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## Research article

# Further generalization of Walker's inequality in acute triangles and its applications 

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#### Abstract

In this paper, we prove a generalization of Walker's inequality in acute (non-obtuse) triangles by using Euler's inequality, Ciamberlini's inequality and a result due to the author, from which a number of corollaries are obtained. We also present three conjectured inequalities involving sides of an acute (non-obtuse) triangle and one exponent as open problems.


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Keywords: Walker's inequality; Euler's inequality; Ciamberlini's inequality; acute (non-obtuse) triangle; circumradius; inradious; semiperimeter; non-negative real number
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## 1. Introduction and main result

Let $A B C$ be a triangle with circumradius $R$, inradius $r$ and semiperimeter $s$. If $A B C$ is an acute (non-obtuse) triangle, then the following linear inequality holds:

$$
\begin{equation*}
s \geq 2 R+r, \tag{1.1}
\end{equation*}
$$

with equality if and only $A B C$ is a right triangle.
Inequality (1.1) is a fundamental inequality for acute triangles. Ciamberlini [1] first noted that it can be obtained clearly from the following identity:

$$
\begin{equation*}
s^{2}-(2 R+r)^{2}=4 R^{2} \cos A \cos B \cos C, \tag{1.2}
\end{equation*}
$$

where $A, B, C$ are the angles of the triangle $A B C$.
In 1975, Walker [2] first proposed the following quadratic inequality:

$$
\begin{equation*}
s^{2} \geq 2 R^{2}+8 R r+3 r^{2} \tag{1.3}
\end{equation*}
$$

with equality if and only if $A B C$ is equilateral or right isosceles.

The monograph [3,p.248] introduced a simple proof of Walker's inequality (3) given by Klamkin, which used a known trigonometric inequality. See [4] for another proof of Walker's inequality (1.3).

In 1996, Yang [5] and Chen [6] obtained one parameter generalizations of Walker's inequality at almost the same time. In [5], Yang established the following inequality:

$$
\begin{equation*}
s^{2} \geq 2(1+k) R^{2}+2[4-(3+\sqrt{2}) k] R r+[3+4(1+\sqrt{2}) k] r^{2}, \tag{1.4}
\end{equation*}
$$

where $k$ is a real number such that $-1 \leq k \leq 1$. In [6], Chen obtained the following result:

$$
\begin{equation*}
s^{2} \geq \lambda R^{2}+[2(7+\sqrt{2})-(3+\sqrt{2}) \lambda] R r-[(1+4 \sqrt{2})-2(1+\sqrt{2}) \lambda] r^{2}, \tag{1.5}
\end{equation*}
$$

where $\lambda$ is a real number such that $2(1-\sqrt{2}) \leq \lambda \leq 4$. Both equalities in (1.4) and (1.5) hold if and only if the triangle $A B C$ is equilateral or right isosceles.

It is easily shown that inequalities (1.4) and (1.5) are equivalent. Putting $k=0$ in (1.4) or $\lambda=2$ in (1.5), Walker's inequality (1.3) follows immediately. From (1.4) or (1.5), some other inequalities similar to Walker's inequality are easily obtained. For example,

$$
\begin{gather*}
s^{2} \geq R^{2}+(11+\sqrt{2}) R r+(1-2 \sqrt{2}) r^{2},  \tag{1.6}\\
s^{2} \geq 3 R^{2}+(5-\sqrt{2}) R r+(5+2 \sqrt{2}) r^{2},  \tag{1.7}\\
s^{2} \geq 4 R^{2}+(2-2 \sqrt{2}) R r+(7+4 \sqrt{2}) r^{2},  \tag{1.8}\\
s^{2} \geq(2-2 \sqrt{2}) R^{2}+(12+6 \sqrt{2}) R r-(5+4 \sqrt{2}) r^{2}, \tag{1.9}
\end{gather*}
$$

etc.(see [5, 6]).
Walker's inequality (1.3) is actually equivalent to the following beautiful trigonometric inequality:

$$
\begin{equation*}
(\cos B+\cos C)^{2}+(\cos C+\cos A)^{2}+(\cos A+\cos B)^{2} \leq 3, \tag{1.10}
\end{equation*}
$$

which can be obtained from a sharped version of the famous Erdös-Mordell inequality (see [7]). Recently, the author [8] obtained a weighted generalization of inequality (1.10).

In this paper, we present a further generalization of Walker's inequality and give its applications. Our main result is the following:

Theorem 1. For any non-negative real numbers $m, n$ and an acute (non-obtuse) triangle $A B C$, we have

$$
\begin{equation*}
s^{2} \geq \frac{4 m R^{3}-4(m-4 n) R^{2} r-(7 m+3 n) R r^{2}-2(m+2 n) r^{3}}{(m+n) R-2 m r} \tag{1.11}
\end{equation*}
$$

If $n=0$ and the triangle $A B C$ is not equilateral, then the equality holds if and only if the triangle $A B C$ is a right triangle. In other cases, the equality holds if and only if the triangle $A B C$ is equilateral or right isosceles.

If $-1 \leq k \leq 1$, then it is easily shown that $R+(\sqrt{2}-1) r-k(R-2 r)>0$ by using Euler's inequality:

$$
\begin{equation*}
R \geq 2 r \tag{1.12}
\end{equation*}
$$

which is valid for any triangle $A B C$. Thus, in (1.11) we may put

$$
m=(k+1) R+(\sqrt{2}-1) r, n=R+(\sqrt{2}-1) r-k(R-2 r),
$$

and inequality (1.4) is obtained from (1.11) after simple calculations. Therefore, inequality (1.11) is a generalization of (1.4) and (1.5).

In fact, a large number of inequalities of the form

$$
\begin{equation*}
s^{2} \geq f(R, r) \tag{1.13}
\end{equation*}
$$

can be obtained from inequality (1.11), which is a further generalization of Walker's inequality (1.3).
We shall discuss applications of inequality (1.11) in Sections 3-6.
Remark 1. In fact, inequality (1.11) is equivalent to the following inequality with one parameter:

$$
\begin{equation*}
s^{2} \geq \frac{4 \lambda R^{3}-4(\lambda-4) R^{2} r-(7 \lambda+3) R r^{2}-2(\lambda+2) r^{3}}{(\lambda+1) R-2 \lambda r} \tag{1.14}
\end{equation*}
$$

where $\lambda$ is a non-negative real number. When $\lambda \rightarrow+\infty$, the inequality becomes $s^{2} \geq(2 R+r)^{2}$ which is equivalent to (1.1).

## 2. Proof of Theorem 1

The proof of Theorem 1 will be used Ciamberlini's inequality (1.1), Euler's inequality (1.12) and inequality (2.1) below.

Lemma 1. In an acute (non-obtuse) triangle $A B C$, we have

$$
\begin{equation*}
s^{2} \geq 16 R r-3 r^{2}-\frac{4 r^{3}}{R} \tag{2.1}
\end{equation*}
$$

with equality if and only if triangle $A B C$ is equilateral or right isosceles.
Inequality (2.1) was established by the author in [9], where two proofs were given. One of the proof used the following important result in any triangle (see [3, 10-15]):

$$
\begin{equation*}
s^{2} \geq 2 R^{2}+10 R r-r^{2}-2(R-2 r) \sqrt{R^{2}-2 R r}, \tag{2.2}
\end{equation*}
$$

with equality if and only if $A B C$ is an isosceles triangle. Here, we present a new proof of inequality (2.1), as follows:

Proof. Inequality (2.1) is equivalent to

$$
\begin{equation*}
s^{3} \geq 16 R r s-s r^{2}\left(3+\frac{4 r}{R}\right) \tag{2.3}
\end{equation*}
$$

We have the following well-known identities:

$$
\begin{align*}
& a b c=4 R r s,  \tag{2.4}\\
& s r^{2}=(s-a)(s-b)(s-c),  \tag{2.5}\\
& \frac{r}{R}=\frac{4(s-a)(s-b)(s-c)}{a b c}, \tag{2.6}
\end{align*}
$$

where $a, b, c$ are the lengths of the sides of $\triangle A B C$. So, inequality (2.3) becomes

$$
s^{3} \geq 4 a b c-(s-a)(s-b)(s-c)\left[3+\frac{16(s-a)(s-b)(s-c)}{a b c}\right],
$$

that is

$$
a b c s^{3}+3 a b c(s-a)(s-b)(s-c)+16(s-a)^{2}(s-b)^{2}(s-c)^{2}-4(a b c)^{2} \geq 0 .
$$

Since $s=(a+b+c) / 2$, one sees that the above inequality is equivalent to

$$
\begin{aligned}
& a b c(a+b+c)^{3}+3 a b c(b+c-a)(c+a-b)(a+b-c) \\
& +2(b+c-a)^{2}(c+a-b)^{2}(a+b-c)^{2}-32(a b c)^{2} \geq 0
\end{aligned}
$$

Dividing both sides of the above inequality by 2 and expanding out gives

$$
\begin{align*}
Q_{0} \equiv & \sum a^{6}-2 \sum a\left(b^{5}+c^{5}\right)-\sum a^{2}\left(b^{4}+c^{4}\right)+5 a b c \sum a^{3} \\
& -a b c \sum a\left(b^{2}+c^{2}\right)+4 \sum b^{3} c^{3}-6(a b c)^{2} \geq 0, \tag{2.7}
\end{align*}
$$

where the symbols $\sum$ denote cyclic sums over the triple ( $a, b, c$ ).
In order to prove $Q_{0} \geq 0$, we may assume that $a \geq b$ and $a \geq c$. Then $b^{2}+c^{2} \geq a^{2}$ (since $A B C$ is acute-angled). After analyzing, we find that $Q_{0}$ can be rewritten in the form:

$$
\begin{align*}
Q_{0}= & a(b+c-a)(a-b)(a-c)\left(b^{2}+c^{2}-a^{2}\right) \\
& +(b+c)(c+a-b)(a+b-c)(2 a-b-c)(b-c)^{2} \\
& +a(b+c-a)\left[a^{2}+b(a-c)+c(a-b)\right](b-c)^{2} . \tag{2.8}
\end{align*}
$$

Thus, we conclude that $Q_{0} \geq 0$ holds for a non-obtuse triangle $A B C$ by the hypothesis. Hence, inequality (2.1) is proved. Also, from identity (2.8), we see that the equality holds exactly under the conditions of the statement of Lemma 1 . This completes the proof of Lemma 1.

Next, we shall prove Theorem 1.
Proof. According to Euler's inequality (1.12), Ciamberlini's inequality (1.1) and inequality (2.1), one sees that if $m \geq 0$ and $n \geq 0$ then

$$
\begin{equation*}
m(R-2 r)\left[s^{2}-(2 R+r)^{2}\right]+n\left(R s^{2}-16 R^{2} r+3 R r^{2}+4 r^{3}\right) \geq 0 \tag{2.9}
\end{equation*}
$$

which is equivalent to

$$
(m R+n R-2 m r) s^{2}-4 m R^{3}+4(m-4 n) R^{2} r+(7 m+3 n) R r^{2}+2(m+2 n) r^{3} \geq 0 .
$$

Therefore, we conclude that inequality (1.11) holds.
It is known that the equality of Euler's inequality occurs only when the triangle is equilateral. Thus, by the equality conditions of (1.1) and (2.1), we know that the equality in (1.11) holds under the exact conditions mentioned in Theorem 1. This completes the proof of Theorem 1.

Remark 2. Based on Lemma 1, we have another proof of Theorem 1, as follows: Let H be the value of the right hand side of (1.11). And, let $H_{1}=(2 R+r)^{2}$ and $H_{2}=16 R r-3 r^{2}-4 r^{3} / R$. Then, it is easy to check the following identities:

$$
\begin{gather*}
H_{1}-H=\frac{4 n(R-r)\left(R^{2}-R r-r^{2}\right)}{(m+n) R-2 m r},  \tag{2.10}\\
H_{2}-H=\frac{-4 m(R-r)(R-2 r)\left(R^{2}-R r-r^{2}\right)}{R[(m+n) R-2 m r]} . \tag{2.11}
\end{gather*}
$$

If $R^{2}-R r-r^{2} \geq 0$, then by $m \geq 0, n \geq 0$, inequality (1.1) and Euler's inequality (1.12) we see that $s^{2} \geq H_{1} \geq H$. If $R^{2}-R r-r^{2}<0$, then by $m \geq 0, n \geq 0$, inequality (2.1) and Euler's inequality (1.12) we see that $s^{2} \geq H_{2} \geq H$. Therefore, for all acute triangles we have $s^{2} \geq H$, i.e., inequality (1.11) holds. Also, it is easy to determine the equality condition of (1.11) from (2.10) and (2.11).

## 3. Applications of Theorem 1 (I)

In this section and in the next three sections, we shall mainly discuss applications of Theorem 1. Let us denote the inequality mentioned in Theorem 1 as follows:

$$
\begin{equation*}
s^{2} \geq \frac{N}{M} \tag{3.1}
\end{equation*}
$$

where

$$
\begin{aligned}
M & =(m+n) R-2 m r, \\
N & =4 m R^{3}-4(m-4 n) R^{2} r-(7 m+3 n) R r^{2}-2(m+2 n) r^{3} .
\end{aligned}
$$

Let $p$ and $q$ be arbitrary non-negative real numbers (not both zero). By Euler's inequality (1.12) we may put $m=p R$ and $n=q(R-r)+p r$ in (3.1), then $M=(p+q) R(R-r)$ and a simple calculation gives

$$
N=(R-r)\left[4 p R^{3}+16 q R^{2} r+3(3 p-q) R r^{2}+4(p-q) r^{3}\right] .
$$

Therefore, we obtain the following corollary:
Corollary 1. For any non-negative real numbers $p, q$ and an acute triangle $A B C$, we have

$$
\begin{equation*}
s^{2} \geq \frac{4 p R^{3}+16 q R^{2} r+3(3 p-q) R r^{2}+4(p-q) r^{3}}{(p+q) R} \tag{3.2}
\end{equation*}
$$

For $p=q$, the above inequality becomes Walker's inequality (1.3).
Remark 3. In this section and in the next two sections, all the conditions for equality in the inequalities given in the corollaries are the same as those of Walker's inequality, i.e., the equalities hold if and only if the triangle $A B C$ is equilateral or right isosceles.
Remark 4. Walker's inequality (1.3) can be obtained from the following identity:

$$
\begin{equation*}
\sum\left(c^{2}+a^{2}-b^{2}\right)\left(a^{2}+b^{2}-c^{2}\right)(b-c)^{2}=32 r^{2} s^{2}\left(s^{2}-2 R^{2}-8 R r-3 r^{2}\right) \tag{3.3}
\end{equation*}
$$

which can be easily proved by using known identities (See [3, 16]), we omit the details here.

Remark 5. The following sharpened version of Walker's inequality

$$
\begin{equation*}
s^{2} \geq 2 R^{2}+8 R r+3 r^{2}+\frac{2(R-2 r)[R-(\sqrt{2}+1) r]^{2}}{R} \tag{3.4}
\end{equation*}
$$

was proved by Wu and Chu in [14]. This inequality can easily be proved by using Theorem 1, as follows:

If $R \geq(\sqrt{2}+1) r$, then we have $(\sqrt{2}+1) R-(3+2 \sqrt{2}) r>0$. Thus, in (3.1) we may put

$$
m=R[R-(\sqrt{2}+1) r], n=r[(\sqrt{2}+1) R-(3+2 \sqrt{2}) r] .
$$

And, it is easy to get that

$$
\begin{aligned}
M & =R(R-r \sqrt{2}-r)(R-r+r \sqrt{2}) \\
N & =(R-r \sqrt{2}-r)(R-r+r \sqrt{2})\left[4 R^{3}-4 \sqrt{2} R^{2} r+(17+12 \sqrt{2}) R r^{2}-4(3+2 \sqrt{2}) r^{3}\right]
\end{aligned}
$$

Thus, we have

$$
s^{2} \geq \frac{4 R^{3}-4 \sqrt{2} R^{2} r+(17+12 \sqrt{2}) R r^{2}-4(3+2 \sqrt{2}) r^{3}}{R}
$$

which is equivalent to (3.4).
If $R<(\sqrt{2}+1) r$, then we have $(\sqrt{2}+1) R-(3+2 \sqrt{2}) r<0$. In this case, in (3.1) we may put

$$
m=-R[R-(\sqrt{2}+1) r], n=-r[(\sqrt{2}+1) R-(3+2 \sqrt{2}) r],
$$

and inequality (3.4) is obtained, too. Therefore, inequality (3.4) is valid for any acute triangle $A B C$.
Inequality (3.2) given in Corollary 1 is an obvious generalization of Walker's inequality (1.3). We shall discuss its applications in the rest of this section.

In (3.2), we take $(p, q)=(1,0),(1,3),(3,1),(5,3),(3,2),(4,3),(1,6)$ respectively, then the following seven inequalities are obtained immediately.
Corollary 2. Let $A B C$ be an acute triangle, then

$$
\begin{gather*}
s^{2} \geq 4 R^{2}+9 r^{2}+\frac{4 r^{3}}{R}  \tag{3.5}\\
s^{2} \geq R^{2}+12 R r-\frac{2 r^{3}}{R}  \tag{3.6}\\
s^{2} \geq 3 R^{2}+4 R r+6 r^{2}+\frac{2 r^{3}}{R},  \tag{3.7}\\
s^{2} \geq \frac{(5 R+2 r)(R+r)^{2}}{2 R}  \tag{3.8}\\
s^{2} \geq \frac{(2 R+r)\left(6 R^{2}+13 R r+4 r^{2}\right)}{5 R}  \tag{3.9}\\
s^{2} \geq \frac{(4 R+r)\left(4 R^{2}+11 R r+4 r^{2}\right)}{7 R}  \tag{3.10}\\
s^{2} \geq \frac{(2 R-r)\left(2 R^{2}+49 R r+20 r^{2}\right)}{7 R} \tag{3.11}
\end{gather*}
$$

Remark 6. In fact, inequality (3.2) in Corollary 1 can be easily obtained from the following linear combined inequality:

$$
\begin{equation*}
p\left(R s^{2}-4 R^{3}-9 R r^{2}-4 r^{3}\right)+q\left(R s^{2}-16 R^{2} r+3 R r^{2}+4 r^{3}\right) \geq 0 \tag{3.12}
\end{equation*}
$$

which is true by $p \geq 0, q \geq 0$, inequalities (3.5) and (2.1). We therefore know that the result of Corollary 1 contains inequality (2.1).

Remark 7. By inequality (3.6) and Euler's inequality $R \geq 2 r$, it is easy to obtain the following quadratic inequality

$$
\begin{equation*}
s^{2} \geq R^{2}+12 R r-r^{2} \tag{3.13}
\end{equation*}
$$

which is given in [6].
According to Euler's inequality, we may take $(p, q)=(4 R-r, r),(3 R-r, R+r),(r, 2 R-r)$ in Corollary 1. Then, it is not difficult to obtain the following three inequalities:

Corollary 3. Let ABC be an acute triangle, then

$$
\begin{align*}
& s^{2} \geq 4 R^{2}-R r+13 r^{2}+\frac{(R-2 r) r^{3}}{R^{2}}  \tag{3.14}\\
& s^{2} \geq 3 R^{2}+3 R r+10 r^{2}-\frac{(R+2 r) r^{3}}{R^{2}}  \tag{3.15}\\
& s^{2} \geq 18 R r-11 r^{2}+\frac{2(R+2 r) r^{3}}{R^{2}} \tag{3.16}
\end{align*}
$$

Remark 8. Euler's inequality $R \geq 2 r$ shows that inequality (3.14) is stronger than

$$
\begin{equation*}
s^{2} \geq 4 R^{2}-R r+13 r^{2} \tag{3.17}
\end{equation*}
$$

which is given in [6]. By inequality (3.15) and Euler's inequality, it is easy to show that

$$
\begin{equation*}
s^{2} \geq 3 R^{2}+3 R r+9 r^{2} \tag{3.18}
\end{equation*}
$$

which is also given in [6].
Remark 9. By inequality (3.16) and Euler's inequality, it is easy to prove that

$$
\begin{equation*}
s^{2} \geq 18 R r-12 r^{2}+\frac{6 r^{3}}{R} \tag{3.19}
\end{equation*}
$$

Furthermore, by combining (1.1) and (3.19) in the same way used to prove inequality (1.11), we obtain the following inequality

$$
\begin{equation*}
s^{2} \geq \frac{4 m R^{3}-2(2 m-9 n) R^{2} r-(7 m+12 n) R r^{2}-2(m-3 n) r^{3}}{(m+n) R-2 m r} \tag{3.20}
\end{equation*}
$$

where the equality holds if and only if $\triangle A B C$ is equilateral.

## 4. Applications of Theorem 1 (II)

In view of Euler's inequality (1.12), we may put $m=2 p(R-r)+q r$ and $n=q r$ in inequality (3.1). Then, it is easy to obtain that

$$
\begin{aligned}
M & =2(R-r)[p R-(2 p-q) r], \\
N & =2(R-r)\left[4 p R^{3}-2(2 p-q) R^{2} r-(7 p-8 q) R r^{2}-(2 p-3 q) r^{3}\right] .
\end{aligned}
$$

Therefore, we have
Corollary 4. For any non-negative real numbers $p, q$ and an acute triangle $A B C$, we have

$$
\begin{equation*}
s^{2} \geq \frac{4 p R^{3}-2(2 p-q) R^{2} r-(7 p-8 q) R r^{2}-(2 p-3 q) r^{3}}{p R-(2 p-q) r} \tag{4.1}
\end{equation*}
$$

Remark 10. The above inequality can be easily obtained from the following

$$
\begin{equation*}
p(R-2 r)\left[s^{2}-(2 R+r)^{2}\right]+q r\left(s^{2}-2 R^{2}-8 R r-3 r^{2}\right) \geq 0, \tag{4.2}
\end{equation*}
$$

which is true by inequalities (1.1),(1.3) and (1.12). Incidentally, it is easily shown that if $m>n$ then inequality (4.1) is equivalent to inequality (1.11).

In (4.1), for $(p, q)=(1,1),(2,1),(3,2),(8,7)$, we get the following four inequalities respectively:
Corollary 5. Let ABC be an acute triangle, then

$$
\begin{align*}
& s^{2} \geq \frac{4 R^{3}-2 R^{2} r+R r^{2}+r^{3}}{R-r}  \tag{4.3}\\
& s^{2} \geq \frac{(4 R+r)\left(2 R^{2}-2 R r-r^{2}\right)}{2 R-3 r},  \tag{4.4}\\
& s^{2} \geq \frac{R\left(12 R^{2}-8 R r-5 r^{2}\right)}{3 R-4 r}  \tag{4.5}\\
& s^{2} \geq \frac{32 R^{3}-18 R^{2} r+5 r^{3}}{8 R-9 r} \tag{4.6}
\end{align*}
$$

Remark 11. Inequality (4.3) and inequality (4.4) can be obtained from the following two identities respectively:

$$
\begin{align*}
& \sum b c(b+c-a)\left(c^{2}+a^{2}-b^{2}\right)\left(a^{2}+b^{2}-c^{2}\right)(b-c)^{2} \\
& =128 r^{3} s^{3}\left[(R-r) s^{2}-4 R^{3}+2 R^{2} r-R r^{2}-r^{3}\right]  \tag{4.7}\\
& \sum\left(c^{2}+a^{2}-b^{2}\right)\left(a^{2}+b^{2}-c^{2}\right)(b-c)^{2}(b+c-a)^{2} \\
& =128 s^{2} r^{3}\left[(2 R-3 r) s^{2}-(4 R+r)\left(2 R^{2}-2 R r-r^{2}\right)\right] \tag{4.8}
\end{align*}
$$

which are both easily proved by using known identities (cf. [3, 16]). Also, inequality (4.5) can be easily obtained from (4.3) and (4.4).

In (4.1), if we take $(p, q)=(r, R),(r, R+r),(29 r, 9 R+22 r),(4 r, 5 R+r),(8 R+8 r, 8 R+5 r)$ in inequality (4.1) respectively, then it is easy to obtain the following five inequalities:

Corollary 6. Let ABC be an acute triangle, then

$$
\begin{gather*}
s^{2} \geq \frac{(R+r)\left(3 R^{2}-R r-r^{2}\right)}{R-r},  \tag{4.9}\\
s^{2} \geq \frac{6 R^{3}+6 R^{2} r+4 R r^{2}+r^{3}}{2 R-r},  \tag{4.10}\\
s^{2} \geq \frac{67 R^{3}+4 r^{3}}{19 R-18 r},  \tag{4.11}\\
s^{2} \geq \frac{(R+r)\left(26 R^{2}-5 r^{2}\right)}{9 R-7 r},  \tag{4.12}\\
s^{2} \geq \frac{\left(2 R^{2}-r^{2}\right)(4 R+r)^{2}}{8 R^{2}-11 r^{2}} . \tag{4.13}
\end{gather*}
$$

Remark 12. Inequality (4.10) can be obtained from the following identity:

$$
\begin{equation*}
\sum b c(b+c)\left(c^{2}+a^{2}-b^{2}\right)\left(a^{2}+b^{2}-c^{2}\right)(b-c)^{2}=128 r^{3} s^{3}\left[(2 R-r) s^{2}-\left(6 R^{3}+6 R^{2} r+4 R r^{2}+r^{3}\right)\right] \tag{4.14}
\end{equation*}
$$

We now turn back to Theorem 1. Note that inequality (1.11) is equivalent to (3.1). By Euler's inequality we may take $m=(6 p+q) R-4 p r$ and $n=q R+2 p r$ in (3.1), then it is easy to get

$$
\begin{aligned}
M & =2(R-r)[(3 p+q) R-4 p r], \\
N & =2 R(R-r)\left[2(6 p+q) R^{2}-8(p-q) R r-(5 p-3 q) r^{2}\right],
\end{aligned}
$$

and we thus have the following corollary:
Corollary 7. For any non-negative real numbers $p, q$ and an acute triangle $A B C$, we have

$$
\begin{equation*}
s^{2} \geq \frac{R\left[2(6 p+q) R^{2}-8(p-q) R r-(5 p-3 q) r^{2}\right]}{(3 p+q) R-4 p r} . \tag{4.15}
\end{equation*}
$$

Remark 13. The above inequality can be obtained from the following inequality:

$$
\begin{equation*}
p\left[(3 R-4 r) s^{2}-R\left(12 R^{2}-8 R r-5 r^{2}\right)\right]+q R\left(s^{2}-2 R^{2}-8 R r-3 r^{2}\right) \geq 0, \tag{4.16}
\end{equation*}
$$

which is clearly true by inequality (4.5) and Walker's inequality (1.3).
In inequality (4.15), putting $(p, q)=(1,1),(1,2),(3,5),(1,4)$, we get the following inequalities respectively:

Corollary 8. Let $A B C$ be an acute triangle, then

$$
\begin{equation*}
s^{2} \geq \frac{R\left(7 R^{2}-r^{2}\right)}{2(R-r)} \tag{4.17}
\end{equation*}
$$

$$
\begin{gather*}
s^{2} \geq \frac{R(4 R+r)^{2}}{5 R-4 r},  \tag{4.18}\\
s^{2} \geq \frac{(23 R+8 r) R^{2}}{7 R-6 r},  \tag{4.19}\\
s^{2} \geq \frac{R(10 R+7 r)(2 R+r)}{7 R-4 r} . \tag{4.20}
\end{gather*}
$$

Remark 14. The above inequality (4.17) can be easily obtained by (4.3) and (4.9) or (4.4) and (4.10). In addition, we have the following simple proof. Since

$$
\begin{equation*}
2(R-r) s^{2}-R\left(7 R^{2}-r^{2}\right)=2 r\left(s^{2}-2 R^{2}-8 R r-3 r^{2}\right)+(R-2 r)\left(2 s^{2}-7 R^{2}-10 R r-3 r^{2}\right), \tag{4.21}
\end{equation*}
$$

we conclude that the value of the right hand is non-negative by Walker's inequality (1.3), Euler's inequality (1.12) and inequality (6.4) below. Hence, inequality (4.17) holds.

Remark 15. A simple proof of inequality (4.18) is as follows: Note that

$$
\begin{equation*}
(5 R-4 r) s^{2}-R(4 R+r)^{2}=6 r\left(s^{2}-2 R^{2}-8 R r-3 r^{2}\right)+(R-2 r)\left(5 s^{2}-16 R^{2}-28 R r-9 r^{2}\right) . \tag{4.22}
\end{equation*}
$$

We thus conclude that the value of the right hand is non-negative by Walker's inequality (1.3), Euler's inequality (1.12) and inequality (6.5) below. Hence, inequality (4.18) holds.

## 5. Applications of Theorem 1 (III)

We now derive a result similar to Corollary 23 from Theorem 1.
For any non-negative $p$ and $q$, we may put $m=q r$ and $n=2 p(R-r)+q r$ in inequality (3.1), then it is easy to get

$$
\begin{aligned}
M & =2(R-r)(p R+q r), \\
N & =2 r(R-r)\left[(16 p+2 q) R^{2}-(3 p-8 q) R r-(4 p-3 q) r^{2}\right] .
\end{aligned}
$$

Thus, we have the following corollary:
Corollary 9. For any non-negative real numbers $p, q$ and an acute triangle $A B C$, we have that

$$
\begin{equation*}
s^{2} \geq \frac{r\left[(16 p+2 q) R^{2}-(3 p-8 q) R r-(4 p-3 q) r^{2}\right]}{p R+q r} \tag{5.1}
\end{equation*}
$$

Remark 16. This inequality can be easily obtained from the following inequality:

$$
\begin{equation*}
p\left(R s^{2}-16 R^{2} r+3 R r^{2}+4 r^{3}\right)+q r\left(s^{2}-2 R^{2}-8 R r-3 r^{2}\right) \geq 0 \tag{5.2}
\end{equation*}
$$

which follows from Walker's inequality and the result of Lemma 1.
In (5.1), for $(p, q)=(1,2),(1,3),(3,4),(8,3)$ respectively, we get the following inequalities:

Corollary 10. Let $A B C$ be an acute triangle, then

$$
\begin{align*}
s^{2} & \geq \frac{r(4 R+r)(5 R+2 r)}{R+2 r} .  \tag{5.3}\\
s^{2} & \geq \frac{r(2 R+r)(11 R+5 r)}{R+3 r} .  \tag{5.4}\\
s^{2} & \geq \frac{(56 R+23 r) R r}{3 R+4 r} .  \tag{5.5}\\
s^{2} & \geq \frac{r\left(134 R^{2}-23 r^{2}\right)}{8 R+3 r} . \tag{5.6}
\end{align*}
$$

Remark 17. Inequality (5.3) can be obtained from the following identity:

$$
\begin{equation*}
\sum\left(c^{2}+a^{2}-b^{2}\right)\left(a^{2}+b^{2}-c^{2}\right)(b-c)^{2} a^{2}=64 R r^{2} s^{2}\left[(R+2 r) s^{2}-r(5 R+2 r)(4 R+r)\right] \tag{5.7}
\end{equation*}
$$

In addition, it is easily shown that

$$
\begin{equation*}
\sum(a-b)(a-c)\left(b^{2}+c^{2}-a^{2}\right) a=8 r s\left[(R+2 r) s^{2}-r(5 R+2 r)(4 R+r)\right] \tag{5.8}
\end{equation*}
$$

So, by inequality (5.3), we have the following inequality involving the sides of the acute triangle $A B C$ :
Corollary 11. Let $A B C$ be an acute triangle, then

$$
\begin{equation*}
\sum(a-b)(a-c)\left(b^{2}+c^{2}-a^{2}\right) a \geq 0 \tag{5.9}
\end{equation*}
$$

We now derive another inequality similar to (5.9). It is easy to prove the following identity:

$$
\begin{equation*}
\sum(a-b)(a-c)\left(b^{2}+c^{2}-a^{2}\right)=2\left[s^{4}-8 r(2 R-r) s^{2}-r^{2}(4 R+r)^{2}\right] \tag{5.10}
\end{equation*}
$$

On the other hand, we have

$$
s^{4}-8 r(2 R-r) s^{2}-r^{2}(4 R+r)^{2}=s^{2}\left(s^{2}-16 R r+3 r^{2}+\frac{4 r^{3}}{R}\right)+\frac{r^{2}}{R}\left[(5 R-4 r) s^{2}-R(4 R+r)^{2}\right] .
$$

Thus, by inequality (2.1) and inequalities (4.18), we have

$$
\begin{equation*}
s^{4}-8 r(2 R-r) s^{2}-r^{2}(4 R+r)^{2} \geq 0 \tag{5.11}
\end{equation*}
$$

Therefore, by identity (5.10) we obtain
Corollary 12. Let $A B C$ be an acute triangle, then

$$
\begin{equation*}
\sum(a-b)(a-c)\left(b^{2}+c^{2}-a^{2}\right) \geq 0 . \tag{5.12}
\end{equation*}
$$

Now, we return to Theorem 1. In inequality (3.1) (which is equivalent to (1.11)), we take $m=$ $2 p(2 R-r)+q r$ and $n=(p+q) R+p r$, then it is easy to get that

$$
\begin{aligned}
& M=(R-r)[(5 p+q) R-2(2 p-q) r] \\
& N=(4 R+r)(R-r)\left[4 p R^{2}+(p+5 q) R r+2 q r^{2}\right] .
\end{aligned}
$$

Hence, we have

Corollary 13. For any non-negative real numbers $p, q$ and an acute triangle $A B C$, we have

$$
\begin{equation*}
s^{2} \geq \frac{(4 R+r)\left[4 p R^{2}+(p+5 q) R r+2 q r^{2}\right]}{(5 p+q) R-2(2 p-q) r} \tag{5.13}
\end{equation*}
$$

Remark 18. The above inequality can be easily obtained from the following inequality:

$$
\begin{equation*}
p\left[(5 R-4 r) s^{2}-R(4 R+r)^{2}\right]+q\left[(R+2 r) s^{2}-r(5 R+2 r)(4 R+r)\right] \geq 0, \tag{5.14}
\end{equation*}
$$

which is clearly true by inequalities (4.18) and (5.3).
Putting $(p, q)=(1,1),(6,5)$ in (5.13) respectively, gives the following two inequalities:
Corollary 14. Let $A B C$ be an acute triangle, then

$$
\begin{gather*}
s^{2} \geq \frac{(R+r)(2 R+r)(4 R+r)}{3 R-r},  \tag{5.15}\\
s^{2} \geq \frac{(4 R+r)(3 R+2 r)(8 R+5 r)}{7(5 R-2 r)} . \tag{5.16}
\end{gather*}
$$

Remark 19. Inequality (5.15) can be obtained from the following identity:

$$
\begin{equation*}
\sum b c\left(c^{2}+a^{2}-b^{2}\right)\left(a^{2}+b^{2}-c^{2}\right)(b-c)^{2}=64 s^{2} r^{3}\left[(3 R-r) s^{2}-(4 R+r)(2 R+r)(R+r)\right] \tag{5.17}
\end{equation*}
$$

If we take $(p, q)=(r, 3 R),(11 R+2 r, 9 R),(21 R+6 r, 23 R+8 r)$ in $(5.13)$, then we can obtain the following three inequalities:

Corollary 15. Let ABC be an acute triangle, then

$$
\begin{gather*}
s^{2} \geq \frac{R r(4 R+r)(19 R+7 r)}{(R+4 r)(3 R-r)} .  \tag{5.18}\\
s^{2} \geq \frac{R(R+r)(11 R+5 r)}{2(2 R-r)}  \tag{5.19}\\
s^{2} \geq \frac{(R+r)(3 R+r)(7 R+4 r)}{2(4 R-r)} . \tag{5.20}
\end{gather*}
$$

We close this section giving another result obtained using Theorem 1 .
By Euler's inequality $R \geq 2 r$ we may take $m=p(2 R-r)$ and $n=2 q(R-r)+p r$ in Theorem 1, after the computations we get the following inequality:

Corollary 16. For any non-negative real numbers $p, q$ and acute triangle $A B C$, it holds that

$$
\begin{equation*}
s^{2} \geq \frac{4 p R^{3}-2(p-8 q) R^{2} r+(p-3 q) R r^{2}+(p-4 q) r^{3}}{(p+q) R-p r} \tag{5.21}
\end{equation*}
$$

Remark 20. This inequality can also be obtained from the following linear combined inequality

$$
\begin{equation*}
p\left[(R-r) s^{2}-\left(4 R^{3}-2 R^{2} r+R r^{2}+r^{3}\right)\right]+q\left(R s^{2}-16 R^{2} r+3 R r^{2}+4 r^{3}\right) \geq 0 \tag{5.22}
\end{equation*}
$$

which follows from the previous inequality (4.3) and Lemma 1.

Remark 21. We remark that the inequalities without parameters given in the previous corollaries (except inequalities (4.4), (4.5), (4.13)) can be derived from inequality (5.21). For example, if we take $(p, q)=(4,1),(2 r, 2 R+r),(4 r, 6 R+r),(11 R-4 r, 5 R-r)$ in $(5.21)$, then we can obtain inequalities (4.18),(5.3), (5.5), (5.19), respectively. In fact, it is easily shown that if $2 m>n$ then inequality (5.21) is equivalent to inequality (1.11).

Remark 22. We have the following obvious conclusion. If $s^{2} \geq f_{1}(R, r)$ and $s^{2} \geq f_{2}(R, r)$ hold for an acute triangle $A B C$, then for any non-negative real numbers $p$ and $q$ we have

$$
\begin{equation*}
s^{2} \geq \frac{p f_{1}(R, r)+q f_{2}(R, r)}{p+q} . \tag{5.23}
\end{equation*}
$$

According to this conclusion, we can establish new inequalities from two known inequalities of type $s^{2} \geq f(R, r)$ in acute triangles. The main result of this paper is actually established in this way. The author finds that this kind of inequalities are a consequence of Theorem 1 in many cases.

## 6. Applications of Theorem 1 (IV)

In this section, we shall continue to derive some inequalities from Theorem 1. All the equalities in these inequalities only occur in the case when the triangle $A B C$ is right isosceles.

Let $k>0$ be a positive number, then we may put $m=2 k(R-r)+R-2 r$ and $n=R-2 r$ in Theorem 1 by Euler's inequality $R \geq 2 r$, and simple calculations gives the following inequality:

Corollary 17. For any positive number $k$ and an acute triangle $A B C$ we have

$$
\begin{equation*}
s^{2} \geq \frac{2(2 k+1) R^{2}+4(k+2) R r+(k+3) r^{2}}{k+1} . \tag{6.1}
\end{equation*}
$$

Remark 23. The above inequality can be easily obtained from Walker' inequality (1.3) and Ciamberlini's inequality (1.1),i.e., it follows from that

$$
\begin{equation*}
s^{2}-\left(2 R^{2}+8 R r+3 r^{2}\right)+k\left[s^{2}-(2 R+r)^{2}\right] \geq 0 \tag{6.2}
\end{equation*}
$$

Clearly, inequality (6.1) can be regarded as a generalization of Walker's inequality if we allow $k=0$.
In (6.1), putting $k=1,3,2 / 3$ respectively, we then obtain the following three inequalities similar to Walker's result (1.1).

Corollary 18. Let ABC be an acute triangle, then

$$
\begin{gather*}
s^{2} \geq 3 R^{2}+6 R r+2 r^{2}  \tag{6.3}\\
2 s^{2} \geq 7 R^{2}+10 R r+3 r^{2}  \tag{6.4}\\
5 s^{2} \geq 16 R^{2}+28 R r+9 r^{2} \tag{6.5}
\end{gather*}
$$

Let $k>0$ and let $t \geq 0$. In Theorem 1, for $m=2 k(R-r) r+(R-2 r) r$ and $n=2(R-r)(R-2 r) t+(R-2 r) r$, then it is easy to obtain the following corollary:

Corollary 19. If $k>0$ and $t \geq 0$, then for an acute triangle $A B C$ we have

$$
\begin{equation*}
s^{2} \geq \frac{r\left[(16 t+4 k+2) R^{2}-(3 t-4 k-8) R r-(4 t-k-3) r^{2}\right]}{t R+(k+1) r} . \tag{6.6}
\end{equation*}
$$

Remark 24. In fact, the above inequality can be obtained from Lemma 1 and Corollary 17. More precisely, it follows easily from

$$
\begin{equation*}
t\left[R s^{2}-\left(16 R^{2} r-3 R r^{2}-4 r^{3}\right)\right]+r\left[(k+1) s^{2}-2(2 k+1) R^{2}-4(k+2) R r-(k+3) r^{2}\right] \geq 0 \tag{6.7}
\end{equation*}
$$

For $t=(k+3) / 4,(4 k+8) / 3$ in (6.6) respectively, we obtain

$$
\begin{align*}
& s^{2} \geq \frac{r R[(32 k+56) R+(13 k+23) r]}{(k+3) R+4(k+1) r},  \tag{6.8}\\
& s^{2} \geq \frac{r\left[(76 k+134) R^{2}-(13 k+23) r^{2}\right]}{(4 k+8) R+3(k+1) r} . \tag{6.9}
\end{align*}
$$

Putting $k=1$ in the above two inequalities, we get
Corollary 20. Let $A B C$ be an acute triangle, then

$$
\begin{align*}
& s^{2} \geq \frac{r R(22 R+9 r)}{R+2 r}  \tag{6.10}\\
& s^{2} \geq \frac{r\left(35 R^{2}-6 r^{2}\right)}{2 R+r} \tag{6.11}
\end{align*}
$$

In (6.6), for $(k, t)=\left(1, \frac{2}{3}\right),\left(1, \frac{1}{3}\right),\left(1, \frac{2 R}{r}\right)$ respectively, we can obtain the following three inequalities.
Corollary 21. Let $A B C$ be an acute triangle, then

$$
\begin{align*}
& s^{2} \geq \frac{r(5 R+r)(5 R+2 r)}{R+3 r},  \tag{6.12}\\
& s^{2} \geq \frac{r(2 R+r)(17 R+8 r)}{R+6 r},  \tag{6.13}\\
& s^{2} \geq \frac{2 r\left(8 R^{3}+R r^{2}+r^{3}\right)}{R^{2}+r^{2}} . \tag{6.14}
\end{align*}
$$

Finally, we give three inequalities which are obtained directly from Theorem 1.
Corollary 22. Let ABC be an acute triangle, then

$$
\begin{gather*}
s^{2} \geq 2 R^{2}+9 R r+r^{2}-\frac{r^{3}}{R},  \tag{6.15}\\
s^{2} \geq \frac{9 R^{3}+9 R^{2} r-r^{3}}{3 R-2 r},  \tag{6.16}\\
s^{2} \geq \frac{R(2 R+r)(3 R+2 r)}{2 R-r} . \tag{6.17}
\end{gather*}
$$

In fact, by Euler's inequality $R \geq 2 r$ we may take

$$
\begin{aligned}
(m, n)= & \left(2 R^{2}-3 R r, 2 R^{2}-5 R r+2 r^{2}\right),\left(9 R^{2}-21 R r+8 r^{2}, 3 R^{2}-5 R r-2 r^{2}\right), \\
& \left(6 R^{2}-13 R r+4 r^{2}, 2 R^{2}-3 R r-2 r^{2}\right)
\end{aligned}
$$

in Theorem 1 respectively, and then inequalities (6.15), (6.16) and (6.17) are easily obtained.

## 7. Three open problems

In any triangle $A B C$, we have the following identity:

$$
\begin{equation*}
\sum(a-b)(a-c)\left(b^{2}+c^{2}-a^{2}\right) a^{2}=16 r^{2} s^{2}\left(s^{2}-2 R^{2}-8 R r-3 r^{2}\right) . \tag{7.1}
\end{equation*}
$$

Thus by Walker's inequality (1.3) we conclude that the following inequality

$$
\begin{equation*}
\sum(a-b)(a-c)\left(b^{2}+c^{2}-a^{2}\right) a^{2} \geq 0 \tag{7.2}
\end{equation*}
$$

holds for the acute triangle $A B C$.
Considering generalizations of inequality (7.2) and the previous inequality (5.9), we present the following conjecture as an open problem:

Conjecture 1. Let $k$ be a real number such that $k \leq 4$, then for an acute triangle $A B C$ we have

$$
\begin{equation*}
\sum(a-b)(a-c)\left(b^{2}+c^{2}-a^{2}\right) a^{k} \geq 0 \tag{7.3}
\end{equation*}
$$

Similarly, we present the following two conjectures checked by a computer:
Conjecture 2. Let $k$ be a real number such that $k \geq-10$, then for an acute triangle $A B C$ we have

$$
\begin{equation*}
\sum(a-b)(a-c)\left(b^{2}+c^{2}-a^{2}\right)(b+c)^{k} \geq 0 \tag{7.4}
\end{equation*}
$$

Conjecture 3. Let $k$ be a real number such that $k \geq-3$, then for an acute triangle $A B C$ we have

$$
\begin{equation*}
\sum(a-b)(a-c)\left(b^{2}+c^{2}-a^{2}\right)(b+c-a)^{k} \geq 0 \tag{7.5}
\end{equation*}
$$

## 8. Conclusions

We have obtained various parallel low bounds of $s^{2}$ in terms of R and r for an acute triangle $A B C$ from Theorem 1. When $R^{2}-R r-r^{2}<0$, the best low bound of these results is given by (2.1), which, as a special case of Theorem 1 when $m=0$, is just the result of Lemma 1 . This shows that inequality (2.1) is such a remarkable result that it might be used to prove some polynomial inequalities $f(s, R, r) \geq 0$ in acute triangles.

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## Conflict of interest

The author declares that there is no competing interest.

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