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Research article

On energy ordering of vertex-disjoint bicyclic sidigraphs

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Abstract: The energy and iota energy of signed digraphs are respectively defined by $E(S) = \sum_{k=1}^{n} |\text{Re}(\rho_k)|$ and $E_c(S) = \sum_{k=1}^{n} |\text{Im}(\rho_k)|$, where $\rho_1, \dots, \rho_n$ are eigenvalues of S, and $\text{Re}(\rho_k)$ and $\text{Im}(\rho_k)$ are respectively real and imaginary values of the eigenvalue ρ_k. Recently, Yang and Wang (2018) found the energy and iota energy ordering of digraphs in $\mathcal{D}_n$ and computed the maximal energy and iota energy, where $\mathcal{D}_n$ denotes the set of vertex-disjoint bicyclic digraphs of a fixed order n. In this paper, we investigate the energy ordering of signed digraphs in $\mathcal{D}_n^s$ and find the maximal energy, where $\mathcal{D}_n^s$ denotes the set of vertex-disjoint bicyclic sidigraphs of a fixed order n.

Keywords: signed digraphs; energy ordering; maximal energy
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1. Introduction

If every arc of a digraph is assigned a weight $+1$ or -1 then it is called a signed digraph (henceforth, sidigraph). Each arc of a sidigraph is called a signed arc. We denote by uw, the arc from a vertex u to a vertex w. The sign of the arc uw is denoted by $\varphi(u, w)$. A directed signed path P_n is a sidigraph on n vertices $\{w_j \mid j = 1, 2, \dots, n\}$ with signed arcs $\{w_j w_{j+1} \mid j = 1, 2, \dots, n-1\}$. A signed directed cycle C_n of order $n \geq 2$ is a sidigraph with vertices $\{w_j \mid j = 1, 2, \dots, n\}$ and signed arcs $\{w_j w_{j+1} \mid j = 1, 2, \dots, n-1\} \cup \{w_n w_1\}$. The product of sign of the arcs of a sidigraph S is called the sign of S. A sidigraphs S is said to be a strongly connected sidigraph if for every pair of vertices v, w, a path from v to w and a path from w to v exist.

A sidigraph with equal number of vertices and arcs and contains only one directed cycle is said to be a unicyclic sidigraph. A sidigraph with connected underlying sigraph and has exactly two directed cycles is said to be bicyclic sidigraph. We denote a positive (respectively, negative) cycle of order n by C_n (respectively, $\boldsymbol{C}_n$). A cycle of order n which is either positive or negative is denoted by C_n. A positive cycle is a cycle with positive sign and a negative cycle is a cycle with negative sign. We denote by $\mathcal{D}_n^s$, the class of vertex-disjoint bicyclic sidigraphs of a fixed order n.

Let S be an n-vertex sidigraphs. Then the adjacency matrix $A(S) = [a_{ij}]_{n\times n}$ of S is given by:

$$a_{ij} = \begin{cases} \varphi(w_i, w_j) & \text{if there is an arc from } w_i \text{ to } w_j, \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $A(S)$ are said to be the eigevalues of S.

Peña and Rada [1] put forward the idea of digraph energy. Let $\rho_1, \ldots, \rho_n$ are the eigenvalues of a sidigraph S. Pirzada and Bhat [2] defined the energy of a sidigraph S as $E(S) = \sum_{k=1}^{n} |\mathrm{Re}(\rho_k)|$, where $\mathrm{Re}(\rho_k)$ denotes the real value of the eigenvalue ρ_k. Khan et al. [3] and Farooq et al. [4] put forward the idea of iota energy of digraph (sidigraph) and defined iota energy as $E_c(S) = \sum_{k=1}^{n} |\mathrm{Im}(\rho_k)|$, where $\mathrm{Im}(\rho_k)$ denotes the imaginary value of the eigenvalue ρ_k. Khan et al. [5,6] found the extremal energy of digraphs and sidigraphs among all vertex-disjoint bicyclic digraphs and sidigraphs of order n. Farooq et al. [7,8] found the extremal iota energy of digraphs and sidigraphs among all vertex-disjoint bicyclic digraphs and sidigraphs of order n. In 2016, Monslave and Rada [9] investigated the general class of bicyclic digraphs and found extremal energy. Hafeez et al. [10] considered the class of all bicyclic sidigraphs and finds extremal energy.

Recently, Yang and Wang [11] determined the energy and iota ordering of digraphs in $\mathcal{D}_n$ and found the extremal energy and iota energy, where $\mathcal{D}_n$ is the class of vertex-disjoint bicyclic digraphs of order n. Yang and Wang [12] considered the problem of finding energy ordering in $\mathcal{D}_n^s$, where both directed cycles are of even length. Yang and Wang [13] also considered the problem of finding iota energy ordering in $\mathcal{D}_n^s$, where both directed cycles are of even length. Motivated by Yang and Wang [11–13], we consider the problem of finding the ordering of sidigraphs in $\mathcal{D}_n^s$ with respect to energy and also investigate extremal energy of sidigraphs in this class, where $\mathcal{D}_n^s$ is the class of vertex-disjoint bicyclic sidigraphs of order n. The results for the case, when both directed cycles are even are obtained in [12]. Therefore we have solved the remaining cases for energy ordering in $\mathcal{D}_n^s$.

2. Some results and notations

Let $p, q \geq 2$ and $D_n^s[p, q]$ be the disjoint union of directed cycles C_p and C_q and $D_n^s[\boldsymbol{p}, \boldsymbol{q}]$ be the disjoint union of directed cycles $\boldsymbol{C}_p$ and $\boldsymbol{C}_q$. Also suppose $D_n^s[\boldsymbol{p}, q]$ denotes the disjoint union of directed cycles $\boldsymbol{C}_p$ and C_q and $D_n^s[p, \boldsymbol{q}]$ denotes the disjoint union of directed cycles C_p and $\boldsymbol{C}_q$ and $D_n^s[\mathfrak{p}, \mathfrak{q}]$ denotes the disjoint union of directed cycles C_p and C_q. Let $\mathcal{D}_n^s[p, q] = \{D_n^s[p, q],\ D_n^s[\boldsymbol{p}, \boldsymbol{q}],\ D_n^s[\boldsymbol{p}, q],\ D_n^s[p, \boldsymbol{q}]\}$.

Let S be a sidigraph with eigenvalues $\rho_1, \ldots, \rho_n$. Then energy of S is defined as $E(S) = \sum_{k=1}^{n} |\mathrm{Re}(\rho_k)|$, where $\mathrm{Re}(\rho_k)$ represents the real value of ρ_k.

Relationship between energy of strong components of a sidigraph S and energy of S is given in the following result.

Lemma 2.1 (Pirzada and Bhat [2])**.** *Let $Q_1, \ldots, Q_k$ are strong components of a sidigraph S. Then $E(S) = \sum\limits_{j=1}^{k} E(Q_j)$.*

Pirzada and Bhat [2] gave the following energy formulae for positive and negative directed cycles of order $n \geq 2$.

$$E(C_n) = \begin{cases} 2\cot\frac{\pi}{n} & \text{if } n \equiv 0(\text{mod}4) \\ 2\csc\frac{\pi}{n} & \text{if } n \equiv 2(\text{mod}4) \\ \csc\frac{\pi}{2n} & \text{if } n \equiv 1(\text{mod}2), \end{cases} \tag{2.1}$$

$$E(\boldsymbol{C}_n) = \begin{cases} 2\csc\frac{\pi}{n} & \text{if } n \equiv 0(\text{mod}4) \\ 2\cot\frac{\pi}{n} & \text{if } n \equiv 2(\text{mod}4) \\ \csc\frac{\pi}{2n} & \text{if } n \equiv 1(\text{mod}2). \end{cases} \tag{2.2}$$

For any $S \in \mathcal{D}_n^s$, its strong components are: a sidigraph from the set $\mathcal{D}_n^s[p,q]$ and few isolated vertices. Therefore using Lemma 2.1, we can only use the energy of strong components to find the energy ordering in $\mathcal{D}_n^s$.

Using Lemma 2.1, we give the following equations.

$$\begin{aligned} E(D_n^s[p,q]) &= E(C_p) + E(C_q), \\ E(D_n^s[\boldsymbol{p},\boldsymbol{q}]) &= E(\boldsymbol{C}_p) + E(\boldsymbol{C}_q), \\ E(D_n^s[\boldsymbol{p},q]) &= E(\boldsymbol{C}_p) + E(C_q), \\ E(D_n^s[p,\boldsymbol{q}]) &= E(C_p) + E(\boldsymbol{C}_q). \end{aligned}$$

Let $n > 4$. In Lemmas 2.2~2.8, we give some results about the monotonicity of some functions which will be used to find the energy ordering of sidigraphs in $\mathcal{D}_n^s$.

Lemma 2.2 (Farooq et al. [7]). *Suppose* $f(z) = 2(\cot\frac{\pi}{z} + \cot\frac{\pi}{n-z})$. *For* $z \in [2, \frac{n}{2}]$, $f(z)$ *is increasing and for* $z \in [\frac{n}{2}, n-2]$, $f(z)$ *is decreasing.*

Lemma 2.3 (Yang and Wang [11]). *Let* $f(z) = 2(\csc\frac{\pi}{z} + \cot\frac{\pi}{n-z})$. *For* $z \in [2, n-2]$, $f(z)$ *is decreasing.*

Lemma 2.4 (Yang and Wang [11]). *Suppose* $f(z) = 2(\csc\frac{\pi}{z} + \csc\frac{\pi}{n-z})$. *For* $z \in [2, \frac{n}{2}]$, $f(z)$ *is decreasing.*

Lemma 2.5 (Farooq et al. [7]). *Let* $f(z) = z\sin\frac{\pi}{z}$. *For* $z \in [2, \infty)$, $f(z)$ *is increasing.*

Lemma 2.6 (Yang and Wang [11]). *Suppose* $f(z) = \frac{\pi}{z^2}\cos\frac{\pi}{z}\csc^2\frac{\pi}{z}$. *For* $z \in [2, n-2]$, $f(z)$ *is increasing.*

The proof of next lemma is similar to the proof of Lemma 2.6 and is thus omitted.

Lemma 2.7. *Suppose* $f(z) = \frac{\pi}{z^2}\cos\frac{\pi}{2z}\csc^2\frac{\pi}{2z}$ *and* $g(z) = \frac{\pi}{z^2}\cos\frac{\pi}{z}\csc^2\frac{\pi}{z}$. *For* $z \in [2, \infty)$, $f(z)$ *and* $g(z)$ *are increasing.*

Now we prove the following results.

Lemma 2.8. *Suppose* $f(z) = 2(\cot\frac{\pi}{z} + \csc\frac{\pi}{n-z})$. *For* $z \in [2, n-2]$, $f(z)$ *is increasing.*

Proof. To prove the result, we will show that for all $z \in [2, n-2]$, $f'(z) \geq 0$.

Now

$$f'(z) = 2\Big(\frac{\pi}{z^2}\csc^2\frac{\pi}{z} - \frac{\pi}{(n-z)^2}\csc\frac{\pi}{n-z}\cot\frac{\pi}{n-z}\Big) \tag{2.3}$$

To prove $f'(z) \geq 0$, we divide the interval in two parts. Firstly let $z \in [\frac{n}{2}, n-2]$. Then $z \geq n-z$. By Lemma 2.6, we know that for $z \in [2, n-2]$, $\frac{\pi}{z^2}\cos\frac{\pi}{z}\csc^2\frac{\pi}{z}$ is increasing. Therefore $\frac{\pi}{(n-z)^2}\csc\frac{\pi}{n-z}\cot\frac{\pi}{n-z} = \frac{\pi}{(n-z)^2}\csc^2\frac{\pi}{n-z}\cos\frac{\pi}{n-z} \leq \frac{\pi}{z^2}\csc^2\frac{\pi}{z}\cos\frac{\pi}{z} < \frac{\pi}{z^2}\csc^2\frac{\pi}{z}$. Hence using (2.3), $f'(z) \geq 0$ for $z \in [\frac{n}{2}, n-2]$.

Now let $z \in [2, \frac{n}{2}]$. Then $z \leq n-z$. By Lemma 2.5, we know that $z \sin\frac{\pi}{z}$ is strictly increasing on $[2,\infty)$. We have $z\sin\frac{\pi}{z} \leq (n-z)\sin\frac{\pi}{n-z}$. From this, we get $\frac{1}{z}\csc\frac{\pi}{n-z} - \frac{1}{n-z}\csc\frac{\pi}{n-z} \geq 0$. Consider

$$\frac{\pi}{z^2}\csc^2\frac{\pi}{z} - \frac{\pi}{(n-z)^2}\csc^2\frac{\pi}{n-z} = \pi\Big(\frac{1}{z}\csc\frac{\pi}{z} + \frac{1}{n-z}\csc\frac{\pi}{n-z}\Big)\Big(\frac{1}{z}\csc\frac{\pi}{z} - \frac{1}{n-z}\csc\frac{\pi}{n-z}\Big).$$

Clearly $\frac{1}{z}\csc\frac{\pi}{z} + \frac{1}{n-z}\csc\frac{\pi}{n-z} > 0$ and $\frac{1}{z}\csc\frac{\pi}{z} - \frac{1}{n-z}\csc\frac{\pi}{n-z} \geq 0$. Hence $\frac{\pi}{z^2}\csc^2\frac{\pi}{z} - \frac{\pi}{(n-z)^2}\csc^2\frac{\pi}{n-z} \geq 0$. This implies that $\frac{\pi}{(n-z)^2}\csc\frac{\pi}{n-z}\cot\frac{\pi}{n-z} = \frac{\pi}{(n-z)^2}\cos\frac{\pi}{n-z}\csc^2\frac{\pi}{n-z} < \frac{\pi}{(n-z)^2}\csc^2\frac{\pi}{n-z} < \frac{\pi}{z^2}\csc^2\frac{\pi}{z}$. Hence using (2.3) and all these facts, $f'(z) \geq 0$ for $z \in [2, \frac{n}{2}]$. Thus $f'(z) \geq 0$ for $z \in [2, n-2]$. This proves the result. □

Lemma 2.9. *Let $z \in [2, n-2]$. The following holds.*

1. *Let $f(z) = \csc\frac{\pi}{2z} + \csc\frac{\pi}{2(n-z)}$ is decreasing on $[2, \frac{n}{2}]$ and increasing on $[\frac{n}{2}, n-2]$.*
2. *The function $f(z) = 2\csc\frac{\pi}{z} + \csc\frac{\pi}{2(n-z)}$ is decreasing on $[2, \frac{2n}{3}]$ and increasing on $[\frac{2n}{3}, n-2]$.*
3. *The function $f(z) = \csc\frac{\pi}{2z} + 2\csc\frac{\pi}{(n-z)}$ is decreasing on $[2, \frac{n}{3}]$ and increasing on $[\frac{n}{3}, n-2]$.*

Proof. (1) To show that $f(z)$ is decreasing on $[2, \frac{n}{2}]$, it is sufficient to show that $f'(z) \leq 0$.

Since $z \leq (n-z)$ for $z \in [2, \frac{n}{2}]$, therefore using Lemma 2.7, we get

$$\begin{aligned} f'(z) &= \frac{\pi}{2z^2}\csc^2\frac{\pi}{2z}\cos\frac{\pi}{2z} - \frac{\pi}{2(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} \\ &\leq \frac{\pi}{2(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} - \frac{\pi}{2(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} = 0 \end{aligned}$$

Hence $f(z)$ is decreasing on $[2, \frac{n}{2}]$.

Now we will show that $f'(z) \geq 0$. Since $z \geq (n-z)$ for $z \in [\frac{n}{2}, n-2]$, therefore using Lemma 2.7, we obtain

$$\begin{aligned} f'(z) &= \frac{\pi}{2z^2}\csc^2\frac{\pi}{2z}\cos\frac{\pi}{2z} - \frac{\pi}{2(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} \\ &\geq \frac{\pi}{2(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} - \frac{\pi}{2(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} = 0 \end{aligned}$$

Hence $f(z)$ is increasing on $[\frac{n}{2}, n-2]$.

(2) To show that $f(z)$ is decreasing on $[2, \frac{2n}{3}]$, it is enough to prove that $f'(z) \leq 0$.

Since $z \leq 2(n-z)$ for $z \in [2, \frac{2n}{3}]$, therefore using Lemma 2.7, we get

$$\begin{aligned} f'(z) &= \frac{2\pi}{z^2}\csc^2\frac{\pi}{z}\cos\frac{\pi}{z} - \frac{\pi}{2(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} \\ &\leq \frac{2\pi}{4(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} - \frac{\pi}{2(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} = 0 \end{aligned}$$

Hence $f(z)$ is decreasing on $[2, \frac{2n}{3}]$.

Now we will show that $f'(z) \geq 0$. Since $z \geq 2(n-z)$ for $z \in [\frac{2n}{3}, n-2]$, therefore using Lemma 2.7, we obtain

$$f'(z) = \frac{2\pi}{z^2}\csc^2\frac{\pi}{z}\cos\frac{\pi}{z} - \frac{\pi}{2(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)}$$

$$\geq \frac{2\pi}{4(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} - \frac{\pi}{2(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} = 0$$

Hence $f(z)$ is increasing on $[\frac{2n}{3}, n-2]$.

Analogously (3) can be proved. □

Lemma 2.10. *Suppose* $f(z) = 2\cot\frac{\pi}{z} + \csc\frac{\pi}{2(n-z)}$. *For* $z \in [2, n-2]$, $f(z)$ *is increasing.*

Proof. We will show that $f'(z) \geq 0$ for $z \in [2, n-2]$.

Since $\cos\frac{\pi}{z} \leq 1$ and $z \geq 2(n-z)$ for $z \in [\frac{2n}{3}, n-2]$, therefore by Lemma 2.7, we have

$$\begin{aligned} f'(z) &= \frac{2\pi}{z^2}\csc^2\frac{\pi}{z} - \frac{\pi}{2(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} \\ &\geq \frac{2\pi}{z^2}\csc^2\frac{\pi}{z}\cos\frac{\pi}{z} - \frac{\pi}{2(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} \\ &\geq \frac{2\pi}{4(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} - \frac{\pi}{2(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} = 0 \end{aligned}$$

Also $-\cos\frac{\pi}{z} \geq -1$ and $z \leq 2(n-z)$ for $z \in [2, \frac{2n}{3}]$, therefore by proof of Lemma 2.4 [7], we see that

$$\begin{aligned} f'(z) &= \frac{2\pi}{z^2}\csc^2\frac{\pi}{z} - \frac{\pi}{2(n-z)^2}\csc^2\frac{\pi}{2(n-z)}\cos\frac{\pi}{2(n-z)} \\ &\geq \frac{2\pi}{z^2}\csc^2\frac{\pi}{z} - \frac{\pi}{2(n-z)^2}\csc^2\frac{\pi}{2(n-z)} \geq 0. \end{aligned}$$

Hence $f(z)$ is increasing on $[2, n-2]$. □

3. Energy ordering

Sidigraphs in $\mathcal{D}_n^s$ are classified into three categories: the sidigraphs whose directed cycles are of even length, the sidigraphs whose directed cycles are of odd length and the sidigraphs whose one directed cycle is of even length and one is of odd length. Yang and Wang [12] investigated the energy ordering in first category where both cycles are of even length. Therefore in the following section, we separately investigate energy ordering in other two categories and find maximal energy.

3.1. Both cycles of even length

Yang and Wang [12] investigated the energy ordering of bicyclic sidigraphs in $\mathcal{D}_n^s$, where each directed cycle is of even length. For details see [12].

Yang and Yang also proved the following theorem about the extremal energy of those bicyclic sidigraphs in the class $\mathcal{D}_n^s$ whose both directed cycles are of even length.

Theorem 3.1 (Xang and Wang [12])**.** *Suppose a sidigraph* $\mathcal{S} \in \mathcal{D}_n^s$ *has even directed cycles.*

(*i*) *For* $n \equiv 0 \pmod 4$, *the largest energy of* $\mathcal{S}$ *is obtained if* $\mathcal{S} \cong D_n^s[2, n-2]$.
(*ii*) *For* $n \equiv 1 \pmod 4$, *the largest energy of* $\mathcal{S}$ *is obtained if* $\mathcal{S} \cong D_n^s[2, n-3]$.
(*iii*) *For* $n \equiv 2 \pmod 4$, *the largest energy of* $\mathcal{S}$ *is obtained if* $\mathcal{S} \cong D_n^s[2, \boldsymbol{n-2}]$.
(*iv*) *For* $n \equiv 3 \pmod 4$, *the largest energy of* $\mathcal{S}$ *is obtained if* $\mathcal{S} \cong D_n^s[2, \boldsymbol{n-3}]$.
(*v*) *The smallest energy of* $\mathcal{S}$ *is obtained if* $\mathcal{S} \cong D_n^s[\mathbf{2}, \mathbf{2}]$.

3.2. Both cycles of odd length

In this section, we find energy ordering of those bicyclic sidigraphs in $\mathcal{D}_n^s$ that contain directed cycles of odd length. Note that for $r \equiv 1 (\text{mod } 2)$, $E(C_r) = E(\boldsymbol{C}_r)$. Hence we only consider the case when both directed cycles are positive.

Lemma 3.2. *Let $n > 5$ and $n \equiv 0(\text{mod } 4)$. Take $r \in [2, n-2]$ satisfying $r \equiv 1(\text{mod } 2)$ and $n - r \equiv 1(\text{mod } 2)$. Then $E(D_n^s[r, n-r])$ has maximum value at $r = 3$. Therefore the following energy ordering holds:*

$$E(D_n^s[3, n-3]) > E(D_n^s[5, n-5]) > \cdots > E(D_n^s[\frac{n-2}{2}, \frac{n+2}{2}]).$$

Proof. Using Eq (2.1), we get

$$E(D_n^s[r, n-r]) = \csc\frac{\pi}{2r} + \csc\frac{\pi}{2(n-r)}.$$

By Part (1) of Lemma 2.9, we see that $\csc\frac{\pi}{2r} + \csc\frac{\pi}{2(n-r)}$ is decreasing on $[2, \frac{n}{2}]$ and increasing on $[\frac{n}{2}, n-2]$. Therefore the smallest odd number in $[2, \frac{n}{2}]$ where $E(D_n^s[r, n-r])$ has maximum value is $r = 3$ and the largest odd number in $[\frac{n}{2}, n-2]$ where $E(D_n^s[r, n-r])$ has maximum value is $r = n-3$. Thus we have

$$E(D_n^s[3, n-3]) > E(D_n^s[5, n-5]) > \cdots > E\left(D_n^s\left[\frac{n}{2} - 1, \frac{n}{2} + 1\right]\right).$$

The proof is complete. □

Similar to Lemma 3.2, the following result can be proved.

Lemma 3.3. *Let $n > 5$ and $n \equiv 2(\text{mod } 4)$. Take $r \in [2, n-2]$ satisfying $r \equiv 1(\text{mod } 2)$ and $n - r \equiv 1(\text{mod } 2)$. Then $E(D_n^s[r, n-r])$ has maximum value at $r = 3$. Therefore the following energy ordering holds:*

$$E(D_n^s[3, n-3]) > E(D_n^s[5, n-5]) > \cdots > E(D_n^s[\frac{n}{2}, \frac{n}{2}]).$$

Lemma 3.4. *Let $n > 5$ and $n \equiv 1(\text{mod } 4)$. Take $r \in [2, n-2]$ satisfying $r \equiv 1(\text{mod } 2)$ and $n - r - 1 \equiv 1(\text{mod } 2)$. Then $E(D_n^s[r, n-r-1])$ has maximum value at $r = 3$. Therefore the following energy ordering holds:*

$$E(D_n^s[3, n-4]) > E(D_n^s[5, n-6]) > \cdots > E(D_n^s[\frac{n-3}{2}, \frac{n+1}{2}]).$$

Proof. Since $r \equiv 1 \ (\text{mod } 2)$ and $n \equiv 1 \ (\text{mod } 4)$, therefore $n - r - 1 \equiv \ (\text{mod } 2)$. Using Eq (2.1), we have

$$E(D_n^s[r, n-r-1] = \csc\frac{\pi}{2r} + \csc\frac{\pi}{2(n-r-1)}.$$

Hence by changing n to $n-1$ is Lemma 3.2, we get the desired result. □

By changing n in Lemma 3.3 to $n-1$, the following result is obtained.

Lemma 3.5. *Let $n > 5$ and $n \equiv 3(\text{mod } 4)$. Take $r \in [2, n-2]$ satisfying $r \equiv 1(\text{mod } 2)$ and $n - r - 1 \equiv 1(\text{mod } 2)$. Then $E(D_n^s[r, n-r-1])$ has maximum value at $r = 3$. Therefore the following energy ordering holds:*

$$E(D_n^s[3, n-4]) > E(D_n^s[5, n-6]) > \cdots > E(D_n^s[\frac{n-1}{2}, \frac{n-1}{2}]).$$

Combining Lemmas 3.2 ∼ 3.5, the following corollary is obtained.

Corollary 3.6. *Suppose* $r \equiv 1 (\bmod 2)$ *and* $n > 5$.

(i) *If* $n \equiv 0 (\bmod 2)$ *then* $E(D_n^s[3, n-3]) \geq E(D_n^s[r, n-r])$.
(ii) *If* $n \equiv 1 (\bmod 2)$ *then* $E(D_n^s[3, n-4]) \geq E(D_n^s[r, n-r-1])$.

Now we give the extremal energy of bicyclic sidigraphs in the class $\mathcal{D}_n^s$.

Theorem 3.7. *Let* $S \in \mathcal{D}_n^s$ *be a sidigraph with odd directed cycles.*

(i) *For* $n \equiv 0 (\bmod 2)$*, the maximal energy of* S *is attained if* $S \cong D_n^s[3, n-3]$.
(ii) *For* $n \equiv 1 (\bmod 2)$*, the maximal energy of* S *is attained if* $S \cong D_n^s[3, n-4]$.
(iii) *The minimal energy of* S *is attained if* $S \cong D_n^s[3, 3]$.

Proof. The proof of Part (i) and (ii) follows from Corollary 3.6.
(iii). As for odd integers r_1 and r_2 with $r_1 \geq r_2 \geq 3$, it holds that $E(C_{r_1}) \geq E(C_{r_2})$. Hence the minimal energy of S is attained if $S \cong D_n^s[3, 3]$. □

In next theorem, we give the energy ordering of those bicyclic sidigraphs in $\mathcal{D}_n^s$ whose both directed cycles are of odd length.

Theorem 3.8. *Let* $n > 5$ *and* $r \in [2, n-2]$.

(i) *If* $n \equiv 0 (\bmod 4)$ *then we have the following energy ordering:*

$$E(D_n^s[3, n-3]) > E(D_n^s[5, n-5]) > \cdots > E(D_n^s[\frac{n-2}{2}, \frac{n+2}{2}]) > E(D_n^s[\frac{n-6}{2}, \frac{n+2}{2}])$$
$$> \cdots > E(D_n^s[3, \frac{n+2}{2}]) > E(D_n^s[3, \frac{n-2}{2}]) > E(D_n^s[3, \frac{n-6}{2}]) > \cdots > E(D_n^s[3, 3]).$$

(ii) *If* $n \equiv 1 (\bmod 4)$ *then we have the following energy ordering:*

$$E(D_n^s[3, n-4]) > E(D_n^s[5, n-6]) > \cdots > E(D_n^s[\frac{n-3}{2}, \frac{n+1}{2}]) > E(D_n^s[\frac{n-7}{2}, \frac{n+1}{2}])$$
$$> \cdots > E(D_n^s[3, \frac{n+1}{2}]) > E(D_n^s[3, \frac{n-3}{2}]) > E(D_n^s[3, \frac{n-7}{2}]) > \cdots > E(D_n^s[3, 3]).$$

(iii) *If* $n \equiv 2 (\bmod 4)$ *then we have the following energy ordering:*

$$E(D_n^s[3, n-3]) > E(D_n^s[5, n-5]) > \cdots > E(D_n^s[\frac{n}{2}, \frac{n}{2}]) > E(D_n^s[\frac{n-4}{2}, \frac{n}{2}])$$
$$> \cdots > E(D_n^s[3, \frac{n}{2}]) > E(D_n^s[3, \frac{n-4}{2}]) > E(D_n^s[3, \frac{n-8}{2}]) > \cdots > E(D_n^s[3, 3]).$$

(iv) *If* $n \equiv 3 (\bmod 4)$ *then we have the following energy ordering:*

$$E(D_n^s[3, n-4]) > E(D_n^s[5, n-6]) > \cdots > E(D_n^s[\frac{n-1}{2}, \frac{n-1}{2}]) > E(D_n^s[\frac{n-5}{2}, \frac{n-1}{2}])$$
$$> \cdots > E(D_n^s[3, \frac{n-1}{2}]) > E(D_n^s[3, \frac{n-5}{2}]) > E(D_n^s[3, \frac{n-9}{2}]) > \cdots > E(D_n^s[3, 3]).$$

Example 3.9. *We illustrate Theorem 3.8 by considering sidigraphs of different order.*

(*i*) *Take* $n = 12$*, that is,* $n \equiv 0(\text{mod}4)$*. Then*

$$\begin{aligned}
E(D_n^s[5, n-5]) &= E(D_n^s[\frac{n-2}{2}, \frac{n+2}{2}]) = E(D_n^s[5,7]) = 7.7300 \\
E(D_n^s[3, n-3]) &= E(D_n^s[3,9]) = 7.7588 \\
E(D_n^s[\frac{n-6}{2}, \frac{n+2}{2}]) &= E(D_n^s[3,5]) = 6.4940 \\
E(D_n^s[3, \frac{n-2}{2}]) &= EE(D_n^s[3,5]) = 5.2361 \\
E(D_n^s[3,3]) &= 4.
\end{aligned}$$

It follows that

$$E(D_n^s[3,9]) > E(D_n^s[5,7]) > E(D_n^s[3,5]) > E(D_n^s[3,3]).$$

(*ii*) *Take* $n = 17$*, that is,* $n \equiv 1(\text{mod}4)$*. Then*

$$\begin{aligned}
E(D_n^s[7, n-7]) &= E(D_n^s[\frac{n-3}{2}, \frac{n+1}{2}]) = E(D_n^s[7,9]) = 10.2527 \\
E(D_n^s[3, n-4]) &= E(D_n^s[3,13]) = 10.2962 \\
E(D_n^s[5, n-6]) &= E(D_n^s[5,11]) = 10.2627 \\
E(D_n^s[\frac{n-7}{2}, \frac{n+1}{2}]) &= E(D_n^s[5,9]) = 8.9948 \\
E(D_n^s[3, \frac{n+1}{2}]) &= E(D_n^s[3,9]) = 7.7588 \\
E(D_n^s[3, \frac{n-3}{2}]) &= E(D_n^s[3,7]) = 6.4940 \\
E(D_n^s[3, \frac{n-7}{2}]) &= E(D_n^s[3,5]) = 5.2361 \\
E(D_n^s[3, \frac{n-11}{2}]) &= E(D_n^s[3,3]) = 4.
\end{aligned}$$

Therefore we get,

$$\begin{aligned}
&E(D_n^s[3,13]) > E(D_n^s[5,11]) > E(D_n^s[7,9]) > E(D_n^s[5,9]) > E(D_n^s[3,9]) \\
&> E(D_n^s[3,7]) > E(D_n^s[3,5]) > E(D_n^s[3,3]).
\end{aligned}$$

(*iii*) *Take* $n = 14$*, that is,* $n \equiv 2(\text{mod}4)$*. Then*

$$\begin{aligned}
E(D_n^s[7, n-7]) &= E(D_n^s[\frac{n}{2}, \frac{n}{2}]) = E(D_n^s[7,7]) = 8.9879 \\
E(D_n^s[\frac{n-8}{2}, \frac{n}{2}]) &= E(D_n^s[3, \frac{n}{2}]) = E(D_n^s[3,7]) = 6.4940 \\
E(D_n^s[3, n-3]) &= E(D_n^s[3,11]) = 9.0267 \\
E(D_n^s[5, n-5]) &= E(D_n^s[5,9]) = 8.9948 \\
E(D_n^s[\frac{n-4}{2}, \frac{n}{2}]) &= E(D_n^s[5,7]) = 7.7300 \\
E(D_n^s[3, \frac{n-4}{2}]) &= E(D_n^s[3,5]) = 5.2361
\end{aligned}$$

$$E(D_n^s[3, \frac{n-8}{2}]) = E(D_n^s[3,3]) = 4.$$

Hence we get,

$$E(D_n^s[3,11]) > E(D_n^s[5,9]) > E(D_n^s[7,7]) > E(D_n^s[5,7]) > E(D_n^s[3,7]) > E(D_n^s[3,5]) > E(D_n^s[3,3]).$$

(iv) *Take* $n = 19$*, that is,* $n \equiv 3(\text{mod}4)$*. Then*

$$\begin{aligned}
E(D_n^s[9, n-9] &= E(D_n^s[\frac{n-1}{2}, \frac{n-1}{2}]) = E(D_n^s[9,9]) = 11.5175 \\
E(D_n^s[3, \frac{n-1}{2}]) &= E(D_n^s[\frac{n-13}{2}, \frac{n-1}{2}]) = E(D_n^s[3,9]) = 7.7588 \\
E(D_n^s[\frac{n-5}{2}, \frac{n-1}{2}]) &= E(D_n^s[7,9]) = 10.2527 \\
E(D_n^s[\frac{n-9}{2}, \frac{n-1}{2}]) &= E(D_n^s[5,9]) = 8.9948 \\
E(D_n^s[3, n-4]) &= E(D_n^s[3,15]) = 11.5688 \\
E(D_n^s[5, n-6]) &= E(D_n^s[5,13]) = 11.5323 \\
E(D_n^s[7, n-8]) &= E(D_n^s[7,11]) = 11.5206 \\
E(D_n^s[3, \frac{n-5}{2}]) &= E(D_n^s[3,7]) = 6.4940 \\
E(D_n^s[3, \frac{n-9}{2}]) &= E(D_n^s[3,5]) = 5.2361 \\
E(D_n^s[3, \frac{n-13}{2}]) &= E(D_n^s[3,3]) = 4.
\end{aligned}$$

Therefore, we obtain

$$E(D_n^s[3,15]) > E(D_n^s[5,13]) > E(D_n^s[7,11]) > E(D_n^s[9,9]) > E(D_n^s[7,9]) > E(D_n^s[5,9]) > E(D_n^s[3,9]) > E(D_n^s[3,7]) > E(D_n^s[3,5]) > E(D_n^s[3,3]).$$

3.3. One cycle of odd length and one cycle of even length

In this section, we find energy ordering of those bicyclic sidigraphs in $\mathcal{D}_n^s$ whose one directed cycle is of even length and one is of odd length. For $n \equiv 0(\text{mod } 2)$, if $r \equiv 1(\text{mod } 2)$ then $n - r \equiv 1(\text{mod } 2)$ and if $r \equiv 0(\text{mod } 2)$ then $n - r \equiv 0(\text{mod } 2)$. So we only consider the case when $n \equiv 1(\text{mod } 2)$. Note that for $r \equiv 0(\text{mod } 2)$ and $n - r \equiv 1(\text{mod } 2)$ then $E(D_n^s[r, n-r]) = E(D_n^s[r, \boldsymbol{n-r}])$. Hence we only have to give the energy ordering of those bicyclic sidigraphs in $\mathcal{D}_n^s$ whose both directed cycles are positive or both directed cycles are negative. The proofs are similar to the proofs of Lemmas 3.2 $\sim$ 3.5 and thus omitted.

Now we give the energy ordering of those bicyclic sidigraphs in $\mathcal{D}_n^s$ whose both directed cycles are positive.

Lemma 3.10. *Suppose* $n > 5$*,* $n \equiv 0(\text{mod } 3)$ *and* $r \in [2, n-2]$ *satisfying* $r \equiv 0(\text{mod } 2)$ *and* $n - r \equiv 1(\text{mod } 2)$*. Then we have the following energy ordering:*

(i) *Let* $r \equiv 2(\mathrm{mod}\ 4)$.

(a) *If* $r \in [2, \frac{2n}{3}]$ *then*

$$E(D_n^s[2, n-2]) > E(D_n^s[6, n-6]) > \cdots > E(D_n^s[\frac{2n}{3}, \frac{n}{3}]).$$

(b) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 2(\mathrm{mod}\ 4)$ *then*

$$E(D_n^s[n-3, 3]) > E(D_n^s[n-7, 7]) > \cdots > E(D_n^s[\frac{2n}{3}, \frac{n}{3}]).$$

(c) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 0(\mathrm{mod}\ 4)$ *then*

$$E(D_n^s[n-5, 5]) > E(D_n^s[n-9, 9]) > \cdots > E(D_n^s[\frac{2n}{3}, \frac{n}{3}]).$$

(ii) *Let* $r \equiv 0(\mathrm{mod}\ 4)$.

(a) *If* $n-3 \equiv 2(\mathrm{mod}\ 4)$ *then*

$$E(D_n^s[n-5, 5]) > E(D_n^s[n-9, 9]) > \cdots > E(D_n^s[4, n-4]).$$

(b) *If* $n-3 \equiv 0(\mathrm{mod}\ 4)$ *then*

$$E(D_n^s[n-3, 3]) > E(D_n^s[n-7, 7]) > \cdots > E(D_n^s[4, n-4]).$$

Lemma 3.11. *Suppose* $n > 5$, $n \equiv 1(\mathrm{mod}\ 3)$ *and* $r \in [2, n-2]$ *satisfying* $r \equiv 0(\mathrm{mod}\ 2)$ *and* $n-r \equiv 1(\mathrm{mod}\ 2)$. *Then we have the following energy ordering:*

(i) *Let* $r \equiv 2(\mathrm{mod}\ 4)$.

(a) *If* $r \in [2, \frac{2n}{3}]$ *then*

$$E(D_n^s[2, n-2]) > E(D_n^s[6, n-6]) > \cdots > E(D_n^s[\frac{2n-8}{3}, \frac{n+8}{3}]).$$

(b) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 2(\mathrm{mod}\ 4)$ *then*

$$E(D_n^s[n-3, 3]) > E(D_n^s[n-7, 7]) > \cdots > E(D_n^s[\frac{2n+4}{3}, \frac{n-4}{3}]).$$

(c) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 0(\mathrm{mod}\ 4)$ *then*

$$E(D_n^s[n-5, 5]) > E(D_n^s[n-9, 9]) > \cdots > E(D_n^s[\frac{2n+4}{3}, \frac{n-4}{3}]).$$

(ii) *Let* $r \equiv 0(\mathrm{mod}\ 4)$.

(a) *If* $n-3 \equiv 2(\mathrm{mod}\ 4)$ *then*

$$E(D_n^s[n-5, 5]) > E(D_n^s[n-9, 9]) > \cdots > E(D_n^s[4, n-4]).$$

(*b*) *If* $n-3 \equiv 0 (\bmod 4)$ *then*

$$E(D_n^s[n-3,3]) > E(D_n^s[n-7,7]) > \cdots > E(D_n^s[4,n-4]).$$

Lemma 3.12. *Suppose* $n > 5$, $n \equiv 2 (\bmod 3)$ *and* $r \in [2, n-2]$ *satisfying* $r \equiv 0 (\bmod 2)$ *and* $n-r \equiv 1 (\bmod 2)$. *Then we have the following energy ordering:*

(*i*) *Let* $r \equiv 2 (\bmod 4)$.

(*a*) *If* $r \in [2, \frac{2n}{3}]$ *then*

$$E(D_n^s[2,n-2]) > E(D_n^s[6,n-6]) > \cdots > E(D_n^s[\frac{2n-4}{3}, \frac{n+4}{3}]).$$

(*b*) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 2 (\bmod 4)$ *then*

$$E(D_n^s[n-3,3]) > E(D_n^s[n-7,7]) > \cdots > E(D_n^s[\frac{2n+8}{3}, \frac{n-8}{3}]).$$

(*c*) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 0 (\bmod 4)$ *then*

$$E(D_n^s[n-5,5]) > E(D_n^s[n-9,9]) > \cdots > E(D_n^s[\frac{2n+8}{3}, \frac{n-8}{3}]).$$

(*ii*) *Let* $r \equiv 0 (\bmod 4)$.

(*a*) *If* $n-3 \equiv 2 (\bmod 4)$ *then*

$$E(D_n^s[n-5,5]) > E(D_n^s[n-9,9]) > \cdots > E(D_n^s[4,n-4]).$$

(*b*) *If* $n-3 \equiv 0 (\bmod 4)$ *then*

$$E(D_n^s[n-3,3]) > E(D_n^s[n-7,7]) > \cdots > E(D_n^s[4,n-4]).$$

Now we give the energy ordering of those bicyclic sidigraphs in $\mathcal{D}_n^s$ whose both directed cycles are negative.

Lemma 3.13. *Suppose* $n > 5$, $n \equiv 0 (\bmod 3)$ *and* $r \in [2, n-2]$ *satisfying* $r \equiv 0 (\bmod 2)$ *and* $n-r \equiv 1 (\bmod 2)$. *Then we have the following energy ordering:*

(*i*) *Let* $r \equiv 0 (\bmod 4)$.

(*a*) *If* $r \in [2, \frac{2n}{3}]$ *then*

$$E(D_n^s[\mathbf{4}, \boldsymbol{n}-\mathbf{4}]) > E(D_n^s[\mathbf{8}, \boldsymbol{n}-\mathbf{8}]) > \cdots > E(D_n^s[\frac{\mathbf{2}\boldsymbol{n}-\mathbf{6}}{\mathbf{3}}, \frac{\boldsymbol{n}+\mathbf{6}}{\mathbf{3}}]).$$

(*b*) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 2 (\bmod 4)$ *then*

$$E(D_n^s[\boldsymbol{n}-\mathbf{5}, \mathbf{5}]) > E(D_n^s[\boldsymbol{n}-\mathbf{9}, \mathbf{9}]) > \cdots > E(D_n^s[\frac{\mathbf{2}\boldsymbol{n}+\mathbf{6}}{\mathbf{3}}, \frac{\boldsymbol{n}-\mathbf{6}}{\mathbf{3}}]).$$

(c) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 0 \pmod 4$ *then*

$$E(D_n^s[\boldsymbol{n-3,3}]) > E(D_n^s[\boldsymbol{n-7,7}]) > \cdots > E(D_n^s[\boldsymbol{\frac{2n+6}{3}, \frac{n-6}{3}}]).$$

(ii) *Let* $r \equiv 2 \pmod 4$.

(a) *If* $n-3 \equiv 2 \pmod 4$ *then*

$$E(D_n^s[\boldsymbol{n-3,3}]) > E(D_n^s[\boldsymbol{n-7,7}]) > \cdots > E(D_n^s[\boldsymbol{2,n-2}]).$$

(b) *If* $n-3 \equiv 0 \pmod 4$ *then*

$$E(D_n^s[\boldsymbol{n-5,5}]) > E(D_n^s[\boldsymbol{n-9,9}]) > \cdots > E(D_n^s[\boldsymbol{2,n-2}]).$$

Lemma 3.14. *Suppose* $n > 5$, $n \equiv 1 \pmod 3$ *and* $r \in [2, n-2]$ *satisfying* $r \equiv 0 \pmod 2$ *and* $n - r \equiv 1 \pmod 2$. *Then we have the following energy ordering:*

(i) *Let* $r \equiv 0 \pmod 4$.

(a) *If* $r \in [2, \frac{2n}{3}]$ *then*

$$E(D_n^s[\boldsymbol{4,n-4}]) > E(D_n^s[\boldsymbol{8,n-8}]) > \cdots > E(D_n^s[\boldsymbol{\frac{2n-2}{3}, \frac{n+2}{3}}]).$$

(b) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 2 \pmod 4$ *then*

$$E(D_n^s[\boldsymbol{n-5,5}]) > E(D_n^s[\boldsymbol{n-9,9}]) > \cdots > E(D_n^s[\boldsymbol{\frac{2n+10}{3}, \frac{n-10}{3}}]).$$

(c) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 0 \pmod 4$ *then*

$$E(D_n^s[\boldsymbol{n-3,3}]) > E(D_n^s[\boldsymbol{n-7,7}]) > \cdots > E(D_n^s[\boldsymbol{\frac{2n+10}{3}, \frac{n-10}{3}}]).$$

(ii) *Let* $r \equiv 2 \pmod 4$.

(a) *If* $n-3 \equiv 2 \pmod 4$ *then*

$$E(D_n^s[\boldsymbol{n-3,3}]) > E(D_n^s[\boldsymbol{n-7,7}]) > \cdots > E(D_n^s[\boldsymbol{2,n-2}]).$$

(b) *If* $n-3 \equiv 0 \pmod 4$ *then*

$$E(D_n^s[\boldsymbol{n-5,5}]) > E(D_n^s[\boldsymbol{n-9,9}]) > \cdots > E(D_n^s[\boldsymbol{2,n-2}]).$$

Lemma 3.15. *Suppose* $n > 5$, $n \equiv 2 \pmod 3$ *and* $r \in [2, n-2]$ *satisfying* $r \equiv 0 \pmod 2$ *and* $n - r \equiv 1 \pmod 2$. *Then we have the following energy ordering:*

(i) *Let* $r \equiv 0 \pmod 4$.

(a) *If* $r \in [2, \frac{2n}{3}]$ *then*

$$E(D_n^s[\boldsymbol{4,n-4}]) > E(D_n^s[\boldsymbol{8,n-8}]) > \cdots > E(D_n^s[\boldsymbol{\frac{2n-10}{3}, \frac{n+10}{3}}]).$$

(*b*) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 2 \pmod 4$ *then*

$$E(D_n^s[\boldsymbol{n-5,5}]) > E(D_n^s[\boldsymbol{n-9,9}]) > \cdots > E(D_n^s[\boldsymbol{\frac{2n+2}{3}, \frac{n-2}{3}}]).$$

(*c*) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 0 \pmod 4$ *then*

$$E(D_n^s[\boldsymbol{n-3,3}]) > E(D_n^s[\boldsymbol{n-7,7}]) > \cdots > E(D_n^s[\boldsymbol{\frac{2n+2}{3}, \frac{n-2}{3}}]).$$

(*ii*) *Let* $r \equiv 2 \pmod 4$.

(*a*) *If* $n-3 \equiv 2 \pmod 4$ *then*

$$E(D_n^s[\boldsymbol{n-3,3}]) > E(D_n^s[\boldsymbol{n-7,7}]) > \cdots > E(D_n^s[\boldsymbol{2,n-2}]).$$

(*b*) *If* $n-3 \equiv 0 \pmod 4$ *then*

$$E(D_n^s[\boldsymbol{n-5,5}]) > E(D_n^s[\boldsymbol{n-9,9}]) > \cdots > E(D_n^s[\boldsymbol{2,n-2}]).$$

Now we give the extremal energy of bicyclic sidigraphs in the class $\mathcal{D}_n^s$.

Theorem 3.16. *Let* $S \in \mathcal{D}_n^s$ *be a sidigraph with one directed cycle of even length and one of odd length.*

(*i*) *For* $n \equiv 1 \pmod 2$*, the maximal energy of* S *is attained if* $S \cong D_n^s[2, n-2]$.
(*ii*) *The minimal energy of* S *is attained if* $S \cong D_n^s[\boldsymbol{2}, 3]$.

Proof. (*i*). For proof, see Theorem 7 [5].
(*ii*). Since for odd integers r_1 and r_2 with $r_1 \geq r_2 \geq 3$, it holds that $E(C_{r_1}) \geq E(C_{r_2})$ and $E(\boldsymbol{C}_2) = 0$. Hence the minimal energy of S is attained if $S \cong D_n^s[\boldsymbol{2}, 3]$. □

In next theorem, we give the energy ordering of those bicyclic sidigraphs in $\mathcal{D}_n^s$ whose one directed cycle is of even length and one is of odd length.

Theorem 3.17. *Let* n *is odd with* $n > 5$ *and* $r \in [2, n-2]$.

(1) *If* $n \equiv 0 \pmod 3$ *then we have the following energy ordering:*

(*i*) *Let* $r \equiv 2 \pmod 4$.

(*a*) *If* $r \in [2, \frac{2n}{3}]$ *then*

$$\begin{aligned} & E(D_n^s[2, n-2]) > E(D_n^s[6, n-6]) > \cdots > E(D_n^s[\frac{2n}{3}, \frac{n}{3}]) \\ > \; & E(D_n^s[\frac{2n-12}{3}, \frac{n}{3}]) > E(D_n^s[\frac{2n-24}{3}, \frac{n}{3}]) > \cdots > E(D_n^s[2, \frac{n}{3}]) \\ > \; & E(D_n^s[2, \frac{n-6}{3}]) > E(D_n^s[2, \frac{n-12}{3}]) > \cdots > E(D_n^s[2, 3]). \end{aligned}$$

(*b*) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 2 \pmod 4$ *then*

$$\begin{aligned} & E(D_n^s[n-3,3]) > E(D_n^s[n-7,7]) > \cdots > E(D_n^s[\frac{2n}{3},\frac{n}{3}]) \\ > \; & E(D_n^s[\frac{2n}{3},\frac{n-6}{3}]) > E(D_n^s[\frac{2n}{3},\frac{n-12}{3}]) > \cdots > E(D_n^s[\frac{2n}{3},3]) \\ > \; & E(D_n^s[\frac{2n-12}{3},3])E(D_n^s[\frac{2n-24}{3},3]) > \cdots > E(D_n^s[2,3]). \end{aligned}$$

(*c*) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 0 \pmod 4$ *then*

$$\begin{aligned} & E(D_n^s[n-5,5]) > E(D_n^s[n-9,9]) > \cdots > E(D_n^s[\frac{2n}{3},\frac{n}{3}]) \\ > \; & E(D_n^s[\frac{2n-12}{3},\frac{n}{3}]) > E(D_n^s[\frac{2n-24}{3},\frac{n}{3}]) > \cdots > E(D_n^s[2,\frac{n}{3}]) \\ > \; & E(D_n^s[2,\frac{n-6}{3}]) > E(D_n^s[2,\frac{n-12}{3}]) > \cdots > E(D_n^s[2,3]). \end{aligned}$$

(*d*) *If* $n-3 \equiv 2 \pmod 4$ *then*

$$\begin{aligned} & E(D_n^s[\boldsymbol{n-3,3}]) > E(D_n^s[\boldsymbol{n-7,7}]) > \cdots > E(D_n^s[\boldsymbol{2,n-2}]) \\ > \; & E(D_n^s[\boldsymbol{2,n-4}]) > E(D_n^s[\boldsymbol{2,n-6}]) > \cdots > E(D_n^s[\boldsymbol{2,3}]) > E(D_n^s[\boldsymbol{2,2}]). \end{aligned}$$

(*e*) *If* $n-3 \equiv 0 \pmod 4$ *then*

$$\begin{aligned} & E(D_n^s[\boldsymbol{n-5,5}]) > E(D_n^s[\boldsymbol{n-9,9}]) > \cdots > E(D_n^s[\boldsymbol{2,n-2}]) \\ > \; & E(D_n^s[\boldsymbol{2,n-4}]) > E(D_n^s[\boldsymbol{2,n-6}]) > \cdots > E(D_n^s[\boldsymbol{2,3}]) > E(D_n^s[\boldsymbol{2,2}]). \end{aligned}$$

(*ii*) *Let* $r \equiv 0 \pmod 4$.

(*a*) *If* $n-3 \equiv 2 \pmod 4$ *then*

$$\begin{aligned} & E(D_n^s[n-5,5]) > E(D_n^s[n-9,9]) > \cdots > E(D_n^s[4,n-4]) \\ > \; & E(D_n^s[4,n-6]) > E(D_n^s[4,n-8]) > E(D_n^s[4,3]) > E(D_n^s[2,3]). \end{aligned}$$

(*b*) *If* $n-3 \equiv 0 \pmod 4$ *then*

$$\begin{aligned} & E(D_n^s[n-3,3]) > E(D_n^s[n-7,5]) > \cdots > E(D_n^s[4,n-4]) \\ > \; & E(D_n^s[4,n-6]) > E(D_n^s[4,n-8]) > E(D_n^s[4,3]) > E(D_n^s[2,3]). \end{aligned}$$

(*c*) *If* $r \in [2, \frac{2n}{3}]$ *then*

$$\begin{aligned} & E(D_n^s[\boldsymbol{4,n-4}]) > E(D_n^s[\boldsymbol{8,n-8}]) > \cdots > E(D_n^s[\boldsymbol{\frac{2n-6}{3},\frac{n+6}{3}}]) \\ > \; & E(D_n^s[\boldsymbol{\frac{2n-18}{3},\frac{n+6}{3}}]) > E(D_n^s[\boldsymbol{\frac{2n-30}{3},\frac{n+6}{3}}]) > \cdots > E(D_n^s[\boldsymbol{4,\frac{n+6}{3}}]) \\ > \; & E(D_n^s[\boldsymbol{4,\frac{n}{3}}]) > E(D_n^s[\boldsymbol{4,\frac{n-6}{3}}]) > \cdots > E(D_n^s[\boldsymbol{4,3}]) > E(D_n^s[\boldsymbol{2,3}]). \end{aligned}$$

(d) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 2 (\mathrm{mod}\ 4)$ *then*

$$\begin{aligned} & E(D_n^s[\mathbf{n-5,5}]) > E(D_n^s[\mathbf{n-9,9}]) > \cdots > E(D_n^s[\mathbf{\frac{2n+6}{3}, \frac{n-6}{3}}]) \\ > \; & E(D_n^s[\mathbf{\frac{2n-6}{3}, \frac{n-6}{3}}]) > E(D_n^s[\mathbf{\frac{2n-18}{3}, \frac{n-6}{3}}]) > \cdots > E(D_n^s[\mathbf{4, \frac{n-6}{3}}]) \\ > \; & E(D_n^s[\mathbf{4, \frac{n-12}{3}}]) > E(D_n^s[\mathbf{4, \frac{n-18}{3}}]) > \cdots > E(D_n^s[\mathbf{4,3}]) > E(D_n^s[\mathbf{2,3}]). \end{aligned}$$

(e) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 0 (\mathrm{mod}\ 4)$ *then*

$$\begin{aligned} & E(D_n^s[\mathbf{n-3,3}]) > E(D_n^s[\mathbf{n-7,7}]) > \cdots > E(D_n^s[\mathbf{\frac{2n+6}{3}, \frac{n-6}{3}}]) \\ > \; & E(D_n^s[\mathbf{\frac{2n+6}{3}, \frac{n-12}{3}}]) > E(D_n^s[\mathbf{\frac{2n+6}{3}, \frac{n-18}{3}}]) > \cdots > E(D_n^s[\mathbf{\frac{2n+6}{3}, 3}]) \\ > \; & E(D_n^s[\mathbf{\frac{2n-6}{3}, 3}]) > E(D_n^s[\mathbf{\frac{2n-18}{3}, 3}]) > \cdots > E(D_n^s[\mathbf{4,3}]) > E(D_n^s[\mathbf{2,3}]). \end{aligned}$$

(2) *If* $n \equiv 1 (\mathrm{mod}\ 3)$ *then we have the following energy ordering:*

(i) *Let* $r \equiv 2 (\mathrm{mod}\ 4)$.

(a) *If* $r \in [2, \frac{2n}{3}]$ *then*

$$\begin{aligned} & E(D_n^s[2, n-2]) > E(D_n^s[6, n-6]) > \cdots > E(D_n^s[\frac{2n-8}{3}, \frac{n+8}{3}]) \\ > \; & E(D_n^s[\frac{2n-20}{3}, \frac{n+8}{3}]) > E(D_n^s[\frac{2n-32}{3}, \frac{n+8}{3}]) > \cdots > E(D_n^s[2, \frac{n+8}{3}]) \\ > \; & E(D_n^s[2, \frac{n+2}{3}]) > E(D_n^s[2, \frac{n-4}{3}]) > \cdots > E(D_n^s[2,3]). \end{aligned}$$

(b) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 2 (\mathrm{mod}\ 4)$ *then*

$$\begin{aligned} & E(D_n^s[n-3, 3]) > E(D_n^s[n-7, 7]) > \cdots > E(D_n^s[\frac{2n+4}{3}, \frac{n-4}{3}]) \\ > \; & E(D_n^s[\frac{2n-8}{3}, \frac{n-4}{3}]) > E(D_n^s[\frac{2n-20}{3}, \frac{n-4}{3}]) > \cdots > E(D_n^s[2, \frac{n-4}{3}]) \\ > \; & E(D_n^s[2, \frac{n-10}{3}]) > E(D_n^s[2, \frac{n-16}{3}]) > \cdots > E(D_n^s[2,3]). \end{aligned}$$

(c) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 0 (\mathrm{mod}\ 4)$ *then*

$$\begin{aligned} & E(D_n^s[n-5, 5]) > E(D_n^s[n-9, 9]) > \cdots > E(D_n^s[\frac{2n+4}{3}, \frac{n-4}{3}]) \\ > \; & E(D_n^s[\frac{2n+4}{3}, \frac{n-10}{3}]) > E(D_n^s[\frac{2n+4}{3}, \frac{n-16}{3}]) > \cdots > E(D_n^s[\frac{2n+4}{3}, 3]) \\ > \; & E(D_n^s[\frac{2n-8}{3}, 3]) > E(D_n^s[\frac{2n-20}{3}, 3]) > \cdots > E(D_n^s[2,3]). \end{aligned}$$

(d) *If* $n-3 \equiv 2 (\mathrm{mod}\ 4)$ *then*

$$\begin{aligned} & E(D_n^s[\mathbf{n-3,3}]) > E(D_n^s[\mathbf{n-7,7}]) > \cdots > E(D_n^s[\mathbf{2,n-2}]) \\ > \ & E(D_n^s[\mathbf{2,n-4}]) > E(D_n^s[\mathbf{2,n-6}]) > \cdots > E(D_n^s[\mathbf{2,3}]). \end{aligned}$$

(e) *If* $n-3 \equiv 0 (\mathrm{mod}\ 4)$ *then*

$$\begin{aligned} & E(D_n^s[\mathbf{n-5,5}]) > E(D_n^s[\mathbf{n-9,9}]) > \cdots > E(D_n^s[\mathbf{2,n-2}]) \\ > \ & E(D_n^s[\mathbf{2,n-4}]) > E(D_n^s[\mathbf{2,n-6}]) > \cdots > E(D_n^s[\mathbf{2,3}]). \end{aligned}$$

(ii) *Let* $r \equiv 0 (\mathrm{mod}\ 4)$.

(a) *If* $n-3 \equiv 2 (\mathrm{mod}\ 4)$ *then*

$$\begin{aligned} & E(D_n^s[n-5,5]) > E(D_n^s[n-9,9]) > \cdots > E(D_n^s[4,n-4]) \\ > \ & E(D_n^s[4,n-6]) > E(D_n^s[4,n-8]) > E(D_n^s[4,3]) > E(D_n^s[2,3]). \end{aligned}$$

(b) *If* $n-3 \equiv 0 (\mathrm{mod}\ 4)$ *then*

$$\begin{aligned} & E(D_n^s[n-3,3]) > E(D_n^s[n-7,7]) > \cdots > E(D_n^s[4,n-4]) \\ > \ & E(D_n^s[4,n-6]) > E(D_n^s[4,n-8]) > E(D_n^s[4,3]) > E(D_n^s[2,3]). \end{aligned}$$

(c) *If* $r \in [2, \frac{2n}{3}]$ *then*

$$\begin{aligned} & E(D_n^s[\mathbf{4,n-4}]) > E(D_n^s[\mathbf{8,n-8}]) > \cdots > E(D_n^s[\mathbf{\tfrac{2n-2}{3},\tfrac{n+2}{3}}]) \\ > \ & E(D_n^s[\mathbf{\tfrac{2n-14}{3},\tfrac{n+2}{3}}]) > E(D_n^s[\mathbf{\tfrac{2n-26}{3},\tfrac{n+2}{3}}]) > \cdots > E(D_n^s[\mathbf{4,\tfrac{n+2}{3}}]) \\ > \ & E(D_n^s[\mathbf{4,\tfrac{n-4}{3}}]) > E(D_n^s[\mathbf{4,\tfrac{n-10}{3}}]) > \cdots > E(D_n^s[\mathbf{4,3}]) > E(D_n^s[\mathbf{2,3}]). \end{aligned}$$

(d) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 2 (\mathrm{mod}\ 4)$ *then*

$$\begin{aligned} & E(D_n^s[\mathbf{n-5,5}]) > E(D_n^s[\mathbf{n-9,9}]) > \cdots > E(D_n^s[\mathbf{\tfrac{2n+10}{3},\tfrac{n-10}{3}}]) \\ > \ & E(D_n^s[\mathbf{\tfrac{2n-2}{3},\tfrac{n-10}{3}}]) > E(D_n^s[\mathbf{\tfrac{2n-14}{3},\tfrac{n-10}{3}}]) > \cdots > E(D_n^s[\mathbf{4,\tfrac{n-10}{3}}]) \\ > \ & E(D_n^s[\mathbf{4,\tfrac{n-16}{3}}]) > E(D_n^s[\mathbf{4,\tfrac{n-22}{3}}]) > \cdots > E(D_n^s[\mathbf{4,3}]) > E(D_n^s[\mathbf{2,3}]). \end{aligned}$$

(e) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 0 (\mathrm{mod}\ 4)$ *then*

$$\begin{aligned} & E(D_n^s[\mathbf{n-3,3}]) > E(D_n^s[\mathbf{n-7,7}]) > \cdots > E(D_n^s[\mathbf{\tfrac{2n+10}{3},\tfrac{n-10}{3}}]) \\ > \ & E(D_n^s[\mathbf{\tfrac{2n+10}{3},\tfrac{n-16}{3}}]) > E(D_n^s[\mathbf{\tfrac{2n+10}{3},\tfrac{n-22}{3}}]) > E(D_n^s[\mathbf{\tfrac{2n+10}{3},3}]) \\ > \ & E(D_n^s[\mathbf{\tfrac{2n-2}{3},3}]) > E(D_n^s[\mathbf{\tfrac{2n-14}{3},3}]) > \cdots > E(D_n^s[\mathbf{4,3}]) > E(D_n^s[\mathbf{2,3}]). \end{aligned}$$

(3) *If* $n \equiv 2 (\text{mod } 3)$ *then we have the following energy ordering:*

(i) *Let* $r \equiv 2 (\text{mod } 4)$.

(a) *If* $r \in [2, \frac{2n}{3}]$ *then*

$$\begin{aligned} & E(D_n^s[2, n-2]) > E(D_n^s[6, n-6]) > \cdots > E(D_n^s[\frac{2n-4}{3}, \frac{n+4}{3}]) \\ > \ & E(D_n^s[\frac{2n-16}{3}, \frac{n+4}{3}]) > E(D_n^s[\frac{2n-26}{3}, \frac{n+4}{3}]) > \cdots > E(D_n^s[2, \frac{n+4}{3}]) \\ > \ & E(D_n^s[2, \frac{n-2}{3}]) > E(D_n^s[2, \frac{n-8}{3}]) > E(D_n^s[2, 3]). \end{aligned}$$

(b) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 2 (\text{mod } 4)$ *then*

$$\begin{aligned} & E(D_n^s[n-3, 3]) > E(D_n^s[n-7, 7]) > \cdots > E(D_n^s[\frac{2n+8}{3}, \frac{n-8}{3}]) \\ > \ & E(D_n^s[\frac{2n-4}{3}, \frac{n-8}{3}]) > E(D_n^s[\frac{2n-16}{3}, \frac{n-8}{3}]) > \cdots > E(D_n^s[2, \frac{n-8}{3}]) \\ > \ & E(D_n^s[2, \frac{n-14}{3}]) > E(D_n^s[2, \frac{n-20}{3}]) > \cdots > E(D_n^s[2, 3]). \end{aligned}$$

(c) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 0 (\text{mod } 4)$ *then*

$$\begin{aligned} & E(D_n^s[n-5, 5]) > E(D_n^s[n-9, 9]) > \cdots > E(D_n^s[\frac{2n+8}{3}, \frac{n-8}{3}]) \\ > \ & E(D_n^s[\frac{2n+8}{3}, \frac{n-14}{3}]) > E(D_n^s[\frac{2n+8}{3}, \frac{n-20}{3}]) > \cdots > E(D_n^s[\frac{2n+8}{3}, 3]) \\ > \ & E(D_n^s[\frac{2n-4}{3}, 3]) > E(D_n^s[\frac{2n-16}{3}, 3]) > \cdots > E(D_n^s[2, 3]). \end{aligned}$$

(d) *If* $n-3 \equiv 2 (\text{mod } 4)$ *then*

$$\begin{aligned} & E(D_n^s[\boldsymbol{n-3, 3}]) > E(D_n^s[\boldsymbol{n-7, 7}]) > \cdots > E(D_n^s[\boldsymbol{2, n-2}]) \\ > \ & E(D_n^s[\boldsymbol{2, n-4}]) > E(D_n^s[\boldsymbol{2, n-6}]) > \cdots > E(D_n^s[\boldsymbol{2, 3}]). \end{aligned}$$

(e) *If* $n-3 \equiv 0 (\text{mod } 4)$ *then*

$$\begin{aligned} & E(D_n^s[\boldsymbol{n-5, 5}]) > E(D_n^s[\boldsymbol{n-9, 9}]) > \cdots > E(D_n^s[\boldsymbol{2, n-2}]) \\ > \ & E(D_n^s[\boldsymbol{2, n-4}]) > E(D_n^s[\boldsymbol{2, n-6}]) > \cdots > E(D_n^s[\boldsymbol{2, 3}]). \end{aligned}$$

(ii) *Let* $r \equiv 0 (\text{mod } 4)$.

(a) *If* $n-3 \equiv 2 (\text{mod } 4)$ *then*

$$\begin{aligned} & E(D_n^s[n-5, 5]) > E(D_n^s[n-9, 9]) > \cdots > E(D_n^s[4, n-4]) \\ > \ & E(D_n^s[4, n-6]) > E(D_n^s[4, n-8]) > E(D_n^s[4, 3]) > E(D_n^s[2, 3]). \end{aligned}$$

(b) *If* $n-3 \equiv 0 (\text{mod } 4)$ *then*

$$\begin{aligned} & E(D_n^s[n-3, 3]) > E(D_n^s[n-7, 7]) > \cdots > E(D_n^s[4, n-4]) \\ > \ & E(D_n^s[4, n-6]) > E(D_n^s[4, n-8]) > E(D_n^s[4, 3]) > E(D_n^s[2, 3]). \end{aligned}$$

(c) *If* $r \in [2, \frac{2n}{3}]$ *then*

$$E(D_n^s[\mathbf{4}, \boldsymbol{n-4}]) > E(D_n^s[\mathbf{8}, \boldsymbol{n-8}]) > \cdots > E(D_n^s[\mathbf{\frac{2n-10}{3}}, \mathbf{\frac{n+10}{3}}])$$
$$> E(D_n^s[\mathbf{\frac{2n-22}{3}}, \mathbf{\frac{n+10}{3}}]) > E(D_n^s[\mathbf{\frac{2n-34}{3}}, \mathbf{\frac{n+10}{3}}]) > \cdots > E(D_n^s[\mathbf{4}, \mathbf{\frac{n+10}{3}}])$$
$$> E(D_n^s[\mathbf{4}, \mathbf{\frac{n+4}{3}}]) > E(D_n^s[\mathbf{4}, \mathbf{\frac{n-2}{3}}]) > \cdots > E(D_n^s[\mathbf{4}, \mathbf{3}]) > E(D_n^s[\mathbf{2}, \mathbf{3}]).$$

(d) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 2(\text{mod } 4)$ *then*

$$E(D_n^s[\boldsymbol{n-5}, \mathbf{5}]) > E(D_n^s[\boldsymbol{n-9}, \mathbf{9}]) > \cdots > E(D_n^s[\mathbf{\frac{2n+2}{3}}, \mathbf{\frac{n-2}{3}}])$$
$$> E(D_n^s[\mathbf{\frac{2n-10}{3}}, \mathbf{\frac{n-2}{3}}]) > E(D_n^s[\mathbf{\frac{2n-22}{3}}, \mathbf{\frac{n-2}{3}}]) > E(D_n^s[\mathbf{4}, \mathbf{\frac{n-2}{3}}])$$
$$> E(D_n^s[\mathbf{4}, \mathbf{\frac{n-8}{3}}]) > E(D_n^s[\mathbf{4}, \mathbf{\frac{n-14}{3}}]) > \cdots > E(D_n^s[\mathbf{4}, \mathbf{3}]) > E(D_n^s[\mathbf{2}, \mathbf{3}]).$$

(e) *If* $r \in [\frac{2n}{3}, n-2]$ *and* $n-3 \equiv 0(\text{mod } 4)$ *then*

$$E(D_n^s[\boldsymbol{n-3}, \mathbf{3}]) > E(D_n^s[\boldsymbol{n-7}, \mathbf{7}]) > \cdots > E(D_n^s[\mathbf{\frac{2n+2}{3}}, \mathbf{\frac{n-2}{3}}])$$
$$> E(D_n^s[\mathbf{\frac{2n+2}{3}}, \mathbf{\frac{n-8}{3}}]) > E(D_n^s[\mathbf{\frac{2n+2}{3}}, \mathbf{\frac{n-14}{3}}]) > \cdots > E(D_n^s[\mathbf{\frac{2n+2}{3}}, \mathbf{3}])$$
$$> E(D_n^s[\mathbf{\frac{2n-10}{3}}, \mathbf{3}]) > E(D_n^s[\mathbf{\frac{2n-22}{3}}, \mathbf{3}]) > \cdots > E(D_n^s[\mathbf{4}, \mathbf{3}]) > E(D_n^s[\mathbf{2}, \mathbf{3}]).$$

Example 3.18. *We illustrate Theorem 3.17 by considering sidigraphs of different order.*

(1) *Take* $n = 15$*, that is,* $n \equiv 0(\text{mod}3)$ *and* $n-3 \equiv 0(\text{mod}4)$.

(i) *Let* $r \equiv 2(\text{mod}4)$.

(a) *Let* $r \in [2, \frac{2n}{3}] = [2, 10]$. *Then*

$$\begin{aligned}
E(D_n^s[10, n-10] &= E(D_n^s[\frac{2n}{3}, \frac{n}{3}]) = E(D_n^s[10, 5]) = 9.7082 \\
E(D_n^s[\frac{2n-24}{3}, \frac{n}{3}]) &= E(D_n^s[2, \frac{n}{3}]) = E(D_n^s[2, 5]) = 5.2361 \\
E(D_n^s[\frac{2n-12}{3}, \frac{n}{3}]) &= E(D_n^s[6, 5]) = 7.2361 \\
E(D_n^s[2, n-2]) &= E(D_n^s[2, 13]) = 10.2962 \\
E(D_n^s[6, n-6]) &= E(D_n^s[6, 9]) = 9.7588 \\
E(D_n^s[2, \frac{n-6}{2}]) &= E(D_n^s[2, 3]) = 4.
\end{aligned}$$

Above calculations yield.

$$E(D_n^s[2, 13]) > E(D_n^s[6, 9]) > E(D_n^s[10, 5]) > E(D_n^s[6, 5]) > E(D_n^s[2, 5]) > E(D_n^s[2, 3]).$$

(*b*) *Let* $r \in [2, \frac{2n}{3}] = [10, 13]$. *Then*

$$\begin{aligned} E(D_n^s[n-3,3] &= E(D_n^s[\frac{2n}{3}, \frac{n}{3}]) = E(D_n^s[10,5]) = 9.7082 \\ E(D_n^s[\frac{2n-12}{3}, \frac{n}{3}]) &= E(D_n^s[2, \frac{n}{3}]) = E(D_n^s[2,5]) = 5.2361 \\ E(D_n^s[\frac{2n-12}{3}, \frac{n}{3}]) &= E(D_n^s[6,5]) = 7.2361 \\ E(D_n^s[2, \frac{n-6}{2}]) &= E(D_n^s[2,3]) = 4. \end{aligned}$$

It follows that

$$E(D_n^s[10,5]) > E(D_n^s[6,5]) > E(D_n^s[2,5]) > E(D_n^s[2,3]).$$

(*c*) *Since* $n-3 = 12 \equiv 0(\mathrm{mod}4)$. *Therefore we have*

$$\begin{aligned} E(D_n^s[\boldsymbol{n-13,13}]) &= E(D_n^s[\mathbf{2}, \boldsymbol{n-2}]) = E(D_n^s[\mathbf{2,13}]) = 8.2962 \\ E(D_n^s[\boldsymbol{n-5,5}]) &= E(D_n^s[\mathbf{10,5}]) = 9.3914 \\ E(D_n^s[\boldsymbol{n-9,9}]) &= E(D_n^s[\mathbf{6,9}]) = 9.2229 \\ E(D_n^s[\mathbf{2}, \boldsymbol{n-4}]) &= E(D_n^s[\mathbf{2,11}]) = 7.0267 \\ E(D_n^s[\mathbf{2}, \boldsymbol{n-6}]) &= E(D_n^s[\mathbf{2,9}]) = 5.7588 \\ E(D_n^s[\mathbf{2}, \boldsymbol{n-8}]) &= E(D_n^s[\mathbf{2,7}]) = 4.4940 \\ E(D_n^s[\mathbf{2}, \boldsymbol{n-10}]) &= E(D_n^s[\mathbf{2,5}]) = 3.2361 \\ E(D_n^s[\mathbf{2}, \boldsymbol{n-12}]) &= E(D_n^s[\mathbf{2,3}]) = 2 \\ E(D_n^s[\mathbf{2,2}]) &= 0. \end{aligned}$$

Hence we get,

$$\begin{aligned} &E(D_n^s[\mathbf{10,5}]) > E(D_n^s[\mathbf{6,9}]) > E(D_n^s[\mathbf{2,13}]) > E(D_n^s[\mathbf{2,11}]) > E(D_n^s[\mathbf{2,9}]) \\ &> E(D_n^s[\mathbf{2,7}]) > E(D_n^s[\mathbf{2,5}]) > E(D_n^s[\mathbf{2,3}]) > E(D_n^s[\mathbf{2,2}]). \end{aligned}$$

(*ii*) *Let* $r \equiv 0(\mathrm{mod}4)$.

(*a*) *Since* $n-3 = 12 \equiv 0(\mathrm{mod}4)$. *Therefore we have*

$$\begin{aligned} E(D_n^s[n-11,11]) &= E(D_n^s[4,n-4]) = E(D_n^s[4,11]) = 9.0267 \\ E(D_n^s[n-3,3]) &= E(D_n^s[12,3]) = 9.4641 \\ E(D_n^s[n-7,7]) &= E(D_n^s[8,7]) = 9.3224 \\ E(D_n^s[4,n-6]) &= E(D_n^s[4,9]) = 7.7588 \\ E(D_n^s[4,n-8]) &= E(D_n^s[4,7]) = 6.4940 \\ E(D_n^s[4,n-10]) &= E(D_n^s[4,5]) = 5.2361 \\ E(D_n^s[4,n-12]) &= E(D_n^s[4,3]) = 4 \\ E(D_n^s[2,3]) &= 2. \end{aligned}$$

Above calculations yield,

$$E(D_n^s[12,3]) > E(D_n^s[8,7]) > E(D_n^s[4,11]) > E(D_n^s[4,9]) > E(D_n^s[4,7]) > E(D_n^s[4,5]) > E(D_n^s[4,3]) > E(D_n^s[2,3]).$$

(b) *Let* $r \in [2, \frac{2n}{3}] = [2,10]$. *Then*

$$\begin{aligned} E(D_n^s[\mathbf{4}, \frac{\boldsymbol{n}}{\mathbf{3}}]) &= E(D_n^s[\mathbf{4},\mathbf{5}]) = 6.0645 \\ E(D_n^s[\mathbf{8}, \boldsymbol{n}-\mathbf{8}]) &= E(D_n^s[\frac{\mathbf{2}\boldsymbol{n}-\mathbf{6}}{\mathbf{3}}, \frac{\boldsymbol{n}+\mathbf{6}}{\mathbf{3}}]) = E(D_n^s[\mathbf{8},\mathbf{7}]) = 9.7202 \\ E(D_n^s[\frac{\mathbf{2}\boldsymbol{n}-\mathbf{18}}{\mathbf{3}}, \frac{\boldsymbol{n}+\mathbf{6}}{\mathbf{3}}]) &= E(D_n^s[\mathbf{4},\mathbf{7}] = 7.3224 \\ E(D_n^s[\mathbf{4}, \frac{\boldsymbol{n}-\mathbf{6}}{\mathbf{3}}]) &= E(D_n^s[\mathbf{4},\mathbf{3}]) = 4.8284 \\ E(D_n^s[\mathbf{4}, \boldsymbol{n}-\mathbf{4}]) &= E(D_n^s[\mathbf{4},\mathbf{11}]) = 9.8551 \\ E(D_n^s[\mathbf{2},\mathbf{3}]) &= 2 \\ E(D_n^s[\mathbf{2},\mathbf{2}]) &= 0. \end{aligned}$$

Therefore, we conclude

$$E(D_n^s[\mathbf{4},\mathbf{11}]) > E(D_n^s[\mathbf{8},\mathbf{7}]) > E(D_n^s[\mathbf{4},\mathbf{7}]) > E(D_n^s[\mathbf{4},\mathbf{5}]) > E(D_n^s[\mathbf{4},\mathbf{3}]) > E(D_n^s[\mathbf{2},\mathbf{3}]) > E(D_n^s[\mathbf{2},\mathbf{2}]).$$

(c) *Let* $r \in [\frac{2n}{3}, n-2] = [10,13]$. *Then*

$$\begin{aligned} E(D_n^s[\boldsymbol{n}-\mathbf{3},\mathbf{3}]) &= E(D_n^s[\frac{\mathbf{2}\boldsymbol{n}+\mathbf{6}}{\mathbf{3}}, \frac{\boldsymbol{n}-\mathbf{6}}{\mathbf{3}}]) = E(D_n^s[\mathbf{12},\mathbf{3}]) = 9.7274 \\ E(D_n^s[\frac{\mathbf{2}\boldsymbol{n}-\mathbf{6}}{\mathbf{3}}, 3]) &= E(D_n^s[\mathbf{8},\mathbf{3}]) = 7.2263 \\ E(D_n^s[\frac{\mathbf{2}\boldsymbol{n}-\mathbf{18}}{\mathbf{3}}, 3]) &= E(D_n^s[\mathbf{4},\mathbf{3}]) = 4.8284. \end{aligned}$$

Hence,

$$E(D_n^s[\mathbf{12},\mathbf{3}]) > E(D_n^s[\mathbf{8},\mathbf{3}]) > E(D_n^s[\mathbf{4},\mathbf{3}]) > E(D_n^s[\mathbf{2},\mathbf{3}]) > E(D_n^s[\mathbf{2},\mathbf{2}]).$$

Analogously, for a fixed n, one can verify the energy ordering of Parts (2) and (3) of Theorem 3.17.

4. Conclusion

Let $\mathcal{D}_n$ denotes the set of vertex-disjoint bicyclic digraphs of a fixed order n. We investigate the energy ordering of signed digraphs in $\mathcal{D}_n^s$ and find the maximal energy. The results for the case, when both directed cycles are even are obtained in [12]. Therefore we have solved the remaining cases for energy ordering in $\mathcal{D}_n^s$.

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Conflict of interest

The authors declare no conflict of interest.

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