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Research article

Applications of a differential operator to a class of harmonic mappings defined by Mittag-leffer functions

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Abstract: Utilizing the concepts of Harmonic analysis and Mittag-Leffler functions we introduce a new subclass of harmonic mappings involving differential operator in domain of Janowski functions. Moreover, we investigate analytic criteria, necessary and sufficient conditions, topological properties, extreme points, radii problems and some applications of this work for the class of functions defined by this operator.

Keywords: harmonic functions; Janowski function; starlike functions; Mittag-Leffer Functions; radii problems
Mathematics Subject Classification: 30C45, 30C50

1. Introduction

1.1. Background of the research

The study of analytic functions has been the core interest of various prominent researchers in the last decade. Much emphasis has been on the aspect of introduction of various concepts in this field. Uralegaddi [1], in 1994, introduced the subclasses of starlike, convex and close-to-convex functions with positive coefficients and opened a new side of Geometric Function Theory. Motivated by his work Dixit and Chandra [2] introduced new subclass of analytic functions with positive coefficients. Continuing the trend Dixit et al. [3], Porwal and Dixit [4] and Porwal et al. [5] made a substantial amount of important theory which illuminated various new directions of this field. One such area is Harmonic Analysis which has vastly influenced and nurtured the branch of Geometric Function Theory.

Dixit and Porwal [6] defined and investigated the class of harmonic univalent functions with positive coefficients. With the introduction of this work many mathematicians generalized various important results with the help of some operators, the work of Pathak et al. [7], Porwal and Aouf [8] and Porwal et al. [9] are worth mentioning here. More recently new subclasses of harmonic starlike and convex functions are introduced and studied by Porwal and Dixit [10], see also [5].

Recently attention has been drawn to Mittag-Leffer functions as these functions can be widely applied across the fields of engineering, chemical, biological, physical sciences as will as in various other applied sciences. Various factors in applying such functions are evident within chaotic, stochastic, dynamic systems, fractional differential equations and distribution of statistics. The geometric characteristics such as convexity, close-to-convexity and starlikeness of the functions investigated here has been broadly examined by many authors. Direct applications of these functions can be seen in a number of fractional calculus tools which includes significant work by [11–20].

1.2. Preliminaries

Before we go into details about our new work we give some basics which will be helpful in understanding the concepts of this research.

A real-valued function $u(x, y)$ is said to be harmonic in a domain $\mathbb{D} \subset \mathbb{C}$ if it has continuous second partial derivative and satisfy the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

and complex-valued function $f = u + iv$ is said to be harmonic in a domain $\mathbb{D}$ if and only if u and v are both real harmonic functions in domain $\mathbb{D}$. Every complex-valued harmonic function f which is harmonic in $\mathbb{D}$, containing the origin, can be represented in the canonical form as

$$f = h(z) + \overline{g(z)}, \tag{1.1}$$

where h and g are analytic functions in $\mathbb{D}$ with $g(0) = 0$.Then functions h and g are known as analytic and co-analytic parts of f respectively. The Jacobian of $f = u + iv$ is given by

$$J_f(z) = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = u_x v_y - v_x u_y ,$$

which can be represented in terms of derivatives with respect to z and $\overline{z}$ as

$$J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 = \left|h'(z)\right|^2 - \left|g'(z)\right|^2 \quad (z \in \mathbb{D}).$$

It can be noted that if f is analytic in $\mathbb{D}$, then $f_{\overline{z}} = 0$ and $f_z(z) = f'(z)$. A well- known result for analytic functions state that an analytic function f is locally univalent at a point z_0 if and only if $J_f(z) \neq 0$ in $\mathbb{D}$ (see for example [21]). In [22], Lewy proved the converse of this theorem which is also true for harmonic mappings. Therefore, f is sense-preserving and locally univalent if and only if

$$\left|h'(z)\right| > \left|g'(z)\right| \quad (z \in \mathbb{D}). \tag{1.2}$$

Let $\mathcal{H}$ denote the class of functions f which are harmonic in the unit disc $\mathfrak{A} := \mathfrak{A}(1)$, where $\mathfrak{A}(r) := \{z \in \mathbb{C} : |z| < r\}$. Also, let $\mathcal{H}_0$ denote the class of functions $f \in \mathcal{H}$ which satisfy the normalization conditions

$$f(0) = f_{\bar{z}}'(0) = f'(0) - 1 = 0.$$

Therefore the analytic functions h and g given by (1.1) can be written in the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ g(z) = \sum_{n=2}^{\infty} b_n z^n \ (z \in \mathfrak{A}),$$

and

$$f(z) = z + \sum_{n=2}^{\infty} \left(a_n z^n + \overline{b_n z^n}\right) \ (z \in \mathfrak{A}). \tag{1.3}$$

Let

$$\mathcal{S}_{\mathcal{H}} := \{f \in \mathcal{H}_0 : f \text{ is univalent and sense-preserving in } \mathfrak{A}\}.$$

It is clear that $\mathcal{S}_{\mathcal{H}}$ reduces to the class $\mathcal{S}$, and by $\mathcal{A}$ whenever the co-analytic part of f vanishes, i.e., $g(z) = 0$ in $\mathfrak{A}$. Clunie and Sheil-Small [23] and Sheil-Small [24] studied $\mathcal{S}_{\mathcal{H}}$ together with some of its geometric subclasses. We say that a function $f \in \mathcal{H}_0$ is said to be harmonic starlike in $\mathfrak{A}$ if it satisfy

$$\mathrm{Re}\frac{\mathcal{D}_{\mathcal{H}} f(z)}{f(z)} > 0,$$

where

$$\mathcal{D}_{\mathcal{H}} f(z) := h'(z) - \overline{g'(z)} \ (z \in \mathfrak{A}).$$

A function $f(z)$ is subordinated to a function $g(z)$ denoted by $f(z) \prec g(z)$, if there is complex-valued function $w(z)$ with $|w(z)| \le 1$ and $g(0) = 0$ such that

$$f(z) = g(w(z)) \quad (z \in \mathfrak{A}).$$

Also, if $g(z)$ is univalent in $\mathfrak{A}$, we have equivalence condition

$$f(z) \prec g(z), \quad z \in \mathfrak{A} \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathfrak{A}) \subset g(\mathfrak{A}).$$

Convolution or Hadamard product of two function f_1 and f_2 is denoted by $f_1 * f_2$ and is defined by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} \left(a_1 a_2 z^n + \overline{b_1 b_2 z^n}\right) \ (z \in \mathfrak{A}). \tag{1.4}$$

In 1973, Janowski [25] introduced the idea of circular domain by introducing Janowski functions as;

A function $k(z)$, analytic in $\mathfrak{A}$ with $k(0) = 1$, is said to be in class $T[A, B]$ if for $-1 \le B < A \le 1$

$$k(z) \prec \frac{1 + Az}{1 + Bz}.$$

Janowski showed that the function k maps $\mathfrak{A}$ onto the domain $\Delta(A, B)$ with centre on real axis and $D_1 = \frac{1-A}{1-B}$ and $D_2 = \frac{1+A}{1+B}$ are diameter end points with $0 < D_1 < 1 < D_2$.

The Mittag-Leffer function is defined as

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\alpha n+1)}. \tag{1.5}$$

The initial two parametric generalizations for the function shown in (1.5) were given by Wiman [26, 27]. It is defined in the following way

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\alpha n+\beta)},$$

where $\alpha,\beta\in\mathbb{C}$, $\operatorname{Re}(\alpha)>0$, $\operatorname{Re}(\beta)>0$ and $\Gamma(z)$ is gamma function.

Now the function $\mathrm{Q}_{\alpha,\beta}$ is defined by

$$\mathrm{Q}_{\alpha,\beta}(z)=z\Gamma(\beta)E_{\alpha,\beta}(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}z^{n}.$$

Using the function $\mathrm{Q}_{\alpha,\beta}$ Elhaddad et al. [28] defined the differential operator for the class of analytic functions as $\mathcal{D}_{\delta}^{m}(\alpha,\beta):\mathcal{A}\rightarrow\mathcal{A}$ as illustrated below :

$$\mathcal{D}_{\delta}^{m}(\alpha,\beta)f(z)=f(z)*\mathrm{Q}_{\alpha,\beta}(z),$$

$$\mathcal{D}_{\delta}^{m}(\alpha,\beta)f(z)=z+\sum_{n=2}^{\infty}[1+(n-1)\delta]^{m}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}a_{n}z^{n}, \tag{1.6}$$

where $m=\mathbb{N}_{0}=\{0,1,2,...\}$, $\delta>0$.

Where the operator $\mathcal{D}_{\delta}^{m}(\alpha,\beta)$ for a function $f\in\mathcal{H}$ given by (1.1) can be defined as below:

$$\mathcal{D}_{\delta}^{m}(\alpha,\beta)f(z)=\mathcal{D}_{\delta}^{m}(\alpha,\beta)h(z)+\overline{\mathcal{D}_{\delta}^{m}(\alpha,\beta)g(z)}\ \ (z\in\mathfrak{A}),$$

where

$$\begin{aligned}\mathcal{D}_{\delta}^{m}(\alpha,\beta)h(z) &= z+\sum_{n=2}^{\infty}[1+(n-1)\delta]^{m}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}a_{n}z^{n},\\ \mathcal{D}_{\delta}^{m}(\alpha,\beta)g(z) &= z+\sum_{n=2}^{\infty}[1+(n-1)\delta]^{m}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}b_{n}z^{n},\end{aligned}$$

for $m\in\mathbb{N}_{0}$.

Motivated by [29, 30] and using the operator $\mathcal{D}_{\delta}^{m}(\alpha,\beta)f(z)$, we introduced the class of harmonic univalent functions as:

Definition Let $-B\leq A<B\leq1$, $0\leq a<1$ and $\mathcal{S}_{\mathcal{H}}^{\alpha,\beta}(m,\delta,A,B)$ denote the class of functions $f\in\mathcal{S}_{\mathcal{H}}$ such that

$$\frac{\mathcal{D}_{\mathcal{H}}\left(\mathcal{D}_{\delta}^{m}(\alpha,\beta)f(z)\right)}{\mathcal{D}_{\delta}^{m}(\alpha,\beta)f(z)}\prec\frac{1+Az}{1+Bz}, \tag{1.7}$$

with

$$\mathcal{D}_{\mathcal{H}}(\mathcal{D}_{\delta}^{m}(\alpha,\beta)f(z)):=\mathcal{D}_{\mathcal{H}}(\mathcal{D}_{\delta}^{m}(\alpha,\beta)h(z))-\overline{\mathcal{D}_{\mathcal{H}}\left(\mathcal{D}_{\delta}^{m}(\alpha,\beta)g(z)\right)}.$$

Note that,

1. $\mathcal{S}_{\mathcal{H}}^{0,1}(0,\delta,A,B)=\mathcal{S}_{\mathcal{H}}^{*}(A,B)$, which was studied by Deziok [29].
2. $\mathcal{S}_{\mathcal{H}}^{0,1}(0,\delta,2a-1,1)=\mathcal{S}_{\mathcal{H}}^{*}(a)$, defined by Jahangiri in [31]
3. $\mathcal{S}_{\mathcal{H}}^{0,1}(1,1,2a-1,1)=\mathcal{S}_{\mathcal{H}}^{c}(a)$, introduced by Jahangiri, see [32] for details.

Let $\mathcal{V}\subset\mathcal{H}_0$, $\mathfrak{A}_0=\mathfrak{A}\setminus\{0\}$. Using Ruscheweyh's approach in [33] we define the dual set of $\mathcal{V}$ by

$$\mathcal{V}^{*}:=\left\{f\in\mathcal{H}_0:*_{g\in\mathcal{V}}(f*g)\neq 0\quad(z\in\mathfrak{A}_0)\right\}$$

2. Main criteria

In this section we prove some important results beginning with necessary and sufficient condition. Then some inequality regarding the coefficients of the functions in their series form are evaluated along with examples for justifications.

2.1. Theorem

Let $f\in\mathcal{H}_0$ and is given by (1.3) is in the class $\mathcal{S}_{\mathcal{H}}^{\alpha,\beta}(m,\delta,A,B)$ if and only if

$$\mathcal{S}_{\mathcal{H}}^{\alpha,\beta}(m,\delta,A,B)=\left\{\mathcal{D}_{\delta}^{m}(\alpha,\beta)\varphi_{\xi}(z);\ |\xi|=1\right\}^{*},$$

where

$$\varphi_{\xi}(z)=z\frac{1+B\xi-(1+A\xi)(1-z)}{(1-z)^2}-\overline{z}\frac{1+B\xi-(1+A\xi)(1-\overline{z})}{(1-\overline{z})^2}\quad(z\in\mathbb{D}).$$

Proof. Let $f\in\mathcal{H}_0$, then $f\in\mathcal{S}_{\mathcal{H}}^{\alpha,\beta}(m,\delta,A,B)$ if and only if the following holds

$$\frac{\mathcal{D}_{\mathcal{H}}\left(\mathcal{D}_{\delta}^{m}(\alpha,\beta)f(z)\right)}{\mathcal{D}_{\delta}^{m}(\alpha,\beta)f(z)}\neq\frac{1+A\xi}{1+B\xi}\quad(\xi\in\mathbb{C},\ |\xi|=1).$$

Now as

$$\mathcal{D}_{\mathcal{H}}\left(\mathcal{D}_{\delta}^{m}(\alpha,\beta)h(z)\right)=\mathcal{D}_{\delta}^{m}(\alpha,\beta)h(z)*\frac{z}{(1-z)^2}$$

and

$$\mathcal{D}_{\delta}^{m}(\alpha,\beta)h(z)=\mathcal{D}_{\delta}^{m}(\alpha,\beta)h(z)*\frac{z}{1-z},$$

thus

$$\begin{aligned}
&(1+B\xi)\mathcal{D}_{\mathcal{H}}\left(\mathcal{D}_{\delta}^{m}(\alpha,\beta)f(z)\right)-(1+A\xi)\mathcal{D}_{\delta}^{m}(\alpha,\beta)f(z)\\
=\;&(1+B\xi)\mathcal{D}_{\mathcal{H}}\left(\mathcal{D}_{\delta}^{m}(\alpha,\beta)h(z)\right)-(1+A\xi)\mathcal{D}_{\delta}^{m}(\alpha,\beta)h(z)\\
&-\left[(1+B\xi)\mathcal{D}_{\mathcal{H}}\left(\overline{\mathcal{D}_{\delta}^{m}(\alpha,\beta)g(z)}\right)+(1+A\xi)\overline{\mathcal{D}_{\delta}^{m}(\alpha,\beta)g(z)}\right]\\
=\;&\mathcal{D}_{\delta}^{m}(\alpha,\beta)h(z)*\left(\frac{(1+B\xi)z}{(1-z)^2}-\frac{(1+A\xi)z}{1-z}\right)\\
&-\overline{\mathcal{D}_{\delta}^{m}(\alpha,\beta)g(z)}*\left(\frac{(1+B\xi)\overline{z}}{(1-\overline{z})^2}+\frac{(1+A\xi)\overline{z}}{1-\overline{z}}\right)\\
=\;&f(z)*\mathcal{D}_{\delta}^{m}(\alpha,\beta)\varphi_{\xi}(z)\neq 0\quad(z\in\mathfrak{A}_0,\ |\xi|=1).
\end{aligned}$$

Thus $f\in\mathcal{S}_{\mathcal{H}}^{\alpha,\beta}(m,\delta,A,B)$ if and only if $f(z)*\mathcal{D}_{\delta}^{m}(\alpha,\beta)\varphi_{\xi}(z)\neq 0$ for $z\in\mathfrak{A}_0$, $|\xi|=1$ i.e. $\mathcal{S}_{\mathcal{H}}^{\alpha,\beta}(m,\delta,A,B)=\left\{\mathcal{D}_{\delta}^{m}(\alpha,\beta)\varphi_{\xi}(z);\ |\xi|=1\right\}^{*}$. □

A sufficient coefficient bound for the class $\mathcal{S}_{\mathcal{H}}^{\alpha,\beta}(m,\delta,A,B)$ is provided in the following.

2.2. Theorem

Let $f \in \mathcal{H}_0$ be of the form (1.3) and satisfies the condition

$$\sum_{n=2}^{\infty}\left(\lambda_n |a_n| + \sigma_n |b_n|\right) \leq B - A, \tag{2.1}$$

with

$$\begin{aligned} \lambda_n &= \left|\frac{[1+(n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right| [(1+B)n-(1+A)], & (2.2)\\ \sigma_n &= \left|\frac{[1+(n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right| [(1+B)n+(1+A)], & (2.3)\end{aligned}$$

then $f \in \mathcal{S}_{\mathcal{H}}^{\alpha,\beta}(m,\delta,A,B)$.

Proof. Obviously the theorem is true for $f(z)=z$. Suppose $f \in \mathcal{H}_0$ given by (1.3) and let there exist $n \geq 2$ such that $a_n \neq 0$ or $b_n \neq 0$. Since

$$\frac{\lambda_n}{B-A} \geq n, \quad \frac{\sigma_n}{B-A} \geq n, \quad n = 2,3,...,$$

by (2.1) we have

$$\sum_{n=2}^{\infty}\left(n\ |a_n| + n\ |b_n|\right) \leq 1 \tag{2.4}$$

and

$$\begin{aligned} \left|h'(z)\right| - \left|g'(z)\right| &\geq 1 - \sum_{n=2}^{\infty} n\ |a_n| |z|^n - \sum_{n=2}^{\infty} n\ |b_n| |z|^n \geq 1 - |z| \sum_{n=2}^{\infty}\left(n\ |a_n| + n\ |b_n|\right)\\ &\geq 1 - \frac{|z|}{B-A}\sum_{n=2}^{\infty}\left(\lambda_n |a_n| + \sigma_n |a_n|\right) \geq 1-|z| > 0 \quad (z \in \mathfrak{A}).\end{aligned}$$

Therefore$\left|h'(z)\right| > \left|g'(z)\right|$ which shows that f is locally univalent and sense-preserving in $\mathfrak{A}$. Moreover if$z_1, z_2 \in \mathfrak{A}$ and $z_1 \neq z_2$ then

$$\left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| = \left|\sum_{k=1}^{n} z_1^{k-1} z_2^{n-k}\right| \leq \sum_{k=1}^{n}\left|z_1^{k-1}\right|\left|z_2^{n-k}\right| < n \quad (n = 2,3,..).$$

Hence by (2.4) we have

$$\begin{aligned} |f(z_1) - f(z_2)| &\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)|\\ &\geq \left|z_1 - z_2 - \sum_{n=2}^{\infty} a_n \left(z_1^n - z_2^n\right)\right| - \left|\sum_{n=2}^{\infty} \overline{b_n\left(z_1^n - z_2^n\right)}\right|\end{aligned}$$

$$
\begin{aligned}
&\geq |z_1 - z_2| - \sum_{n=2}^{\infty} |a_n| \left|z_1^n - z_2^n\right| - \sum_{n=2}^{\infty} |b_n| \left|z_1^n - z_2^n\right| \\
&= |z_1 - z_2| \left(1 - \sum_{n=2}^{\infty} |a_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right| - \sum_{n=2}^{\infty} |b_n| \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right|\right) \\
&> |z_1 - z_2| \left(1 - \sum_{n=2}^{\infty} n\ |a_n| - \sum_{n=2}^{\infty} n\ |b_n|\right) \geq 0.
\end{aligned}
$$

This shows that f is univalent, i.e. $f \in \mathcal{S}_{\mathcal{H}}$. Therefore $f \in \mathcal{S}_{\mathcal{H}}^{\alpha,\beta}(m, \delta, A, B)$ if and only if there exists a complex-valued function ω, $\omega(0) = 0$, $|\omega(z)| < 1$ $(z \in \mathfrak{U})$, such that

$$
\frac{\mathcal{D}_{\mathcal{H}}\left(\mathcal{D}_{\delta}^{m}(\alpha,\beta) f(z)\right)}{\mathcal{D}_{\delta}^{m}(\alpha,\beta) f(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (z \in \mathfrak{U}),
$$

or equivalently

$$
\left|\frac{\mathcal{D}_{\mathcal{H}}\left(\mathcal{D}_{\delta}^{m}(\alpha,\beta) f(z)\right) - \mathcal{D}_{\delta}^{m}(\alpha,\beta) f(z)}{B\mathcal{D}_{\mathcal{H}}\left(\mathcal{D}_{\delta}^{m}(\alpha,\beta) f(z)\right) - A\mathcal{D}_{\delta}^{m}(\alpha,\beta) f(z)}\right| < 1 \quad (z \in \mathfrak{U}). \tag{2.5}
$$

Thus, it is sufficient to prove that

$$
\left|\mathcal{D}_{\mathcal{H}}\left(\mathcal{D}_{\delta}^{m}(\alpha,\beta) f(z)\right) - \mathcal{D}_{\delta}^{m}(\alpha,\beta) f(z)\right| - \left|B\mathcal{D}_{\mathcal{H}}\left(\mathcal{D}_{\delta}^{m}(\alpha,\beta) f(z)\right) - A\mathcal{D}_{\delta}^{m}(\alpha,\beta) f(z)\right| < 0,
$$

where $z \in \mathfrak{U} \setminus \{0\}$, now by putting $|z| = r$, $r \in (0, 1)$ we get

$$
\begin{aligned}
&\left|\mathcal{D}_{\mathcal{H}}\left(\mathcal{D}_{\delta}^{m}(\alpha,\beta) f(z)\right) - \mathcal{D}_{\delta}^{m}(\alpha,\beta) f(z)\right| - \left|B\mathcal{D}_{\mathcal{H}}\left(\mathcal{D}_{\delta}^{m}(\alpha,\beta) f(z)\right) - A\mathcal{D}_{\delta}^{m}(\alpha,\beta) f(z)\right| \\
= &\left|\sum_{n=2}^{\infty} \frac{[1 + (n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)} (n-1) a_n z^n - \sum_{n=2}^{\infty} \frac{[1 + (n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)} (n+1) \overline{b_n z^n}\right| \\
&- \left|(B - A) z + \sum_{n=2}^{\infty} \frac{[1 + (n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)} (Bn - A) a_n z^n - \frac{[1 + (n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)} (Bn + A) \overline{b_n z^n}\right|, \\
\leq &\sum_{n=2}^{\infty} \left|\frac{[1 + (n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)}\right| (n-1) |a_n| r^n + \sum_{n=2}^{\infty} \left|\frac{[1 + (n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)}\right| (n+1) |b_n| r^n \\
&- (B - A) r + \sum_{n=2}^{\infty} \left|\frac{[1 + (n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)}\right| (Bn - A) |a_n| r^n \\
&\qquad + \sum_{n=2}^{\infty} \left|\frac{[1 + (n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)}\right| (Bn + A) |b_n| r^n \\
\leq &\ r\left\{\sum_{n=2}^{\infty} (\lambda_n |a_n| + \sigma_n |b_n|) r^n - (B - A)\right\} < 0,
\end{aligned}
$$

hence $f \in \mathcal{S}_{\mathcal{H}}^{\alpha,\beta}(m, \delta, A, B)$. □

2.3. Example

For function

$$f(z) = z + \sum_{n=2}^{\infty} p_n \frac{B-A}{\lambda_n} z^n + \sum_{n=2}^{\infty} q_n \frac{B-A}{\sigma_n} \overline{z}^n \ \ (z \in \mathfrak{U}),$$

such that $\sum_{n=2}^{\infty} (|p_n| + |q_n|) = 1$, we have

$$\begin{aligned} \sum_{n=2}^{\infty} (\lambda_n |a_n| + \sigma_n |b_n|) &= \sum_{n=2}^{\infty} (|p_n| (B-A) + |q_n| (B-A)) \\ &= (B-A) \sum_{n=2}^{\infty} (|p_n| + |q_n|) = (B-A). \end{aligned}$$

Thus $f \in \mathcal{S}_{\mathcal{H}}^{\alpha,\beta}(m, \delta, A, B)$ and above inequality (2.1) is sharp for this function.

Motivated from Silverman [34], we introduce the class τ for functions $f \in \mathcal{H}_0$ of the form (1.3) such that $a_n = -|a_n|,\ b_n = |b_n|\ (n = 2, 3, ...)$, i.e.

$$f(z) = h(z) + \overline{g(z)},\ h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n,\ g(z) = \sum_{n=2}^{\infty} |b_n| \overline{z}^n \ \ (z \in \mathfrak{U}). \tag{2.6}$$

Further, let us define

$$\mathcal{S}_{\tau}^{\alpha,\beta}(m, \delta, A, B) = \tau \cap \mathcal{S}_{\mathcal{H}}^{\alpha,\beta}(m, \delta, A, B).$$

Where $\alpha = 0, \beta = 1$ and $m = 0$ the class is studied by Dziok see [29].

2.4. Theorem

Let $f \in \tau$ and of the form (2.6). Then $f \in \mathcal{S}_{\tau}^{\alpha,\beta}(m, \delta, A, B)$ if and only if condition (2.1) holds true.

Proof. In Theorem 2.2 we need only to show that each function $f \in \mathcal{S}_{\tau}^{\alpha,\beta}(m, \delta, A, B)$ satisfies coefficient inequality (2.1). If $f \in \mathcal{S}_{\tau}^{\alpha,\beta}(m, \delta, A, B)$ then it is of the form (2.6) and satisfies (2.5) or equivalently

$$\left| \frac{-\sum_{n=2}^{\infty} \frac{[1+(n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} (n-1) a_n z^n - \sum_{n=2}^{\infty} \frac{[1+(n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} (n+1) \overline{b_n z^n}}{(B-A) z - \sum_{n=2}^{\infty} \frac{[1+(n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} (Bn - A) a_n z^n - \sum_{n=2}^{\infty} \frac{[1+(n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} (Bn + A) \overline{b_n z^n}} \right| < 1,$$

where $z \in \mathfrak{U}$, therefore by putting $z = r$, $r \in [0, 1)$, we get

$$\frac{\sum_{n=2}^{\infty} \left| \frac{[1+(n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \right| [(n-1)|a_n| + (n+1)|b_n|] r^{n-1}}{(B-A) + \sum_{n=2}^{\infty} \left| \frac{[1+(n-1)\delta]^m \Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \right| \{(Bn - A)|a_n| + (Bn + A)|b_n|\}} < 1. \tag{2.7}$$

It is clear that the denominator of the left hand side cannot vanishes for $r \in (0, 1)$. Moreover, it is positive for $r = 0$, and in consequence for $r \in (0, 1)$. Thus, by (2.7) we have

$$\sum_{n=2}^{\infty} (\lambda_n |a_n| + \sigma_n |b_n|) r^{n-1} \le B - A \ \ r \in [0, 1). \tag{2.8}$$

The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=2}^{\infty}(\lambda_n|a_n| + \sigma_n|b_n|)$ is a non-decreasing sequence. Moreover, by (2.8) it is bounded by $B - A$. Hence, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=2}^{\infty}(\lambda_n|a_n| + \sigma_n|b_n|)\,r^{n-1} = \lim_{n\to\infty} S_n \leq B - A,$$

which yields assertion (2.1). □

2.5. Example

For function

$$f(z) = z - \sum_{n=2}^{\infty} c_n \frac{B-A}{\lambda_n} z^n + \sum_{n=2}^{\infty} d_n \frac{B-A}{\sigma_n} \overline{z}^n \quad (z \in \mathfrak{U}),$$

such that $\sum_{n=2}^{\infty}(|c_n| + |d_n|) = 1$, we have

$$\begin{aligned}\sum_{n=2}^{\infty}(\lambda_n|a_n| + \sigma_n|b_n|) &= \sum_{n=2}^{\infty}(|c_n|(B-A) + |d_n|(B-A)) \\ &= (B-A)\sum_{n=2}^{\infty}(|c_n| + |d_n|) = (B-A).\end{aligned}$$

Thus $f \in \mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)$.

3. Topological properties

we consider the usual topology on $\mathcal{H}$ in which a sequence $\{f_n\}$ in $\mathcal{H}$ converges to f if and only if it converges to f uniformly on each compact subset of $\mathfrak{U}$. The metric induces the usual topology on $\mathcal{H}$. It is to verify that the obtained topological space is complete.

Let $\mathcal{F}$ be a subclass of the class $\mathcal{H}$. A function $f \in \mathcal{F}$ is called an extreme point of $\mathcal{F}$ if the condition

$$f = \gamma f_1 + (1-\gamma) f_2 \quad (f_1, f_2 \in \mathcal{F},\ 0 < \gamma < 1)$$

implies $f_1 = f_2 = f$. We shall use the notation $E\mathcal{F}$ to denote the set of all extreme points of $\mathcal{F}$. It is clear that $E\mathcal{F} \subset \mathcal{F}$.

We say that $\mathcal{F}$ is locally uniformly bounded if for each r, $0 < r < 1$, there is a real constant $M = M(r)$ so that

$$|f(z)| \leq M \quad (f \in \mathcal{F},\ |z| \leq r).$$

We say that a class $\mathcal{F}$ is convex if

$$\gamma f + (1-\gamma) g \in \mathcal{F} \quad (f, g \in \mathcal{F},\ 0 \leq \gamma \leq 1).$$

Moreover, we define the closed convex hull of $\mathcal{F}$ as the intersection of all closed convex subsets of $\mathcal{H}$ that contain $\mathcal{F}$. We denote the closed convex hull of $\mathcal{F}$ by $\overline{co}\mathcal{F}$.

A real-valued function $\mathfrak{J} : \mathcal{H} \to \mathbb{R}$ is called convex on a convex class $\mathcal{F} \subset \mathcal{H}$ if

$$\mathfrak{J}(\gamma f + (1-\gamma) g) \leq \gamma \mathfrak{J}(f) + (1-\gamma)\mathfrak{J}(g) \quad (f, g \in \mathcal{F},\ 0 \leq \gamma \leq 1).$$

The Krein-Milman theorem (see [35]) is fundamental in the theory of extreme points. In particular, it implies the following lemma.

3.1. Lemma

Let $\mathcal{F}$ be a non-empty convex compact subclass of the class $\mathcal{H}$ and let $\mathfrak{J} : \mathcal{H} \to \mathbb{R}$ be a real-valued, continuous and convex function on $\mathcal{F}$. Then

$$\max\{\mathfrak{J}(f) : f \in \mathcal{F}\} = \max\{\mathfrak{J}(f) : f \in E\mathcal{F}\}.$$

3.2. Lemma

A class $\mathcal{F} \subset \mathcal{H}$ is compact if and only if $\mathcal{F}$ is closed and locally uniformly bounded.

Since $\mathcal{H}$ is complete metric space, Montel's theorem (see [36]) implies the following lemma.

3.3. Lemma

Let $\mathcal{F}$ be a non-empty compact subclass of the class $\mathcal{H}$, then $E\mathcal{F}$ is non-empty and $\overline{co}E\mathcal{F} = \overline{co}\mathcal{F}$.

3.4. Theorem

The class $\mathcal{S}_{\tau}^{\alpha,\beta}(m, \delta, A, B)$ is a convex and compact subset of $\mathcal{H}$.

Proof. Let $f_l \in \mathcal{S}_{\tau}^{\alpha,\beta}(m, \delta, A, B)$ be a functions of the form

$$f_l(z) = z - \sum_{n=2}^{\infty}\left(\left|a_{l,n}\right| z^n - \left|b_{l,n}\right| \overline{z}^n\right) \quad (z \in \mathfrak{A},\, l \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{3.1}$$

and $0 \leq \gamma \leq 1$. Since

$$\gamma f_1(z) + (1-\gamma) f_2(z) =$$

$$z - \sum_{n=2}^{\infty}\left\{\left(\gamma\left|a_{1,n}\right| + (1-\gamma)\left|a_{2,n}\right|\right) z^n + \left(\gamma\left|b_{1,n}\right| + (1-\gamma)\left|b_{2,n}\right|\right)\right\} \overline{z}^n$$

and by Theorem 2.4, we have

$$\begin{aligned}
& \sum_{n=2}^{\infty}\left\{\alpha_n\left(\gamma\left|a_{1,n}\right| + (1-\gamma)\left|a_{2,n}\right|\right) z^n + \beta_n\left(\gamma\left|b_{1,n}\right| + (1-\gamma)\left|b_{2,n}\right|\right)\right\} \\
= \;& \gamma\sum_{n=2}^{\infty}\left\{\alpha_n\left|a_{1,n}\right| + \beta_n\left|b_{1,n}\right|\right\} + (1-\gamma)\sum_{n=2}^{\infty}\left\{\alpha_n\left|a_{2,n}\right| + \beta_n\left|b_{2,n}\right|\right\} \\
\leq \;& \gamma(B-A) + (1-\gamma)(B-A) = B - A,
\end{aligned}$$

the function $\Psi = \gamma f_1 + (1-\gamma) f_2 \in \mathcal{S}_{\tau}^{\alpha,\beta}(m, \delta, A, B)$. Hence, the class is convex. Furthermore, for $f \in \mathcal{S}_{\tau}^{\alpha,\beta}(m, \delta, A, B)$, $|z| \leq r$ $r \in (0, 1)$, we have

$$|f(z)| \leq r + \sum_{n=2}^{\infty}(|a_n| + |b_n|) r^n \leq r + \sum_{n=2}^{\infty}(\alpha_n|a_n| + \beta_n|b_n|) \leq r + (B-A). \tag{3.2}$$

Thus, we conclude that the class $\mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)$ is locally uniformly bounded. By Lemma 3.2, we need only to show that it is closed, i.e. if $f_l \to f$, then $f \in \mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)$. Let f_l and f be given by (3.1) and (2.6), respectively. Using Theorem 2.4, we have

$$\sum_{n=2}^{\infty}\left(\alpha_n\left|a_{i,n}\right|+\beta_n\left|b_{i,n}\right|\right)\le B-A \quad (i\in\mathbb{N}). \tag{3.3}$$

Since $f_i \to f$, we conclude that $\left|a_{i,n}\right| \to |a_n|$ and $\left|b_{i,n}\right| \to |b_n|$ as $i\to\infty$ $(n\in\mathbb{N})$. The sequence of partial sums $\{S_n\}$ associated with the series $\sum_{n=2}^{\infty}\left(\alpha_n\left|a_{i,n}\right|+\beta_n\left|b_{i,n}\right|\right)$ is non-decreasing sequence. Moreover, by (3.3) it is bounded by $B-A$. Therefore, the sequence $\{S_n\}$ is convergent and

$$\sum_{n=2}^{\infty}\left(\alpha_n\left|a_{i,n}\right|+\beta_n\left|b_{i,n}\right|\right)=\lim_{n\to\infty}\{S_n\}\le B-A.$$

This gives condition (2.1) and in consequence, $f \in \mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)$, which complete the proof. □

3.5. Theorem

We have

$$E\mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)=\{h_n^*: n\in\mathbb{N}\}\cup\{g_n^*: n\in\{2,3,...\}\},$$

where

$$h_1^*(z)=z,\ h_n^*(z)=z-\frac{B-A}{\lambda_n}z^n,\ g_n^*(z)=z+\frac{B-A}{\sigma_n}\overline{z}^n\ \ (n=2,3,...,\ z\in\mathfrak{A}) \tag{3.4}$$

Proof. Suppose that $0<\gamma<1$ and

$$g_n^*(z)=\gamma f_1+(1-\gamma)f_2,$$

where f_1,$f_2 \in \mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)$ are functions of the form (3.1). Then, by (2.1) we have

$$\left|b_{1,n}\right|=\left|b_{2,n}\right|=\frac{B-A}{\sigma_n},$$

and, in consequence , $a_{1,i}=a_{2,i}=0$ for $i\in\{2,3,...\}$ and $b_{1,i}=b_{2,i}=0$ for $i\in\{2,3,...\}\setminus\{n\}$. It follows that $g_n^*=f_1=f_2$, and consequently $g_n^*\in E\mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)$. Similarly, we verify that the functions h_n^* of the form (3.4) are extreme points of the class $\mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)$. Now, suppose that $f\in E\mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)$ and f is not of the form (3.4). Then there exists $i\in\{2,3,...\}$ such that

$$0<|a_i|<\frac{B-A}{\lambda_i}\ \text{ or }\ 0<|b_i|<\frac{B-A}{\sigma_i}.$$

If $0<|a_i|<\frac{B-A}{\lambda_i}$, then putting

$$\gamma=\frac{|a_i|\lambda_i}{B-A},\ \Phi=\frac{1}{1-\gamma}\left(f-\gamma h_i^*\right),$$

we have that $0<\gamma<1$, $h_i^*,\Phi\in\mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)$, $h_i^*\ne\Phi$ and

$$f=\gamma h_i^*+(1-\gamma)\Phi.$$

Thus, $f \notin E\mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)$. Similarly, if $0<|b_i|<\frac{B-A}{\sigma_i}$, then putting

$$\gamma=\frac{|b_i|\,\sigma_i}{B-A},\ \ \Psi=\frac{1}{1-\gamma}\left(f-\gamma g_i^*\right),$$

we have that $0<\gamma<1$, $g_i^*, \Phi\in\mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)$, $g_i^*\neq\Psi$ and

$$f=\gamma g_i^*+(1-\gamma)\Psi.$$

It follows that $f\notin E\mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)$, and this completes the proof. □

4. Radii of starlikeness and convexity

A function $f\in\mathcal{H}_0$ is said to be starlike of order α in $\mathfrak{A}(r)$ if

$$\frac{\partial}{\partial t}\left(\arg f\left(\rho e^{it}\right)\right)>\alpha,\ 0\le t\le 2\pi,\ 0<\rho<r<1. \tag{4.1}$$

Also, A function $f\in\mathcal{H}_0$ is said to be convex of order α in $\mathfrak{A}(r)$ if

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left(\arg f\left(\rho e^{it}\right)\right)\right)>\alpha, 0\le t\le 2\pi, 0<\rho<r<1.$$

It easy to verify that for function $f\in\tau$ the condition (4.1) is equivalent to the following

$$\mathrm{Re}\frac{\mathcal{D}_{\mathcal{H}}f(z)}{f(z)}>\alpha\quad(z\in\mathfrak{A}(r)),$$

or equivalently

$$\left|\frac{\mathcal{D}_{\mathcal{H}}f(z)-(1+\alpha)f(z)}{\mathcal{D}_{\mathcal{H}}f(z)+(1+\alpha)f(z)}\right|<1\ \ (z\in\mathfrak{A}(r)). \tag{4.2}$$

Let $\mathcal{B}$ be a subclass of the class $\mathcal{H}_0$. We define the radius of starlikeness and convexity

$$\begin{aligned} R_\alpha^*(\mathcal{B}) &= \inf_{f\in\mathcal{B}}\left(\sup\{r\in(0,1]: f \text{ is starlike of order } \alpha \text{ in } \mathfrak{A}(r)\}\right),\\ R_\alpha^c(\mathcal{B}) &= \inf_{f\in\mathcal{B}}\left(\sup\{r\in(0,1]: f \text{ is convex of order } \alpha \text{ in } \mathfrak{A}(r)\}\right). \end{aligned}$$

In simple word these show the subregion of the open unit disc where the functions would behave starlike and convex of order α.

4.1. Theorem

The radii of starlikeness of order α for the class $\mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)$ is given by

$$R_\alpha^*\left(\mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)\right)=\inf_{n\ge2}\left(\frac{1-\alpha}{B-A}\min\left\{\frac{\lambda_n}{n-\alpha},\frac{\sigma_n}{n+\alpha}\right\}\right)^{\frac{1}{n-1}}, \tag{4.3}$$

where λ_n and σ_n are define in (2.2) and (2.3) respectively.

Proof. Let $f \in \mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)$ be of the form (2.6). Then, for $|z| = r < 1$ we have

$$\begin{aligned}\left|\frac{\mathcal{D}_{\mathcal{H}} f(z) - (1+\alpha) f(z)}{\mathcal{D}_{\mathcal{H}} f(z) + (1+\alpha) f(z)}\right| &= \left|\frac{-\alpha z + \sum_{n=2}^{\infty}\left((n-1-\alpha)|a_n| z^n - (n+1+\alpha)|b_n|\overline{z}^n\right)}{(2-\alpha) z + \sum_{n=2}^{\infty}\left((n+1-\alpha)|a_n| z^n - (n-1+\alpha)|b_n|\overline{z}^n\right)}\right| \\ &\leq \frac{\alpha + \sum_{n=2}^{\infty}\left((n-1-\alpha)|a_n| - (n+1+\alpha)|b_n|\right) r^{n-1}}{(2-\alpha) - \sum_{n=2}^{\infty}\left((n+1-\alpha)|a_n| - (n-1+\alpha)|b_n|\right) r^{n-1}}.\end{aligned}$$

Thus the condition (4.2) is true if and only if

$$\sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) r^{n-1} \leq 1. \tag{4.4}$$

By Theorem 2.2, we have

$$\sum_{n=2}^{\infty}\left(\frac{\lambda_n}{B-A}|a_n| + \frac{\sigma_n}{B-A}|b_n|\right) \leq 1, \tag{4.5}$$

where λ_n and σ_n are defined by (2.2) and (2.3) respectively. Thus the conditions (4.4) is true if

$$\frac{n-\alpha}{1-\alpha} r^{n-1} \leq \frac{\lambda_n}{B-A}, \frac{n+\alpha}{1-\alpha} r^{n-1} \leq \frac{\sigma_n}{B-A} \ (n = 2, 3, ...),$$

i.e.,

$$r \leq \left(\frac{1-\alpha}{B-A} \min\left\{\frac{\lambda_n}{n-\alpha}, \frac{\sigma_n}{n+\alpha}\right\}\right)^{\frac{1}{n-1}} \ (n = 2, 3, ...).$$

It follows that the function f is starlike of order α in the disc $\mathfrak{A}(r^*)$, where r^*

$$r^* := \inf\left(\frac{1-\alpha}{B-A} \min\left\{\frac{\lambda_n}{n-\alpha}, \frac{\sigma_n}{n+\alpha}\right\}\right)^{\frac{1}{n-1}}.$$

The functions h_n^* and g_n^* are define by (3.4) realize equality in (4.5), and the radius r^* cannot be larger, thus we have (4.3). □

The following theorem may be proved in much same fashion as Theorem 4.1..

4.2. Theorem

The radii of convexity of order α for the class $\mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)$ is given by

$$R_{\alpha}^{c}\left(\mathcal{S}_{\tau}^{\alpha,\beta}(m,\delta,A,B)\right) = \inf_{n\geq 2}\left(\frac{1-\alpha}{B-A} \min\left\{\frac{\lambda_n}{n-\alpha}, \frac{\sigma_n}{n+\alpha}\right\}\right)^{\frac{1}{n-1}},$$

where λ_n and σ_n are define in (2.2) and (2.3) respectively.

5. Applications

In this section we give some applications of the work discussed in this article in the form of some results and examples. It is clear that if the class

$$\mathcal{F} = \{f_n \in \mathcal{H} : n \in \mathbb{N}\}$$

is locally uniformly bounded, then

$$\overline{co}\mathcal{F} = \left\{ \sum_{n=1}^{\infty} \gamma_n f_n : \sum_{n=2}^{\infty} \gamma_n = 1, \gamma_n \geq 0 \ \ (n \in \mathbb{N}) \right\} \tag{5.1}$$

5.1. Corollary

$$\mathcal{S}_{\tau}^{\alpha,\beta}(m, \delta, A, B) = \left\{ \sum_{n=2}^{\infty} (\gamma_n h_n + \delta_n g_n) : \sum_{n=2}^{\infty} (\gamma_n + \delta_n) = 1, \delta_1 = 0, \ \gamma_n, \delta_n \geq 0 \ \ (n \in \mathbb{N}) \right\}, \tag{5.2}$$

where h_n and g_n are defined by Eq (3.4) .

Proof. By Theorem 3.4 and Lemma 3.3 we have

$$\mathcal{S}_{\tau}^{\alpha,\beta}(m, \delta, A, B) = \overline{co}\mathcal{S}_{\tau}^{\alpha,\beta}(m, \delta, A, B) = \overline{co}E\mathcal{S}_{\tau}^{\alpha,\beta}(m, \delta, A, B) .$$

Thus, by Theorem 3.5 and by (5.1) we have Eq (5.2) . □

We observe, that for each $n \in \mathbb{N}$, $z \in \mathfrak{A}$, the following real-valued functionals are continuous and convex on $\mathcal{H}$:

$$\mathcal{J}(f) = |a_n| , \ \mathcal{J}(f) = |b_n| , \ \mathcal{J}(f) = |f(z)| , \ \mathcal{J}(f) = |\mathcal{D}_{\mathcal{H}} f(z)| \ \ (f \in \mathcal{H}) ,$$

and

$$\mathcal{J}(f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^{\gamma} d\theta \right)^{\frac{1}{\gamma}} \ \ (f \in \mathcal{H}, \gamma \geq 1, 0 < r < 1) .$$

Therefore, using Lemma 3.1 and Theorem 3.5 we have the following corollaries.

5.2. Corollary

Let $f \in \mathcal{S}_{\tau}^{\alpha,\beta}(m, \delta, A, B)$ be a function of the form (2.6) . Then

$$|a_n| \leq \frac{B - A}{\lambda_n}, \ \ |b_n| \leq \frac{B - A}{\sigma_n} \ \ (n = 2, 3, ...) . \tag{5.3}$$

where λ_n and σ_n are defined by (2.2) and (2.3) respectively. The result is sharp. The function h_n^* and g_n^* of the form (3.4) are extremal functions.

Proof. Since for the extremal functions h_n^* and g_n^* we have $|a_n| = \frac{B-A}{\lambda_n}$ and $|b_n| = \frac{B-A}{\sigma_n}$. Thus, by Lemma 3.1 we have Eq (5.3) . □

5.3. Example

Since $\frac{B-A+2}{\lambda_2} > \frac{B-A}{\lambda_2}$ the polynomial

$$k(z) = z - \frac{B-A+2}{\lambda_2} z^2 \ \ (z \in \mathfrak{U}),$$

by Corollary 5.2, clearly $k(z)$ does not belong to $\mathcal{S}_\tau^{\alpha,\beta}(m,\delta,A,B)$.

5.4. Corollary

Let $f \in \mathcal{S}_\tau^{\alpha,\beta}(m,\delta,A,B)$, $|z| = r < 1$. Then

$$\begin{aligned} r - \left|\frac{\Gamma(\alpha+\beta)}{[1+\delta]^m \Gamma(\beta)}\right| \frac{B-A}{(1+2B-A)} r^2 &\leq |f(z)| \leq r + \left|\frac{\Gamma(\alpha+\beta)}{[1+\delta]^m \Gamma(\beta)}\right| \frac{B-A}{(1+2B-A)} r^2, \\ r - \left|\frac{\Gamma(\alpha+\beta)}{[1+\delta]^m \Gamma(\beta)}\right| \frac{2(B-A)}{(1+2B-A)} r^2 &\leq |\mathcal{D}_{\mathcal{H}} f(z)| \leq r + \left|\frac{\Gamma(\alpha+\beta)}{[1+\delta]^m \Gamma(\beta)}\right| \frac{2(B-A)}{(1+2B-A)} r^2. \end{aligned}$$

Due to Littlewood [37] we consider the integral means inequalities for functions from the class $\mathcal{S}_\tau^{\alpha,\beta}(m,\delta,A,B)$.

5.5. Lemma

Let $f, g \in \mathcal{A}$. If $f \prec g$, then

$$\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^\gamma d\theta \leq \int_0^{2\pi} \left|g\left(re^{i\theta}\right)\right|^\gamma d\theta.$$

5.6. Theorem

Let $0 < r < 1$, $\gamma > 0$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \left|h_n^*\left(re^{i\theta}\right)\right|^\gamma d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \left|h_2^*\left(re^{i\theta}\right)\right|^\gamma d\theta \ \ (n = 2,3,...), \tag{5.4}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \left|g_n^*\left(re^{i\theta}\right)\right|^\gamma d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \left|h_2^*\left(re^{i\theta}\right)\right|^\gamma d\theta \ \ (n = 2,3,...), \tag{5.5}$$

where h_n^* and $g_n^*$is defined by Eq (3.4).

Proof. Let h_n^* and g_n^* are define by Eq (3.4) and let $\widetilde{g_n}(z) = z + \frac{B-A}{\sigma_n} z^n$ $(n = 2,3,...)$. Since $\frac{h_n^*}{z} \prec \frac{h_2^*}{z}$ and $\frac{\widetilde{g_n}}{z} \prec \frac{h_2^*}{z}$, by Lemma 5.5 we have

$$\int_0^{2\pi} \left|h_n^*\left(re^{i\theta}\right)\right|^\gamma d\theta \leq \int_0^{2\pi} \left|h_2^*\left(re^{i\theta}\right)\right|^\gamma d\theta,$$

and

$$\int_0^{2\pi} \left|g_n^*\left(re^{i\theta}\right)\right|^\gamma d\theta = \int_0^{2\pi} \left|\widetilde{g_n}\left(re^{i\theta}\right)\right|^\gamma d\theta \leq \int_0^{2\pi} \left|h_2^*\left(re^{i\theta}\right)\right|^\gamma d\theta,$$

which complete the proof. □

5.7. Corollary

If $f \in \mathcal{S}_{\tau}^{\alpha,\beta}(m, \delta, A, B)$ then

$$\frac{1}{2\pi}\int_{0}^{2\pi}\left|f\left(re^{i\theta}\right)\right|^{\gamma}d\theta \leq \frac{1}{2\pi}\int_{0}^{2\pi}\left|h_{2}^{*}\left(re^{i\theta}\right)\right|^{\gamma}d\theta,$$

and

$$\frac{1}{2\pi}\int_{0}^{2\pi}\left|\mathcal{D}_{\mathcal{H}}f\left(re^{i\theta}\right)\right|^{\gamma}d\theta \leq \frac{1}{2\pi}\int_{0}^{2\pi}\left|\mathcal{D}_{\mathcal{H}}h_{2}^{*}\left(re^{i\theta}\right)\right|^{\gamma}d\theta,$$

where $\gamma \geq 1$, $0 < r < 1$ and h_2^* is the function defined by Eq (3.4) .

6. Conclusions

With the use of Mittag-Leffer functions, we introduced a new subclass of harmonic mappings in Janowski domain. We studied some useful results, like necessary and sufficient conditions, coefficient inequality, topological properties, radii problems, distortion bounds and integral mean of inequality for newly defined classes of functions. It can be seen that our defined class not only generalizes various well known classes and their respective results but also give new direction to this field by the introduction of Mittag-Leffer functions here. Further using the concepts of Mittag-Leffer functions these problems can be studied for classes of meromorphic harmonic functions, Bazilevi´c harmonic functions and for p-valent harmonic functions as well.

Conflict of interest

The authors declare that they have no competing interests.

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