192}\right)^{\frac{1}{q}}\right] \tag{5.6}$$

holds for all $z\in[c,d]$, where $\Upsilon_1(c,d)$ is given in (2.9).

Proof. Making use of Lemma 5.1 and changing the integration order, we get

$$\begin{aligned}
&\left|\frac{1}{d-c}\int_c^d e^{\varphi(z)}e^{w(z)}dz-\frac{1}{d-c}e^{\varphi\left(\frac{c+d}{2}\right)}\int_c^d e^{w(z)}dz\right|\\
&\leq (d-c)\left[\left\{\int_0^{\frac12}\left(\int_0^{\xi}e^{w(cs+(1-s)d)}ds\right)\left[e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)\right]d\xi\right\}\right.\\
&\qquad\left.+\left\{\int_{\frac12}^1\left(\int_{\xi}^1 e^{w(cs+(1-s)d)}ds\right)\left[e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)\right]d\xi\right\}\right]\\
&=(d-c)\left[\left\{\int_0^{\frac12}\int_s^{\frac12}e^{w(cs+(1-s)d)}e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)d\xi ds\right\}\right.\\
&\qquad\left.+\left\{\int_{\frac12}^1\int_{\frac12}^s e^{w(cs+(1-s)d)}e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)d\xi ds\right\}\right].
\end{aligned}$$

By Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{1}{d-c}\int_c^d e^{\varphi(z)}e^{w(z)}dz-\frac{1}{d-c}e^{\varphi\left(\frac{c+d}{2}\right)}\int_c^d e^{w(z)}dz\right|\\
&\leq (d-c)\left\{\left(\int_0^{\frac12}\int_s^{\frac12}e^{w(cs+(1-s)d)}d\xi ds\right)^{1/p}\left(\int_0^{\frac12}\int_s^{\frac12}\left|e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)\right|^q d\xi ds\right)^{1/q}\right.\\
&\qquad\left.+\left(\int_{\frac12}^1\int_{\frac12}^s e^{w(cs+(1-s)d)}d\xi ds\right)^{1/p}\left(\int_{\frac12}^s\int_s^{\frac12}\left|e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)\right|^q d\xi ds\right)^{1/q}\right\}.
\end{aligned}$$

It follows from the convexity of $\left|(e^{\varphi})'\right|^q$ that

$$\left|e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)\right|^q$$

$$\leq \left[\xi \left|e^{\varphi(c)}\right|^q + (1-\xi)\left|e^{\varphi(d)}\right|^q\right]\left[\xi\left|\varphi'(c)\right|^q + (1-\xi)\left|\varphi'(d)\right|^q\right]$$
$$\leq \left[\xi^2\left|e^{\varphi(c)}\varphi'(c)\right|^q + (1-\xi)^2\left|e^{\varphi(d)}\varphi'(d)\right|^q + \xi(1-\xi)\left\{\left|e^{\varphi(c)}\varphi'(d)\right|^q + \left|e^{\varphi(d)}\varphi'(c)\right|^q\right\}\right]$$
$$\leq \left[\xi^2\left|e^{\varphi(c)}\varphi'(c)\right|^q + (1-\xi)^2\left|e^{\varphi(d)}\varphi'(d)\right|^q + \xi(1-\xi)\Upsilon_1(c,d)\right]$$

for $\xi \in [0,1]$.

Therefore, one has

$$\left|\frac{1}{d-c}\int_c^d e^{\varphi(z)}e^{w(z)}dz - \frac{1}{d-c}e^{\varphi\left(\frac{c+d}{2}\right)}\int_c^d e^{w(z)}dz\right|$$
$$\leq (d-c)\Bigg[\Bigg(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} e^{w(cs+(1-s)d)}d\xi ds\Bigg)^{1/p}\Bigg(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}\Big[\xi^2\left|e^{\varphi(c)}\varphi'(c)\right|^q + (1-\xi)^2\left|e^{\varphi(d)}\varphi'(d)\right|^q$$
$$+ \xi(1-\xi)\Upsilon_1(c,d)\Big]d\xi ds\Bigg)^{1/q} + \Bigg(\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^s e^{w(cs+(1-s)d)}d\xi ds\Bigg)^{\frac{1}{p}}\Bigg(\int_{\frac{1}{2}}^s\int_s^{\frac{1}{2}}\Big[\xi^2\left|e^{\varphi(c)}\varphi'(c)\right|^q$$
$$+ (1-\xi)^2\left|e^{\varphi(d)}\varphi'(d)\right|^q + \xi(1-\xi)\Upsilon_1(c,d)\Big]d\xi ds\Bigg)^{\frac{1}{q}}\Bigg] = S_1 + S_2. \tag{5.7}$$

Note that

$$S_1 = \left(\frac{1}{2(d-c)^2}\int_{\frac{c+d}{2}}^d e^{pw(z)}(2z-c-d)dz\right)^{1/p}\left(\frac{3\left|e^{\varphi(c)}\varphi'(c)\right|^q + 11\left|e^{\varphi(d)}\varphi'(d)\right|^q + 6\Upsilon_1(c,d)}{192}\right)^{1/q} \tag{5.8}$$

and

$$S_2 = \left(\frac{1}{2(d-c)^2}\int_c^{\frac{c+d}{2}} e^{pw(z)}(c+d-2z)dz\right)^{1/p}\left(\frac{11\left|e^{\varphi(c)}\varphi'(c)\right|^q + 3\left|e^{\varphi(d)}\varphi'(d)\right|^q + 6\Upsilon_1(c,d)}{192}\right)^{1/q}. \tag{5.9}$$

Therefore, inequality (5.3) follows from (5.7)–(5.9). This completes the proof. □

Lemma 5.4. *Let $\varphi : \mathcal{I} \mapsto \mathbb{R}$ be a differentiable and exponentially convex function on $\mathcal{I}^\circ$ such that $c, d \in \mathcal{I}^\circ$ with $c < d$ and $(e^\varphi)' \in L_1([c,d])$. Also, let $w : [c,d] \mapsto \mathbb{R}$ be a differentiable mapping such that w is symmetric with respect to $(c+d)/2$. Then the identity*

$$\frac{1}{d-c}\frac{e^{\varphi(c)}+e^{\varphi(d)}}{2}\int_c^d e^{w(z)}dz - \frac{1}{d-c}\int_c^d e^{w(z)}e^{\varphi(z)}dz = \frac{d-c}{2}\int_0^1 \mu(\xi)e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)d\xi$$

holds for all $z \in [c,d]$, where

$$\mu(\xi) = \int_\xi^1 e^{w(cs+(1-s)d)}ds - \int_0^\xi e^{w(cs+(1-s)d)}ds.$$

Proof. We clearly see that

$$\begin{aligned} U &= \int_0^1 \mu(\xi) e^{\varphi(\xi c+(1-\xi)d)} \varphi'(\xi c + (1-\xi)d) d\xi \\ &= \int_0^1 \Big(\int_\xi^1 e^{w(cs+(1-s)d)} ds \Big) e^{\varphi(\xi c+(1-\xi)d)} \varphi'(\xi c + (1-\xi)d) d\xi \\ &= c_1 + c_2. \end{aligned}$$

It follows from integration by parts that

$$\begin{aligned} c_1 &= \Big(\int_\xi^1 e^{w(cs+(1-s)d)} ds \Big) \frac{e^{\varphi(\xi c+(1-\xi)d)}}{c-d} \Big|_0^1 + \int_0^1 e^{w(c\xi+(1-\xi)d)} \frac{e^{\varphi(\xi c+(1-\xi)d)}}{c-d} d\xi \\ &= \Big(- \int_0^1 e^{w(cs+(1-s)d)} ds \Big) \frac{e^{\varphi(d)}}{c-d} + \int_0^1 e^{w(c\xi+(1-\xi)d)} \frac{e^{\varphi(\xi c+(1-\xi)d)}}{c-d} d\xi. \end{aligned}$$

Similarly, we have

$$c_2 = \Big(- \int_0^1 e^{w(cs+(1-s)d)} ds \Big) \frac{e^{\varphi(c)}}{c-d} + \int_0^1 e^{w(c\xi+(1-\xi)d)} \frac{e^{\varphi(\xi c+(1-\xi)d)}}{c-d} d\xi.$$

Thus, we get

$$U = c_1 + c_2 = -\frac{e^{\varphi(c)} + e^{\varphi(d)}}{c-d} \Big(\int_0^1 e^{w(cs+(1-s)d)} ds \Big) + 2 \int_0^1 e^{w(c\xi+(1-\xi)d)} \frac{e^{\varphi(\xi c+(1-\xi)d)}}{c-d} d\xi.$$

Therefore, the desired result can be obtained by use of the change of variable technique and multiplying the both sided by $(d-c)/2$. □

Theorem 5.5. *Let $\varphi : \mathcal{I} \mapsto \mathbb{R}$ be a differentiable and exponentially convex function on $\mathcal{I}^\circ$ such that $c, d \in \mathcal{I}^\circ$ with $c < d$. Also, let $w : [c,d] \mapsto \mathbb{R}$ be a differentiable mapping and symmetric with respect to $(c+d)/2$. If $\left|(e^\varphi)'\right|^q$ is convex on $\mathcal{I}$ for some $q > 1$ with $p^{-1} + q^{-1} = 1$. Then the inequality*

$$\begin{aligned} &\left| \frac{1}{d-c} \frac{e^{\varphi(c)} + e^{\varphi(d)}}{2} \int_c^d e^{w(z)} dz - \frac{1}{d-c} \int_c^d e^{\varphi(z)} e^{w(z)} dz \right| \\ &\leq \frac{1}{2} \Big(\int_0^1 (\eta(z))^p d\xi \Big)^{1/p} \Big(\frac{2(\left|e^{\varphi(c)} \varphi'(c)\right|^q + \left|e^{\varphi(d)} \varphi'(d)\right|^q) + \Upsilon_1(c,d)}{6} \Big) \end{aligned}$$

holds for all $z \in [c,d]$, where $\Upsilon_1(c,d)$ is given in (2.9) *and*

$$\eta(z) = \left| \int_{c+(d-c)\xi}^{d-(d-c)\xi} e^{w(z)} dz \right|.$$

Proof. From Lemma 5.4 and the hypothesis of Theorem 5.5, we get

$$\begin{aligned}
&\left|\frac{1}{d-c}\frac{e^{\varphi(c)}+e^{\varphi(d)}}{2}\int_c^d e^{w(z)}dz-\frac{1}{d-c}\int_c^d e^{\varphi(z)}e^{w(z)}dz\right|\\
&\leq\frac{d-c}{2}\left[\int_0^1\left|\int_\xi^1 e^{w(cs+(1-s)d)}ds-\int_0^\xi e^{w(cs+(1-s)d)}ds\right|\left|e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)\right|d\xi\right]\\
&\leq\frac{1}{2}\left[\int_0^1\left|\int_c^{d-(d-c)\xi} e^{w(z)}dz-\int_{d-(d-c)\xi}^d e^{w(z)}dz\right|\left|e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)\right|d\xi\right].
\end{aligned}\tag{5.10}$$

Since $w(z)$ is symmetric to $z=(c+d)/2$, we get

$$\int_c^{d-(d-c)\xi} e^{w(z)}dz-\int_{d-(d-c)\xi}^d e^{w(z)}dz=\int_{c+(d-c)\xi}^{d-(d-c)\xi} e^{w(z)}dz\tag{5.11}$$

for $\xi\in[0,1/2]$, and

$$\int_c^{d-(d-c)\xi} e^{w(z)}dz-\int_{d-(d-c)\xi}^d e^{w(z)}dz=-\int_{d-(d-c)\xi}^{c+(d-c)\xi} e^{w(z)}dz\tag{5.12}$$

for $\xi\in[1/2,1]$.

It follows from (5.10)–(5.12) that

$$\begin{aligned}
\left|\frac{1}{d-c}\frac{e^{\varphi(c)}+e^{\varphi(d)}}{2}\int_c^d e^{w(z)}dz-\frac{1}{d-c}\int_c^d e^{\varphi(z)}e^{w(z)}dz\right|&\leq\frac{1}{2}\int_0^1\eta(z)\left|e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)\right|d\xi\\
&\leq\frac{1}{2}\left\{\int_0^1(\eta(z))^p d\xi\right\}^{\frac{1}{p}}\left\{\int_0^1\left|e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)\right|^q d\xi\right\}^{\frac{1}{q}}.
\end{aligned}$$

It follows from the convexity of $\left|(e^{\varphi})'\right|^q$ that

$$\begin{aligned}
&\left|e^{\varphi(\xi c+(1-\xi)d)}\varphi'(\xi c+(1-\xi)d)\right|^q\\
&\leq\left[\xi\left|e^{\varphi(c)}\right|^q+(1-\xi)\left|e^{\varphi(d)}\right|^q\right]\left[\xi|\varphi'(c)|^q+(1-\xi)|\varphi'(d)|^q\right]\\
&\leq\left[\xi^2\left|e^{\varphi(c)}\varphi'(c)\right|^q+(1-\xi)^2\left|e^{\varphi(d)}\varphi'(d)\right|^q+\xi(1-\xi)\left\{\left|e^{\varphi(c)}\varphi'(d)\right|^q+\left|e^{\varphi(d)}\varphi'(c)\right|^q\right\}\right]\\
&\leq\left[\xi^2\left|e^{\varphi(c)}\varphi'(c)\right|^q+(1-\xi)^2\left|e^{\varphi(d)}\varphi'(d)\right|^q+\xi(1-\xi)\Upsilon_1(c,d)\right].
\end{aligned}$$

Therefore,

$$\left|\frac{1}{d-c}\frac{e^{\varphi(c)}+e^{\varphi(d)}}{2}\int_c^d e^{w(z)}dz-\frac{1}{d-c}\int_c^d e^{\varphi(z)}e^{w(z)}dz\right|$$

$$\leq \frac{1}{2}\left[\int_0^1 (\eta(z))^p d\xi\right]^{1/p} \left\{\int_0^1 \left[\xi^2 \left|e^{\varphi(c)}\varphi'(c)\right|^q + (1-\xi)^2 \left|e^{\varphi(d)}\varphi'(d)\right|^q + \xi(1-\xi)\Upsilon_1(c,d)\right] d\xi\right\}^{1/q}$$

$$= \frac{1}{2}\left[\int_0^1 (\eta(z))^p d\xi\right]^{1/p} \left[\frac{2\left(\left|e^{\varphi(c)}\varphi'(c)\right|^q + \left|e^{\varphi(d)}\varphi'(d)\right|^q\right) + \Upsilon_1(c,d)}{6}\right],$$

which completes the proof. □

6. Applications

A real-valued function $M : (0,\infty)\times(0,\infty) \mapsto (0,\infty)$ is said to be a bivariate mean if $\min\{c,d\} \leq M(c,d) \leq \max\{c,d\}$ for all $c,d \in (0,\infty)$. Recently, the inequalities for the bivariate means have attracted the attention of many researchers.

Let $c,d > 0$ with $c \neq d$ and $n \in \mathbb{Z}\setminus\{-1,0\}$. Then the classical arithmetic mean $A(c,d)$ and n-th generalized logarithmic mean $L_n(c,d)$ are defined by

$$A(c,d) = \frac{c+d}{2}, \quad L_n(c,d) = \left[\frac{d^{n+1}-c^{n+1}}{(n+1)(d-c)}\right]^{1/n}.$$

In this section, we shall establish some inequalities for the arithmetic and generalized logarithmic means by use of our results obtained in section 5.

Theorem 6.1. *Let $c,d > 0$ with $d > c$ and $n \in \mathbb{Z}$ with $n \geq 2$. Then the inequality*

$$\left|\int_c^d z^n e^{w(z)}dz - A^n(c,d)\int_c^d e^{w(z)}dz\right| \leq (d-c)\left[\frac{1}{6(d-c)^3}\int_{\frac{c+d}{2}}^d e^{w(z)}dz\right]$$
$$\times\left\{8n[(z-c)^3-(d-z)^3]\left[A\left(c^{2n-1},d^{2n-1}\right)\right] + \Upsilon_1(c,d)\left[(d-c)^3 - 2(d-z)^2(3d-c-2z)\right]\right\} \quad (6.1)$$

holds for all $z \in [c,d]$, where $\Upsilon_1(c,d)$ is given in (2.9).

Proof. By taking $\varphi(z) = n\log z$ in Theorem 5.2, we get the desired result. □

Let $w = 1$. Then inequality (6.1) leads to Corollary 6.2 immediately.

Corollary 6.2. *Let $c,d > 0$ with $d > c$ and $n \in \mathbb{Z}$ with $n \geq 2$. Then*

$$\left|L_n^n(c,d) - \frac{1}{d-c}A^n(c,d)\right| \leq \left[\frac{1}{12(d-c)^2}\right]\left\{8n[(z-c)^3-(d-z)^3]\left[A(c^{2n-1},d^{2n-1})\right]\right.$$
$$\left. + \Upsilon_1(c,d)\left\{(d-c)^3 - 2(d-z)^2(3d-c-2z)\right\}\right\}.$$

Theorem 6.3. *Let $p,q > 1$ with $p^{-1}+q^{-1} = 1$, $c,d > 0$ with $d > c$ and $n \in \mathbb{Z}$ with $n \geq 2$. Then the inequality*

$$\left|\int_c^d z^n e^{w(z)}dz - A^n(c,d))\int_c^d e^{w(z)}dz\right|$$

$$\leq (d-c)^2\left[\frac{1}{(d-c)^2}\int_{\frac{c+d}{2}}^{d} e^{pw(z)}\left(z-\frac{(c+d)}{2}\right)dz\right]^{1/p}\left[\left(\frac{n^q 6A\left(c^{(2n-1)q},d^{(2n-1)q}\right)+8n^q d^{(2n-1)q}+6\Upsilon_1(c,d)}{192}\right)^{1/q}\right.$$
$$\left.+\left(\frac{n^q 6A\left(c^{(2n-1)q},d^{(2n-1)q}\right)+8n^q c^{(2n-1)q}+6\Upsilon_1(c,d)}{192}\right)^{1/q}\right] \tag{6.2}$$

holds for all $z \in [c,d]$.

Proof. By taking $\varphi(z) = n\ln z$ in Theorem (5.3), we get the desired result. □

Let $w = 1$. Then (6.2) leads to Corollary 6.4 immediately.

Corollary 6.4. *Let* $p,q > 1$ *with* $p^{-1}+q^{-1} = 1$, $c,d > 0$ *with* $d > c$ *and* $n \in \mathbb{Z}$ *with* $n \geq 2$. *Then*

$$\left|L_n^n(c,d)-\frac{1}{d-c}A^n(c,d)\right| \leq \frac{d-c}{2^{-3p}}\left[\left(\frac{n^q 6A\left(c^{(2n-1)q},d^{(2n-1)q}\right)+8n^q d^{(2n-1)q}+6\Upsilon_1(c,d)}{192}\right)^{1/q}\right.$$
$$\left.+\left(\frac{n^q 6A\left(c^{(2n-1)q},d^{(2n-1)q}\right)+8n^q c^{(2n-1)q}+6\Upsilon_1(c,d)}{192}\right)^{1/q}\right].$$

Theorem 6.5. *Let* $p,q > 1$ *with* $p^{-1}+q^{-1} = 1$, $c,d > 0$ *with* $d > c$ *and* $n \in \mathbb{Z}$ *with* $n \geq 2$. *Then the inequality*

$$\left|\frac{1}{d-c}A(c^n,d^n)\int_c^d e^{w(z)}dz-\frac{1}{d-c}\int_c^d z^n e^{w(z)}dz\right|$$
$$\leq \frac{1}{2}\left(\int_0^1 (\eta(z))^p d\xi\right)^{1/p}\left(\frac{4n^q A\left(c^{(2n-1)q},d^{(2n-1)q}\right)+\Upsilon_1(c,d)}{6}\right) \tag{6.3}$$

holds for $z \in [c,d]$.

Proof. By taking $\varphi(z) = n\ln z$ in Theorem 5.5, we get the desired result. □

Let $w = 1$. Then (6.3) leads to Corollary 6.6 immediately.

Corollary 6.6. *Let* $u,v > 0$ *with* $v > u$ *and* $n \in \mathbb{Z}$ *with* $n \geq 2$. *Then the inequality*

$$\left|\frac{1}{d-c}\left(A(c^n,d^n)\right)-L_n^n(c,d)\right| \leq \frac{d-c}{2e}\left[\frac{4n^q A\left(c^{(2n-1)q},d^{(2n-1)q}\right)+\Upsilon_1(c,d)}{6}\right]$$

holds for all $q > 1$.

7. Conclusions

Using the $\mathcal{K}$-conformable fractional integrals, certain inequalities related to the Hermite-Hadamard inequalities for exponentially convex functions are established. The inequalities are parameterized by the parameters δ, γ and $\mathcal{K}$. These inequalities generalize and extend parts of the results for Riemann-Liouville and Hadamard fractional integrals. Also, we have derived the weighted Hermite-Hadamard inequalities for exponentially convex functions in the classical sense. Some applications of the obtained results to special means are also presented. With these contributions, we hope to motivate the interested researcher to further explore this enchanting field of the fractional integral inequalities and exponentially convexity based on these techniques and the ideas developed in the present paper.

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Conflict of interest

The authors declare that they have no competing interests.

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