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Research article

Exact divisibility by powers of the integers in the Lucas sequence of the first kind

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Abstract: Lucas sequence of the first kind is an integer sequence $(U_n)_{n\geq 0}$ which depends on parameters $a, b \in \mathbb{Z}$ and is defined by the recurrence relation $U_0 = 0$, $U_1 = 1$, and $U_n = aU_{n-1} + bU_{n-2}$ for $n \geq 2$. In this article, we obtain exact divisibility results concerning U_n^k for all positive integers n and k. This extends many results in the literature from 1970 to 2020 which dealt only with the classical Fibonacci and Lucas numbers ($a = b = 1$) and the balancing and Lucas-balancing numbers ($a = 6, b = -1$).

Keywords: Lucas sequence; Lucas number; Fibonacci number; exact divisibility; p-adic valuation
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1. Introduction

Throughout this article, let a and b be relatively prime integers and let $(U_n)_{n\geq 0}$ be the Lucas sequence of the first kind which is defined by the recurrence relation $U_0 = 0$, $U_1 = 1$, $U_n = aU_{n-1} + bU_{n-2}$ for $n \geq 2$. To avoid triviality, we also assume that $b \neq 0$ and α/β is not a root of unity where α and β are the roots of the characteristic polynomial $x^2 - ax - b$. In particular, this implies that $\alpha \neq \beta$ and the discriminant $D = a^2 + 4b \neq 0$. If $a = b = 1$, then $(U_n)_{n\geq 0}$ reduces to the sequence of Fibonacci numbers F_n; if $a = 6$ and $b = -1$, then $(U_n)_{n\geq 0}$ becomes the sequence of balancing numbers; if $a = 2$ and $b = 1$, then $(U_n)_{n\geq 0}$ is the sequence of Pell numbers; and many other famous integer sequences are just special cases of the Lucas sequence of the first kind.

The divisibility by powers of the Fibonacci numbers has attracted some attentions because it is used in Matijasevich's solution to Hilbert's 10th problem [7–9]. More precisely, Matijasevich shows that

$$F_n^2 \mid F_{nm} \quad \text{if and only if} \quad F_n \mid m. \tag{1.1}$$

Hoggatt and Bicknell-Johnson [3] give a generalization of (1.1) by replacing F_n^2 by F_n^3, and for a general k, they prove that

$$\text{if } F_n^k \mid m, \text{ then } F_n^{k+1} \mid F_{nm}. \tag{1.2}$$

Benjamin and Rouse [1], and Seibert and Trojovský [27] also provide a different proof of (1.2). Then the investigation on the exact divisibility results for a subsequence of $(F_n)_{n\geq 1}$ begin with the work of Tangboonduangjit et. al [12, 29] and is generalized by Onphaeng and Pongsriiam [10]. The most general results in this direction are obtained by Pongsriiam [18] where (1.2) is extended to include the divisibility and exact divisibility for both the Fibonacci and Lucas numbers. Finally, Onphaeng and Pongsriiam [11] have recently given the converse of the results in [18] which completely answers this kind of questions for the Fibonacci and Lucas numbers. Then Panraksa and Tangboonduangjit [13] initiate the investigation on a special subsequence of $(U_n)_{n\geq 0}$. Patra, Panda, and Khemaratchatakumthorn [14] also obtain the analogue of those results for the balancing and Lucas-balancing numbers. For other related and recent results on Fibonacci, Lucas, balancing, and Lucas-balancing numbers, see for example in [2, 4–6, 15–17, 19–25, 28] and references there in.

In this article, we extend all results in the literature to the Lucas sequence of the first kind. We organize this article as follows. In Section 2, we give some auxiliary results which are needed later. In Section 3, we give main theorems and some related examples. Remark that the corresponding results for other generalizations of the Fibonacci sequence have not been discovered. For example, the question on exact divisibility by powers of the Tribonacci numbers T_n is wide open, where T_n is given by $T_0 = 0$, $T_1 = T_2 = 1$, and $T_k = T_{k-1} + T_{k-2} + T_{k-3}$ for $k \geq 3$. We leave this problem to the interested readers.

2. Preliminaries and Lemmas

In this section, we recall some well-known results and give some useful lemmas for the reader's convenience. The order (or the rank) of appearance of $n \in \mathbb{N}$ in the Lucas sequence $(U_n)_{n\geq 0}$ is defined as the smallest positive integer m such that $n \mid U_m$ and is denoted by $\tau(n)$. The exact divisibility $m^k \parallel n$ means that $m^k \mid n$ and $m^{k+1} \nmid n$. For a prime p and $n \in \mathbb{N}$, the p-adic valuation of n, denoted by $v_p(n)$ is the power of p in the prime factorization of n. We sometimes write the expression such as $a \mid b \mid c = d$ to mean that $a \mid b$, $b \mid c$, and $c = d$. We let $D = a^2 + 4b$ be the discriminant and let α and β be the roots of the characteristic polynomial $x^2 - ax - b$. It is well known that if $D \neq 0$, then the Binet formula $U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ holds for all $n \geq 0$. Next, we recall Sanna's result [26] on the p-adic valuation of Lucas sequence of the first kind.

Lemma 1. [26, Theorem 1.5] *Let p be a prime number such that $p \nmid b$. Then, for each positive integer n,*

$$v_p(U_n) = \begin{cases} v_p(n) + v_p(U_p) - 1 & \text{if } p \mid D \text{ and } p \mid n, \\ 0 & \text{if } p \mid D \text{ and } p \nmid n, \\ v_p(n) + v_p(U_{p\tau(p)}) - 1 & \text{if } p \nmid D,\ \tau(p) \mid n, \text{ and } p \mid n, \\ v_p(U_{\tau(p)}) & \text{if } p \nmid D,\ \tau(p) \mid n, \text{ and } p \nmid n, \\ 0 & \text{if } p \nmid D \text{ and } \tau(p) \nmid n. \end{cases}$$

In fact, we use Lemma 1 only for $p = 2$, because there is a more suitable version of Lemma 1 when

p is odd as given by Panraksa and Tangboonduangjit [13] in their calculation concerning a special subsequence of $(U_n)_{n\geq 0}$. We state it in the next lemma.

Lemma 2. [13, Lemma 2.3] *Let $m, n \geq 1$ and p a prime factor of U_n such that $p \nmid b$. Then, if* (i) *p is odd, or* (ii) *$p = 2$ and n is even, or* (iii) *$p = 2$ and m is odd, we have*

$$v_p(U_{nm}) = v_p(m) + v_p(U_n).$$

Lemma 3. *Let a and b be odd integers. Then, for each positive integer n,*

$$v_2(U_n) = \begin{cases} v_2(n) + v_2(U_6) - 1 & \text{if } n \equiv 0 \pmod 6, \\ v_2(U_3) & \text{if } n \equiv 3 \pmod 6, \\ 0 & \text{if } n \equiv 1, 2, 4, 5 \pmod 6. \end{cases}$$

Proof. Since $U_1 = 1$, $U_2 = a$ are odd and $U_3 = a^2 + b$ is even, we have $\tau(2) = 3$. Applying Lemma 1 for $p = 2$, we obtain the desired result. □

The next two lemmas are also important tools in proving exact divisibility by U_n^k for all $n, k \in \mathbb{N}$.

Lemma 4. [10, Lemma 2.3] *Let k, ℓ, m be positive integers, s nonzero integer, and $s^k \mid m$. Then $s^{k+\ell} \mid \binom{m}{j} s^j$ for all $1 \leq j \leq m$ satisfying $2^{j-\ell+1} > j$. In particular, $s^{k+1} \mid \binom{m}{j} s^j$ for all $1 \leq j \leq m$, and $s^{k+2} \mid \binom{m}{j} s^j$ for all $3 \leq j \leq m$.*

Proof. The statement in [10, Lemma 2.3] is given for $s \geq 1$ but it is easy to see that if $s \leq -1$, then we can replace s by $-s$ and every divisibility relation still holds. Therefore this is true for all $s \neq 0$. □

Lemma 5. *Let $m, n \geq 1$ and $r \geq 0$ be integers. Then*

(i) $U_{mn+r} = \sum_{j=0}^{m} \binom{m}{j} U_n^j (bU_{n-1})^{m-j} U_{j+r}$,
(ii) $U_{mn} = \sum_{j=1}^{m} \binom{m}{j} U_n^j (bU_{n-1})^{m-j} U_j$.

Proof. By Binet's formula, we obtain $\alpha^n = \alpha U_n + bU_{n-1}$, $\beta^n = \beta U_n + bU_{n-1}$, and

$$\begin{aligned} U_{mn+r} &= \frac{\alpha^{mn+r} - \beta^{mn+r}}{\alpha - \beta} \\ &= \frac{1}{\alpha - \beta} \left((\alpha U_n + bU_{n-1})^m \alpha^r - (\beta U_n + bU_{n-1})^m \beta^r \right) \\ &= \frac{1}{\alpha - \beta} \left(\sum_{j=0}^{m} \binom{m}{j} (\alpha U_n)^j (bU_{n-1})^{m-j} \alpha^r - \sum_{j=0}^{m} \binom{m}{j} (\beta U_n)^j (bU_{n-1})^{m-j} \beta^r \right) \\ &= \frac{1}{\alpha - \beta} \sum_{j=o}^{m} \left(\binom{m}{j} U_n^j (bU_{n-1})^{m-j} \left(\alpha^{j+r} - \beta^{j+r} \right) \right) \\ &= \sum_{j=0}^{m} \binom{m}{j} U_n^j (bU_{n-1})^{m-j} U_{j+r}. \end{aligned}$$

This proves (i). Since $U_0 = 0$, (ii) follows immediately from (i) by substituting $r = 0$. □

Recall that we assume throughout this article that $(a, b) = 1$. This is necessary for the proof of the following lemmas.

Lemma 6. *Suppose* $(a, b) = 1$. *Then* $(U_m, U_n) = U_{(m,n)}$ *and in particular* $(U_n, U_{n+1}) = 1$ *for each* $m, n \in \mathbb{N}$.

Proof. This is well known. □

Lemma 7. *Let* $n \geq 1$ *and* $(a, b) = 1$. *If* p *is a prime factor of* U_n, *then* $p \nmid b$. *Consequently,* $(U_n, b) = 1$ *for all* $n \geq 1$.

Proof. Suppose for a contradiction that $(a, b) = 1$, $n \geq 1$, $p \mid U_n$, and $p \mid b$. We can choose n to be the smallest such integer. Since $U_1 = 1$, $U_2 = a$, we see that $n \geq 3$. Since $p \mid U_n = aU_{n-1} + bU_{n-2}$ and $p \mid b$, we have $p \mid aU_{n-1}$. By the choice of n, $p \nmid U_{n-1}$. So $p \mid a$. Therefore $p \mid (a, b) = 1$, a contradiction. □

Lemma 8. *Let* a *and* b *be odd,* $(a, b) = 1$, *and* $v_2(U_6) \geq v_2(U_3) + 2$. *Then* $v_2(U_3) = 1$.

Proof. Since $U_3 = a^2 + b$ is even and $U_6 = a(a^2 + 3b)U_3$, we obtain $v_2(U_3) \geq 1$ and

$$v_2(U_6) = v_2(U_3) + v_2(a^2 + 3b). \tag{2.1}$$

If $v_2(U_3) \geq 2$, then $4 \mid a^2 + b$, and so $b \equiv 3 \pmod 4$ and (2.1) implies $v_2(U_6) = v_2(U_3) + 1$ contradicting $v_2(U_6) \geq v_2(U_3) + 2$. Thus $v_2(U_3) = 1$.

□

3. Main results

We begin with the simplest main result of this paper.

Theorem 9. *Let* k, m, *and* n *be positive integers. If* $U_n^k \mid m$, *then* $U_n^{k+1} \mid U_{nm}$.

Proof. If $U_n^k \mid m$, then we obtain by Lemma 4 that, $U_n^{k+1} \mid \binom{m}{j} U_n^j$ for all $1 \leq j \leq m$, which implies $U_n^{k+1} \mid U_{nm}$, by Lemma 5. □

Next, we extend Theorem 9 to include exact divisibility. The proof of Theorem 10 is much longer than that of Theorem 9 since we would like to cover all possible cases. Although many cases can be combined, it is more convenient to state them separately. Recall that for $x \in \mathbb{R}$, the largest integer which is less than or equal to x is denoted by $\lfloor x \rfloor$.

Theorem 10. *Let* $k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, $n \geq 2$, *and* $U_n^k \parallel m$. *Then*

(i) *if* a *is odd and* b *is even, then* $U_n^{k+1} \parallel U_{nm}$;
(ii) *if* a *is even and* b *is odd, then* $U_n^{k+1} \parallel U_{nm}$;
(iii) *if* a *and* b *are odd and* $n \not\equiv 3 \pmod 6$, *then* $U_n^{k+1} \parallel U_{nm}$;
(iv) *if* a *and* b *are odd,* $n \equiv 3 \pmod 6$, *and* $\frac{U_n^{k+1}}{2} \nmid m$, *then* $U_n^{k+1} \parallel U_{nm}$;
(v) *if* a *and* b *are odd,* $n \equiv 3 \pmod 6$, $\frac{U_n^{k+1}}{2} \mid m$, *and* $2 \parallel a^2 + 3b$, *then* $U_n^{k+1} \parallel U_{nm}$;
(vi) *if* a *and* b *are odd,* $n \equiv 3 \pmod 6$, $\frac{U_n^{k+1}}{2} \mid m$, *and* $4 \mid a^2 + 3b$, *then* $U_n^{k+t+1} \parallel U_{nm}$, *where*

$$t = \min(\{v_2(U_6) - 2\} \cup \{y_p - k \mid p \text{ is an odd prime factor of } U_n\})$$

and $y_p = \left\lfloor \frac{v_p(m)}{v_p(U_n)} \right\rfloor$ *for each odd prime* p *dividing* U_n.

Proof. By Theorem 9, we obtain $U_n^{k+1} \mid U_{nm}$. So for (i) to (v), it is enough to show that $U_n^{k+2} \nmid U_{nm}$. We divide the calculation into several cases.

Case 1. a is odd and b is even. Since U_1 and U_2 are odd and $U_r = aU_{r-1} + bU_{r-2} \equiv U_{r-1} \pmod 2$ for $r \geq 3$, it follows by induction that U_n is odd. From the assumption $U_n^k \parallel m$, we have $U_n^{k+1} \nmid m$, and so there exists a prime p dividing U_n such that $v_p(U_n^{k+1}) > v_p(m)$. Since U_n is odd, p is also odd. In addition, $p \nmid b$ by Lemma 7. So we can apply Lemma 2(i) to obtain

$$v_p(U_{nm}) = v_p(m) + v_p(U_n) < v_p(U_n^{k+1}) + v_p(U_n) = v_p(U_n^{k+2}),$$

which implies $U_n^{k+2} \nmid U_{nm}$, as required. This proves (i).

Case 2. a is even and b is odd. Similar to Case 1, we have U_1 is odd, U_2 is even, $U_r \equiv U_{r-2} \pmod 2$ for $r \geq 3$, and so U_n is even if and only if n is even. In addition, there exists a prime p such that $p \mid U_n$, $v_p(U_n^{k+1}) > v_p(m)$, and $p \nmid b$. So if $2 \nmid n$, then U_n is odd, p is odd, and we obtain by Lemma 2(i) that

$$v_p(U_{nm}) = v_p(m) + v_p(U_n) < v_p(U_n^{k+1}) + v_p(U_n) = v_p(U_n^{k+2}), \tag{3.1}$$

which implies $U_n^{k+2} \nmid U_{nm}$. If $2 \mid n$, then we can still use either Lemma 2(i) or Lemma 2(ii) to obtain (3.1), which leads to the same conclusion $U_n^{k+2} \nmid U_{nm}$. This proves (ii).

Case 3. a and b are odd. Similar to Case 1, there is a prime p such that $p \mid U_n$, $v_p(U_n^{k+1}) > v_p(m)$, and $p \nmid b$.

Case 3.1 $n \not\equiv 3 \pmod 6$. If $n \equiv 1, 2, 4, 5 \pmod 6$, then we obtain by Lemmas 3 and 2, respectively that p is odd and

$$v_p(U_{nm}) = v_p(U_n) + v_p(m) < v_p(U_n) + v_p(U_n^{k+1}) = v_p(U_n^{k+2}). \tag{3.2}$$

If $n \equiv 0 \pmod 6$, then n is even and Lemma 2(i) or Lemma 2(ii) can still be used to obtain (3.2). In any case, $U_n^{k+2} \nmid U_{nm}$. This proves (iii).

Case 3.2 $n \equiv 3 \pmod 6$ and $\frac{U_n^{k+1}}{2} \nmid m$. Since $U_n^k \parallel m$, we can write $m = cU_n^k$ where $c \geq 1$ and $U_n \nmid c$. By Lemma 4, $U_n^{k+2} \mid \binom{m}{j} U_n^j$ for $3 \leq j \leq m$. Then we obtain by Lemma 5 that

$$U_{nm} = U_{mn} \equiv mU_n(bU_{n-1})^{m-1} + \frac{m(m-1)}{2} U_n^2 (bU_{n-1})^{m-2} a \pmod{U_n^{k+2}}.$$

By Lemma 3, we know that $v_2(U_n) = v_2(U_3) \geq 1$. Since $\frac{U_n^{k+1}}{2} \nmid m$ and $m = cU_n^k$, we see that $\frac{U_n}{2}$ does not c. Let $d = bU_{n-1} + \frac{U_n}{2}(m-1)a$. By Lemmas 6 and 7, we obtain $\left(\frac{U_n}{2}, d\right) = \left(\frac{U_n}{2}, bU_{n-1}\right) = 1$. Then

$$U_{nm} \equiv mU_n b^{m-2} U_{n-1}^{m-2} \left(bU_{n-1} + \frac{U_n}{2}(m-1)a\right) \equiv cU_n^{k+1} b^{m-2} U_{n-1}^{m-2} d \pmod{U_n^{k+2}}.$$

By Lemmas 6 and 7, we obtain $U_n^{k+2} \mid U_{nm}$ if and only if $U_n \mid cd$. But if $U_n \mid cd$, then $\frac{U_n}{2} \mid cd$ which implies $\frac{U_n}{2} \mid c$, a contradiction. So $U_n \nmid cd$ and therefore $U_n^{k+2} \nmid U_{nm}$. This proves (iv). To prove (v) and (vi), we first assume that a and b are odd, $n \equiv 3 \pmod 6$, and $\frac{U_n^{k+1}}{2} \mid m$. (The other condition will be assumed later). Then $v_p(U_n^{k+1}) \leq v_p(m)$ for all odd primes p and $v_2(U_n^{k+1}) - 1 \leq v_2(m)$. If $v_2(U_n^{k+1}) - 1 < v_2(m)$, then $v_2(U_n^{k+1}) \leq v_2(m)$, and so $v_p(U_n^{k+1}) \leq v_p(m)$ for all primes p, which implies $U_n^{k+1} \mid m$ contradicting the assumption $U_n^k \parallel m$. Hence

$$v_2(U_n^{k+1}) - 1 = v_2(m) \text{ and } v_p(U_n^{k+1}) \leq v_p(m) \text{ for every odd prime } p \tag{3.3}$$

We now separate the consideration into two cases according to the additional conditions in (v) and (vi). Observe that $v_2(a^2+3b)=1$ is equivalent to $2 \parallel a^2+3b$.

Case 4. $v_2(a^2+3b)=1$. Since $U_6 = a(a^2+3b)U_3$, we obtain $v_2(U_6)=v_2(U_3)+1$. Recall that $n \equiv 3 \pmod 6$ and $U_n^k \mid m$. So n is odd, m is even, and $nm \equiv 0 \pmod 6$. If $U_n^{k+2} \mid U_{nm}$, then we obtain by Lemma 3 and (3.3) that

$$\begin{aligned} v_2(U_n^{k+1}) + v_2(U_n) = v_2(U_n^{k+2}) \le v_2(U_{nm}) &= v_2(n) + v_2(m) + v_2(U_6) - 1 \\ &= v_2(U_n^{k+1}) - 1 + v_2(U_3) \\ &= v_2(U_n^{k+1}) + v_2(U_n) - 1, \end{aligned}$$

which is a contradiction. Therefore $U_n^{k+2} \nmid U_{nm}$. This proves (v).

Case 5. $v_2(a^2+3b) \ge 2$. Then $v_2(U_6) = v_2(U_3)+v_2(a^2+3b) \ge v_2(U_3)+2$. By Lemma 8, $v_2(U_3)=1$ and so $v_2(U_6) = x+2$ where $x = v_2(a^2+3b)-1 \in \mathbb{N}$. For each odd prime p dividing U_n, let $y_p = \left\lfloor \frac{v_p(m)}{v_p(U_n)} \right\rfloor$ be the largest integer which is less than or equal to $\frac{v_p(m)}{v_p(U_n)}$. Since $U_n^k \mid m$, we have $y_p \ge k$ for all odd $p \mid U_n$. Let

$$t = \min(\{x\} \cup \{y_p - k \mid p \text{ is an odd prime factor of } U_n\}).$$

Then $t \ge 0$. By Lemma 3 and (3.3), $v_2(m) = (k+1)v_2(U_3) - 1 = k$ and

$$v_2(U_{nm}) = v_2(m) + v_2(U_6) - 1 = k + x + 1 \ge k + t + 1 = v_2(U_n^{k+t+1}). \tag{3.4}$$

By the definition of y_p, we have $v_p(m) \ge y_p v_p(U_n)$. So by Lemma 2, if p is an odd prime dividing U_n, then

$$v_p(U_{nm}) = v_p(m) + v_p(U_n) \ge (y_p+1)v_p(U_n) \ge (k+t+1)v_p(U_n) = v_p(U_n^{k+t+1}). \tag{3.5}$$

By (3.4) and (3.5), $v_p(U_{nm}) \ge v_p(U_n^{k+t+1})$ for all primes p dividing U_n. This show that $U_n^{k+t+1} \mid U_{nm}$. It remains to show that $U_n^{k+t+2} \nmid U_{nm}$. If $t = y_p - k$ for some odd prime p dividing U_n, then we recall the definition of y_p and apply Lemma 2 to obtain

$$v_p(U_{nm}) = v_p(m) + v_p(U_n) < (y_p+2)v_p(U_n) = (k+t+2)v_p(U_n) = v_p(U_n^{k+t+2}).$$

If $t = x = v_2(U_6) - 2$, then we use Lemma 3 to get

$$v_2(U_{nm}) = v_2(m) + v_2(U_6) - 1 = k + t + 1 < v_2(U_n^{k+t+2}).$$

In any case, $U_n^{k+t+2} \nmid U_{nm}$. This completes the proof. □

The next example shows that the integer t in Theorem 10(vi) can be any odd positive integer.

Example 11. Let $M \in \mathbb{N}$ be given. We show that there are positive integers k, m, n, a, b satisfying the conditions in Theorem 10(vi) with $t = M$. Choose $a = 1$ and $b = \left(2^{4M} - 1\right)/3$. Then a and b are odd integers, $(a,b)=1$, and $v_2(a^2+3b) = 4M > 2$. Next choose any $k, n \in \mathbb{N}$ such that $n \equiv 3 \pmod 6$. Since $v_2(U_6) = v_2(U_3) + v_2(a^2+3b) \ge v_2(U_3)+2$, we obtain by Lemmas 3 and 8 that $v_2(U_n) = v_2(U_3) = 1$ and $v_2(U_6) = 4M+1$. Since $U_n \ge U_3 = a^2 + b > 2$ and $v_2(U_n) = 1$, we can write $U_n = 2p_1^{a_1}p_2^{a_2}\cdots p_s^{a_s}$ where $s \ge 1$, $p_1, p_2, \ldots, p_s$ are distinct odd primes, and $a_1, a_2, \ldots, a_s$ are positive integers. Next, choose $m = 2^k p_1^{a_1(k+M)} p_2^{a_2(k+M)} \cdots p_s^{a_s(k+M)}$. Then $U_n^k \parallel m$ and $\frac{U_n^{k+1}}{2} \mid m$. Therefore k, m, n, a, b satisfy all the conditions in Theorem 10(vi). Finally, we have

$$v_2(U_6) - 2 = v_2(a^2+3b) - 1 = 4M - 1$$

and $y_p - k = M$ for all $p \in \{p_1, p_2, \ldots, p_s\}$, and therefore $t = \min\{4M-1, M\} = M$, as desired.

Next, we prove the converse of Theorem 10.

Theorem 12. *$k, m, n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, $n \geq 2$, and $U_n^{k+1} \parallel U_{nm}$. Then*

(i) *if a is odd and b is even, then $U_n^k \parallel m$;*
(ii) *if a is even and b is odd, then $U_n^k \parallel m$;*
(iii) *if a and b are odd and $n \not\equiv 3 \pmod 6$, then $U_n^k \parallel m$;*
(iv) *if a and b are odd, $n \equiv 3 \pmod 6$, and $2 \parallel a^2 + 3b$, then $U_n^k \parallel m$;*
(v) *if a and b are odd, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$, and $v_2(m) \geq k$, then $U_n^k \parallel m$;*
(vi) *if a and b are odd, $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$, and $v_2(m) < k$, then*

$$m \text{ is even, } v_2(m) \geq k + 1 - v_2(a^2 + 3b), \text{ and } U_n^{v_2(m)} \parallel m.$$

Proof. Some parts of the proof are similar to those of Theorem 10, so we skip some details.
Case 1. a is odd and b is even. Similar to Case 1 of Theorem 10, we have U_n is odd. For any prime $p \mid U_n$, we obtain by Lemma 2 that

$$v_p(U_n^k) + v_p(U_n) = v_p(U_n^{k+1}) \leq v_p(U_{nm}) = v_p(U_n) + v_p(m), \tag{3.6}$$

which implies $U_n^k \mid m$. If $U_n^{k+1} \mid m$, then by Theorem 9, we have $U_n^{k+2} \mid U_{nm}$ which contradicts $U_n^{k+1} \parallel U_{nm}$. Therefore $U_n^{k+1} \nmid m$, and thus $U_n^k \parallel m$.
Case 2. a is even and b is odd. Then U_n is even if and only if n is even. So if $2 \nmid n$, then for any prime $p \mid U_n$, we have p is odd, (3.6) holds, and so $U_n^k \mid m$. If $2 \mid n$, then we can still apply Lemma 2(i) or Lemma 2(ii) to obtain (3.6) and conclude that $U_n^k \mid m$. If $U_n^{k+1} \mid m$, then by Theorem 9, we have $U_n^{k+2} \mid U_{nm}$ which contradicts $U_n^{k+1} \parallel U_{nm}$. So $U_n^{k+1} \nmid m$ and therefore $U_n^k \parallel m$.

We now assume throughout that a and b are odd and divide the consideration into four cases according to the additional conditions in (iii) to (vi).
Case 3. $n \not\equiv 3 \pmod 6$. If $n \equiv 1, 2, 4, 5 \pmod 6$, then we apply Lemma 3 to obtain $v_2(U_n^k) = 0 \leq v_2(m)$, and use Lemma 2 to show that for any odd prime $p \mid U_n$,

$$v_p(U_n) + v_p(U_n^k) = v_p(U_n^{k+1}) \leq v_p(U_{nm}) = v_p(m) + v_p(U_n). \tag{3.7}$$

If $n \equiv 0 \pmod 6$, then n is even and we can apply Lemma 2(i) or Lemma 2(ii) to obtain (3.7) for any prime $p \mid U_n$. In any case, we have $U_n^k \mid m$. Again, by Theorem 9, we have $U_n^{k+1} \nmid m$, and so $U_n^k \parallel m$. This proves (iii).
Case 4. $n \equiv 3 \pmod 6$ and $2 \parallel a^2 + 3b$. Similar to Case 4 in the proof of Theorem 10 we have $v_2(U_6) = v_2(U_3) + 1$. If m is odd, then $nm \equiv 3 \pmod 6$ and we obtain by Lemma 3 that $v_2(U_{nm}) = v_2(U_3) < (k+1)v_2(U_3) = v_2(U_n^{k+1})$, which contradicts the assumption $U_n^{k+1} \mid U_{nm}$. So m is even, and thus $nm \equiv 0 \pmod 6$. By Lemma 3 and the fact that $n \equiv 3 \pmod 6$ is odd, we obtain $v_2(m) + v_2(U_6) - 1 = v_2(U_{nm}) \geq v_2(U_n^{k+1}) = v_2(U_n^k) + v_2(U_n) = v_2(U_n^k) + v_2(U_3) = v_2(U_n^k) + v_2(U_6) - 1$, which implies $v_2(m) \geq v_2(U_n^k)$. If p is odd and $p \mid U_n$, then we apply Lemma 2 to obtain (3.7) Therefore $v_p(U_n^k) \leq v_p(m)$ for every prime p dividing U_n. Thus $U_n^k \mid m$. By Theorem 9, $U_n^{k+1} \nmid m$. Hence $U_n^k \parallel m$. This proves (iv).
Case 5. $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$, and $v_2(m) \geq k$. Then $U_3 = a^2 + b = (a^2 + 3b) - 2b \equiv 2 \pmod 4$, and so $v_2(U_3) = 1$. By Lemma 3, we obtain $v_2(m) \geq kv_2(U_3) = kv_2(U_n) = v_2(U_n^k)$. By Lemma 2, if p is

an odd prime dividing U_n, then (3.6) holds, and so we conclude that $v_p(U_n^k) \le v_p(m)$ for every prime p dividing U_n. Therefore $U_n^k \mid m$. By Theorem 9, $U_n^{k+1} \nmid m$ and so $U_n^k \parallel m$. This proves (v).

Case 6. $n \equiv 3 \pmod 6$, $4 \mid a^2 + 3b$, and $v_2(m) < k$. For convenience, let $t = v_2(m)$. Similar to Case 4, we have m is even. In addition, $v_2(U_6) = v_2(U_3) + v_2(a^2 + 3b) = 1 + v_2(a^2 + 3b)$. So $k > t \ge 1$ and $v_2(m) = tv_2(U_3) = tv_2(U_n) = v_2(U_n^t)$. By Lemma 2, if p is odd and $p \mid U_n$, then

$$v_p(U_n) + v_p(U_n^t) \le v_p(U_n) + v_p(U_n^k) = v_p(U_n^{k+1}) \le v_p(U_{nm}) = v_p(m) + v_p(U_n).$$

From the above inequalities, we obtain that $v_p(U_n^t) \le v_p(m)$ for every prime p dividing U_n. Therefore $U_n^t \mid m$. If $U_n^{t+1} \mid m$, then we obtain by Lemma 3 that $t = v_2(m) \ge v_2(U_n^{t+1}) = t + 1$, which is false. So $U_n^{t+1} \nmid m$. Therefore $U_n^t \parallel m$. From $U_n^{k+1} \parallel U_{nm}$, we also obtain $k + 1 = v_2(U_n^{k+1}) \le v_2(U_{nm}) = v_2(m) + v_2(U_6) - 1 = v_2(m) + v_2(a^2 + 3b)$, which implies $v_2(m) \ge k + 1 - v_2(a^2 + 3b)$. This completes the proof. □

The next example shows that $v_2(m)$ in Theorem 12(vi) can be any positive integer in $[1, k)$.

Example 13. Let $k \ge 1$ and $1 \le M < k$ be integers. We show that there are m, n, a, b satisfying the conditions in Theorem 12(vi) with $v_2(m) = M$. Choose $n \in \mathbb{N}$ and $n \equiv 3 \pmod 6$.

Case 1. $k - M$ is odd. Choose $a = 1$, $b = \frac{2^{k-M+1}-1}{3}$, and $m = \frac{U_n^k}{2^{k-M}}$. Then a and b are odd integers, $(a, b) = 1$, and $v_2(a^2 + 3b) = k - M + 1 \ge 2$. Since $v_2(U_6) = v_2(U_3) + v_2(a^2 + 3b) \ge v_2(U_3) + 2$, we obtain by Lemmas 3 and 8 that $v_2(U_n) = v_2(U_3) = 1$ and $v_2(U_6) = k - M + 2$. By Lemma 2, for $p > 2$ and $p \mid U_n$ we obtain

$$v_p(U_{nm}) = v_p(m) + v_p(U_n) = v_p(U_n^k) + v_p(U_n) = v_p(U_n^{k+1}).$$

By Lemma 3, we have

$$v_2(m) = v_2(U_n^k) - v_2(2^{k-M}) = k - k + M = M$$

and

$$v_2(U_{nm}) = v_2(m) + v_2(U_6) - 1 = M + k - M + 2 - 1 = v_2(U_n^{k+1}).$$

From these, we obtain $U_n^{k+1} \parallel U_{nm}$ and $U_n^M \parallel m$. Therefore k, m, n, a, b satisfy all the conditions in Theorem 12(vi).

Case 2. $k - M$ is even. Choose $a = 1$, $b = \frac{5 \cdot 2^{k-M+1}-1}{3}$, and $m = \frac{U_n^k}{2^{k-M}}$. The verification is the same as that in Case 1. So we leave the details to the reader.

Substituting $a = b = 1$ in Theorems 10 and 12, (U_n) becomes the Fibonacci sequence $(F_n)_{n \ge 0}$ and we obtain our previous results [11, 18] as a corollary.

Corollary 14. [18, Theorem 2] and [11, Theorem 3.2] *Let $n \ge 3$. Then the following statements hold:*

(i) *if $F_n^k \parallel m$ and $n \not\equiv 3 \pmod 6$, then $F_n^{k+1} \parallel F_{nm}$;*
(ii) *if $F_n^k \parallel m$, $n \equiv 3 \pmod 6$ and $\frac{F_n^{k+1}}{2} \nmid m$, then $F_n^{k+1} \parallel F_{nm}$;*
(iii) *if $F_n^k \parallel m$, $n \equiv 3 \pmod 6$ and $\frac{F_n^{k+1}}{2} \mid m$, then $F_n^{k+2} \parallel F_{nm}$;*
(iv) *if $F_n^{k+1} \parallel F_{nm}$ and $n \not\equiv 3 \pmod 6$, then $F_n^k \parallel m$;*
(v) *if $F_n^{k+1} \parallel F_{nm}$, $n \equiv 3 \pmod 6$, and $2^k \mid m$, then $F_n^k \parallel m$;*
(vi) *if $F_n^{k+1} \parallel F_{nm}$, $n \equiv 3 \pmod 6$, and $2^k \nmid m$, then $F_n^{k-1} \parallel m$.*

Substituting $a = 6$ and $b = -1$, in our theorems, (U_n) reduces to the sequence (B_n) of balancing numbers and we obtain the results of Patra, Panda, and Khemaratchatakumthorn.

Corollary 15. [14, Theorem 9] *For all $k \geq 1$ and $m, n \geq 2$, we obtain $B_n^k \parallel m$ if and only if $B_n^{k+1} \parallel B_{nm}$.*

Similarly by, substituting $a = 2$ and $b = 1$ in our theorems, we obtain the exact divisibility results for the Pell sequence $(P_n)_{n\geq 0}$ as follows.

Corollary 16. *For all $k \geq 1$ and $m, n \geq 2$, we obtain $P_n^k \parallel m$ if and only if $P_n^{k+1} \parallel P_{nm}$.*

We also plan to solve this problem for the Lucas sequence of the second kind in the future. The answers will appear in our next article.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

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