Mathematics

## Research article

# Double-framed soft $\boldsymbol{h}$-semisimple hemirings 

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#### Abstract

In dearth of parameterization, various uncertain-ordinary theories like the theory of fuzzy sets and the theory of probability, which failed to address the emergence of modern day sophisticated, complex, and unpredictable problems of various disciplines such as economics and engineering. We aim to provide an appropriate mathematical tool for resolving such complicated problems with the initiation and conceptualization of the notion of double-framed soft sets in hemirings. In the structural theory, $h$-ideals of hemirings play a key role, therefore, new types of $h$-ideals of a hemiring $R$ known as double-framed soft $h$-interior ideals and double-framed soft $h$-ideals are determined. It is shown that every double-framed soft $h$-ideal of $R$ is double-framed soft $h$-interior ideal but the converse is not true which is verified through suitable examples. Further, the conditions under which both these concepts coincide are provided. More precisely, if a hemiring $R$ is $h$-hemiregular, (resp. $h$-intra-hemiregular, $h$-semisimple), then every double-framed soft $h$-interior ideal of $R$ will be double-framed soft $h$-ideal. Several classes of hemirings such as $h$-hemiregular, $h$-intra-hemiregular, $h$-simple and $h$-semisimple are characterized by the notion of double-framed soft $h$-interior ideals.


Keywords: DFS sets; DFS $h$-ideal; DFS $h$-interior ideal; $h$-hemiregular hemiring; $h$-simple hemiring; $h$-semisimple hemiring
Mathematics Subject Classification: 08Axx, 08A72, 16Wxx

## 1. Introduction

In contemporary globalized world, economic and technological advancement inevitably plays a significant role in the development of a country. Unlike advanced developed countries, most of the countries are left behind due to lack of the high-quality research in advanced fields: engineering, computer science and data analysis, decision-making problems, error correcting codes, economics,
forecasting, and robotics. Despite the fact advanced countries are spending a huge budget on these domains, but these fields are facing some complicated problems involving uncertainties. These obscured problems cannot be dealt with through classical methods of problems resolving mechanisms. There are certain types of theories such as the theory of probability, the theory of fuzzy sets, and the theory of rough sets, which could help resolve these modern-day complex problems. Nonetheless, all of these theories have their significance and inherent limitations: incompatibility with the parameterization tools is one of the main problems associated with these theories. In order to overcome these implied challenges, Molodtsov [1] initiated the icebreaking concept of soft set theory. The soft sets theory is a novel mathematical approach for dealing with the uncertainties. This contemporary approach is free from the difficulties pointed out in the other theories of uncertainties, which usually use membership function. From the last decade, the soft sets' conception gained reputation for its parameterization nature, which is free of membership function. Due to its dynamical nature, the soft sets successfully made its place and now extensively used in various applied fields. For example, soft sets are used in decision making problems [2,3], soft derivatives, soft integrals and soft numbers along with their applications in [4]. In international trade, soft sets are used for forecasting the export and import volumes [5]. Maji et al. [6] presented several operations of algebraic structures in terms of soft sets which is further extended by Ali et al. [7,8].

Several fields like computer programming, coding theory, fuzzy automata, optimization, formal languages, graph theory uses semirings [9] for many purposes. Semirings are algebraic structures with two binary operation. Semirings with commutative addition and zero element are known as hemirings. Among the above mentioned fields some of them like theory of automata, formal languages and computer sciences uses these hemirings [10-12]. Further, ideals of hemirings play a key role in structure theory for many purposes. Torre [13] determined $h$-ideals and $k$-ideals in hemirings with several classification of hemirings are discussed in terms of these ideals. The $h$-hemiregularity are investigated by Yin and Li [14]. They also determined $h$-intra hemiregular hemirings and presented various characterization theorems of hemirings in terms of these notions. Droste and Kuich [15] discuss hemirings in automata domain. Moreover, Ma and Zhan [16] characterized hemiregular hemirings by the properties of new type of soft union sets. The concept of cubic $h$-semisimple hemiring is presented by Khan et al. [17].

Recently, Jun and Ahn [18] introduced the notion of double-framed soft sets and defined doubleframed soft subalgebra of a BCK/BCI-algebra. Beside this, Jun et al. [19] also determined doubleframed soft ideals of BCK/BCI-algebra. Khan et al. [20] applied the notion of double-framed soft sets to AG-groupoids and investigated various results. Moreover, double-framed soft sets are further elaborated in LA-semigroups [21]. Further, several researchers applied the notion of doubleframed soft sets in diverse fields of algebra. For instance, Asif et al. [22] discussed ideal theory in ordered AG-groupoid based on double-framed soft sets. Also, Asif and coauthors [23] determined fully prime double-framed soft ordered semigrouops. Iftikhar and Mahmood [24] investigated several results on lattice ordered double-framed soft semirings, Bordar et al. [25] applied the said notion to hyper BCK-algebra. In addition, Jayaraman et al. [26] developed double-framed soft lattices, distributive double-framed soft lattice and double-framed soft chain. Khan and Mahmood [27] initiated the notion of double-framed T-soft fuzzy set and applied the concept in BCK/BCI-algebra. Park [28] introduced double-framed soft deductive system in subtraction algebra while Hussain [29] discussed the application aspect of doubel-framed soft ideal in gamma near-rings. Also, Hussain and
coauthors [30] developed double-framed fuzzy quotient lattices. For further study on double-famed soft sets, the readers refer to [31-34].

In this paper, we define double-framed soft $h$-interior ideals and double-framed soft $h$-ideals of a hemiring $R$. Further, these notions are elaborated through suitable examples. It is shown that every double-framed soft $h$-ideal of a hemiring $R$ is double-framed soft $h$-interior ideal of $R$ but the converse not hold in general. The converse statement will hold under some conditions. More precisely, if a hemiring $R$ is $h$-hemiregular, (resp. $h$-intra-hemiregular, $h$-semisimple), then every double-framed soft $h$-interior ideal of $R$ will be double-framed soft $h$-ideal of $R$. Finally, ordinary $h$-interior ideals are linked with double-framed soft $h$-interior ideals and various classes like $h$-hemiregular, $h$-intrahemiregular, $h$-simple and $h$-semisimple hemirings are characterized by the properties of these newly developed double-framed soft $h$-interior ideals of $R$.

## 2. Double-framed soft sets (Basic operations)

Definition 2.1 [18] A double-framed soft set of $A$ over $U$ is a pair $\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$, such that $f_{A}^{+}$ and $f_{A}^{-}$both are mappings from $A$ to $P(U)$ where $P(U)$ is the set of all subsets of $U$. It is denoted by $D F S$-set of $A$.

The set of all $D F S$-set of $A$ over $U$ is denoted by $D F S(U)$.
$\gamma$-inclusive set: If $\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is a $D F S$-set of $A$ and $\gamma$ is a subset of $U$, then the $\gamma$-inclusive set is denoted by $i_{A}\left(f_{A}^{+} ; \gamma\right)$ and is defined as

$$
i_{A}\left(f_{A}^{+} ; \gamma\right)=\left\{x \in A \mid f_{A}^{+}(x) \supseteq \gamma\right\}
$$

$\delta$-exclusive set: If $\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is a $D F S$-set of $A$ and $\delta$ is a subset of $U$, then the $\delta$-exclusive set is denoted by $e_{A}\left(f_{A}^{-} ; \delta\right)$ and is defined as

$$
e_{A}\left(f_{A}^{-} ; \delta\right)=\left\{x \in A \mid f_{A}^{-}(x) \subseteq \delta\right\} .
$$

Note that a double-framed soft including set is of the form

$$
D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{(\gamma, \delta)}=\left\{x \in A \mid f_{A}^{+}(x) \supseteq \gamma, f_{A}^{-}(x) \subseteq \delta\right\}
$$

clearly, $D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{(\gamma, \delta)}=i_{A}\left(f_{A}^{+} ; \gamma\right) \cap e_{A}\left(f_{A}^{-} ; \delta\right)$.
In the following, the double-framed soft sum briefly $h$-sum and soft product ( $h$-product) for two double-framed soft sets of hemirings are introduced.

Definition 2.2 Let $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ and $g_{A}=\left\langle\left(g_{A}^{+}, g_{A}^{-}\right) ; A\right\rangle$ be two double-framed soft sets of a hemiring $R$ over $U$. Then the $h$-sum $f_{A} \widetilde{\oplus} g_{A}$ is a double-framed soft set of $R$ over $U$ denoted by $f_{A} \widetilde{\oplus} g_{A}=\left\langle\left(f_{A}^{+} \oplus g_{A}^{+}, f_{A}^{-} \boxplus g_{A}^{-}\right) ; A\right\rangle$. Where $f_{A}^{+} \oplus g_{A}^{+}$and $f_{A}^{-} \boxplus g_{A}^{-}$are called soft mappings from $R$ to $P(U)$ which are defined as follows

$$
f_{A}^{+} \oplus g_{A}^{+}: x \longmapsto\left\{\left\{\begin{array}{c}
\widetilde{\bigcup} \quad\left\{f_{A}^{+}\left(a_{1}\right) \cap f_{A}^{+}\left(a_{2}\right) \cap g_{A}^{+}\left(b_{1}\right) \cap g_{A}^{+}\left(b_{2}\right)\right\} \\
\\
\\
\\
\\
x+a_{1}+b_{1}+z=a_{2}+b_{2}+z \\
\text { if } x \text { does not expressed as } x+a_{1}+b_{1}+z=a_{2}+b_{2}+z
\end{array}\right.\right.
$$

$$
f_{A}^{-} \boxplus g_{A}^{-}: x \longmapsto\left\{\begin{array}{c}
\widetilde{\cup} \quad\left\{f_{A}^{-}\left(a_{1}\right) \cup f_{A}^{-}\left(a_{2}\right) \cup g_{A}^{-}\left(b_{1}\right) \cup g_{A}^{-}\left(b_{2}\right)\right\} \\
\\
U+a_{1}+b_{1}+z=a_{2}+b_{2}+z \\
\text { if } x \text { is expressed as } x+a_{1}+b_{1}+z=a_{2}+b_{2}+z \\
\text { if } x \text { does not expressed as } x+a_{1}+b_{1}+z=a_{2}+b_{2}+z .
\end{array}\right.
$$

Definition 2.3 Let $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ and $g_{A}=\left\langle\left(g_{A}^{+}, g_{A}^{-}\right) ; A\right\rangle$ be two double-framed soft sets of a hemiring $R$ over $U$. If $f_{A}^{+} \otimes g_{A}^{+}$and $f_{A}^{-} \otimes g_{A}^{-}$are called soft mappings from $R$ to $P(U)$ defined as follows

$$
\begin{aligned}
& f_{A}^{+} \otimes g_{A}^{+}: x \longmapsto\left\{\begin{array}{c}
\widetilde{\bigcup} \quad\left\{f_{A}^{+}\left(a_{1}\right) \cap f_{A}^{+}\left(a_{2}\right) \cap g_{A}^{+}\left(b_{1}\right) \cap g_{A}^{+}\left(b_{2}\right)\right\} \\
x+a_{1} b_{1}+z=a_{2} b_{2}+z \\
\text { if } x \text { is expressed as } x+a_{1} b_{1}+z=a_{2} b_{2}+z \\
\emptyset \quad \text { if } x \text { does not expressed as } x+a_{1} b_{1}+z=a_{2} b_{2}+z .
\end{array}\right. \\
& f_{A}^{-} \boxtimes g_{A}^{-}: x \longmapsto\left\{\begin{array}{c}
\widetilde{\bigcup} \quad\left\{f_{A}^{-}\left(a_{1}\right) \cup f_{A}^{-}\left(a_{2}\right) \cup g_{A}^{-}\left(b_{1}\right) \cup g_{A}^{-}\left(b_{2}\right)\right\} \\
x+a_{1} b_{1}+z=a_{2} b_{2}+z \\
\text { if } x \text { is expressed as } x+a_{1} b_{1}+z=a_{2} b_{2}+z \\
U \\
\text { if } x \text { does not expressed as } x+a_{1} b_{1}+z=a_{2} b_{2}+z .
\end{array}\right.
\end{aligned}
$$

Then the $h$-product of $f_{A}$ and $g_{A}$ is denoted by $f_{A} \widetilde{\triangleleft} g_{A}$ which is defined as $\left(f_{A} \widetilde{\nabla} g_{A}\right)(x)=$ $\left\{\left(\left(f_{A}^{+} \otimes g_{A}^{+}\right)(x),\left(f_{A}^{-} \boxtimes g_{A}^{-}\right)(x)\right): x \in R\right\}$. Note that $f_{A} \widetilde{\nabla} g_{A}$ will also be double-framed soft sets of a hemiring $R$.
Definition 2.4 Let $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ and $g_{B}=\left\langle\left(g_{B}^{+}, g_{B}^{-}\right) ; B\right\rangle$ be two double-framed soft sets over $U$. Then, $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is called a double-framed soft subset of $g_{B}=\left\langle\left(g_{B}^{+}, g_{B}^{-}\right) ; B\right\rangle$ denoted by $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle \widetilde{\sqsubseteq} g_{B}=\left\langle\left(g_{B}^{+}, g_{B}^{-}\right) ; B\right\rangle$ if $A$ is the subset of $B, f_{A}^{+}(x) \subseteq g_{B}^{+}(x)$ and $f_{A}^{-}(x) \supseteq g_{B}^{-}(x)$ for all $x \in A$. Also two double-framed soft sets $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ and $g_{B}=\left\langle\left(g_{B}^{+}, g_{B}^{-}\right) ; B\right\rangle$ are equal denoted by $\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle=\left\langle\left(g_{B}^{+}, g_{B}^{-}\right) ; B\right\rangle$, if $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle \cong g_{B}=\left\langle\left(g_{B}^{+}, g_{B}^{-}\right) ; B\right\rangle$ and $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle \cong g_{B}=$ $\left\langle\left(g_{B}^{+}, g_{B}^{-}\right) ; B\right\rangle$ both hold.
Definition 2.5 Let $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ and $g_{A}=\left\langle\left(g_{A}^{+}, g_{A}^{-}\right) ; A\right\rangle$ be two double-framed soft sets of a hemiring $R$ over $U$. Then the DFS int-uni set of $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ and $g_{A}=\left\langle\left(g_{A}^{+}, g_{A}^{-}\right) ; A\right\rangle$ is defined as a DFS set $\left\langle\left(f_{A}^{+} \widetilde{\cap} g_{A}^{+}, f_{A}^{-} \widetilde{\cup} g_{A}^{-}\right) ; A\right\rangle$ where $f_{A}^{+} \widetilde{\cap} g_{A}^{+}$and $f_{A}^{-} \widetilde{\cup} g_{A}^{-}$are mappings from $A$ to $P(U)$ such that $\left(f_{A}^{+} \widetilde{\cap} g_{A}^{+}\right)(x)=f_{A}^{+}(x) \cap g_{A}^{+}(x)$ and $\left(f_{A}^{-} \widetilde{\cup} g_{A}^{-}\right)(x)=f_{A}^{-}(x) \cup g_{A}^{-}(x)$. It is denoted by $\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle \bar{\Pi}\left\langle\left(g_{A}^{+}, g_{A}^{-}\right) ; A\right\rangle=\left\langle\left(f_{A}^{+} \bar{\cap} g_{A}^{+}, f_{A}^{-} \widetilde{\cup} g_{A}^{-}\right) ; A\right\rangle$.
Lemma 2.6 [34] Suppose that $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle, g_{B}=\left\langle\left(g_{B}^{+}, g_{B}^{-}\right) ; B\right\rangle$ and $h_{C}=\left\langle\left(h_{C}^{+}, h_{C}^{-}\right) ; C\right\rangle$ are double-framed soft sets in a hemiring $R$, then the following hold.
(1) $f_{A} \widetilde{\oplus}\left(g_{B} \widetilde{\Pi} h_{C}\right)=\left(f_{A} \widetilde{\oplus} g_{B}\right) \widetilde{\Pi}\left(f_{A} \widetilde{\oplus} h_{C}\right)$.
(2) $\quad f_{A} \widetilde{\diamond}\left(g_{B} \widetilde{\square} h_{C}\right)=\left(f_{A} \widetilde{\diamond} g_{B}\right) \widetilde{\Pi}\left(f_{A} \widetilde{\nabla} h_{C}\right)$.

Definition 2.7 Suppose that $A$ is a non-empty subset of a hemiring $R$, then the characteristic double-framed soft mapping of $A$ is a double-framed soft set denoted by $\mathbb{C}_{A}=\left\langle\left(\mathbb{C}_{A}^{+}, \mathbb{C}_{A}^{-}\right) ; A\right\rangle$ where $\mathbb{C}_{A}^{+}, \mathbb{C}_{A}^{-}$are soft mappings from $R$ to $P(U)$ and are defined as follows

$$
\mathbb{C}_{A}^{+}: x \longmapsto \begin{cases}U & \text { if } x \in A, \\ \emptyset & \text { if } x \notin A .\end{cases}
$$

and

$$
\mathbb{C}_{A}^{-}: x \longmapsto \begin{cases}\emptyset & \text { if } x \in A, \\ U & \text { if } x \notin A .\end{cases}
$$

It is important to note that the identity double-framed soft mapping is denoted by $\mathbb{C}_{R}=\left\langle\left(\mathbb{C}_{R}^{+}, \mathbb{C}_{R}^{-}\right) ; R\right\rangle$ where $\mathbb{C}_{R}^{+}: x \longmapsto U$ and $\mathbb{C}_{R}^{-}: x \longmapsto \emptyset$ for all $x \in R$.
Theorem 2.8 [34] Suppose that $A$ and $B$ are two non-empty subsets of a hemiring $R$, then the following axioms for characteristic double-framed soft mapping are holds:
(1) $\quad A \subseteq B$ if and only if $\mathbb{C}_{A} \widetilde{\subseteq} \mathbb{C}_{B}$, i.e., $A \subseteq B \Longleftrightarrow \mathbb{C}_{A}^{+}(x) \subseteq \mathbb{C}_{B}^{+}(x)$ and $\mathbb{C}_{A}^{-}(x) \supseteq \mathbb{C}_{B}^{-}(x)$ for all $x \in A$.

$$
\begin{equation*}
\mathbb{C}_{A} \widetilde{\sqcap} \mathbb{C}_{B}=\mathbb{C}_{A \cap B} \text {, i.e., }\left\langle\mathbb{C}_{A}^{+} \widetilde{\cap} \mathbb{C}_{B}^{+}, \mathbb{C}_{A}^{-} \widetilde{\cup} \mathbb{C}_{B}^{-}\right\rangle=\left\langle\mathbb{C}_{A \cap B}^{+}, \mathbb{C}_{A \cap B}^{-}\right\rangle \tag{2}
\end{equation*}
$$

$$
\begin{equation*}
\mathbb{C}_{A} \widetilde{\oplus \mathbb{C}_{B}}=\mathbb{C}_{\overline{A+B}}, \text { i.e., }\left\langle\mathbb{C}_{A}^{+} \oplus \mathbb{C}_{B}^{+}, \mathbb{C}_{A}^{-} \boxplus \mathbb{C}_{B}^{-}\right\rangle=\left\langle\mathbb{C}_{\overline{A+B}}^{+}, \mathbb{C}_{A+B}^{-}\right\rangle . \tag{3}
\end{equation*}
$$

$$
\begin{equation*}
\mathbb{C}_{A} \widetilde{\triangleright} \mathbb{C}_{B}=\mathbb{C}_{\overline{A B}}, \text { i.e., }\left\langle\mathbb{C}_{A}^{+} \otimes \mathbb{C}_{B}^{+}, \mathbb{C}_{A}^{-} \otimes \mathbb{C}_{B}^{-}\right\rangle=\left\langle\mathbb{C}_{\overline{A B}}^{+}, \mathbb{C}_{\overline{A B}}^{-}\right\rangle . \tag{4}
\end{equation*}
$$

## 3. Double-framed soft $h$-interior ideals

Since from the last decade, soft sets gain reputation due to the diverse applications in multi disciplines. The $h$-ideals of a hemirings play a central role in structural theory. Therefore, the concept of double-framed soft $h$-interior ideals of $R$ is introduced and hemirings are characterized by the properties of these double-framed soft $h$-interior ideals.
Definition 3.1 A DFS-set $\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ of a hemiring $R$ is said to be a double-framed soft left (resp. right) $h$-ideal of $R$ if for all $a, b \in R$, the following conditions hold.
(1a) $f_{A}^{+}(a+b) \supseteq f_{A}^{+}(a) \cap f_{A}^{+}(b)$
(1b) $\quad f_{A}^{-}(a+b) \subseteq f_{A}^{-}(a) \cup f_{A}^{-}(b)$
(2a) $\quad f_{A}^{+}(a b) \supseteq f_{A}^{+}(b)\left(\right.$ resp. $\left.f_{A}^{+}(a b) \supseteq f_{A}^{+}(a)\right)$
(2b) $\quad f_{A}^{-}(a b) \subseteq f_{A}^{-}(b)\left(\right.$ resp. $\left.f_{A}^{-}(a b) \subseteq f_{A}^{-}(a)\right)$

$$
\begin{equation*}
(\forall a, b, x, z \in R)\left(x+a+z=b+z \longrightarrow f_{A}^{+}(x) \supseteq f_{A}^{+}(a) \cap f_{A}^{+}(b)\right) \tag{3a}
\end{equation*}
$$

(3b) $\quad(\forall a, b, x, z \in R)\left(x+a+z=b+z \longrightarrow f_{A}^{-}(x) \subseteq f_{A}^{-}(a) \cup f_{A}^{-}(b)\right)$.
Definition 3.2 A DFS-set $\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ of a hemiring $R$ is said to be a double-framed soft interior ideal of $R$ if for all $a, b, c \in R$, the following conditions hold.
(4a) $\quad f_{A}^{+}(a+b) \supseteq f_{A}^{+}(a) \cap f_{A}^{+}(b)$
(4b) $\quad f_{A}^{-}(a+b) \subseteq f_{A}^{-}(a) \cup f_{A}^{-}(b)$
(5a) $\quad f_{A}^{+}(a b) \supseteq f_{A}^{+}(a) \cap f_{A}^{+}(b)$
(5b) $\quad f_{A}^{-}(a b) \subseteq f_{A}^{-}(a) \cup f_{A}^{-}(b)$
(6a) $f_{A}^{+}(a b c) \supseteq f_{A}^{+}(b)$
(6b) $\quad f_{A}^{-}(a b c) \subseteq f_{A}^{-}(b)$
Example 3.3 Suppose that $R=\{0, x, y, z\}$ is a set with addition and multiplication defined in the following tables:

| + | 0 | $x$ | $y$ | $z$ |  | $\cdot$ | 0 | $x$ | $y$ | $z$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | $x$ | $y$ | $z$ |  | 0 | 0 | 0 | 0 | 0 |
| $x$ | $x$ | $x$ | $y$ | $z$ |  | $x$ | 0 | 0 | 0 | 0 |
| $y$ | $y$ | $y$ | $y$ | $z$ |  | $y$ | 0 | 0 | 0 | $x$ |
| $z$ | $z$ | $z$ | $z$ | $z$ |  | $z$ | 0 | 0 | $x$ | $y$ |

Define a double-framed soft $\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ in $R$ over $U=\mathbb{Z}^{-}$as follows:

| $R$ | 0 | $x$ | $y$ | $z$ |
| :--- | :--- | :--- | :--- | :--- |
| $f_{A}^{+}(a)$ | $\{-1,-2, \ldots,-10\}$ | $\{-1,-3,-5,-7,-9\}$ | $\{-3,-5,-7\}$ | $\{-3,-5\}$ |
| $f_{A}^{-}(a)$ | $\{-6\}$ | $\{-2,-4,-6,-10\}$ | $\{-2,-6\}$ | $\{-2,-4,-6,-8,-10\}$ |

where $a \in R$, using Definition 3.2, $\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is a double-framed soft h-interior ideal of $R$ over $\mathbb{Z}^{-}$.
Definition 3.4 A double-framed soft interior ideal of $R$ is called a double-framed soft $h$-interior ideal of $R$ if the following conditions hold.
(7a)

$$
\begin{array}{ll}
\text { (7a) } & (\forall a, b, x, z \in R)\left(x+a+z=b+z \longrightarrow f_{A}^{+}(x) \supseteq f_{A}^{+}(a) \cap f_{A}^{+}(b)\right) \\
\text { (7b) } & (\forall a, b, x, z \in R)\left(x+a+z=b+z \longrightarrow f_{A}^{-}(x) \subseteq f_{A}^{-}(a) \cup f_{A}^{-}(b)\right) .
\end{array}
$$

Note that a Double-framed soft left $h$-ideal $\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ (resp. DFS $h$-interior ideal) of a semiring $R$ with zero element satisfies the inequalities $f_{A}^{+}(0) \supseteq f_{A}^{+}(a), f_{A}^{-}(0) \subseteq f_{A}^{-}(a)$ for all $a \in R$.
Example 3.5 Let $\mathbb{N}_{0}$ be the set of all non-negative integers, then $\mathbb{N}_{0}$ is a hemiring with usual addition and multiplication. Define a double-framed soft set $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ of $\mathbb{N}_{0}$ as follows

| $\mathbb{N}_{0}$ | $x \in\langle 4\rangle$ | $x \in\langle 2\rangle-\langle 4\rangle$ | otherwise |
| :--- | :--- | :--- | :--- |
| $f_{A}^{+}(x)$ | $\{-1,-2, \ldots,-10\}$ | $\{-2,-4,-6,-8\}$ | $\{-2,-4\}$ |
| $f_{A}^{-}(x)$ | $\{-4\}$ | $\{-4,-8\}$ | $\{-2,-4,-8\}$ |

Then $f_{A}$ is a double-framed soft $h$-interior ideal of $\mathbb{N}_{0}$.
Lemma 3.6 Every double-framed soft $h$-ideal of a hemiring $R$ is a double-framed soft $h$-interior ideal of $R$.

Proof. Suppose $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is a double-framed soft $h$-ideal $R$, If $a, b, c \in R$, then $f_{A}^{+}(a b) \supseteq$ $f_{A}^{+}(b) \supseteq f_{A}^{+}(a) \cap f_{A}^{+}(b)\left(f_{A}\right.$ being DFS left $h$-ideal $)$. Also, $f_{A}^{-}(a b) \subseteq f_{A}^{-}(b) \subseteq f_{A}^{-}(a) \cup f_{A}^{-}(b)$. Similarly, $f_{A}^{+}(a b c)=f_{A}^{+}\left(a(b c) \supseteq f_{A}^{+}(b c) \supseteq f_{A}^{+}(b)\right.$ and $f_{A}^{-}(a b c)=f_{A}^{-}\left(a(b c) \subseteq f_{A}^{-}(b c) \subseteq f_{A}^{-}(b)\right.$. All other conditions of DFS $h$-ideal and DFS $h$-interior ideal of $R$ are same. Hence, $f_{A}$ is DFS $h$-interior ideal of $R$.

The following example shows that the converse of Lemma 3.6 is not true in general.
Example 3.7 Consider $R=\{0, x, y, z\}$ with addition and multiplication defined as in Example 3.3,
for all $a \in R$, define a double-framed soft $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ in $R$ as follows:

| $R$ | 0 | $x$ | $y$ | $z$ |
| :--- | :--- | :--- | :--- | :--- |
| $f_{A}^{+}(a)$ | $\{-1,-2, \ldots,-10\}$ | $\{-1,-3,-5,-7\}$ | $\{-3,-5,-7\}$ | $\{-3,-5\}$ |
| $f_{A}^{-}(a)$ | $\{-6\}$ | $\{-2,-4,-6\}$ | $\{-2,-6\}$ | $\{-2,-4,-6,-10\}$ |

then $\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is a double-framed soft $h$-interior ideal of $R$ over $\mathbb{Z}^{-}$but not a double-framed soft $h$-ideal of $R$ because $f_{A}^{-}(z y)=f_{A}^{-}(x)=\{-2,-4,-6\} \nsubseteq f_{A}^{-}(y)=\{-2,-6\}$. Hence $f_{A}$ is not a doubleframed soft left $h$-ideal of $R$.

In the following propositions, double-framed soft including sets and characteristic double-framed soft functions are used to connect ordinary $h$-interior ideals with DFS $h$-interior ideals of hemiring $R$.
Proposition 3.8 For a hemiring $R$ with $A \subseteq R$, the following conditions are equivalent:

1. $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is double-framed soft $h$-interior ideal of $R$.
2. A non-empty double-framed soft including set $D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{(\gamma, \delta)}$ is $h$-interior ideal of $R$.

Proof. (1) $\Longrightarrow(2)$. Assume that $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is double-framed soft $h$-interior ideal of $R$, Consider $a, b \in D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{(\gamma, \delta)}$, then $f_{A}^{+}(a) \supseteq \gamma, f_{A}^{+}(b) \supseteq \gamma$ and $f_{A}^{-}(a) \subseteq \delta, f_{A}^{-}(b) \subseteq \delta$. Since $\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is DFS $h$-interior ideal of $R$, so $f_{A}^{+}(a+b) \supseteq f_{A}^{+}(a) \cap f_{A}^{+}(b) \supseteq \gamma \cap \gamma=\gamma$ implies that $f_{A}^{+}(a+b) \supseteq \gamma$ and $f_{A}^{-}(a+b) \subseteq f_{A}^{-}(a) \cup f_{A}^{-}(b) \subseteq \delta \cup \delta=\delta$, hence $f_{A}^{-}(a+b) \subseteq \delta$. Thus $a+b \in D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{(\gamma, \delta)}$. Similarly, for $a, b \in D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{(\gamma, \delta)}, f_{A}^{+}(a b) \supseteq f_{A}^{+}(a) \cap f_{A}^{+}(b) \supseteq \gamma \cap \gamma=\gamma, f_{A}^{-}(a b) \subseteq f_{A}^{-}(a) \cup$ $f_{A}^{-}(b) \subseteq \delta \cup \delta=\delta$ leads to $a b \in D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{(\gamma, \delta)}$. Also, for $x \in D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{(\gamma, \delta)}$ and $a, b \in R$, implies that $f_{A}^{+}(x) \supseteq \gamma$ and $f_{A}^{-}(x) \subseteq \delta$ then $f_{A}^{+}(a x b) \supseteq f_{A}^{+}(x) \supseteq \gamma, f_{A}^{-}(a x b) \subseteq f_{A}^{-}(x) \subseteq \delta$ leads to axb $\in$ $D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{(\gamma, \delta)}$. Finally, assume $a, b \in D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{(\gamma, \delta)}, x, z \in R$ with the expression $x+a+z=b+z$, then $f_{A}^{+}(x) \supseteq f_{A}^{+}(a) \cap f_{A}^{+}(b) \supseteq \gamma$ and $f_{A}^{-}(x) \subseteq f_{A}^{-}(a) \cup f_{A}^{-}(b) \subseteq \delta$ means that $f_{A}^{+}(x) \supseteq \gamma$ and $f_{A}^{-}(x) \subseteq \delta$, hence $x \in D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{(\gamma, \delta)}$. Therefore, $D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{(\gamma, \delta)}$ is an $h$-interior ideal of $R$.
(2) $\Longrightarrow(1)$. Suppose $\emptyset \neq D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{(\gamma, \delta)} \in R$ be an $h$-interior ideal, if there exist $a, b \in R$ such that $f_{A}^{+}(a+b) \subset f_{A}^{+}(a) \cap f_{A}^{+}(b)=\gamma_{1}$ and $f_{A}^{-}(a+b) \supset f_{A}^{-}(a) \cup f_{A}^{-}(b)=\delta_{1}$ for some $\gamma_{1}, \delta_{1}$ are subsets of $U$. Then $a, b \in D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{\left(\gamma_{1}, \delta_{1}\right)}$ but $a+b \notin D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{\left(\gamma_{1}, \delta_{1}\right)}$ which is contradiction to the fact that $D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{\left(\gamma_{1}, \delta_{1}\right)}$ is an $h$-interior ideal. Hence $f_{A}^{+}(a+b) \supseteq f_{A}^{+}(a) \cap f_{A}^{+}(b)$ and $f_{A}^{-}(a+b) \subseteq$ $f_{A}^{-}(a) \cup f_{A}^{-}(b)$ hold for all $a, b \in R$. Let $a, b \in R$, assume that

$$
f_{A}^{+}(a b) \subset f_{A}^{+}(a) \cap f_{A}^{+}(b)=\gamma_{2} \subseteq\left\{\begin{array}{l}
f_{A}^{+}(a), \\
f_{A}^{+}(b)
\end{array}\right.
$$

and

$$
, f_{A}^{-}(a b) \supset f_{A}^{-}(a) \cup f_{A}^{-}(b)=\delta_{2} \supseteq\left\{\begin{array}{l}
f_{A}^{+}(a), \\
f_{A}^{+}(b)
\end{array}\right.
$$

Then $a, b \in D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{\left(\gamma_{2}, \delta_{2}\right)}$ but $a b \notin D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{\left(\gamma_{2}, \delta_{2}\right)}$ which is contradiction, thus $f_{A}^{+}(a b) \supseteq f_{A}^{+}(a) \cap$ $f_{A}^{+}(b)$ and $f_{A}^{-}(a b) \subseteq f_{A}^{-}(a) \cup f_{A}^{-}(b)$ is true for all $a, b \in R$. Further, Let $a, b, c \in R$, be such that $f_{A}^{+}(a b c) \subset f_{A}^{+}(b)=\gamma$ and $f_{A}^{-}(a b c) \supset f_{A}^{-}(b)=\delta$. Then $b \in D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{(\gamma, \delta)}$ but $a b c \notin D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{(\gamma, \delta)}$
which is contradiction, thus $f_{A}^{+}(a b c) \supseteq f_{A}^{+}(b)$ and $f_{A}^{-}(a b c) \subseteq f_{A}^{-}(b)$ is true for all $a, b, c \in R$. Lastly, if there exist $x, a, b, z \in R$ with the expression $x+a+z=b+z$ such that

$$
f_{A}^{+}(x) \subset f_{A}^{+}(a) \cap f_{A}^{+}(b)=\gamma_{3} \subseteq\left\{\begin{array}{l}
f_{A}^{+}(a), \\
f_{A}^{+}(b)
\end{array}\right.
$$

and

$$
f_{A}^{-}(x) \supset f_{A}^{-}(a) \cup f_{A}^{-}(b)=\delta_{3} \supseteq\left\{\begin{array}{l}
f_{A}^{+}(a), \\
f_{A}^{+}(b) .
\end{array}\right.
$$

Then, $a, b \in D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{\left(\gamma_{3}, \delta_{3}\right)}$ but $x \notin D F_{A}\left(f_{A}^{+}, f_{A}^{-}\right)_{\left(\gamma_{3}, \delta_{3}\right)}$ leads to contradiction again. Thus, $f_{A}^{+}(x) \supseteq$ $f_{A}^{+}(a) \cap f_{A}^{+}(b)$ and $f_{A}^{-}(x) \subseteq f_{A}^{-}(a) \cup f_{A}^{-}(b)$ hold for all $x, a, b, z \in R$. Consequently, $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is double-framed soft $h$-interior ideal of $R$.

Proposition 3.9 For a hemiring $R$ with $A \subseteq R$, the following conditions are equivalent:

1. $A$ is left (resp. right, interior) $h$-ideal of $R$.
2. $\quad \mathbb{C}_{A}=\left\langle\left(\mathbb{C}_{A}^{+}, \mathbb{C}_{A}^{-}\right) ; A\right\rangle$ is DFS left (resp. right, interior) $h$-ideal of $R$.

Proof. Follows from Proposition 3.8.
Theorem 3.10 $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is a double-framed soft $h$-interior ideal of $R$ if and only if:
(1) $f_{A} \widetilde{\oplus} f_{A} \widetilde{\sqsubseteq} f_{A}$,
(2) $f_{A} \widetilde{\diamond} f_{A} \widetilde{\square} f_{A}$,
(3) $\mathbb{C}_{R} 厄 f_{A} \widetilde{\triangleright} \mathbb{C}_{R} \widetilde{\square} f_{A}$,
(4) $(\forall a, b, x, z \in R)\left(x+a+z=b+z \longrightarrow f_{A}^{+}(x) \supseteq f_{A}^{+}(a) \cap f_{A}^{+}(b)\right.$ and

$$
\left.f_{A}^{-}(x) \subseteq f_{A}^{-}(a) \cup f_{A}^{-}(b)\right)
$$

Proof. Let $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ be a double-framed soft $h$-interior ideal of $R$. Then Condition (4) directly follows from definition. Now for Condition (1) let $x \in R$ be such that it can not be expressed in the form $x+\left(a_{1}+b_{1}\right)+z=\left(a_{2}+b_{2}\right)+z$, then $\left(f_{A}^{+} \oplus f_{A}^{+}\right)(x)=\emptyset \subseteq f_{A}^{+}(x)$ and $\left(f_{A}^{-} \boxplus f_{A}^{-}\right)(x)=U \supseteq f_{A}^{-}(x)$. So $f_{A} \widetilde{\oplus} f_{A} \widetilde{\sqsubseteq} f$. Now if $x$ can be expressed in the form $x+\left(a_{1}+b_{1}\right)+z=\left(a_{2}+b_{2}\right)+z$, then

$$
\begin{aligned}
\left(f_{A}^{+} \oplus f_{A}^{+}\right)(x) & =\widetilde{\bigcup}_{x+\left(a_{1}+b_{1}\right)+z=\left(a_{2}+b_{2}\right)+z}\left\{f_{A}^{+}\left(a_{1}\right) \cap f_{A}^{+}\left(a_{2}\right) \cap f_{A}^{+}\left(b_{1}\right) \cap f_{A}^{+}\left(b_{2}\right)\right\} \\
& \subseteq \prod_{x+\left(a_{1}+b_{1}\right)+z=\left(a_{2}+b_{2}\right)+z}\left\{f_{A}^{+}\left(a_{1}+a_{2}\right) \cap f_{A}^{+}\left(b_{1}+b_{2}\right)\right\} \text { by }(4 a) \\
& \subseteq \prod_{x+\left(a_{1}+b_{1}\right)+z=\left(a_{2}+b_{2}\right)+z} f_{A}^{+}(x) \text { by }(7 a) \\
& =f_{A}^{+}(x),
\end{aligned}
$$

also,

$$
\begin{aligned}
\left(f_{A}^{-} \boxplus f_{A}^{-}\right)(x) & =\widetilde{\bigcup}_{x+\left(a_{1}+b_{1}\right)+z=\left(a_{2}+b_{2}\right)+z}\left\{f_{A}^{-}\left(a_{1}\right) \cup f_{A}^{-}\left(a_{2}\right) \cup f_{A}^{-}\left(b_{1}\right) \cup f_{A}^{-}\left(b_{2}\right)\right\} \\
& \supseteq \varlimsup_{x+\left(a_{1}+b_{1}\right)+z=\left(a_{2}+b_{2}\right)+z}\left\{f_{A}^{-}\left(a_{1}+a_{2}\right) \cup f_{A}^{-}\left(b_{1}+b_{2}\right)\right\} \text { by }(4 b)
\end{aligned}
$$

$$
\begin{aligned}
& \supseteq \widetilde{\bigcup}_{x+\left(a_{1}+b_{1}\right)+z=\left(a_{2}+b_{2}\right)+z} f_{A}^{-}(x) \text { by }(7 b) \\
& =f_{A}^{-}(x) .
\end{aligned}
$$

Hence, $f_{A} \widetilde{\oplus} f_{A} \widetilde{\sqsubseteq} f_{A}$.
Now, for Condition (2), if $x \in R$ is not expressed as in the form $x+\left(a_{1} b_{1}\right)+z=\left(a_{2} b_{2}\right)+z$, then $\left(f_{A}^{+} \otimes f_{A}^{+}\right)(x)=\emptyset \subseteq f_{A}^{+}(x)$ and $\left(f_{A}^{-} \boxtimes f_{A}^{-}\right)(x)=U \supseteq f_{A}^{-}(x)$. So $f_{A} \widetilde{\diamond} f_{A} \widetilde{\sqsubseteq} f_{A}$. If $x$ is expressed as in the form $x+\left(a_{1} b_{1}\right)+z=\left(a_{2} b_{2}\right)+z$, then

$$
\begin{aligned}
\left(f_{A}^{+} \otimes f_{A}^{+}\right)(x) & =\widetilde{\bigcup}_{x+\left(a_{1} b_{1}\right)+z=\left(a_{2} b_{2}\right)+z}\left\{f_{A}^{+}\left(a_{1}\right) \cap f_{A}^{+}\left(a_{2}\right) \cap f_{A}^{+}\left(b_{1}\right) \cap f_{A}^{+}\left(b_{2}\right)\right\} \\
& \subseteq \prod_{x+\left(a_{1} b_{1}\right)+z=\left(a_{2} b_{2}\right)+z}\left\{f_{A}^{+}\left(a_{1} a_{2}\right) \cap f_{A}^{+}\left(b_{1} b_{2}\right)\right\} \text { by }(5 a) \\
& \subseteq \prod_{x+\left(a_{1} b_{1}\right)+z=\left(a_{2} b_{2}\right)+z} f_{A}^{+}(x) \\
& =f_{A}^{+}(x),
\end{aligned}
$$

also,

$$
\begin{aligned}
\left(f_{A}^{-} \boxtimes f_{A}^{-}\right)(x) & =\widetilde{\bigcup}_{x+\left(a_{1} b_{1}\right)+z=\left(a_{2} b_{2}\right)+z}\left\{f_{A}^{-}\left(a_{1}\right) \cup f_{A}^{-}\left(a_{2}\right) \cup f_{A}^{-}\left(b_{1}\right) \cup f_{A}^{-}\left(b_{2}\right)\right\} \\
& \supseteq \varlimsup_{x+\left(a_{1} b_{1}\right)+z=\left(a_{2} b_{2}\right)+z}\left\{f_{A}^{-}\left(a_{1} a_{2}\right) \cup f_{A}^{-}\left(b_{1} b_{2}\right)\right\} \text { by }(5 b) \\
& \supseteq \prod_{x+\left(a_{1} b_{1}\right)+z=\left(a_{2} b_{2}\right)+z} f_{A}^{-}(x) \\
& =f_{A}^{-}(x) .
\end{aligned}
$$

Thus $f_{A} \widetilde{\diamond} f_{A} \widetilde{\sqsubseteq} f_{A}$. For the proof of Condition (3), since $\mathbb{C}_{R} \widetilde{\diamond} f_{A} \widetilde{\sqsubseteq} f_{A}$ and $f_{A} \widetilde{\diamond} \mathbb{C}_{R} \widetilde{\sqsubseteq} f_{A}$, then by from (2) $\mathbb{C}_{R} \widetilde{\diamond} f_{A} \widetilde{\triangleright} \mathbb{C}_{R} \widetilde{\square} f_{A}$.

Conversely, assume that conditions (1)-(4) hold. First to show that $f_{A}^{+}(0) \supseteq f_{A}^{+}(x)$ and $f_{A}^{-}(0) \subseteq$ $f_{A}^{-}(x)$ for all $x \in R$.

$$
\begin{aligned}
f_{A}^{+}(0) & \supseteq\left(f_{A}^{+} \oplus f_{A}^{+}\right)(0) \\
& =\prod_{0+\left(a_{1}+b_{1}\right)+z=\left(a_{2}+b_{2}\right)+z}\left\{f_{A}^{+}\left(a_{1}\right) \cap f_{A}^{+}\left(a_{2}\right) \cap f_{A}^{+}\left(b_{1}\right) \cap f_{A}^{+}\left(b_{2}\right)\right\} \\
& \supseteq\left\{f_{A}^{+}(x) \cap f_{A}^{+}(x) \cap f_{A}^{+}(x) \cap f_{A}^{+}(x)\right\} \\
& =f_{A}^{+}(x) \quad \text { as } 0+x+x+z=x+x+z,
\end{aligned}
$$

also

$$
\begin{aligned}
f_{A}^{-}(0) & \subseteq\left(f_{A}^{-} \boxplus f_{A}^{-}\right)(0) \\
& =\overbrace{0+\left(a_{1}+b_{1}\right)+z=\left(a_{2}+b_{2}\right)+z}\left\{f_{A}^{-}\left(a_{1}\right) \cup f_{A}^{-}\left(a_{2}\right) \cup f_{A}^{-}\left(b_{1}\right) \cup f_{A}^{-}\left(b_{2}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \subseteq\left\{f_{A}^{-}(x) \cup f_{A}^{-}(x) \cup f_{A}^{-}(x) \cup f_{A}^{-}(x)\right\} \\
& =f_{A}^{-}(x)
\end{aligned}
$$

Hence $f_{A}^{+}(0) \supseteq f_{A}^{+}(x)$ and $f_{A}^{-}(0) \subseteq f_{A}^{-}(x)$ hold for all $x \in R$. Now,

$$
\begin{aligned}
f_{A}^{+}(x+y) & \supseteq\left(f_{A}^{+} \oplus f_{A}^{+}\right)(x+y) \\
& =\prod_{x+y+\left(a_{1}+b_{1}\right)+z=\left(a_{2}+b_{2}\right)+z}\left\{f_{A}^{+}\left(a_{1}\right) \cap f_{A}^{+}\left(a_{2}\right) \cap f_{A}^{+}\left(b_{1}\right) \cap f_{A}^{+}\left(b_{2}\right)\right\} \\
& \supseteq\left\{f_{A}^{+}(0) \cap f_{A}^{+}(0) \cap f_{A}^{+}(x) \cap f_{A}^{+}(y)\right\}
\end{aligned}
$$

$$
\begin{aligned}
\text { now as } x+y+0+0+z & =x+y+z, \text { so } \\
f_{A}^{+}(x+y) & \supseteq f_{A}^{+}(x) \cap f_{A}^{+}(y)
\end{aligned}
$$

also,

$$
\begin{aligned}
f_{A}^{-}(x+y) & \subseteq\left(f_{A}^{-} \boxplus f_{A}^{-}\right)(x+y) \\
& =\prod_{x+y+\left(a_{1}+b_{1}\right)+z=\left(a_{2}+b_{2}\right)+z}\left\{f_{A}^{-}\left(a_{1}\right) \cup f_{A}^{-}\left(a_{2}\right) \cup f_{A}^{-}\left(b_{1}\right) \cup f_{A}^{-}\left(b_{2}\right)\right\} \\
& \subseteq\left\{f_{A}^{-}(0) \cup f_{A}^{-}(0) \cup f_{A}^{-}(x) \cup f_{A}^{-}(y)\right\} \\
& =f_{A}^{-}(x) \cup f_{A}^{-}(y) .
\end{aligned}
$$

Further,

$$
\begin{aligned}
f_{A}^{+}(x) & \supseteq\left(f_{A}^{+} \oplus f_{A}^{+}\right)(x) \\
& =\prod_{x+\left(a_{1}+b_{1}\right)+z=\left(a_{2}+b_{2}\right)+z}\left\{f_{A}^{+}\left(a_{1}\right) \cap f_{A}^{+}\left(a_{2}\right) \cap f_{A}^{+}\left(b_{1}\right) \cap f_{A}^{+}\left(b_{2}\right)\right\} \\
\text { If } x+a+z & =b+z, \text { then } x+a+0+z=b+0+z, \text { therefore, } \\
f_{A}^{+}(x) & \supseteq\left\{f_{A}^{+}(a) \cap f_{A}^{+}(0) \cap f_{A}^{+}(b) \cap f_{A}^{+}(0)\right\} \\
& =\left\{f_{A}^{+}(a) \cap f_{A}^{+}(b)\right\}
\end{aligned}
$$

similarly,

$$
\begin{aligned}
f_{A}^{-}(x) & \subseteq\left(f_{A}^{-} \boxplus f_{A}^{-}\right)(x) \\
& =\prod_{x+\left(a_{1}+b_{1}\right)+z=\left(a_{2}+b_{2}\right)+z}\left\{f_{A}^{-}\left(a_{1}\right) \cup f_{A}^{-}\left(a_{2}\right) \cup f_{A}^{-}\left(b_{1}\right) \cup f_{A}^{-}\left(b_{2}\right)\right\} \\
& \subseteq\left\{f_{A}^{-}(a) \cup f_{A}^{-}(0) \cup f_{A}^{-}(b) \cup f_{A}^{-}(0)\right\} \\
& =f_{A}^{-}(a) \cup f_{A}^{-}(b) .
\end{aligned}
$$

From condition (3), $\mathbb{C}_{R} \widetilde{\diamond} f_{A} \widetilde{\diamond} \mathbb{C}_{R} \widetilde{\sqsubseteq} f_{A}$ implies that $\mathbb{C}_{R} \widetilde{\diamond} f_{A} \widetilde{\sqsubseteq} f_{A}$. Now for any $x, z \in R, x+\left(a_{1} b_{1}\right)+z=$ $\left(a_{2} b_{2}\right)+z$, then

$$
\begin{aligned}
f_{A}^{+}(x y) & \supseteq\left(\mathbb{C}_{R}^{+} \otimes f_{A}^{+}\right)(x y) \\
& =\prod_{x y+\left(a_{1} b_{1}\right)+z=\left(a_{2} b_{2}\right)+z}\left\{\mathbb{C}_{R}^{+}\left(a_{1}\right) \cap \mathbb{C}_{R}^{+}\left(a_{2}\right) \cap f_{A}^{+}\left(b_{1}\right) \cap f_{A}^{+}\left(b_{2}\right)\right\}
\end{aligned}
$$

$$
\supseteq f_{A}^{+}(x) \cap f_{A}^{+}(y):(\text { as } x y+0 x+0=x y+0)
$$

and

$$
\begin{aligned}
f_{A}^{-}(x y) & \subseteq\left(\mathbb{C}_{R}^{-} \boxtimes f_{A}^{-}\right)(x y) \\
& =\prod_{x y+\left(a_{1} b_{1}\right)+z=\left(a_{2} b_{2}\right)+z}\left\{\mathbb{C}_{R}^{-}\left(a_{1}\right) \cup \mathbb{C}_{R}^{-}\left(a_{2}\right) \cup f_{A}^{-}\left(b_{1}\right) \cup f_{A}^{-}\left(b_{2}\right)\right\} \\
& \subseteq f_{A}^{-}(x) \cup f_{A}^{-}(y) .
\end{aligned}
$$

The rest of the conditions can be proved in a similar manner. Hence, $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is a doubleframed soft $h$-interior ideal of $R$.

## 4. Double-framed soft $h$-hemiregular hemirings

In this section, we introduce the concept of double-framed soft $h$-hemiregular hemirings. Various conditions are provided under which each double-framed soft $h$-interior ideal is a double-framed soft $h$-ideal of $R$.
Definition 4.1 [35] A hemiring $R$ is said to be $h$-hemiregular if for all $x \in R$, there exist $a, b, z \in R$ such that $x+x a x+z=x b x+z$.
Example 4.2 Suppose that $R$ is a non-negative integers with $\infty \geq x \in N_{0}$. Define $a+b=\max \{a, b\}$ and $a \cdot b=\min \{a, b\}$, then $(R,+, \cdot)$ is a $h$-hemiregular hemring. Define $f_{A}^{+}$and $f_{A}^{-}$in $R$ over $Z^{-}$by

| $R$ | $x \in\left\{2 n \mid n \in N_{0}\right\}$ | otherwise |
| :--- | :--- | :--- |
| $f_{A}^{+}(a)$ | $\{-4,-6,-8,-10\}$ | $\{-4,-8,-10\}$ |
| $f_{A}^{-}(a)$ | $\{-4\}$ | $\{-4,-8\}$. |

Then $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is a double-framed soft set of $h$-hemiregular hemirings over $Z^{-}$.
Lemma 4.3 [35] A hemiring $R$ is $h$-hemiregular if and only if for any right $h$-ideal $M$ and left $h$-ideal $N$ of $R, \overline{M N}=M \cap N$.
Proposition 4.4 Every double-framed soft $h$-interior ideal of an $h$-hemiregular hemiring $R$ is a double-framed soft $h$-ideal of $R$.

Proof. Let $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ be a double-framed soft $h$-interior ideal of an $h$-hemiregular hemiring $R$ and $x \in R$, then there exist $a, b, z \in R$ such that $x+x a x+z=x b x+z$. Therefore, $y x+y x a x+y z=y x b x+y z$. Thus,

$$
\begin{aligned}
f_{A}^{+}(y x) & \supseteq f_{A}^{+}(y x a x) \cap f_{A}^{+}(y x b x) \\
& =f_{A}^{+}(y(x) a x) \cap f_{A}^{+}(y(x) b x) \\
& \supseteq f_{A}^{+}(x) \cap f_{A}^{+}(x):\left(f_{A} \text { being DFS } h \text {-interior ideal }\right) \\
& =f_{A}^{+}(x),
\end{aligned}
$$

and

$$
\begin{aligned}
f_{A}^{-}(y x) & \subseteq f_{A}^{-}(y x a x) \cup f_{A}^{-}(y x b x) \\
& =f_{A}^{-}(y(x) a x) \cup f_{A}^{-}(y(x) b x) \\
& \subseteq f_{A}^{-}(x) \cup f_{A}^{-}(x):\left(f_{A} \text { being DFS } h \text {-interior ideal }\right) \\
& =f_{A}^{-}(x) .
\end{aligned}
$$

Hence, $f_{A}$ is DFS left $h$-ideal of $R$. The case for right ideal can be proved in a similar way. Thus, $f_{A}$ is DFS $h$-ideal of $R$.

Corollary 4.5 In $h$-hemiregular hemiring the concept of double-framed soft $h$-ideal and doubleframed soft $h$-interior ideals are coincide.
Definition 4.6 A hemiring $R$ is said to be $h$-intra-hemiregular if for all $x \in R$, there exist $a_{i}, b_{j}, a_{i}^{\prime}, b_{j}^{\prime}, z \in R$ such that $x+\sum_{i=1}^{m} a_{i} x^{2} a_{i}^{\prime}+z=\sum_{j=1}^{n} b_{j} x^{2} b_{j}^{\prime}+z$.
Theorem 4.7 Every double-framed soft $h$-interior ideal of an $h$-intra-hemiregular hemiring $R$ is a double-framed soft $h$-ideal of $R$.

Proof. Let $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ be a double-framed soft $h$-interior ideal of an $h$-intra-hemiregular if for $x, a_{i}, b_{j}, a_{i}^{\prime}, b_{j}^{\prime}, z \in R$ such that $x+\sum_{i=1}^{\widetilde{m}} a_{i} x^{2} a_{i}^{\prime}+z=\sum_{j=1}^{n} b_{j} x^{2} b_{j}^{\prime}+z$. Then $y x+\sum_{i=1}^{m} y a_{i} x^{2} a_{i}^{\prime}+y z=$ $\widetilde{\sum}_{j=1}^{n} y b_{j} x^{2} b_{j}^{\prime}+y z$
$j=1$

$$
\begin{aligned}
f_{A}^{+}(y x) & \supseteq f_{A}^{+}\left(\sum_{i=1}^{\widetilde{\Sigma}} y a_{i} x^{2} a_{i}^{\prime}\right) \cap f_{A}^{+}\left(\sum_{j=1}^{n} y b_{j} x^{2} b_{j}^{\prime}\right) \\
& =f_{A}^{+}\left(\sum_{i=1}^{\widetilde{\Sigma}}\left(y a_{i}\right) x\left(x a_{i}^{\prime}\right)\right) \cap f_{A}^{+}\left(\sum_{j=1}^{\widetilde{\sum}}\left(y b_{j}\right) x\left(x b_{j}^{\prime}\right)\right) \\
& \supseteq f_{A}^{+}\left(\left(y a_{i}\right) x\left(x a_{i}^{\prime}\right)\right) \cap f_{A}^{+}\left(\left(y b_{j}\right) x\left(x b_{j}^{\prime}\right)\right) \\
& \supseteq f_{A}^{+}(x) \cap f_{A}^{+}(x) \\
& =f_{A}^{+}(x),
\end{aligned}
$$

and

$$
f_{A}^{-}(y x) \subseteq f_{A}^{-}\left({\underset{\sum}{i=1}}_{m}^{\infty} y a_{i} x^{2} a_{i}^{\prime}\right) \cup f_{A}^{-}\left({\underset{\sum}{j=1}}_{n} y b_{j} x^{2} b_{j}^{\prime}\right)
$$

$$
\begin{aligned}
& =f_{A}^{-}\left(\sum_{i=1}^{m}\left(y a_{i}\right) x\left(x a_{i}^{\prime}\right)\right) \cup f_{A}^{-}\left(\sum_{j=1}^{n}\left(y b_{j}\right) x\left(x b_{j}^{\prime}\right)\right) \\
& \subseteq f_{A}^{-}\left(\left(y a_{i}\right) x\left(x a_{i}^{\prime}\right)\right) \cup f_{A}^{-}\left(\left(y b_{j}\right) x\left(x b_{j}^{\prime}\right)\right) \\
& \subseteq f_{A}^{+}(x) \cup f_{A}^{-}(x) \\
& =f_{A}^{-}(x)
\end{aligned}
$$

Hence, $f_{A}$ is DFS left $h$-ideal of $R$. The case for right ideal can be proved in a similar way. Therefore, $f_{A}$ is DFS $h$-ideal of $R$.

Corollary 4.8 In $h$-intra-hemiregular hemiring the concept of double-framed soft $h$-ideal and double-framed soft $h$-interior ideals are coincide.

If $\left\{f_{A_{i}}: i \in \Omega\right\}$ be the indexed family of double-framed soft sets in a hemiring $R$, then $\widetilde{\sim}{ }_{i \in \Omega} f_{A_{i}}$ set is denoted by $\widetilde{\tilde{T}_{i \in \Omega}} f_{A_{i}}=\left\langle\left(\widetilde{\widetilde{\in}} \tilde{S}^{+} f_{A_{i}}^{+}, \widetilde{\cup} f_{i \in \Omega}^{-}\right) ; A\right\rangle$ where $\widetilde{A_{i}} f_{A_{i}}^{+}(x)=\cap\left\{f_{A_{i}}^{+}(x): i \in \Omega, x \in R\right\}$ and $\widetilde{\sim}{ }_{i \in \Omega} f_{A_{i}}^{-}(x)=$ $\cup\left\{f_{A_{i}}^{-}(x): i \in \Omega, x \in R\right\}$.
Theorem 4.9 Suppose that $\left\{f_{A_{i}}: i \in \Omega\right\}$ is an indexed family of double-framed soft $h$-interior ideals of a hemiring $R$. Then, $\widetilde{\overbrace{i \in \Omega}} f_{A_{i}}$ is a double-framed soft $h$-interior ideal of $R$ if $\widetilde{\overbrace{i \in \Omega}} f_{A_{i}} \neq \emptyset$.
Proof. Assume that $a, b \in R$, then

$$
\begin{aligned}
\widetilde{\cap} \overbrace{i \in \Omega}^{+} f_{A_{i}}^{+}(a+b) & \supseteq \widetilde{\cap}\left\{f_{A_{i}}^{+}(a) \cap f_{A_{i}}^{+}(b): i \in \Omega\right\} \text { : by Condition (4a) } \\
& =\left(\widetilde{\bigcap} \widetilde{n}_{\in \Omega} f_{A_{i}}^{+}(a)\right) \cap\left(\widetilde{\cap} f_{i \in \Omega}^{+}(b)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{U} \tilde{U}_{\Omega} f_{A_{i}}^{-}(a+b) & \subseteq \widetilde{\widetilde{U}_{i \in \Omega}}\left\{f_{A_{i}}^{-}(a) \cup f_{A_{i}}^{-}(b): i \in \Omega\right\}: \text { by Condition (4b) } \\
& =\left(\widetilde{\cup} \widetilde{U}_{i \in \Omega}^{-} f_{A_{i}}^{-}(a)\right) \cap\left(\widetilde{\cup} f_{i \in \Omega}^{-}(b)\right),
\end{aligned}
$$

also for $a, b \in R$,

$$
\begin{aligned}
\widetilde{\cap} f_{A_{A}}^{+}(a b) & \supseteq \widetilde{\cap}\left\{f_{A_{i}}^{+}(a) \cap f_{A_{i}}^{+}(b): i \in \Omega\right\}: \text { by Condition (5a) } \\
& =(\widetilde{\overbrace{i \in \Omega}} f_{A_{i}}^{+}(a)) \cap\left(\widetilde{\cap} f_{i \in \Omega}^{+}(b)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\bigcup_{i \in \Omega}} f_{A_{i}}^{-}(a b) & \subseteq \widetilde{\widetilde{U}_{i \in \Omega}}\left\{f_{A_{i}}^{-}(a) \cup f_{A_{i}}^{-}(b): i \in \Omega\right\} \text { : by Condition (5b) } \\
& =\left(\widetilde{U_{i \in \Omega}} f_{A_{i}}^{-}(a)\right) \cap\left(\widetilde{\cup}{ }_{i \in \Omega} f_{A_{i}}^{-}(b)\right) .
\end{aligned}
$$

Now let $a, b, c \in R$,

$$
\widetilde{\cap}_{i \in \Omega} f_{A_{i}}^{+}(a b c) \supseteq \widetilde{\cap}_{i \in \Omega}\left\{f_{A_{i}}^{+}(b): i \in \Omega\right\}: \text { by Condition (6a) }
$$

$$
=\widetilde{\cap} f_{i \in \Omega} f_{A_{i}}^{+}(b)
$$

and

$$
\begin{aligned}
\widetilde{\widetilde{U}_{\in \Omega}} f_{A_{i}}^{-}(a b c) & \subseteq \widetilde{\widetilde{U} \in \Omega}\left\{f_{A_{i}}^{-}(b): i \in \Omega\right\}: \text { by Condition (6b) } \\
& =\widetilde{U_{i \in \Omega}} f_{A_{i}}^{-}(b)
\end{aligned}
$$

Lastly, let $a, b, x, z \in R$ such that $x+a+z=b+z$, then,

$$
\begin{aligned}
\widetilde{\cap}_{i \in \Omega} f_{A_{i}}^{+}(x) & \supseteq \widetilde{\cap}\left\{\tilde{n}_{i \in \Omega}\left\{f_{A_{i}}^{+}(a) \cap f_{A_{i}}^{+}(b): i \in \Omega\right\}\right. \text { : by Condition (7a) } \\
& =\left(\widetilde{\cap} f_{i \in \Omega}^{+}(a)\right) \cap\left(\widetilde{A_{i}} f_{A_{i}}^{+}(b)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \widetilde{{ }_{i \in \Omega}} f_{A_{i}}^{-}(x) \subseteq \widetilde{i} \underset{i \in \Omega}{\widetilde{u}}\left\{f_{A_{i}}^{-}(a) \cup f_{A_{i}}^{-}(b): i \in \Omega\right\} \text { : by Condition (7b) } \\
& =\left(\widetilde{U_{i \in \Omega}} f_{A_{i}}^{-}(a)\right) \cap\left(\widetilde{U_{i \in \Omega}} f_{A_{i}}^{-}(b)\right) .
\end{aligned}
$$

Hence, $\widetilde{\widetilde{i}}_{i \in \Omega} f_{A_{i}}=\left\langle\left(\widetilde{त}_{i \in \Omega} f_{A_{i}}^{+}, \widetilde{U}_{i \in \Omega} f_{A_{i}}^{-}\right) ; A\right\rangle$ is double-framed soft $h$-interior ideal of $R$.

## 5. Double-framed soft $h$-simple hemirings

In this section, the concept of double-framed soft $h$-simple hemirings is determined. Further, double-framed soft $h$-interior ideal of $R$ are used to classify double-framed soft $h$-simple hemirings.
Definition 5.1 A double-framed soft set $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ of a hemiring $R$ is a constant function if for all $x, y \in R, f_{A}^{+}(x)=f_{A}^{+}(y)$ and $f_{A}^{-}(x)=f_{A}^{-}(y)$.
Definition 5.2 A hemiring $R$ is double-framed soft left (right) $h$-simple if every double-framed soft left (right) $h$-ideal of $R$ is a constant function. A hemiring $R$ is double-framed soft $h$-simple if it is both DFS left and DFS right $h$-simple.
Example 5.3 Let $R=\{0, a, 1\}$ be a set with two binary opertions " + " and "." defined by the following tables

| + | 0 | $a$ | 1 | . | 0 | $a$ | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | $a$ | 1 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | $a$ | $a$ | $a$ | 0 | $a$ | $a$ |
| 1 | 1 | $a$ | 1 | 1 | 0 | $a$ | 1 |

then the only $h$-ideal of $R$ is $R$ itself. Any double-framed soft $h$-ideal is constant function. Therefore, $R$ is double-framed soft $h$-simple.

For a hemiring $R$, a subset $I_{a}$ where $a \in R$ is define by

$$
I_{a}=\left\{x \in R: f_{A}^{+}(x) \supseteq f_{A}^{+}(a) \text { and } f_{A}^{-}(x) \subseteq f_{A}^{-}(a)\right\} .
$$

Theorem 5.4 If $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is a double-framed soft left $h$-ideal of a hemiring $R$, then $I_{a}$ is a double-framed soft left $h$-ideal of $R$.

Proof. Suppose $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is a double-framed soft left $h$-ideal of a hemiring $R$. Since for any $a \in R, f_{A}^{+}(a) \supseteq f_{A}^{+}(a)$ and $f_{A}^{-}(a) \subseteq f_{A}^{-}(a)$. Therefore, $a \in I_{a}$ so $I_{a} \neq \emptyset$. Let $x, y \in I_{a}$, then $f_{A}^{+}(x) \supseteq f_{A}^{+}(a), f_{A}^{-}(x) \subseteq f_{A}^{-}(a)$ and $f_{A}^{+}(y) \supseteq f_{A}^{+}(a), f_{A}^{-}(y) \subseteq f_{A}^{-}(a)$. Thus

$$
\begin{aligned}
f_{A}^{+}(x+y) & \supseteq f_{A}^{+}(x) \cap f_{A}^{+}(y): \quad \text { by }(1 a) \\
& \supseteq f_{A}^{+}(a) \cap f_{A}^{+}(a)=f_{A}^{+}(a)
\end{aligned}
$$

and

$$
\begin{aligned}
f_{A}^{-}(x+y) & \subseteq f_{A}^{-}(x) \cup f_{A}^{-}(y): \quad \text { by }(1 \mathrm{~b}) \\
& \subseteq f_{A}^{-}(a) \cup f_{A}^{-}(a)=f_{A}^{-}(a) .
\end{aligned}
$$

Hence $x+y \in I_{a}$. Now if $x \in R, y \in I_{a}$, then $f_{A}^{+}(y) \supseteq f_{A}^{+}(a), f_{A}^{-}(y) \subseteq f_{A}^{-}(a)$ so by hypothesis, $f_{A}^{+}(x y) \supseteq$ $f_{A}^{+}(y) \supseteq f_{A}^{+}(a)$ and $f_{A}^{-}(x y) \subseteq f_{A}^{-}(y) \subseteq f_{A}^{-}(a)$ implies that $x y \in I_{a}$. Finally, let $x \in R, a_{1}, b_{1} \in I_{a}$ be such that $x+a_{1}+z=b_{1}+z$ for some $z \in R$. Then $f_{A}^{+}\left(a_{1}\right) \supseteq f_{A}^{+}(a), f_{A}^{-}\left(a_{1}\right) \subseteq f_{A}^{-}(a)$ and $f_{A}^{+}\left(b_{1}\right) \supseteq f_{A}^{+}(a)$, $f_{A}^{-}\left(b_{1}\right) \subseteq f_{A}^{-}(a)$, therefore,

$$
\begin{aligned}
f_{A}^{+}(x) & \supseteq f_{A}^{+}\left(a_{1}\right) \cap f_{A}^{+}\left(b_{1}\right): & \text { by (3a) } \\
& \supseteq f_{A}^{+}(a) \cap f_{A}^{+}(a)=f_{A}^{+}(a) &
\end{aligned}
$$

and

$$
\begin{aligned}
f_{A}^{-}(x) & \subseteq f_{A}^{-}\left(a_{1}\right) \cup f_{A}^{-}\left(b_{1}\right): \quad \text { by }(3 \mathrm{~b}) \\
& \subseteq f_{A}^{-}(a) \cup f_{A}^{-}(a)=f_{A}^{-}(a) .
\end{aligned}
$$

Implies that $x \in I_{a}$. Thus $I_{a}$ is a double-framed soft left $h$-ideal of $R$.
Theorem 5.5 If $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is a double-framed soft right $h$-ideal of a hemiring $R$, then $I_{a}$ is a double-framed soft right $h$-ideal of $R$.

Proof. Follows from Theorem 5.4.
From Theorem 5.4 and Theorem 5.5, the following corollary is obtained.
Corollary 5.6 If $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is a double-framed soft $h$-ideal of a hemiring $R$, then $I_{a}$ is a double-framed soft $h$-ideal of $R$.

The relationship between $h$-simple and double-framed soft $h$-simple hemirings is constructed in the following theorem.
Theorem 5.7 For a hemiring $R$, the following conditions are equivalent:
(i) $\quad R$ is $h$-simple hemiring.
(ii) $\quad R$ is double-framed soft $h$-simple hemiring.

Proof. (i) $\Longrightarrow$ (ii). Let $R$ is $h$-simple hemiring, $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ is a double-framed soft $h$-ideal of $R$ and $a, b \in R$. As $f_{A}$ is DFS $h$-ideal of $R$, so by Corollary 5.6, $I_{a}$ is an $h$-ideal of $R$. By assumption $I_{a}=R$ so $b \in I_{a}$. Therefore, $f_{A}^{+}(b) \supseteq f_{A}^{+}(a), f_{A}^{-}(b) \subseteq f_{A}^{-}(a)$. Also, $I_{b}=R$ so $a \in I_{b}$. Therefore, $f_{A}^{+}(a) \supseteq f_{A}^{+}(b), f_{A}^{-}(a) \subseteq f_{A}^{-}(b)$. Hence, $f_{A}^{+}(b)=f_{A}^{+}(a), f_{A}^{-}(b)=f_{A}^{-}(a)$. Thus, every DFS left (resp. right) $h$-ideal is constant function so $R$ is double-framed soft $h$-simple.
(ii) $\Longrightarrow$ (i). Suppose that $R$ is not a double-framed soft $h$-simple. That is $R$ contains proper $h$-ideal $I$ of $R$ such that $I \neq R$. As $I$ is an $h$-ideal of $R$ so by Proposition 3.9, the characteristic DFS function $\mathbb{C}_{I}$ of $I$ is DFS $h$-ideal of $R$. Let $x \in R$, then by assumption, we have $f_{\mathbb{C}_{I}}^{+}(x)=f_{\mathbb{C}_{I}}^{+}(b), f_{\mathbb{C}_{I}}^{-}(x)=f_{\mathbb{C}_{I}}^{-}(b)$ for all $b \in R$. Since $I$ is proper $h$-ideal of $R$ so non-empty. Let $a \in I$, then $f_{\mathrm{C}_{I}}^{+}(x)=f_{\mathrm{C}_{I}}^{+}(a)=U$, $f_{\mathbb{C}_{I}}^{-}(x)=f_{\mathbb{C}_{I}}^{-}(a)=\emptyset$. Thus $x \in I$, so $R \subset I$ which is a contradiction. Therefore, $I=R$ implies that $R$ is $h$-simple hemiring.

## 6. Double-framed soft $h$-semisimple hemirings

In the aforementioned section, it is shown that every double-framed soft $h$-ideal of a hemiring is double-framed soft $h$-interior ideal of $R$ but the converse is not true in general. The aim of the present section is to provide the conditions under which the converse of the said statement hold. More precisely, if a hemiring $R$ is an $h$-semisimple, then every DFS $h$-interior ideal of $R$ will be DFS $h$-ideal. Also, the notions of double-framed soft transformation and inverse double-framed soft transformation in hemiring $R$ are introduced and hemirings are characterized by the properties of these newly developed notions.
Definition 6.1 A double-framed soft $h$-ideal $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ of a hemiring $R$ is idempotent in $R$, if $f_{A} \widetilde{f_{A}}=f_{A}$, i.e., $f_{A}^{+} \otimes f_{A}^{+}=f_{A}^{+}$and $f_{A}^{-} \boxtimes f_{A}^{-}=f_{A}^{-}$.
Example 6.2 Let $R=\{0, x, 1\}$ be the set with multiplication and addition as define in the following tables,

| + | 0 | $x$ | 1 |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | $x$ | 1 |  |  |  |  |  |
| $x$ | $x$ | $x$ | $x$ |  | 0 | 0 | $x$ | 1 |
| 1 | 0 | $x$ | 1 |  | 0 | $x$ | $x$ |  |
| 1 | 1 | 0 | $x$ | 1 |  |  |  |  |

Then, $(R,+, \cdot)$ is a commutative hemiring with identity. The set $\{0, x\}$ is the only proper ideal which is not $h$-ideal. The only $h$-ideal of $R$ is $R$ itself. This ideal is an idempotent. Note that $0=0 x=$ $x 0=01=10$, now for any double-framed soft ideal $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ of $R, f_{A}^{+}(0) \supseteq f_{A}^{+}(x)$, $f_{A}^{+}(0) \supseteq f_{A}^{+}(1)$ and $f_{A}^{+}(x)=f_{A}^{+}(1 x) \supseteq f_{A}^{+}(1)$. Hence, $f_{A}^{+}(0) \supseteq f_{A}^{+}(x) \supseteq f_{A}^{+}(1)$. If $f_{A}$ is doubelframed soft $h$-ideal, then $1+0+1=0+1$ which implies that $f_{A}^{+}(1) \supseteq f_{A}^{+}(0) \cap f_{A}^{+}(0)=f_{A}^{+}(0)$ shows that doubel-framed soft $h$-ideal of $R$ is constant function. So $f_{A}^{+} \otimes f_{A}^{+}=f_{A}^{+}$. Now $f_{A}^{-}(0) \subseteq f_{A}^{-}(x)$, $f_{A}^{-}(0) \subseteq f_{A}^{-}(1)$ and $f_{A}^{-}(x)=f_{A}^{-}(1 x) \subseteq f_{A}^{-}(1)$. Thus, $f_{A}^{-}(0) \subseteq f_{A}^{-}(x) \subseteq f_{A}^{-}(1), 1+0+1=0+1$ implies that $f_{A}^{-}(1) \subseteq f_{A}^{-}(0) \cap f_{A}^{-}(0)=f_{A}^{-}(0)$. It means that $f_{A}^{-}(0)=f_{A}^{-}(x)=f_{A}^{-}(1)$. Consequently, $f_{A}^{-} \boxtimes f_{A}^{-}=f_{A}^{-}$. Hence, $f_{A} \oslash f_{A}=f_{A}$ so $f_{A}$ is idempotent.
Definition 6.3 A hemiring $R$ is $h$-semisimple, if every double-framed soft $h$-ideal is idempotent.
Proposition 6.4 If $R$ is an $h$-semisimple hemiring, then every double-framed soft $h$-interior ideal of $R$ is double-framed soft $h$-ideal of $R$.

Proof. Follows from Theorem 4.7.

Lemma 6.5 [36] A hemiring $R$ is $h$-semisimple if and only if it satisfy one of the following condition:
(1) There exist $c_{i}, d_{i}, e_{i}, f_{i}, c_{j}, d_{j}, e_{j}, f_{j}, z \in R$ such that $x+\underset{i=1}{\underset{\sum}{m}} c_{i} x d_{i} e_{i} x f_{i}+z=$ $\widetilde{\Sigma}^{n} \quad c_{j} x d_{j} e_{j} x f_{j}+z$ for all $x \in R$.
$j=1$
(2) $x \in \overline{R x R x R}$ for all $x \in R$.
(3) $A \subseteq \overline{R A R A R}$ for all $A \subseteq R$.

Lemma 6.6 [36] If $R$ is a hemiring, then the following conditions are equivalent.
(1) $\quad R$ is $h$-semisimple.
(2) $A \cap B=\overline{A B}$ for all $h$-ideals $A$ and $B$ of $R$.
(3) $A=\overline{A^{2}}$ for every $h$-ideals $A$ of $R$.
(3) $\quad A(a)=\overline{A(a)^{2}}$ for every $a$ of $R$.

Theorem 6.7 If $R$ is a hemiring, then the following conditions are equivalent:
(1) $\quad R$ is $h$-semisimple.
(2) For any double-framed soft $h$-interior ideals $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ and $f_{B}=\left\langle\left(f_{B}^{+}, f_{B}^{-}\right) ; B\right\rangle$ of $R, f_{A} \widetilde{\square} f_{B} \widetilde{\sqsubseteq} f_{A} \widetilde{\diamond} f_{B}$ hold.

Proof. (1) $\Longrightarrow$ (2). Let $R$ be $h$-semisimple and $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle, f_{B}=\left\langle\left(f_{B}^{+}, f_{B}^{-}\right) ; B\right\rangle$ be double-framed soft $h$-interior ideals of $R$. Suppose $x \in R$, then there exist $c_{i}, d_{i}, e_{i}, f_{i}, c_{j}, d_{j}, e_{j}, f_{j}, z \in R$ such that $x+\sum_{i=1}^{m} c_{i} x d_{i} e_{i} x f_{i}+z=\sum_{j=1}^{n} c_{j} x d_{j} e_{j} x f_{j}+z$. Therefore, we have

$$
\begin{aligned}
& \left(f_{A}^{+} \otimes f_{B}^{+}\right)(x)=\quad \widetilde{\bigcup} \quad\left\{f_{A}^{+}\left(a_{i}\right) \cap f_{A}^{+}\left(a_{j}\right) \cap f_{B}^{+}\left(b_{i}\right) \cap f_{B}^{+}\left(b_{j}\right)\right\} \\
& { }^{x+}{\underset{\sum}{i=1}}_{\widetilde{N}}^{a_{i} b_{i}+z=}{\underset{\sum}{j=1}}_{n} a_{j} b_{j}+z \\
& \supseteq\left\{f_{A}^{+}\left(c_{i} x d_{i}\right) \cap f_{A}^{+}\left(c_{j} x d_{j}\right) \cap f_{B}^{+}\left(e_{i} x f_{i}\right) \cap f_{B}^{+}\left(e_{j} x f_{j}\right)\right\} \\
& \supseteq\left\{f_{A}^{+}(x) \cap f_{B}^{+}(x)\right\}=\left(f_{A}^{+} \widetilde{\cap} f_{B}^{+}\right)(x),
\end{aligned}
$$

and

$$
\begin{aligned}
\left(f_{A}^{-} \boxtimes f_{B}^{-}\right)(x) & =\widetilde{\bigcup}\left\{f_{A}^{-}\left(a_{i}\right) \cup f_{A}^{-}\left(a_{j}\right) \cup f_{B}^{-}\left(b_{i}\right) \cup f_{B}^{-}\left(b_{j}\right)\right\} \\
& \underset{\sum_{i=1}^{m} a_{i} b_{i}+z=}{\sum_{j=1}} a_{a_{j} b_{j}+z} \\
& \subseteq\left\{f_{A}^{-}\left(c_{i} x d_{i}\right) \cup f_{A}^{-}\left(c_{j} x d_{j}\right) \cup f_{B}^{-}\left(e_{i} x f_{i}\right) \cup f_{B}^{-}\left(e_{j} x f_{j}\right)\right\} \\
& \subseteq\left\{f_{A}^{-}(x) \cup f_{B}^{-}(x)\right\}=\left(f_{A}^{-} \cup f_{B}^{-}\right)(x) .
\end{aligned}
$$

Hence, $f_{A} \widetilde{\square} f_{B} \widetilde{\sqsubseteq} f_{A} \widetilde{\triangleright} f_{B}$.
(2) $\Longrightarrow(1)$. Suppose Condition (2) hold, let $M$ and $N$ be any double-framed soft $h$-interior ideals of $R$, then the characteristic double-framed soft functions $\mathbb{C}_{A}$ and $\mathbb{C}_{B}$ of $A$ and $B$ respectively are doubleframed soft $h$-interior ideals of $R$. So, using Condition (2) and Theorem 2.8, $\mathbb{C}_{A \cap B}=\mathbb{C}_{A} \widetilde{\square} \mathbb{C}_{B} \widetilde{\boxed{ }} \mathbb{C}_{A} \widetilde{\rightharpoonup} \mathbb{C}_{B}=$ $\mathbb{C}_{\overline{A B}}$, implies that $A \cap B \subseteq \overline{A B}$, but $\overline{A B} \subseteq A \cap B$ (always). Thus $\overline{A B}=A \cap B$, therefore, by Lemma 6.6, $R$ is $h$-semisimple.

Theorem 6.8 A hemiring $R$ is $h$-semisimple if and only if every double-framed soft $h$-ideal $f_{A}=$ $\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ of $R$ is idempotent.
Proof. The Proof follows from Theorem 6.7.
Theorem 6.9 If $R$ is a hemiring, then the following conditions are equivalent:
(1) $\quad R$ is $h$-semisimple.
(2) Every double-framed soft $h$-interior ideal of $R$ is idempotent.
(3) For any double-framed soft $h$-interior ideals $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ and $f_{B}=\left\langle\left(f_{B}^{+}, f_{B}^{-}\right) ; B\right\rangle$ of $R, f_{A} \widetilde{\sqcap} f_{B}=f_{A} \widetilde{\diamond} f_{B}$.

Suppose $\Omega(X)$ be the family of all double-framed soft sets in $X$. Assume that $X$ and $Y$ be any given classical sets. Two mappings are induces by $f: X \longrightarrow Y$ i.e., $\Omega_{f}: \Omega(X) \longrightarrow \Omega(Y)$ where $A \longmapsto \Omega_{f}(A)$ and $\Omega_{f}^{-1}: \Omega(Y) \longrightarrow \Omega(X)$ where $B \longmapsto \Omega_{f}^{-1}(B) . \Omega_{f}(A)$ is defined as

$$
\begin{aligned}
& \Omega_{f}\left(f_{A}^{+}\right)(y)=\left\{\begin{array}{lc}
\widetilde{\bigcup} \widetilde{U}_{y=f(x)} f_{A}^{+}(x) & \text { if } f^{-1}(y) \neq \emptyset \\
\emptyset & \text { otherwise },
\end{array}\right. \\
& \Omega_{f}\left(f_{A}^{-}\right)(y)=\left\{\begin{array}{lc}
\widetilde{\bigcup} \tilde{U}_{y=f(x)} f_{A}^{+}(x) & \text { if } f^{-1}(y) \neq \emptyset \\
U & \text { otherwise },
\end{array}\right.
\end{aligned}
$$

for all $y \in Y$ and $\Omega_{f}^{-1}(B)$ is defined by $\Omega_{f}^{-1}\left(f_{B}^{+}\right)(x)=f_{B}^{+}(f(x))$ and $\Omega_{f}^{-1}\left(f_{B}^{-}\right)(x)=f_{B}^{-}(f(x))$ for all $x \in X$. The mappings $\Omega_{f}$ and $\Omega_{f}^{-1}$ are called double-framed soft transformation and inverse double-framed soft transformation induced by $f$.

Note that a double-framed soft $h$-interior ideal $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ of $X$ has double-framed soft property if for any $P$ of $X$ there exists $p \in P$ such that $f_{A}^{+}(p)=\widetilde{\bigcup_{p \in P}} f_{A}^{+}(p)$ and $f_{A}^{-}(p)=\widetilde{\bigcup_{p \in P}} f_{A}^{-}(p)$.
Example 6.10 Let $R=\{0, x, 1\}$ be the set with multiplication and addition as define in Example 6.2 , then $(R,+, \cdot)$ is a commutative hemiring with identity. Since every $h$-ideal is $h$-interior ideal of $R$. Therefore, using Lemma 3.6 and Example 6.2, every double-framed soft $h$-interior ideal of $R$ is idempotent i.e., $f_{A} \widetilde{\diamond} f_{A}=f_{A}$. Moreover, $f_{A} \widetilde{\square} f_{A} \widetilde{\sqsubseteq} f_{A}=f_{A} \widetilde{\diamond} f_{A}$ and on the other hand by Theorem 3.10 (2), $f_{A} \widetilde{\diamond} f_{A} \widetilde{\sqsubseteq} f_{A}$ implies that $\left(f_{A} \widetilde{\diamond} f_{A}\right) \widetilde{\Pi}\left(f_{A} \widetilde{\diamond} f_{A}\right) \widetilde{\sqsubseteq} f_{A} \widetilde{\square} f_{A}$ which shows that $f_{A} \widetilde{\square} f_{A}=f_{A} \widetilde{\nabla} f_{A}$.
Theorem 6.11 If $f: X \longrightarrow Y$ is a homomorphism of a hemiring $R$ with $\Omega_{f}: \Omega(X) \longrightarrow \Omega(Y)$ and $\Omega_{f}^{-1}: \Omega(Y) \longrightarrow \Omega(X)$ be the double-framed soft transformation and inverse double-framed soft transformation induced by $f$ respectively, then
(1) If $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle \in \Omega(X)$ is a double-framed soft $h$-interior ideal of $X$ with doubleframed soft property, then $\Omega_{f}(A)$ is a double-framed soft $h$-interior ideal of $Y$.
(2) If $f_{B}=\left\langle\left(f_{B}^{+}, f_{B}^{-}\right) ; B\right\rangle \in \Omega(Y)$ is a double-framed soft $h$-interior ideal of $Y$, then $\Omega_{f}^{-1}(B)$ is a double-framed soft $h$-interior ideal of $X$.

Proof. To prove (1), let $f(x), f(y) \in f(X), p \in f^{-1}(f(x))$ and $q \in f^{-1}(f(y))$ be such that

$$
f_{A}^{+}(p)=\varlimsup_{a \in f^{-1}(f(x))} f_{A}^{+}(a), f_{A}^{-}(p)=\bigcup_{a \in f^{-1}(f(x))} f_{A}^{-}(a)
$$

and

$$
f_{A}^{+}(q)=\widetilde{\bigcup}_{b \in f^{-1}(f(x))} f_{A}^{+}(b), f_{A}^{-}(q)=\widetilde{\bigcup}_{b \in f^{-1}(f(x))} f_{A}^{-}(b) .
$$

Then

$$
\begin{aligned}
\Omega_{f}\left(f_{A}^{+}\right)(f(x)+f(y)) & =\widetilde{\bigcup}_{z \in f^{-1}(f(x)+f(y))} f_{A}^{+}(z) \\
& \supseteq f_{A}^{+}(p+q) \supseteq f_{A}^{+}(p) \cap f_{A}^{+}(q) \\
& \left.=\widetilde{\bigcup}_{a \in f^{-1}(f(x))} f_{A}^{+}(a)\right) \cap\left(\widetilde{\bigcup}_{b \in f^{-1}(f(x))} f_{A}^{+}(b)\right) \\
& =\Omega_{f}\left(f_{A}^{+}\right)(f(x)) \cap \Omega_{f}\left(f_{A}^{+}\right)(f(y)), \\
\Omega_{f}\left(f_{A}^{-}\right)(f(x)+f(y)) & =\widetilde{\bigcup_{z \in f^{-1}(f(x)+f(y))} f_{A}^{-}(z)} \\
& \subseteq f_{A}^{-}(p+q) \subseteq f_{A}^{-}(p) \cup f_{A}^{-}(q) \\
& =\left(\widetilde{\left.\bigcup_{a \in f^{-1}(f(x))} f_{A}^{-}(a)\right) \cup\left(\widetilde{\bigcup}_{b \in f^{-1}(f(x))} f_{A}^{-}(b)\right)}\right. \\
& =\Omega_{f}\left(f_{A}^{-}\right)(f(x)) \cup \Omega_{f}\left(f_{A}^{-}\right)(f(y)),
\end{aligned}
$$

Also for $f(x), f(y) \in f(X)$,

$$
\begin{aligned}
\Omega_{f}\left(f_{A}^{+}\right)(f(x) f(y)) & =\widetilde{\Xi}_{z \in f^{-1}(f(x) f(y))} f_{A}^{+}(z) \\
& \supseteq f_{A}^{+}(p q) \supseteq f_{A}^{+}(p) \cap f_{A}^{+}(q) \\
& =\left(\widetilde{\bigcup}_{a \in f^{-1}(f(x))} f_{A}^{+}(a)\right) \cap\left(\widetilde{\bigcup}_{b \in f^{-1}(f(x))} f_{A}^{+}(b)\right) \\
& =\Omega_{f}\left(f_{A}^{+}\right)(f(x)) \cap \Omega_{f}\left(f_{A}^{+}\right)(f(y)), \\
\Omega_{f}\left(f_{A}^{-}\right)(f(x) f(y)) & =\widetilde{\bigcup_{z \in f^{-1}(f(x) f(y))} f_{A}^{-}(z)} \\
& \subseteq f_{A}^{-}(p+q) \subseteq f_{A}^{-}(p) \cup f_{A}^{-}(q) \\
& =\left(\widetilde{\left.\bigcup_{a \in f^{-1}(f(x))} f_{A}^{-}(a)\right) \cup\left(\widetilde{\bigcup}_{b \in f^{-1}(f(x))} f_{A}^{-}(b)\right)}\right.
\end{aligned}
$$

$$
=\Omega_{f}\left(f_{A}^{-}\right)(f(x)) \cup \Omega_{f}\left(f_{A}^{-}\right)(f(y))
$$

Now let $f(x) \in f(X), a, b \in R$ and $p \in f^{-1}(f(x))$ such that $f_{A}^{+}(p)=\underset{a \in f^{-1}(f(x))}{\widetilde{U}} f_{A}^{+}(a), f_{A}^{-}(p)=$ $\underset{a \in f^{-1}(f(x))}{\widetilde{u}} f_{A}^{-}(a)$ then,

$$
\begin{aligned}
\Omega_{f}\left(f_{A}^{+}\right)(a f(x) b) & =\widetilde{\bigcup}_{z \in f^{-1}(a f(x) b)} f_{A}^{+}(z) \\
& \supseteq f_{A}^{+}(a p b) \supseteq f_{A}^{+}(p)=\widetilde{\bigcup}_{a \in f^{-1}(f(x))} f_{A}^{+}(a) \\
& =\Omega_{f}\left(f_{A}^{+}\right)(f(x)) . \\
\Omega_{f}\left(f_{A}^{-}\right)(a f(x) b) & =\widetilde{\bigcup}_{z \in f^{-1}(a f(x) b)} f_{A}^{-}(z) \\
& \subseteq f_{A}^{-}(a p b) \subseteq f_{A}^{-}(p)=\widetilde{\bigcup_{a \in f^{-1}(f(x))} f_{A}^{-}(a)} \\
& =\Omega_{f}\left(f_{A}^{-}\right)(f(x)) .
\end{aligned}
$$

Further, if $x+a+z=b+z$, then $f(x)+f(a)+f(z)=f(b)+f(z)$. Therefore,

$$
\begin{aligned}
\Omega_{f}\left(f_{A}^{+}\right)(f(x)) & =\widetilde{\bigcup}_{z \in f^{-1}(f(x))} f_{A}^{+}(z) \\
& \supseteq f_{A}^{+}(p) \\
& \supseteq\left(\widetilde{\bigcup}_{a \in f^{-1}(f(x))} f_{A}^{+}(a)\right) \cap\left(\widetilde{\bigcup}_{b \in f^{-1}(f(x))} f_{A}^{+}(b)\right) \\
& \supseteq \Omega_{f}\left(f_{A}^{+}\right)(f(a)) \cap \Omega_{f}\left(f_{A}^{+}\right)(f(b)), \\
\Omega_{f}\left(f_{A}^{-}\right)(f(x)) & =\widetilde{\bigcup_{z \in f^{-1}(f(x))} f_{A}^{-}(z)} \\
& \subseteq f_{A}^{-}(p) \\
& \subseteq\left(\widetilde{\bigcup}_{a \in f^{-1}(f(x))} f_{A}^{-}(a)\right) \cup\left(\widetilde{\bigcup}_{b \in f^{-1}(f(x))} f_{A}^{-}(b)\right) \\
& \subseteq \Omega_{f}\left(f_{A}^{-}\right)(f(a)) \cup \Omega_{f}\left(f_{A}^{-}\right)(f(b)) .
\end{aligned}
$$

Thus, $\Omega_{f}(A)$ is a double-framed soft $h$-interior ideal of $Y$.
To prove (2), if $x, y \in X$, then

$$
\begin{aligned}
\Omega_{f}^{-1}\left(f_{B}^{+}\right)(x+y) & =f_{B}^{+}(f(x+y))=f_{B}^{+}(f(x)+f(y)) \\
& \supseteq f_{B}^{+}(f(x)) \cap f_{B}^{+}(f(y))
\end{aligned}
$$

$$
=\Omega_{f}^{-1}\left(f_{B}^{+}\right)(x) \cap \Omega_{f}^{-1}\left(f_{B}^{+}\right)(y)
$$

and

$$
\begin{aligned}
\Omega_{f}^{-1}\left(f_{B}^{-}\right)(x+y) & =f_{B}^{-}(f(x+y))=f_{B}^{-}(f(x)+f(y)) \\
& \subseteq f_{B}^{-}(f(x)) \cup f_{B}^{-}(f(y)) \\
& =\Omega_{f}^{-1}\left(f_{B}^{-}\right)(x) \cup \Omega_{f}^{-1}\left(f_{B}^{-}\right)(y),
\end{aligned}
$$

by similar way all other conditions of a DFS $h$-interior ideal are hold. Consequently, $\Omega_{f}^{-1}(B)$ is a double-framed soft $h$-interior ideal of $X$.

Example 6.12 Assume that $N_{0}$ and $N_{0} /(10)$ denote the hemirings of non-negative integers and the hemiring of non-negative integers module 10 , respectively. Define a mapping $\Omega: N_{0} \longrightarrow N_{0} /(10)$ by $\Omega(x)=[x]$. Then $\Omega$ is homomorphism from $N_{0}$ Onto $N_{0} /(10)$ define a double-framed soft set $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle$ of $N_{0}$ by

| $N_{0}$ | $x=[(10)]$ | otherwise |
| :--- | :--- | :--- |
| $f_{A}^{+}(a)$ | $Z^{-}$ | $\{2,-4, \ldots,-10\}$ |
| $f_{A}^{-}(a)$ | $\{-1,-3, \ldots,-9\}$ | $Z^{-}$ |

then by Theorem 6.11, $\Omega_{f}(A)$ is a double-framed soft $h$-interior ideal of $N_{0} /(10)$.
Theorem 6.13 If $f: X \longrightarrow Y$ is a homomorphism of a hemiring $R$ with $\Omega_{f}: \Omega(X) \longrightarrow \Omega(Y)$ and $\Omega_{f}^{-1}: \Omega(Y) \longrightarrow \Omega(X)$ be the double-framed soft transformation and inverse double-framed soft transformation induced by $f$ respectively, then
(1) If $f_{A}=\left\langle\left(f_{A}^{+}, f_{A}^{-}\right) ; A\right\rangle \in \Omega(X)$ is a double-framed soft left (resp. right) $h$-ideal of $X$ with double-framed soft property, then $\Omega_{f}(A)$ is a double-framed soft left (resp. right ) $h$-ideal of $Y$.
(2) If $f_{B}=\left\langle\left(f_{B}^{+}, f_{B}^{-}\right) ; B\right\rangle \in \Omega(Y)$ is a double-framed soft left (resp. right ) $h$-ideal of $Y$, then $\Omega_{f}^{-1}(B)$ is a double-framed soft left (resp. right ) $h$-ideal of $X$.
Proof. Follow from Theorem 6.11.

## 7. Conclusion

In this research, we developed a new type of soft $h$-ideal theory known as double-framed soft $h$ interior ideals and double-framed soft $h$-ideals of a hemiring $R$. Further, several classes of hemirings like, $h$-hemiregular, $h$-simple and $h$-semisimple hemirings are characterized by the properties of double-framed soft $h$-interior ideals and double-framed soft $h$-ideals of $R$. Moreover, ordinary $h$-interior ideals are linked with double-framed soft $h$-interior ideals. Keeping in view the vitality and potential of the present work, the notion of double-framed soft sets can be applied to semigroups and ordered semigroups to investigate various type of double-framed soft ideals in the said algebraic structures. Since, gamma hemirings are the generalization of classical hemirings, therefore, the notion of doubleframed soft sets can also be applied to gamma hemirings where different type of $h$-gamma-ideals can be
studied using double-framed soft sets. In addition, gamma semirings, gamma semigroups and gamma ordered semigroups can also be characterized by the properties of double-framed soft sets with diverse applications in various applied fields of science.

## Acknowledgments

The authors would like to thank reviewers and the editor for their constructive and valuable suggestions which undoubtedly elevated the presentation of this paper. This research was supported by National Natural Science Foundation of China (No.11501450) and Shaanxi Province Postdoctoral Research Project Funding.

## Conflict of interest

The authors declare no conflicts of interest in this paper.

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