



Research article

Orlicz mixed chord-integrals

Chang-Jian Zhao*

Department of Mathematics, China Jiliang University, Hangzhou 310018, P. R. China

* **Correspondence:** Email: chjzhao@163.com, chjzhao@cjlu.edu.cn.

Abstract: In this paper, we introduce an affine geometric quantity and call it *Orlicz mixed chord integral* by defining a new Orlicz chord addition, which generalizes the mixed chord integrals to Orlicz space. The Minkowski and Brunn-Minkowski inequalities for the Orlicz mixed chord integrals are established. The new inequalities in special cases yield L_p -Minkowski and Brunn-Minkowski inequalities for the chord integrals. The related concepts and inequalities of L_p -mixed chord integrals are derived. As an application, a new isoperimetric inequality for the chord integrals is given. As extensions, Orlicz multiple mixed chord integrals and Orlicz-Aleksandrov-Fenchel inequality for the Orlicz multiple mixed chord integrals are also derived here for the first time.

Keywords: star body; first order variation; chord integral; mixed chord integral; L_p -mixed chord integral; Orlicz chord addition; Orlicz mixed chord integral

Mathematics Subject Classification: 46E30, 52A30, 52A40

1. Introduction

The radial addition $K\widetilde{+}L$ of star sets K and L can be defined by

$$\rho(K\widetilde{+}L, \cdot) = \rho(K, \cdot) + \rho(L, \cdot),$$

where a star set is a compact set that is star-shaped at o and contains o and $\rho(K, \cdot)$ denotes the radial function of star set K . The radial function is defined by

$$\rho(K, u) = \max\{c \geq 0 : cu \in K\}, \tag{1.1}$$

for $u \in S^{n-1}$, where S^{n-1} denotes the surface of the unit ball centered at the origin. The initial study of the radial addition can be found in [1, p. 235]. K is called a star body if $\rho(K, \cdot)$ is positive and continuous, and let \mathcal{S}^n denote the set of star bodies. The radial addition and volume are the basis

and core of the dual Brunn-Minkowski theory (see, e.g., [2–10]). It is important that the dual Brunn-Minkowski theory can count among its successes the solution of the Busemann-Petty problem in [3, 11–14]. Recently, it has turned to a study extending from L_p -dual Brunn-Minkowski theory to Orlicz dual Brunn-Minkowski theory. The Orlicz dual Brunn-Minkowski theory and its dual have attracted people's attention [15–28].

For $K \in \mathcal{S}^n$ and $u \in S^{n-1}$, the half chord of K in the direction u is defined by

$$d(K, u) = \frac{1}{2}(\rho(K, u) + \rho(K, -u)).$$

If there exists a constant $\lambda > 0$ such that $d(K, u) = \lambda d(L, u)$, for all $u \in S^{n-1}$, then star bodies K, L are said to have similar chord (see Gardner [1] or Schneider [29]). Lu [30] introduced the i -th chord integral of star bodies: For $K \in \mathcal{S}^n$ and $0 \leq i < n$, the i -th chord integral of K , is denoted by $B_i(K)$, is defined by

$$B_i(K) = \frac{1}{n} \int_{S^{n-1}} d(K, u)^{n-i} dS(u). \quad (1.2)$$

Obviously, for $i = 0$, $B_i(K)$ becomes the chord integral $B(K)$.

The main aim of the present article is to generalize the chord integrals to Orlicz space. We introduce a new affine geometric quantity which we shall call Orlicz mixed chord integrals. The fundamental notions and conclusions of the chord integral and related isoperimetric inequalities for the chord integral are extended to an Orlicz setting. The new inequalities in special cases yield the L_p -dual Minkowski and L_p -dual Brunn-Minkowski inequalities for the L_p -mixed chord integrals. The related concepts and inequalities of L_p -mixed chord integrals are derived. As extensions, Orlicz multiple mixed chord integrals and Orlicz-Aleksandrov-Fenchel inequality for the Orlicz multiple mixed chord integrals are also derived.

In Section 3, we introduce the following new notion of Orlicz chord addition of star bodies.

Orlicz chord addition Let K and L be star bodies, the Orlicz chord addition of K and L , is denoted by $K \check{+}_\phi L$, is defined by

$$\phi\left(\frac{d(K, u)}{d(K \check{+}_\phi L, u)}, \frac{d(L, u)}{d(K \check{+}_\phi L, u)}\right) = 1, \quad (1.3)$$

where $u \in S^{n-1}$, and $\phi \in \Phi_2$, which is the set of convex functions $\phi : [0, \infty)^2 \rightarrow (0, \infty)$ that are decreasing in each variable and satisfy $\phi(0, 0) = \infty$ and $\phi(\infty, 1) = \phi(1, \infty) = 1$.

The particular instance of interest corresponds to using (1.3) with $\phi(x_1, x_2) = \phi_1(x_1) + \varepsilon\phi_2(x_2)$ for $\varepsilon > 0$ and some $\phi_1, \phi_2 \in \Phi$, which are the sets of convex functions $\phi_1, \phi_2 : [0, \infty) \rightarrow (0, \infty)$ that are decreasing and satisfy $\phi_1(0) = \phi_2(0) = \infty$, $\phi_1(\infty) = \phi_2(\infty) = 0$ and $\phi_1(1) = \phi_2(1) = 1$.

In accordance with the spirit of Aleksandrov [31], Fenchel and Jessen's [32] introduction of mixed quermassintegrals, and introduction of Lutwak's [33] L_p -mixed quermassintegrals, we are based on the study of first-order variations of the chord integrals. In Section 4, we prove that the first order Orlicz variation of the mixed chord integral can be expressed as: For $K, L \in \mathcal{S}^n$, $\phi_1, \phi_2 \in \Phi$, $0 \leq i < n$ and $\varepsilon > 0$,

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0^+} B_i(K \check{+}_{\phi_1 + \varepsilon\phi_2} L) = (n-i) \cdot \frac{1}{(\phi_1)'_r(1)} \cdot B_{\phi_2, i}(K, L), \quad (1.4)$$

where $(\phi_1)'_r(1)$ denotes the value of the right derivative of convex function ϕ_1 at point 1. In this first order variational equation (1.4), we find a new geometric quantity. Based on this, we extract the

required geometric quantity, denoted by $B_{\phi,i}(K, L)$ which we shall call Orlicz mixed chord integrals of K and L , as follows

$$B_{\phi_2,i}(K, L) = \frac{1}{n-i} \cdot (\phi_1)'_r(1) \cdot \frac{d}{d\varepsilon} \Big|_{\varepsilon=0^+} B_i(K \dot{\times}_{\phi} \varepsilon \cdot L). \quad (1.5)$$

We show also that the new affine geometric quantity has an integral representation as follows:

$$B_{\phi,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} \phi \left(\frac{d(L, u)}{d(K, u)} \right) d(K, u)^{n-i} dS(u). \quad (1.6)$$

When $\phi(t) = t^{-p}$ and $p \geq 1$, the new affine geometric quantity becomes a new L_p -mixed chord integrals of K and L , denoted by $B_{p,i}(K, L)$, which as is in (2.7).

In Section 5, we establish an Orlicz Minkowski inequality for the mixed chord and Orlicz mixed chord integrals.

Orlicz Minkowski inequality for the Orlicz mixed chord integrals *If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $\phi \in \Phi$, then*

$$B_{\phi,i}(K, L) \geq B_i(K) \cdot \phi \left(\left(\frac{B_i(L)}{B_i(K)} \right)^{1/(n-i)} \right). \quad (1.7)$$

If ϕ is strictly convex, the equality holds if and only if K and L are similar chord.

When $\phi(t) = t^{-p}$ and $p \geq 1$, (1.7) becomes a new L_p -Minkowski inequality (2.8) for the L_p -mixed chord integrals.

In Section 6, as an application, we establish an Orlicz Brunn-Minkowski inequality for the Orlicz chord additions and the mixed chord integrals:

Orlicz Brunn-Minkowski inequality for the Orlicz chord additions *If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $\phi \in \Phi_2$, then*

$$1 \geq \phi \left(\left(\frac{B_i(K)}{B_i(K \dot{+}_{\phi} L)} \right)^{1/(n-i)} \cdot \left(\frac{B_i(L)}{B_i(K \dot{+}_{\phi} L)} \right)^{1/(n-i)} \right). \quad (1.8)$$

If ϕ is strictly convex, the equality holds if and only if K and L are similar chord.

When $\phi(t) = t^{-p}$ and $p \geq 1$, (1.8) becomes a new L_p -Brunn-Minkowski inequality (2.9) for the mixed chord integrals.

A new isoperimetric inequality for the chord integrals is given in Section 7. In Section 8, Orlicz multiple mixed chord integrals is introduced and Orlicz-Aleksandrov-Fenchel inequality for the Orlicz multiple mixed chord integrals is established.

2. Preliminaries

The setting for this paper is n -dimensional Euclidean space \mathbb{R}^n . A body in \mathbb{R}^n is a compact set equal to the closure of its interior. For a compact set $K \subset \mathbb{R}^n$, we write $V(K)$ for the (n -dimensional) Lebesgue measure of K and call this the volume of K . Associated with a compact subset K of \mathbb{R}^n which is star-shaped with respect to the origin and contains the origin, its radial function is $\rho(K, \cdot) : S^{n-1} \rightarrow [0, \infty)$ is defined by

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}.$$

Note that the class (star sets) is closed under union, intersection, and intersection with subspace. The radial function is homogeneous of degree -1 , that is (see e.g. [1]),

$$\rho(K, ru) = r^{-1} \rho(K, u),$$

for all $u \in S^{n-1}$ and $r > 0$. Let $\tilde{\delta}$ denote the radial Hausdorff metric, as follows: if $K, L \in \mathcal{S}^n$, then

$$\tilde{\delta}(K, L) = |\rho(K, u) - \rho(L, u)|_{\infty}.$$

From the definition of the radial function, it follows immediately that for $A \in GL(n)$ the radial function of the image $AK = \{Ay : y \in K\}$ of K is given by (see e.g. [29])

$$\rho(AK, x) = \rho(K, A^{-1}x), \quad (2.1)$$

for all $x \in \mathbb{R}^n$.

For $K_i \in \mathcal{S}^n, i = 1, \dots, m$, define the real numbers R_{K_i} and r_{K_i} by

$$R_{K_i} = \max_{u \in S^{n-1}} d(K_i, u), \quad \text{and} \quad r_{K_i} = \min_{u \in S^{n-1}} d(K_i, u). \quad (2.2)$$

Obviously, $0 < r_{K_i} < R_{K_i}$, for all $K_i \in \mathcal{S}^n$. Writing $R = \max\{R_{K_i}\}$ and $r = \min\{r_{K_i}\}$, where $i = 1, \dots, m$.

2.1. Mixed chord integrals

If $K_1, \dots, K_n \in \mathcal{S}^n$, the mixed chord integral of K_1, \dots, K_n , is denoted by $B(K_1, \dots, K_n)$, is defined by (see [30])

$$B(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} d(K_1, u) \cdots d(K_n, u) dS(u).$$

If $K_1 = \cdots = K_{n-i} = K, K_{n-i+1} = \cdots = K_n = L$, the mixed chord integral $B(K_1, \dots, K_n)$ is written as $B_i(K, L)$. If $L = B$ (B is the unit ball centered at the origin), the mixed chord integral $B_i(K, L) = B_i(K, B)$ is written as $B_i(K)$ and called the i -th chord integral of K . Obviously, For $K \in \mathcal{S}^n$ and $0 \leq i < n$, we have

$$B_i(K) = \frac{1}{n} \int_{S^{n-1}} d(K, u)^{n-i} dS(u). \quad (2.3)$$

If $K_1 = \cdots = K_{n-i-1} = K, K_{n-i} = \cdots = K_{n-1} = B$ and $K_n = L$, the mixed chord integral $B(\underbrace{K, \dots, K}_{n-i-1}, \underbrace{B, \dots, B}_i, L)$ is written as $B_i(K, L)$ and called the i -th mixed chord integral of K and L .

For $K, L \in \mathcal{S}^n$ and $0 \leq i < n$, it is easy to see that

$$B_i(K, L) = \frac{1}{n} \int_{S^{n-1}} d(K, u)^{n-i-1} d(L, u) dS(u). \quad (2.4)$$

This integral representation (2.4), together with the Hölder inequality, immediately give the Minkowski inequality for the i -th mixed chord integral: If $K, L \in \mathcal{S}^n$ and $0 \leq i < n$, then

$$B_i(K, L)^{n-i} \leq B_i(K)^{n-i-1} B_i(L), \quad (2.5)$$

with equality if and only if K and L are similar chord.

2.2. L_p -mixed chord integrals

Definition 2.1 (The L_p -chord addition) Let $K, L \in \mathcal{S}^n$ and $p \geq 1$, the L_p chord addition $\check{+}_p$ of star bodies K and L , is defined by

$$d(K\check{+}_p L, u)^{-p} = d(K, u)^{-p} + d(L, u)^{-p}, \quad (2.6)$$

for $u \in S^{n-1}$.

Obviously, putting $\phi(x_1, x_2) = x_1^{-p} + x_2^{-p}$ and $p \geq 1$ in (1.3), (1.3) becomes (2.6). The following result follows immediately from (2.6) with $p \geq 1$.

$$-\frac{np}{n-i} \lim_{\varepsilon \rightarrow 0^+} \frac{B_i(K\check{+}_p \varepsilon \cdot L) - B_i(L)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} d(K, u)^{n-i+p} d(L, u)^{-p} dS(u).$$

Definition 2.2 (The L_p -mixed chord integrals) Let $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, the L_p -mixed chord integral of star K and L , denoted by $B_{p,i}(K, L)$, is defined by

$$B_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} d(K, u)^{n-i+p} d(L, u)^{-p} dS(u). \quad (2.7)$$

Obviously, when $K = L$, the L_p -mixed chord integral $B_{p,i}(K, K)$ becomes the i -th chord integral $B_i(K)$. This integral representation (2.7), together with the Hölder inequality, immediately gives:

Proposition 2.3 If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, then

$$B_{p,i}(K, L)^{n-i} \geq B_i(K)^{n-i+p} B_i(L)^{-p}, \quad (2.8)$$

with equality if and only if K and L are similar chord.

Proposition 2.4 If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, then

$$B_i(K\check{+}_p L)^{-p/(n-i)} \geq B_i(K)^{-p/(n-i)} + B_i(L)^{-p/(n-i)}, \quad (2.9)$$

with equality if and only if K and L are similar chord.

Proof From (2.6) and (2.7), it is easily seen that the L_p -chord integrals is linear with respect to the L_p -chord addition, and together with inequality (2.8), we have for $p \geq 1$

$$\begin{aligned} B_{p,i}(Q, K\check{+}_p L) &= B_{p,i}(Q, K) + B_{p,i}(Q, L) \\ &\geq B_i(Q)^{(n-i+p)/(n-i)} (B_i(K)^{-p/(n-i)} + B_i(L)^{-p/(n-i)}), \end{aligned}$$

with equality if and only if K and L are similar chord.

Take $K\check{+}_p L$ for Q , recall that $B_{p,i}(Q, Q) = B_i(Q)$, inequality (2.9) follows easily. \square

3. Orlicz chord addition

Throughout this paper, the standard orthonormal basis for \mathbb{R}^n will be $\{e_1, \dots, e_n\}$. Let Φ_n , $n \in \mathbb{N}$, denote the set of convex functions $\phi : [0, \infty)^n \rightarrow (0, \infty)$ that are strictly decreasing in each variable and satisfy $\phi(0) = \infty$ and $\phi(e_j) = 1$, $j = 1, \dots, n$. When $n = 1$, we shall write Φ instead of Φ_1 . The left

derivative and right derivative of a real-valued function f are denoted by $(f)'_l$ and $(f)'_r$, respectively. We first define the Orlicz chord addition.

Definition 3.1 (The Orlicz chord addition) Let $m \geq 2$, $\phi \in \Phi_m$, $K_j \in \mathcal{S}^n$ and $j = 1, \dots, m$, the Orlicz chord addition of K_1, \dots, K_m , is denoted by $\check{\check{\phi}}(K_1, \dots, K_m)$, is defined by

$$d(\check{\check{\phi}}(K_1, \dots, K_m), u) = \sup \left\{ \lambda > 0 : \phi \left(\frac{d(K_1, u)}{\lambda}, \dots, \frac{d(K_m, u)}{\lambda} \right) \leq 1 \right\}, \quad (3.1)$$

for $u \in S^{n-1}$. Equivalently, the Orlicz chord addition $\check{\check{\phi}}(K_1, \dots, K_m)$ can be defined implicitly by

$$\phi \left(\frac{d(K_1, u)}{d(\check{\check{\phi}}(K_1, \dots, K_m), u)}, \dots, \frac{d(K_m, u)}{d(\check{\check{\phi}}(K_1, \dots, K_m), u)} \right) = 1, \quad (3.2)$$

for all $u \in S^{n-1}$.

An important special case is obtained when

$$\phi(x_1, \dots, x_m) = \sum_{j=1}^m \phi_j(x_j),$$

for some fixed $\phi_j \in \Phi$ such that $\phi_1(1) = \dots = \phi_m(1) = 1$. We then write $\check{\check{\phi}}(K_1, \dots, K_m) = K_1 \check{\check{\phi}} \dots \check{\check{\phi}} K_m$. This means that $K_1 \check{\check{\phi}} \dots \check{\check{\phi}} K_m$ is defined either by

$$d(K_1 \check{\check{\phi}} \dots \check{\check{\phi}} K_m, u) = \sup \left\{ \lambda > 0 : \sum_{j=1}^m \phi_j \left(\frac{d(K_j, u)}{\lambda} \right) \leq 1 \right\}, \quad (3.3)$$

for all $u \in S^{n-1}$, or by the corresponding special case of (3.2).

Lemma 3.2 The Orlicz chord addition $\check{\check{\phi}} : (\mathcal{S}^n)^m \rightarrow \mathcal{S}^n$ is monotonic.

Proof This follows immediately from (3.1). □

Lemma 3.3 The Orlicz chord addition $\check{\check{\phi}} : (\mathcal{S}^n)^m \rightarrow \mathcal{S}^n$ is $GL(n)$ covariant.

Proof From (2.1), (3.1) and let $A \in GL(n)$, we obtain

$$\begin{aligned} & d(\check{\check{\phi}}(AK_1, AK_2, \dots, AK_m), u) \\ &= \sup \left\{ \lambda > 0 : \phi \left(\frac{d(AK_1, u)}{\lambda}, \frac{d(AK_2, u)}{\lambda}, \dots, \frac{d(AK_m, u)}{\lambda} \right) \leq 1 \right\} \\ &= \sup \left\{ \lambda > 0 : \phi \left(\frac{d(K_1, A^{-1}u)}{\lambda}, \frac{d(K_2, A^{-1}u)}{\lambda}, \dots, \frac{d(K_m, A^{-1}u)}{\lambda} \right) \leq 1 \right\} \\ &= d(\check{\check{\phi}}(K_1, \dots, K_m), A^{-1}u) \\ &= d(\check{\check{\phi}}(K_1, \dots, K_m), u). \end{aligned}$$

This shows Orlicz chord addition $\check{\check{\phi}}$ is $GL(n)$ covariant. □

Lemma 3.4 Suppose $K_1, \dots, K_m \in \mathcal{S}^n$. If $\phi \in \Phi$, then

$$\phi \left(\frac{d(K_1, u)}{t} \right) + \dots + \phi \left(\frac{d(K_m, u)}{t} \right) = 1$$

if and only if

$$d(\check{\ddagger}_\phi(K_1, \dots, K_m), u) = t$$

Proof This follows immediately from Definition 3.1. \square

Lemma 3.5 Suppose $K_1, \dots, K_m \in \mathcal{S}^n$. If $\phi \in \Phi$, then

$$\frac{r}{\phi^{-1}(\frac{1}{m})} \leq d(\check{\ddagger}_\phi(K_1, \dots, K_m), u) \leq \frac{R}{\phi^{-1}(\frac{1}{m})}.$$

Proof Suppose $d(\check{\ddagger}_\phi(K_1, \dots, K_m), u) = t$, from Lemma 3.4 and noting that ϕ is strictly decreasing on $(0, \infty)$, we have

$$\begin{aligned} 1 &= \phi\left(\frac{d(K_1, u)}{t}\right) + \dots + \phi\left(\frac{d(K_m, u)}{t}\right) \\ &\leq \phi\left(\frac{r_{K_1}}{t}\right) + \dots + \phi\left(\frac{r_{K_m}}{t}\right) \\ &= m\phi\left(\frac{r}{t}\right). \end{aligned}$$

Noting that the inverse ϕ^{-1} is strictly decreasing on $(0, \infty)$, we obtain the lower bound for $d(\check{\ddagger}_\phi(K_1, \dots, K_m), u)$:

$$t \geq \frac{r}{\phi^{-1}(\frac{1}{m})}.$$

To obtain the upper estimate, observe the fact from the Lemma 3.4, together with the convexity and the fact ϕ is strictly decreasing on $(0, \infty)$, we have

$$\begin{aligned} 1 &= \phi\left(\frac{d(K_1, u)}{t}\right) + \dots + \phi\left(\frac{d(K_m, u)}{t}\right) \\ &\geq m\phi\left(\frac{d(K_1, u) + \dots + d(K_m, u)}{mt}\right) \\ &\geq m\phi\left(\frac{R}{t}\right). \end{aligned}$$

Then we obtain the upper estimate:

$$t \leq \frac{R}{\phi^{-1}(\frac{1}{m})}.$$

\square

Lemma 3.6 The Orlicz chord addition $\check{\ddagger}_\phi : (\mathcal{S}^n)^m \rightarrow \mathcal{S}^n$ is continuous.

Proof To see this, indeed, let $K_{ij} \in \mathcal{S}^n$, $i \in \mathbb{N} \cup \{0\}$, $j = 1, \dots, m$, be such that $K_{ij} \rightarrow K_{0j}$ as $i \rightarrow \infty$. Let

$$d(\check{\ddagger}_\phi(K_{i1}, \dots, K_{im}), u) = t_i.$$

Then Lemma 3.5 shows

$$\frac{r_{ij}}{\phi^{-1}(\frac{1}{m})} \leq t_i \leq \frac{R_{ij}}{\phi^{-1}(\frac{1}{m})},$$

where $r_{ij} = \min\{r_{K_{ij}}\}$ and $R_{ij} = \max\{R_{K_{ij}}\}$. Since $K_{ij} \rightarrow K_{0j}$, we have $R_{K_{ij}} \rightarrow R_{K_{0j}} < \infty$ and $r_{K_{ij}} \rightarrow r_{K_{0j}} > 0$, and thus there exist a, b such that $0 < a \leq t_i \leq b < \infty$ for all i . To show that the bounded

sequence $\{t_i\}$ converges to $d(\check{\times}_\phi(K_{01}, \dots, K_{0m}), u)$, we show that every convergent subsequence of $\{t_i\}$ converges to $d(\check{\times}_\phi(K_{01}, \dots, K_{0m}), u)$. Denote any subsequence of $\{t_i\}$ by $\{t_i\}$ as well, and suppose that for this subsequence, we have

$$t_i \rightarrow t_*.$$

Obviously $a \leq t_* \leq b$. Noting that ϕ is a continuous function, we obtain

$$\begin{aligned} t_* &\rightarrow \sup \left\{ t_* > 0 : \phi \left(\frac{d(K_{01}, u)}{t_*}, \dots, \frac{d(K_{0m}, u)}{t_*} \right) \leq 1 \right\} \\ &= d(\check{\times}_\phi(K_{01}, \dots, K_{0m}), u). \end{aligned}$$

Hence

$$d(\check{\times}_\phi(K_{i1}, \dots, K_{im}), u) \rightarrow d(\check{\times}_\phi(K_{01}, \dots, K_{0m}), u)$$

as $i \rightarrow \infty$.

This shows that the Orlicz chord addition $\check{\times}_\phi : (\mathcal{S}^n)^m \rightarrow \mathcal{S}^n$ is continuous. \square

Next, we define the Orlicz chord linear combination for the case $m = 2$.

Definition 3.7 (The Orlicz chord linear combination) The Orlicz chord linear combination, denoted by $\check{\times}_\phi(K, L, \alpha, \beta)$ for $K, L \in \mathcal{S}^n$, and $\alpha, \beta \geq 0$ (not both zero), is defined by

$$\alpha \cdot \phi_1 \left(\frac{d(K, u)}{d(\check{\times}_\phi(K, L, \alpha, \beta), u)} \right) + \beta \cdot \phi_2 \left(\frac{d(L, u)}{d(\check{\times}_\phi(K, L, \alpha, \beta), u)} \right) = 1, \quad (3.4)$$

for $\phi_1, \phi_2 \in \Phi$ and all $u \in S^{n-1}$.

We shall write $K\check{\times}_{\phi, \varepsilon}L$ instead of $\check{\times}_\phi(K, L, 1, \varepsilon)$, for $\varepsilon \geq 0$ and assume throughout that this is defined by (3.1), if $\alpha = 1, \beta = \varepsilon$ and $\phi \in \Phi$. We shall write $K\check{\times}_\phi L$ instead of $\check{\times}_\phi(K, L, 1, 1)$ and call it the Orlicz chord addition of K and L .

4. Orlicz mixed chord integrals

In order to define Orlicz mixed chord integrals, we need the following Lemmas 4.1-4.4.

Lemma 4.1 Let $\phi \in \Phi$ and $\varepsilon > 0$. If $K, L \in \mathcal{S}^n$, then $K\check{\times}_{\phi, \varepsilon}L \in \mathcal{S}^n$.

Proof Let $u_0 \in S^{n-1}$, and $\{u_i\} \subset S^{n-1}$ be any subsequence such that $u_i \rightarrow u_0$ as $i \rightarrow \infty$.

Let

$$d(K\check{\times}_{\phi, \varepsilon}L, u_i) = \lambda_i.$$

Then Lemma 3.5 shows

$$\frac{r}{\phi^{-1}(\frac{1}{2})} \leq \lambda_i \leq \frac{R}{\phi^{-1}(\frac{1}{2})},$$

where $R = \max\{R_K, R_L\}$ and $r = \min\{r_K, r_L\}$.

Since $K, L \in \mathcal{S}^n$, we have $0 < r_K \leq R_K < \infty$ and $0 < r_L \leq R_L < \infty$, and thus there exist a, b such that $0 < a \leq \lambda_i \leq b < \infty$ for all i . To show that the bounded sequence $\{\lambda_i\}$ converges to $d(K\check{\times}_{\phi, \varepsilon}L, u_0)$, we show that every convergent subsequence of $\{\lambda_i\}$ converges to $d(K\check{\times}_{\phi, \varepsilon}L, u_0)$. Denote any subsequence of $\{\lambda_i\}$ by $\{\lambda_i\}$ as well, and suppose that for this subsequence, we have

$$\lambda_i \rightarrow \lambda_0.$$

Obviously $a \leq \lambda_0 \leq b$. From (3.4) and note that ϕ_1, ϕ_2 are continuous functions, so ϕ_1^{-1} is continuous, we obtain

$$\lambda_i \rightarrow \frac{d(K, u_0)}{\phi_1^{-1}\left(1 - \varepsilon\phi_2\left(\frac{d(L, u_0)}{\lambda_0}\right)\right)}$$

as $i \rightarrow \infty$. Hence

$$\phi_1\left(\frac{d(K, u_0)}{\lambda_0}\right) + \varepsilon\phi_2\left(\frac{d(L, u_0)}{\lambda_0}\right) = 1.$$

Therefore

$$\lambda_0 = d(K \check{\times}_{\phi} \varepsilon \cdot L, u_0).$$

That is

$$d(K \check{\times}_{\phi} \varepsilon \cdot L, u_i) \rightarrow d(K \check{\times}_{\phi} \varepsilon \cdot L, u_0).$$

as $i \rightarrow \infty$.

This shows that $K \check{\times}_{\phi} \varepsilon \cdot L \in \mathcal{S}^n$. □

Lemma 4.2 *If $K, L \in \mathcal{S}^n$, $\varepsilon > 0$ and $\phi \in \Phi$, then*

$$K \check{\times}_{\phi} \varepsilon \cdot L \rightarrow K \tag{4.1}$$

as $\varepsilon \rightarrow 0^+$.

Proof This follows immediately from (3.4). □

Lemma 4.3 *If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $\phi_1, \phi_2 \in \Phi$, then*

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0^+} d(K \check{\times}_{\phi} \varepsilon \cdot L, u)^{n-i} = \frac{n-i}{(\phi_1)'_r(1)} \cdot \phi_2\left(\frac{d(L, u)}{d(K, u)}\right) \cdot d(K, u)^{n-i}. \tag{4.2}$$

Proof From (3.4), Lemma 4.2 and notice that ϕ_1^{-1}, ϕ_2 are continuous functions, we obtain for $0 \leq i < n$

$$\begin{aligned} & \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0^+} d(K \check{\times}_{\phi} \varepsilon \cdot L, u)^{n-i} \\ &= \lim_{\varepsilon \rightarrow 0^+} (n-i) d(K, u)^{n-i-1} \left(d(K, u) \phi_2\left(\frac{d(L, u)}{d(K \check{\times}_{\phi} \varepsilon \cdot L, u)}\right) \right) \\ & \times \lim_{y \rightarrow 1^-} \frac{\phi_1^{-1}(y) - \phi_1^{-1}(1)}{y - 1} \\ &= \frac{n-i}{(\phi_1)'_r(1)} \cdot \phi_2\left(\frac{d(L, u)}{d(K, u)}\right) \cdot d(K, u)^{n-i}, \end{aligned}$$

where

$$y = 1 - \varepsilon\phi_2\left(\frac{d(L, u)}{d(K \check{\times}_{\phi} \varepsilon \cdot L, u)}\right),$$

and note that $y \rightarrow 1^-$ as $\varepsilon \rightarrow 0^+$. □

Lemma 4.4 *If $\phi \in \Phi_2$, $0 \leq i < n$ and $K, L \in \mathcal{S}^n$, then*

$$\frac{(\phi_1)'_r(1)}{n-i} \cdot \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0^+} B_i(K \check{\times}_{\phi} \varepsilon \cdot L) = \frac{1}{n} \int_{S^{n-1}} \phi_2\left(\frac{d(L, u)}{d(K, u)}\right) \cdot d(K, u)^{n-i} dS(u). \tag{4.3}$$

Proof This follows immediately from (2.1) and Lemma 4.3. \square

Denoting by $B_{\phi,i}(K, L)$, for any $\phi \in \Phi$ and $0 \leq i < n$, the integral on the right-hand side of (4.3) with ϕ_2 replaced by ϕ , we see that either side of the equation (4.3) is equal to $B_{\phi_2,i}(K, L)$ and hence this new Orlicz mixed chord integrals $B_{\phi,i}(K, L)$ has been born.

Definition 4.5 (The Orlicz mixed chord integral) For $\phi \in \Phi$ and $0 \leq i < n$, Orlicz mixed chord integral of star bodies K and L , $B_{\phi,i}(K, L)$, is defined by

$$B_{\phi,i}(K, L) =: \frac{1}{n} \int_{S^{n-1}} \phi \left(\frac{d(L, u)}{d(K, u)} \right) \cdot d(K, u)^{n-i} dS(u). \quad (4.4)$$

Lemma 4.6 If $\phi_1, \phi_2 \in \Phi$, $0 \leq i < n$ and $K, L \in \mathcal{S}^n$, then

$$B_{\phi_2,i}(K, L) = \frac{(\phi_1)'_r(1)}{n-i} \lim_{\varepsilon \rightarrow 0^+} \frac{B_i(K \check{\times}_{\phi} \varepsilon \cdot L) - B_i(K)}{\varepsilon}. \quad (4.5)$$

Proof This follows immediately from Lemma 4.4 and (4.4). \square

Lemma 4.7 If $K, L \in \mathcal{S}^n$, $\phi \in \Phi$ and any $A \in SL(n)$, then for $\varepsilon > 0$

$$A(K \check{\times}_{\phi} \varepsilon \cdot L) = (AK) \check{\times}_{\phi} \varepsilon \cdot (AL). \quad (4.6)$$

Proof This follows immediately from (2.1) and (3.3). \square

We find easily that $B_{\phi,i}(K, L)$ is invariant under simultaneous unimodular centro-affine transformation.

Lemma 4.8 If $\phi \in \Phi$, $0 \leq i < n$ and $K, L \in \mathcal{S}^n$, then for $A \in SL(n)$,

$$B_{\phi,i}(AK, AL) = B_{\phi,i}(K, L). \quad (4.7)$$

Proof This follows immediately from Lemmas 4.6 and 4.7. \square

5. Orlicz chord Minkowski inequality

In this section, we will define a Borel measure in S^{n-1} , denoted by $B_{n,i}(K, \nu)$, which we shall call the chord measure of star body K .

Definition 5.1 (The chord measure) Let $K \in \mathcal{S}^n$ and $0 \leq i < n$, the chord measure of star body K , denoted by $B_{n,i}(K, \nu)$, is defined by

$$dB_{n,i}(K, \nu) = \frac{1}{n} \cdot \frac{d(K, \nu)^{n-i}}{B_i(K)} dS(\nu). \quad (5.1)$$

Lemma 5.2 (Jensen's inequality) Let μ be a probability measure on a space X and $g : X \rightarrow I \subset \mathbb{R}$ be a μ -integrable function, where I is a possibly infinite interval. If $\psi : I \rightarrow \mathbb{R}$ is a convex function, then

$$\int_X \psi(g(x)) d\mu(x) \geq \psi \left(\int_X g(x) d\mu(x) \right). \quad (5.2)$$

If ψ is strictly convex, the equality holds if and only if $g(x)$ is constant for μ -almost all $x \in X$ (see [34, p. 165]).

Lemma 5.3 Suppose that $\phi : [0, \infty) \rightarrow (0, \infty)$ is decreasing and convex with $\phi(0) = \infty$. If $K, L \in \mathcal{S}^n$ and $0 \leq i < n$, then

$$\frac{1}{nB_i(K)} \int_{S^{n-1}} \phi \left(\frac{d(L, u)}{d(K, u)} \right) d(K, u)^{n-i} dS(u) \geq \phi \left(\left(\frac{B_i(L)}{B_i(K)} \right)^{1/(n-i)} \right). \quad (5.3)$$

If ϕ is strictly convex, the equality holds if and only if K and L are similar chord.

Proof For $K \in \mathcal{S}^{n-1}$, $0 \leq i < n$ and any $u \in S^{n-1}$, the chord measure $\frac{d(K, u)^{n-i}}{nB_i(K)} dS(u)$ is a probability measure on S^{n-1} . Hence, from (2.4), (2.5), (5.1) and by using Jensen's inequality, and in view of ϕ is decreasing, we obtain

$$\begin{aligned} & \frac{1}{nB_i(K)} \int_{S^{n-1}} \phi \left(\frac{d(L, u)}{d(K, u)} \right) d(K, u)^{n-i} dS(u) \\ &= \int_{S^{n-1}} \phi \left(\frac{d(L, u)}{d(K, u)} \right) dB_{n,i}(K, u) \\ &\geq \phi \left(\frac{B_i(K, L)}{B_i(K)} \right) \\ &\geq \phi \left(\left(\frac{B_i(L)}{B_i(K)} \right)^{1/(n-i)} \right). \end{aligned}$$

Next, we discuss the equality in (5.3). If ϕ is strictly convex, suppose the equality holds in (5.3), form the equality necessary conditions of Jensen's inequality and (2.5), it follows that $d(L, u)/d(K, u)$ is constant, and K and L are similar chord, respectively. This yields that there exists $r > 0$ such that $d(L, u) = rd(K, u)$, for all $u \in S^{n-1}$. On the other hand, suppose that K and L are similar chord, i.e. there exists $\lambda > 0$ such that $d(L, u) = \lambda d(K, u)$ for all $u \in S^{n-1}$. Hence

$$\begin{aligned} & \frac{1}{nB_i(K)} \int_{S^{n-1}} \phi \left(\frac{d(L, u)}{d(K, u)} \right) d(K, u)^{n-i} dS(u) \\ &= \frac{1}{nB_i(K)} \int_{S^{n-1}} \phi \left(\left(\frac{B_i(L)}{B_i(K)} \right)^{1/(n-i)} \right) d(K, u)^{n-i} dS(u) \\ &= \phi \left(\left(\frac{B_i(L)}{B_i(K)} \right)^{1/(n-i)} \right). \end{aligned}$$

This implies the equality in (5.3) holds. □

Theorem 5.4 (Orlicz chord Minkowski inequality) If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $\phi \in \Phi$, then

$$B_{\phi,i}(K, L) \geq B_i(K) \cdot \phi \left(\left(\frac{B_i(L)}{B_i(K)} \right)^{1/(n-i)} \right). \quad (5.4)$$

If ϕ is strictly convex, the equality holds if and only if K and L are similar chord.

Proof This follows immediately from (4.4) and Lemma 5.3. □

Corollary 5.5 *If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, then*

$$B_{p,i}(K, L)^{n-i} \geq B_i(K)^{n-i+p} B_i(L)^{-p}, \quad (5.5)$$

with equality if and only if K and L are similar chord.

Proof This follows immediately from Theorem 5.4 with $\phi_1(t) = \phi_2(t) = t^{-p}$ and $p \geq 1$. \square

Taking $i = 0$ in (5.5), this yields L_p -Minkowski inequality: If $K, L \in \mathcal{S}^n$ and $p \geq 1$, then

$$B_p(K, L)^n \geq B(K)^{n+p} B(L)^{-p},$$

with equality if and only if K and L are similar chord.

Corollary 5.6 *Let $K, L \in \mathcal{M} \subset \mathcal{S}^n$, $0 \leq i < n$ and $\phi \in \Phi$, and if either*

$$B_{\phi,i}(Q, K) = B_{\phi,i}(Q, L), \text{ for all } Q \in \mathcal{M} \quad (5.6)$$

or

$$\frac{B_{\phi,i}(K, Q)}{B_i(K)} = \frac{B_{\phi,i}(L, Q)}{B_i(L)}, \text{ for all } Q \in \mathcal{M}, \quad (5.7)$$

then $K = L$.

Proof Suppose (5.6) holds. Taking K for Q , then from (2.3), (4.4) and (5.3), we obtain

$$B_i(K) = B_{\phi,i}(K, L) \geq B_i(K) \phi \left(\left(\frac{B_i(L)}{B_i(K)} \right)^{1/(n-i)} \right)$$

with equality if and only if K and L are similar chord. Hence

$$B_i(K) \leq B_i(L),$$

with equality if and only if K and L are similar chord. On the other hand, if taking L for Q , by similar arguments, we get $B_i(K) \geq B_i(L)$, with equality if and only if K and L are similar chord. Hence $B_i(K) = B_i(L)$, and K and L are similar chord, it follows that K and L must be equal.

Suppose (5.7) holds. Taking L for Q , then from from (2.3), (4.4) and (5.3), we obtain

$$1 = \frac{B_{\phi,i}(K, L)}{B_i(K)} \geq \phi \left(\left(\frac{B_i(L)}{B_i(K)} \right)^{1/(n-i)} \right),$$

with equality if and only if K and L are similar chord. Hence

$$B_i(K) \leq B_i(L),$$

with equality if and only if K and L are similar chord. On the other hand, if taking K for Q , by similar arguments, we get $B_i(K) \geq B_i(L)$, with equality if and only if K and L are similar chord. Hence $B_i(K) = B_i(L)$, and K and L have similar chord, it follows that K and L must be equal. \square

When $\phi_1(t) = \phi_2(t) = t^{-p}$ and $p \geq 1$, Corollary 5.6 becomes the following result.

Corollary 5.7 *Let $K, L \in \mathcal{M} \subset \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, and if either*

$$B_{p,i}(K, Q) = B_{p,i}(L, Q), \text{ for all } Q \in \mathcal{M}$$

or

$$\frac{B_{p,i}(K, Q)}{B_i(K)} = \frac{B_{p,i}(L, Q)}{B_i(L)}, \text{ for all } Q \in \mathcal{M},$$

then $K = L$.

6. Orlicz chord Brunn-Minkowski inequality

Lemma 6.1 *If $K, L \in \mathcal{S}^n$, $0 \leq i < n$, and $\phi_1, \phi_2 \in \Phi$, then*

$$B_i(K \check{\times}_{\phi} L) = B_{\phi_1, i}(K \check{\times}_{\phi} L, K) + B_{\phi_2, i}(K \check{\times}_{\phi} L, L). \quad (6.1)$$

Proof From (3.1), (3.4) and (4.4), we have for $K \check{\times}_{\phi} L = Q \in \mathcal{S}^n$

$$\begin{aligned} & B_{\phi_1, i}(Q, K) + B_{\phi_2, i}(Q, L) \\ &= \frac{1}{n} \int_{\mathcal{S}^{n-1}} \phi \left(\frac{d(K, u)}{d(Q, u)}, \frac{d(L, u)}{d(Q, u)} \right) d(Q, u)^{n-i} dS(u) \\ &= B_i(Q). \end{aligned} \quad (6.2)$$

The completes the proof. \square

Lemma 6.2 *Let $K, L \in \mathcal{S}^n$, $\varepsilon > 0$ and $\phi \in \Phi$.*

(1) *If K and L are similar chord, then K and $K \check{\times}_{\phi \varepsilon} \cdot L$ are similar chord.*

(2) *If K and $K \check{\times}_{\phi \varepsilon} \cdot L$ are similar chord, then K and L are similar chord.*

Proof Suppose exist a constant $\lambda > 0$ such that $d(L, u) = \lambda d(K, u)$, we have

$$\phi \left(\frac{d(K, u)}{d(K \check{\times}_{\phi \varepsilon} \cdot L, u)} \right) + \varepsilon \phi \left(\frac{\lambda d(K, u)}{d(K \check{\times}_{\phi \varepsilon} \cdot L, u)} \right) = 1.$$

On the other hand, the exist unique constant $\delta > 0$ such that

$$\phi \left(\frac{d(K, u)}{d(\delta K, u)} \right) + \varepsilon \phi \left(\frac{\lambda d(K, u)}{d(\delta K, u)} \right) = 1,$$

where δ satisfies that

$$\phi \left(\frac{1}{\delta} \right) + \varepsilon \phi \left(\frac{\lambda}{\delta} \right) = 1.$$

This shows that $d(K \check{\times}_{\phi \varepsilon} \cdot L, u) = \delta d(K, u)$.

Suppose exist a constant $\lambda > 0$ such that $d(K \check{\times}_{\phi \varepsilon} \cdot L, u) = \lambda d(K, u)$. Then

$$\phi \left(\frac{1}{\lambda} \right) + \varepsilon \phi \left(\frac{d(L, u)}{d(K \check{\times}_{\phi \varepsilon} \cdot L, u)} \right) = 1.$$

This shows that

$$\frac{d(L, u)}{d(K \check{\times}_{\phi \varepsilon} \cdot L, u)}$$

is a constant. This yields that $K \check{\times}_{\phi \varepsilon} \cdot L$ and L are similar chord. Namely K and L are similar chord. \square

Theorem 6.3 (Orlicz chord Brunn-Minkowski inequality) *If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $\phi \in \Phi_2$, then*

$$1 \geq \phi \left(\left(\frac{B_i(K)}{B_i(K \check{\times}_{\phi} L)} \right)^{1/(n-i)}, \left(\frac{B_i(L)}{B_i(K \check{\times}_{\phi} L)} \right)^{1/(n-i)} \right). \quad (6.3)$$

If ϕ is strictly convex, the equality holds if and only if K and L are similar chord.

Proof From (5.4) and Lemma 6.1, we have

$$\begin{aligned} B_i(K \check{+}_\phi L) &= B_{\phi_1, i}(K \check{+}_\phi L, K) + B_{\phi_2, i}(K \check{+}_\phi L, L) \\ &\geq B_i(K \check{+}_\phi L) \left(\phi_1 \left(\left(\frac{B_i(K)}{B_i(K \check{+}_\phi L)} \right)^{1/(n-i)} \right) + \phi_2 \left(\left(\frac{B_i(L)}{k B_i(K \check{+}_\phi L)} \right)^{1/(n-i)} \right) \right) \\ &= B_i(K \check{+}_\phi L) \phi \left(\left(\frac{B_i(K)}{B_i(K \check{+}_\phi L)} \right)^{1/(n-i)}, \left(\frac{B_i(L)}{B_i(K \check{+}_\phi L)} \right)^{1/(n-i)} \right). \end{aligned}$$

This is just inequality (6.3). From the equality condition of (5.4) and Lemma 6.3, it yields that if ϕ is strictly convex, equality in (6.3) holds if and only if K and L are similar chord. \square

Corollary 6.4 *If $K, L \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, then*

$$B_i(K \check{+}_p L)^{-p/(n-i)} \geq B_i(K)^{-p/(n-i)} + B_i(L)^{-p/(n-i)}, \quad (6.4)$$

with equality if and only if K and L are similar chord.

Proof This follows immediately from Theorem 6.2 with $\phi(x_1, x_2) = x_1^{-p} + x_2^{-p}$ and $p \geq 1$. \square

Taking $i = 0$ in (6.4), this yields the L_p -Brunn-Minkowski inequality for the chord integrals. If $K, L \in \mathcal{S}^n$ and $p \geq 1$, then

$$B(K \check{+}_p L)^{-p/n} \geq B(K)^{-p/n} + B(L)^{-p/n},$$

with equality if and only if K and L are similar chord.

7. The isoperimetric inequality for chord integrals

As an application, in this section, we give a new isoperimetric inequality for chord integrals. As we all know, the isoperimetric inequality for convex bodies can be stated below (see e.g. [26], p. 318).

The isoperimetric inequality *If K is a convex body in \mathbb{R}^n , then*

$$\left(\frac{V(K)}{V(B)} \right)^{n-1} \leq \left(\frac{S(K)}{S(B)} \right)^n, \quad (7.1)$$

with equality if and only if K is an n -ball.

Here B is the unit ball centered at the origin, $V(K)$ denotes the volume of K and $S(K)$ is the surface area of K , defined by (see [26], p. 318)

$$S(K) = \lim_{\varepsilon \rightarrow 0} \frac{V(K + \varepsilon B) - V(K)}{\varepsilon} = nV_1(K, B),$$

where $+$ is the usual Minkowski sum. Here, the mixed volume of convex bodies K and L , $V_1(K, L)$, defined by (see e.g. [1])

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u) dS(K, u). \quad (7.2)$$

Next, we give some new isoperimetric inequalities for mixed chord integrals by using the Orlicz chord Minkowski inequality established in Section 5.

Theorem 7.1 (The L_p -isoperimetric inequality for mixed chord integrals) *If $K \in \mathcal{S}^n$, $0 \leq i < n$ and $p \geq 1$, then*

$$\left(\frac{\widetilde{B}_{p,i}(K)}{S(B)} \right)^{n-i} \geq \left(\frac{B_i(K)}{V(B)} \right)^{n-i+p}, \quad (7.3)$$

with equality if and only if K is an n -ball, where $\widetilde{B}_{p,i}(K) = nB_{p,i}(K, B)$.

Proof Putting $L = B$, $\phi(t) = t^{-p}$ and $p \geq 1$ in Orlicz chord Minkowski inequality (5.4)

$$\frac{B_{p,i}(K, B)}{B_i(K)} \geq \left(\frac{B_i(B)}{B_i(K)} \right)^{-p/(n-i)}.$$

That is

$$\left(\frac{B_{p,i}(K, B)}{B_i(K)} \right)^{n-i} \geq \left(\frac{B_i(K)}{V(B)} \right)^p.$$

Hence

$$\left(\frac{nB_{p,i}(K, B)}{S(B)} \right)^{n-i} \geq \left(\frac{B_i(K)}{V(B)} \right)^{n-i+p}.$$

From the equality of (5.4), we find that the equality in (7.3) holds if and only if K and B are similar chord. This yields that the equality in (7.3) holds if and only if K is an n -ball. \square

Theorem 7.2 (The isoperimetric inequality for the chord integrals) *If $K \in \mathcal{S}^n$, then*

$$\left(\frac{\widetilde{B}(K)}{S(B)} \right)^n \geq \left(\frac{B(K)}{V(B)} \right)^{n+1}, \quad (7.4)$$

with equality if and only if K is an n -ball, where $\widetilde{B}(K) = nB_1(K, B)$.

Proof This follows immediately from (7.3) with $p = 1$ and $i = 0$. \square

This is just a similar form of the classical isoperimetric inequality (7.1).

8. Extensions

As extensions, in the Section, the Orlicz mixed chord integral of K and L , $B_\phi(K, L)$, is generalized into *Orlicz multiple mixed chord integral* of $(n + 1)$ star bodies L_1, K_1, \dots, K_n . Further, we generalize the Orlicz-Minkowski inequality into Orlicz-Aleksandrov-Fenchel inequality for the Orlicz multiple mixed chord integrals.

Theorem 8.1 *If $L_1, K_1, \dots, K_n \in \mathcal{S}^n$ and $\phi_1, \phi_2 \in \Phi$, then*

$$\begin{aligned} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0^+} B(L_1 \dot{+}_{\phi} \varepsilon \cdot K_1, K_2, \dots, K_n) &= \frac{1}{n(\phi_1)'_r(1)} \\ &\times \int_{S^{n-1}} \phi_2 \left(\frac{d(K_1, u)}{d(L_1, u)} \right) d(L_1, u) d(K_2, u) \cdots d(K_n, u) dS(u). \end{aligned} \quad (8.1)$$

Proof This may yield by using a generalized idea and method of proving Lemma 4.4. Here, we omit the details. \square

Obviously, (4.3) is a special case of (8.1). Moreover, from Theorem 8.1, we can find the following definition:

Definition 8.2 (Orlicz multiple mixed chord integrals) Let $L_1, K_1, \dots, K_n \in \mathcal{S}^n$ and $\phi \in \Phi$, the Orlicz multiple mixed chord integral of $(n + 1)$ star bodies L_1, K_1, \dots, K_n , is denoted by $B_\phi(L_1, K_1, \dots, K_n)$, is defined by

$$B_\phi(L_1, K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} \phi \left(\frac{d(K_1, u)}{d(L_1, u)} \right) d(L_1, u) d(K_2, u) \cdots d(K_n, u) dS(u). \quad (8.2)$$

When $L_1 = K_1$, $B_\phi(L_1, K_1, \dots, K_n)$ becomes the well-known mixed chord integral $B(K_1, \dots, K_n)$. Obviously, for $0 \leq i < n$, $B_{\phi,i}(K, L)$ is also a special case of $B_\phi(L_1, K_1, \dots, K_n)$.

Corollary 8.3 If $L_1, K_1, \dots, K_n \in \mathcal{S}^n$ and $\phi_1, \phi_2 \in \Phi$, then

$$B_{\phi_2}(L_1, K_1, \dots, K_n) = (\phi_1)'_r(1) \cdot \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} B(L_1 +_\phi \varepsilon \cdot K_1, K_2, \dots, K_n). \quad (8.3)$$

Proof This yields immediately from (8.1) and (8.2). \square

Similar to the proof of Theorem 5.4, we may establish an Orlicz-Aleksandrov-Fenchel inequality as follows:

Theorem 8.4 (Orlicz-Aleksandrov-Fenchel inequality for the Orlicz multiple mixed chord integrals) If $L_1, K_1, \dots, K_n \in \mathcal{S}^n$, $\phi \in \Phi$ and $1 \leq r \leq n$, then

$$B_\phi(L_1, K_1, K_2, \dots, K_n) \geq B(L_1, K_2, \dots, K_n) \cdot \phi \left(\frac{\prod_{i=1}^r B(K_i \dots, K_i, K_{r+1}, \dots, K_n)^{\frac{1}{r}}}{B(L_1, K_2, \dots, K_n)} \right). \quad (8.4)$$

If ϕ is strictly convex, equality holds if and only if L_1, K_1, \dots, K_r are all of similar chord.

Proof This yields immediately by using a generalized idea and method of proving Theorem 5.4. Here, we omit the details. \square

Obviously, the Orlicz-Minkowski inequality (5.4) is a special case of the Orlicz-Aleksandrov-Fenchel inequality (8.4). Moreover, when $L_1 = K_1$, (8.4) becomes the following Aleksandrov-Fenchel inequality for the mixed chords.

Corollary 8.5 (Aleksandrov-Fenchel inequality for the mixed chord integrals) If $K_1, \dots, K_n \in \mathcal{S}^n$ and $1 \leq r \leq n$, then

$$B(K_1, \dots, K_n) \leq \prod_{i=1}^r B(K_i \dots, K_i, K_{r+1}, \dots, K_n)^{\frac{1}{r}}. \quad (8.5)$$

with equality if and only if K_1, \dots, K_r are all of similar chord.

Finally, it is worth mentioning: when $\phi(t) = t^{-p}$ and $p \geq 1$, $B_\phi(L_1, K_1, \dots, K_n)$ written as $B_p(L_1, K_1, \dots, K_n)$ and call it L_p -multiple mixed chord integrals of $(n + 1)$ star bodies L_1, K_1, \dots, K_n . So, the new concept of L_p -multiple mixed chord integrals and L_p -Aleksandrov-Fenchel inequality for the L_p -multiple mixed chord integrals may be also derived. Here, we omit the details of all derivations.

Acknowledgments

Research is supported by National Natural Science Foundation of China (11371334, 10971205).

Conflict of interest

The author declares that no conflicts of interest in this paper.

References

1. R. J. Gardner, *Geometric Tomography*, Cambridge Univ. Press, New York, 1996.
2. G. Berck, *Convexity of L_p -intersection bodies*, *Adv. Math.*, **222** (2009), 920–936.
3. Y. D. Burago, V. A. Zalgaller, *Geometric Inequalities*, Springer-Verlag, Berlin, 1988.
4. C. Haberl, *L_p intersection bodies*, *Adv. Math.*, **217** (2008), 2599–2624.
5. C. Haberl, M. Ludwig, *A characterization of L_p intersection bodies*, *Int. Math. Res. Not.*, **2006** (2006), Art. ID 10548.
6. A. Koldobsky, *Fourier analysis in convex geometry*, Mathematical Surveys and Monographs 116, American Mathematical Society, Providence, RI, 2005.
7. M. Ludwig, *Intersection bodies and valuations*, *Amer. J. Math.*, **128** (2006), 1409–1428.
8. E. Lutwak, *Centroid bodies and dual mixed volumes*, *Proc. London Math. Soc.*, **3** (1990), 365–391.
9. E. M. Werner, *Rényi divergence and L_p -affine surface area for convex bodies*, *Adv. Math.*, **230** (2012), 1040–1059.
10. E. Lutwak, *Dual mixed volumes*, *Pacific J. Math.*, **58** (1975), 531–538.
11. R. J. Gardner, *A positive answer to the Busemann-Petty problem in three dimensions*, *Ann. Math.*, **140** (1994), 435–447.
12. R. J. Gardner, A. Koldobsky, T. Schlumprecht, *An analytic solution to the Busemann-Petty problem on sections of convex bodies*, *Ann. Math.*, **149** (1999), 691–703.
13. F. E. Schuster, *Valuations and Busemann-Petty type problems*, *Adv. Math.*, **219** (2008), 344–368.
14. E. Lutwak, *Intersection bodies and dual mixed volumes*, *Adv. Math.*, **71** (1988), 232–261.
15. R. J. Gardner, D. Hug, W. Weil, *The Orlicz-Brunn-Minkowski theory: a general framework, additions, and inequalities*, *J. Differ. Geom.*, **97** (2014), 427–476.
16. E. Lutwak, D. Yang, G. Zhang, *Orlicz projection bodies*, *Adv. Math.*, **223** (2010), 220–242.
17. E. Lutwak, D. Yang, G. Zhang, *Orlicz centroid bodies*, *J. Differ. Geom.*, **84** (2010), 365–387.
18. D. Xi, H. Jin, G. Leng, *The Orlicz Brunn-Minkowski inequality*, *Adv. Math.*, **260** (2014), 350–374.
19. B. He, Q. Huang, *On the Orlicz Minkowski problem for polytopes*, *Discrete Comput. Geom.*, **48** (2012), 281–297.
20. C. Haberl, E. Lutwak, D. Yang, et al., *The even Orlicz Minkowski problem*, *Adv. Math.*, **224** (2010), 2485–2510.
21. J. Li, D. Ma, *Laplace transforms and valuations*, *J. Func. Anal.*, **272** (2017), 738–758.
22. Y. Lin, *Affine Orlicz Pólya-Szegő principle for log-concave functions*, *J. Func. Anal.*, **273** (2017), 3295–3326.
23. C. J. Zhao, *On the Orlicz-Brunn-Minkowski theory*, *Balkan J. Geom. Appl.*, **22** (2017), 98–121.

24. C. J. Zhao, *Orlicz dual mixed volumes*, *Results Math.*, **68** (2015), 93–104.
25. C. J. Zhao, *Orlicz dual affine quermassintegrals*, *Forum Math.*, **30** (2018), 929–945.
26. C. J. Zhao, *The dual logarithmic Aleksandrov-Fenchel inequality*, *Balkan J. Geom. Appl.*, **25** (2020), 157–169.
27. C. J. Zhao, *Orlicz mixed affine surface areas*, *Balkan J. Geom. Appl.*, **24** (2019), 100–118.
28. C. J. Zhao, *Orlicz-Aleksandrov-Fenchel inequality for Orlicz multiple mixed volumes*, *J. Func. Spaces*, **2018** (2018), Ar. ID 9752178.
29. R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Second Edition, Cambridge Univ. Press, 2014.
30. F. Lu, *Mixed chord-integrals of star bodies*, *J. Korean Math. Soc.*, **47** (2010), 277–288.
31. A. D. Aleksandrov, *Zur Theorie der gemischten Volumina von konvexen Körpern, I: Verallgemeinerung einiger Begriffe der Theorie der konvexen Körper*, *Mat. Sbornik N. S.*, **2** (1937), 947–972.
32. W. Fenchel, B. Jessen, *Mengenfunktionen und konvexe Körper*, *Danske Vid Selskab Mat-fys Medd*, **16** (1938), 1–31.
33. E. Lutwak, *The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem*, *J. Differ. Geom.*, **38** (1993), 131–150.
34. J. Hoffmann-Jørgensen, *Probability With a View Toward Statistics*, Vol. I, Chapman and Hall, New York, 1994.



AIMS Press

© 2020 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)