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Research article

On some fractional integral inequalities for generalized strongly modified h-convex functions

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Abstract: Convex functions play an important role in many areas of mathematics. They are especially important in the study of optimization problems where they are distinguished by a number of convenient properties. Many generalizations of convex functions exists in literature. The main aim of the article is to develop fractional integral inequalities for generalized strongly modified h-convex functions. Based on obtained fractional type integral inequalities we give some applications to the means. Our results are extension and generalization of many existing results.

Keywords: generalized convex functions; strongly convex functions; modified h-convex function
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1. Introduction

Linear functions are considered as simplest functions in linear spaces. The class of functions and sets that are just a step more complicated then linear ones namely convex functions and convex sets.

The subset C of $\mathbb{R}^n$ is said to be convex if

$$px + qy \in C$$

$\forall x, y \in C$, $p \in (0, 1)$ and $q = 1 - p$. The function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if its epigraph is convex subset of $\mathbb{R}$.The convexity of sets and functions are the the objects of many studies during the past few decades. The convexity of a function and set make it so special because of its interesting properties like convex function has global minima, it has non-empty relative interior and convex set is connected having feasible directions at any point.

Some of early contributions to convex analysis were made by Holder, Jensen and Minkowski. The importance of convex analysis is well known in optimization theory [2,3], inspite of many applications, many recent problems in economics and engineering the notion of convexity does not longer suffices. Hence it is always necessary to extend the notion of convexity to some general form to meet recent problems see [4–10], for further reading on fractional integral inequalities we refer [11–17]. Moreover, the new inequalities in analysis is always appreciable. The present paper is organized as follow: in the second section, we give some preliminary material. In the third section, we derive some fractional integral inequalities for generalized strongly modified h-convex function, whereas in the fourth section. we present applications of results to the mean. Finally, we conclude our results.

2. Preliminaries

We start from some preliminaries material and basic definitions.

Definition 2.1. [18] Let $f : \varphi \to \mathbb{R}$ be an extended-real-valued function define on a convex set $\varphi \subset \mathbb{R}^n$. Then the function f is convex on φ if

$$f(tb_1 + (1-t)b_2) \leq tf(b_1) + (1-t)f(b_2), \tag{2.1}$$

for all $b_1, b_2 \in \varphi$ and $t \in (0, 1)$.

Definition 2.2. [19] Choose the functions $f, h : J \subset \mathbb{R} \to \mathbb{R}$ are non-negative. Then f is called h-convex function if

$$f(tb_1 + (1-t)b_2) \leq h(t)f(b_1) + h(1-t)f(b_2), \tag{2.2}$$

for all $b_1, b_2 \in J$ and $t \in [0, 1]$.

Definition 2.3. [20] Choose the functions $f, h : J \subset \mathbb{R} \to \mathbb{R}$ are non-negative. Then f is called modified h-convex function if

$$f(tb_1 + (1-t)b_2) \leq h(t)f(b_1) + (1-h(t))f(b_2), \tag{2.3}$$

for all $b_1, b_2 \in J$ and $t \in [0, 1]$.

Definition 2.4. [21] Let φ be an interval in real line $\mathbb{R}$. A function $f : \varphi = [b_1, b_2] \to \mathbb{R}$ is said to be generalized convex with respect to an arbitrary bifunction $\eta(b_1, b_2) : \mathbb{E} \times \mathbb{E} \to \mathbb{F}$ where $E, F \in \mathbb{R}$ if

$$f(tb_1 + (1-t)b_2) \leq f(b_2) + t\eta\left(f(b_1), f(b_2)\right), \tag{2.4}$$

for all $b_1, b_2 \in \varphi, t \in [0, 1]$.

Definition 2.5. A function $f : \varphi = [b_1, b_2] \to \mathbb{R}$ is called η_h convex function if

$$f(tb_1 + (1-t)b_2) \leq f(b_2) + h(t)\eta\left(f(b_1), f(b_2)\right), \tag{2.5}$$

for all $b_1, b_2 \in \varphi, t \in [0, 1]$ and $h : J \to \mathbb{R}$ is a non-negative function.

Definition 2.6. [22] A function $f : \varphi = [b_1, b_2] \to \mathbb{R}$ is called strongly convex function with modulus μ on φ, where $\mu \geq 0$ if

$$f(tb_1 + (1-t)b_2) \leq tf(b_1) + (1-t)f(b_2) - \mu t(1-t)(b_1 - b_2)^2, \tag{2.6}$$

for all $b_1, b_2 \in \varphi$ and $t \in [0, 1]$.

Definition 2.7. [23] A function $f : J \subset \mathbb{R} \to \mathbb{R}$ is said to be strongly η-convex function with respect to $\eta : \mathbb{E} \times \mathbb{E} \to \mathbb{F}$ where $E, F \in \mathbb{R}$ and modulus $\mu \geq 0$, if

$$f(tb_1 + (1-t)b_2) \leq f(b_2) + t\eta(f(b_1), f(b_2)) - \mu t(1-t)(b_1 - b_2)^2, \tag{2.7}$$

for all $b_1, b_2 \in J, t \in [0, 1]$.

Definition 2.8. [24] Choose the functions $f, h : J \subset \mathbb{R} \to \mathbb{R}$ are non-negative. Then f is called generalized strongly modified h-convex function if

$$f(tb_1 + (1-t)b_2) \leq f(b_2) + h(t)\eta(f(b_1), f(b_2)) - \mu t(1-t)(b_1 - b_2)^2, \tag{2.8}$$

for all $b_1, b_2 \in J$ and $t \in [0, 1]$.

Definition 2.9. [25] Let $0 < s \leq 1$. A function $f : J \subset \mathbb{R} \to \mathbb{R}$ is called s-ϕ *-convex* with respect to bifunction $\phi : \mathbb{E} \times \mathbb{E} \to \mathbb{F}$ where $E, F \in \mathbb{R}$ (briefly ϕ*-convex*) if

$$f(tb_1 + (1-t)b_2) \leq f(b_2) + t^s \phi\left(f(b_1), f(b_2)\right), \tag{2.9}$$

The next remark provides the relations among the convexities.

Remark 1. 1. If $\eta(b_1, b_2) = b_1 - b_2$ then,(2.4) reduces to (2.1);
2. If $h(t) = t$ then, (2.5) reduces to (2.4);
3. If $h(t) = t$ and $\eta(b_1, b_2) = b_1 - b_2$ then,(2.5) reduces to(2.1);
4. If $\eta(b_1, b_2) = b_1 - b_2$ then, (2.5) reduces to (2.3);
5. If $\mu = 0$ and $\eta(b_1, b_2) = b_1 - b_2$ then,(2.8) reduces to (2.3);
6. If $\mu = 0$, $\eta(b_1, b_2) = b_1 - b_2$ and $h(t) = t$ then,(2.8) reduces to(2.1);
7. If $\mu = 0$ then,(2.8) reduces to (2.5);
8. If $h(t) = t$ then, (2.8) reduces to (2.7);
9. If $\mu = 0$ and $h(t) = t^s$ then, (2.8) reduces to (2.9).

Utilization of more complicated convex functions

Most of the modern problems in engineering and other applied sciences are non-convex in nature. So it is difficult to reach at favorite results by only the classical convexity. That's why the convexity is generalized in many directions. To understand the generalization of convexity it may categorize as:

Some generalization are made to change the form of defining e.g. quasi convex [26], pseudo convex [27] and strongly convex [28].

Some generalizations are made by expanding the domain e.g. [29] and some generalization are made by changing the range set of convex functions e.g. [30]. So generalizations the convex is always appreciable.

The next lemmas are useful in proving the main results.

Lemma 2.10. *[31] Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on $\mathring{J}$ such that $f' \in L^1[b_1, b_2]$, where $b_1, b_2 \in J$ with $b_1 < b_2$. If $\alpha, \beta \in \mathbb{R}$, then*

$$\frac{\alpha f(b_1) + \beta f(b_2)}{2} + \frac{2-\alpha-\beta}{2} f\left(\frac{b_1+b_2}{2}\right) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx$$
$$= \frac{b_2-b_1}{4}\int_0^1 \left[(1-\alpha-t) f'\left(tb_1 + (1-t)\frac{b_1+b_2}{2}\right) + (\beta - t) f'\left(t\frac{b_1+b_2}{2} + (1-t) b_2\right)\right] \quad (2.10)$$

Lemma 2.11. *[31] For $s > 0$ and $0 \le \varepsilon \le 1$, we have*

$$\int_0^1 |\varepsilon - t|^s \, dt = \frac{\varepsilon^{s+1} + (1-\varepsilon)^{s+1}}{s+1}, \quad (2.11)$$
$$\int_0^1 t\,|\varepsilon - t|^s \, dt = \frac{\varepsilon^{s+2} + (s+1+\varepsilon)(1-\varepsilon)^{s+1}}{s+1} \quad (2.12)$$
$$\int_0^1 t^2\,|\varepsilon - t|^s = \frac{-2(\varepsilon - t)^{s+3} + (1-\varepsilon)^{s+1}(s+2)(s+3) - 2(1-\varepsilon)^{s+2}(s+3) + 2(t-\varepsilon)^{s+3}}{(s+1)(s+2)(s+3)}.$$

Lemma 2.12. *[32] Let $f : J \to \mathbb{R}$, $J \subseteq \mathbb{R}$ be a differentiable mapping on $\mathring{J}$ with $f'' \in L^1[b_1, b_2]$, where $b_1, b_2 \in J, b_1 < b_2$, then*

$$\frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx - f\left(\frac{b_1+b_2}{2}\right)$$
$$= \frac{(b_2-b_1)^2}{16}\left[\int_0^1 t^2 f''\left(t\frac{b_1+b_2}{2} + (1-t) b_1\right)dt + \int_0^1 (t-1)^2 f''\left(tb_2 + (1-t)\frac{b_1+b_2}{2}\right)dt\right]. \quad (2.13)$$

Lemma 2.13. *[23] If f^n for $n \epsilon \mathbb{N}$ exists and is integrable on $[b_1, b_2]$, then*

$$\frac{f(b_1) + f(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx - \Sigma_{k=2}^{n-1}\frac{(k-1)(b_2-b_1)^k}{2(k+1)!} f^{(k)}(b_1)$$
$$= \frac{(b_2-b_1)^n}{2n!}\int_0^1 t^{n-1}(n-2t) f^{(n)}(tb_1 + (1-t)b_2)\,dt. \quad (2.14)$$

Lemma 2.14. *[25] Suppose that $f : [b_1, b_2] \to \mathbb{R}$ is a differentiable function, $g : [b_1, b_2] \to \mathbb{R}^+$ is a continuous function and symmetric about $\frac{b_1+b_2}{2}$ and f' is an integrable function on $[b_1, b_2]$. Then*

$$\frac{f(b_1)+f(b_2)}{2}\int_{b_1}^{b_2} g(x)dx - \int_{b_1}^{b_2} f(x)g(x)dx$$
$$= \frac{b_2-b_1}{4}\left\{\int_0^1 \left(\int_{\frac{1+t}{2}b_1+\frac{1-t}{2}b_2}^{\frac{1-t}{2}b_1+\frac{1+t}{2}b_2} g(u)du\right) f'\left(\frac{1+t}{2}b_1 + \frac{1-t}{2}b_2\right)dt\right.$$
$$\left.+ \int_0^1 \left(\int_{\frac{1+t}{2}b_1+\frac{1-t}{2}b_2}^{\frac{1-t}{2}b_1+\frac{1+t}{2}b_2} g(u)du\right) f'\left(\frac{1-t}{2}b_1 + \frac{1+t}{2}b_2\right)dt\right\}.$$

3. Fractional integral inequalities

Theorem 3.1. *Let* $f : J \to \mathbb{R}$*,* $J \subseteq \mathbb{R}$ *be a differentiable mapping on* $\mathring{J}$ *with* $f' \in L^1[b_1, b_2]$*, where* $b_1, b_2 \in J, b_1 < b_2$*. If* $|f'(x)|^q$ *for* $q \geq 1$ *and* $0 \leq \alpha, \beta \leq 1$*, is generalized strongly modified h-convex function on* $[b_1, b_2]$*, then*

$$\left|\frac{\alpha f(b_1) + \beta f(b_2)}{2} + \frac{2-\alpha-\beta}{2} f(\frac{b_1+b_2}{2}) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx\right|$$
$$\leq \left(\frac{b_2-b_1}{8}\right)(2)^{\frac{1}{q}} \left\{\left(1-2\alpha+2\alpha^2\right)^{1-\frac{1}{q}} \left[\frac{1}{2}\left(1-2\alpha+2\alpha^2\right)|f'(b_2)|^q\right.\right.$$
$$+ \int_0^1 |1-\alpha-t| \left(h\left(\frac{1+t}{2}\right)\eta\left(|f'(b_1)|^q + |f'(b_2)|^q\right)\right)dt$$
$$\left.- \frac{\mu}{4}(b_1-b_2)^2\left(\frac{-2\alpha^4+8\alpha^3-8\alpha+5}{12}\right)\right]^{\frac{1}{q}} + \left(1-2\beta+2\beta^2\right)^{1-\frac{1}{q}}$$
$$\times\left[\frac{1}{2}\left(1-2\beta+2\beta^2\right)|f'(b_2)|^q + \int_0^1 |\beta-t|\, h(\frac{t}{2})\eta\left(|f'(b_1)|^q, |f'(b_2)|^q\right)dt\right.$$
$$\left.\left.- \frac{\mu}{4}(b_1-b_2)^2\left(\frac{-2\beta^4+8\beta^3-8\beta+5}{12}\right)\right]^{\frac{1}{q}}\right\} \tag{3.1}$$

Proof. The proof begins with $f'(x) \in [b_1, b_2]$, then using Lemma (2.10), and power mean inequality we have for $q > 1$

$$\left|\frac{\alpha f(b_1) + \beta f(b_2)}{2} + \frac{2-\alpha-\beta}{2} f(\frac{b_1+b_2}{2}) - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx\right|$$
$$\leq \frac{b_2-b_1}{4}\left[\int_0^1 |1-\alpha-t|\left|f'\left(tb_1 + (1-t)\frac{b_1+b_2}{2}\right)\right|dt + \int_0^1 |\beta-t|\left|f'\left(t\frac{b_1+b_2}{2} + (1-t)b_2\right)\right|dt\right]$$
$$\leq \frac{b_2-b_1}{4}\left\{\left(\int_0^1 |1-\alpha-t|\,dt\right)^{1-\frac{1}{q}}\left[\int_0^1 |1-\alpha-t|\left(|f'(b_2)|^q + h\left(\frac{1+t}{2}\right)\right.\right.\right.$$
$$\left.\left.\times\ \eta\left(|f'(b_1)|^q, |f'(b_2)|^q\right) - \mu\frac{1+t}{2}\left(1-\frac{1+t}{2}\right)(b_1-b_2)^2\,dt\right)\right]^{\frac{1}{q}} + \left(\int_0^1 |\beta-t|\,dt\right)^{1-\frac{1}{q}}$$
$$\left.\times\left[\int_0^1 |\beta-t|\left(|f'(b_2)|^q + h\left(\frac{t}{2}\right)\eta\left(|f'(b_1)|^q, |f'(b_2)|^q\right)\ -\mu\frac{t}{2}\left(1-\frac{t}{2}\right)(b_1-b_2)^2\,dt\right)\right]^{\frac{1}{q}}\right\}. \tag{3.2}$$

Using Lemma (2.11), we have

$$\mu(b_1-b_2)^2\int_0^1 |1-\alpha-t|\left(\frac{1+t}{2}\right)\left(\frac{1-t}{2}\right)dt = \frac{\mu}{4}(b_1-b_2)^2\left(\frac{-2\alpha^4+8\alpha^3-8\alpha+5}{12}\right). \tag{3.3}$$

And

$$\mu(b_1-b_2)^2\int_0^1 |\beta-t|\,\frac{t}{2}\left(1-\frac{t}{2}\right)dt = \frac{\mu}{4}(b_1-b_2)^2\left(\frac{-2\beta^4+8\beta^3-8\beta+5}{12}\right). \tag{3.4}$$

Substituting values from Eqs (3.3), (3.4) in inequality (3.2), we obtain

$$\left|\frac{\alpha f(b_1)+\beta f(b_2)}{2}+\frac{2-\alpha-\beta}{2}f(\frac{b_1+b_2}{2})-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx\right|$$
$$\leq\left(\frac{b_2-b_1}{8}\right)(2)^{\frac{1}{q}}\left\{\left(1-2\alpha+2\alpha^2\right)^{1-\frac{1}{q}}\left[\frac{1}{2}\left(1-2\alpha+2\alpha^2\right)|f'(b_2)|^q\right.\right.$$
$$+\int_0^1|1-\alpha-t|\left(h\left(\frac{1+t}{2}\right)\eta\left(|f'(b_1)|^q+|f'(b_2)|^q\right)\right)dt$$
$$\left.-\frac{\mu}{4}(b_1-b_2)^2\left(\frac{-2\alpha^4+8\alpha^3-8\alpha+5}{12}\right)\right]^{\frac{1}{q}}+\left(1-2\beta+2\beta^2\right)^{1-\frac{1}{q}}$$
$$\times\left[\frac{1}{2}\left(1-2\beta+2\beta^2\right)|f'(b_2)|^q+\int_0^1|\beta-t|\,h(\frac{t}{2})\eta\left(|f'(b_1)|^q,|f'(b_2)|^q\right)dt\right.$$
$$\left.\left.-\frac{\mu}{4}(b_1-b_2)^2\left(\frac{-2\beta^4+8\beta^3-8\beta+5}{12}\right)\right]^{\frac{1}{q}}\right\}.$$

For $q=1$, using Lemma (2.10) and Lemma (2.11), we have

$$\left|\frac{\alpha f(b_1)+\beta f(b_2)}{2}+\frac{2-\alpha-\beta}{2}f(\frac{b_1+b_2}{2})-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx\right|$$
$$\leq\left(\frac{b_2-b_1}{4}\right)\left\{\frac{1}{2}\left(1-2\alpha+2\alpha^2\right)|f'(b_2)|\right.$$
$$+\int_0^1|1-\alpha-t|\left(h\left(\frac{1+t}{2}\right)\eta\left(|f'(b_1)|,|f'(b_2)|\right)\right)dt-\frac{\mu}{4}(b_1-b_2)^2\left(\frac{-2\alpha^4+8\alpha^3-8\alpha+5}{12}\right)$$
$$+\frac{1}{2}\left(1-2\beta+2\beta^2\right)|f'(b_2)|+\int_0^1|\beta-t|\,h(\frac{t}{2})\eta\left(|f'(b_1)|,|f'(b_2)|\right)dt$$
$$\left.-\frac{\mu}{4}(b_1-b_2)^2\left(\frac{-2\beta^4+8\beta^3-8\beta+5}{12}\right)\right\}, \tag{3.5}$$

This completes the proof. □

Remark 2. If we take $h(t)=t$ and $\mu=0$ then inequality (3.1) reduces to inequality (13) in [33].

Taking $\alpha=\beta$ in Theorem (3.1), we have following corollary.

Corollary 1. *Let $f:J\to\mathbb{R}$, $J\subseteq\mathbb{R}$ be a differentiable mapping on $\mathring{J}$ with $f'\in L^1[b_1,b_2]$, where $b_1,b_2\in J, b_1<b_2$. If $|f'(x)|^q$ for $q\geq 1$ is generalized strongly modified h-convex function on $[b_1,b_2]$ and $0\leq\alpha,\beta\leq 1$, then*

$$
\begin{aligned}
&\left|\frac{\alpha}{2}\left[f(b_1)+f(b_2)\right]+(1-\alpha)f\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx\right|\\
&\leq\left(\frac{b_2-b_1}{4}\right)\left(\frac{1-2\alpha+2\alpha^2}{2}\right)^{1-\frac{1}{q}}\left\{\left[\left(\frac{1-2\alpha+2\alpha^2}{2}\right)|f'(b_2)|^q+\int_0^1|1-\alpha-t|\right.\right.\\
&\times h\left(\frac{1+t}{2}\right)\eta\left(|f'(b_1)|^q,|f'(b_2)|^q\right)dt-\frac{\mu}{4}(b_1-b_2)^2\left(\frac{-2\alpha^4+8\alpha^3-8\alpha+5}{12}\right)\Bigg]^{\frac{1}{q}}+\Bigg[\left(\frac{1-2\alpha+2\alpha^2}{2}\right)\\
&\left.\times\left(|f'(b_2)|^q\right)+\int_0^1|\alpha-t|\,h(\frac{t}{2})\eta\left(|f'(b_1)|^q,|f'(b_2)|^q\right)dt-\frac{\mu}{4}(b_1-b_2)^2\left(\frac{-2\alpha^4+8\alpha^3-8\alpha+5}{12}\right)\Bigg]^{\frac{1}{q}}\right\}\\
&=\left(\frac{b_2-b_1}{8}\right)(2)^{\frac{1}{q}}\left(1-2\alpha+2\alpha^2\right)^{1-\frac{1}{q}}\left\{\left[\left(\frac{1-2\alpha+2\alpha^2}{2}\right)|f'(b_2)|^q+\int_0^1|1-\alpha-t|\right.\right.\\
&h\left(\frac{1+t}{2}\right)\eta\left(|f'(b_1)|^q,\times|f'(b_2)|^q\right)dt-\frac{\mu}{4}(b_1-b_2)^2\left(\frac{-2\alpha^4+8\alpha^3-8\alpha+5}{12}\right)\Bigg]^{\frac{1}{q}}+\Bigg[\left(\frac{1-2\alpha+2\alpha^2}{2}\right)\\
&\left.\times\left(|f'(b_2)|^q\right)+\int_0^1|\alpha-t|\,h(\frac{t}{2})\eta\left(|f'(b_1)|^q,|f'(b_2)|^q\right)dt-\frac{\mu}{4}(b_1-b_2)^2\left(\frac{-2\alpha^4+8\alpha^3-8\alpha+5}{12}\right)\Bigg]^{\frac{1}{q}}\right\}.
\end{aligned}
\tag{3.6}
$$

Remark 3. If we take $h(t) = t$ and $\mu = 0$ then inequality (3.6) reduces to inequality (16) in [33].

By choosing $\alpha = \beta = \frac{1}{2}, \frac{1}{3}$ in Theorem (3.1) respectively, we obtain following corollary.

Corollary 2. *Let $f : J \to \mathbb{R}$, $J \subseteq \mathbb{R}$ be a differentiable mapping on $\mathring{J}$ with $f' \in L^1[b_1, b_2]$, where $b_1, b_2 \in J, b_1 < b_2$. If $|f'(x)|^q$ for $q \geq 1$ is generalized strongly modified h-convex function on $[b_1, b_2]$ and $0 \leq \alpha, \beta \leq 1$, then*

$$
\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(b_1)+f(b_2)}{2}+f\left(\frac{b_1+b_2}{2}\right)\right]-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx\right|\\
&\leq\left(\frac{b_2-b_1}{16}\right)(2)^{\frac{2}{q}}\left\{\left[\int_0^1\left|\frac{1}{2}-t\right|h\left(\frac{1+t}{2}\right)\eta\left(|f'(b_1)|^q,|f'(b_2)|^q\right)dt\right.\right.\\
&+\frac{1}{4}|f'(b_2)|^q-\frac{\mu}{4}(b_1-b_2)^2\left(\frac{5}{128}\right)\Bigg]^{\frac{1}{q}}+\Bigg[\frac{1}{4}\left(|f'(b_2)|^q\right)+\int_0^1\left|\frac{1}{2}-t\right|\\
&\left.\times h\left(\frac{t}{2}\right)\eta\left(|f'(b_1)|^q,|f'(b_2)|^q\right)dt-\frac{\mu}{4}(b_1-b_2)^2\left(\frac{5}{128}\right)\Bigg]^{\frac{1}{q}}\right\},
\end{aligned}
\tag{3.7}
$$

and

$$\left|\frac{1}{6}\left[f(b_1)+f(b_2)+4f\left(\frac{b_1+b_2}{2}\right)\right]-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx\right|$$
$$\leq\frac{5}{72}(b_2-b_1)\left(\frac{18}{5}\right)^{\frac{1}{q}}\left\{\left[\frac{5}{18}|f'(b_2)|^q+\int_0^1\left|\frac{2}{3}-t\right|\left(h\left(\frac{1+t}{2}\right)\eta\left(|f'(b_1)|^q,|f'(b_2)|^q\right)\right)dt\right.\right.$$
$$\left.\left.-\frac{211}{324}\mu(b_1-b_2)^2\right]^{\frac{1}{q}}+\left[\frac{5}{18}|f'(b_2)|^q+\int_0^1\left|\frac{1}{3}-t\right|\left(h\left(\frac{t}{2}\right)\eta\left(|f'(b_1)|^q,|f'(b_2)|^q\right)\right)dt\ -\frac{211}{324}\mu(b_1-b_2)^2\right]^{\frac{1}{q}}\right\}. \tag{3.8}$$

Remark 4. Setting $q = 1$ in Corollary (2), we have the following result.

Corollary 3. *Let $f : J \to \mathbb{R}$, $J \subseteq \mathbb{R}$ be a differentiable mapping on $\mathring{J}$ with $f' \in L^1[b_1, b_2]$, where $b_1, b_2 \in I, b_1 < b_2$. If $|f'(x)|^q$ for $q \geq 1$ is generalized strongly modified h-convex function on $[b_1, b_2]$ and $0 \leq \alpha, \beta \leq 1$, then*

$$\left|\frac{1}{2}\left[\frac{f(b_1)+f(b_2)}{2}+f\left(\frac{b_1+b_2}{2}\right)\right]-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx\right|$$
$$\leq\left(\frac{b_2-b_1}{4}\right)\left\{\frac{1}{2}|f'(b_2)|+\eta\left(|f'(b_1)|,|f'(b_2)|\right)\int_0^1\left[h\left(\frac{1+t}{2}\right)+h\left(\frac{t}{2}\right)\right]\left|\frac{1}{2}-t\right|dt-\frac{\mu}{2}(b_1-b_2)^2\left(\frac{5}{128}\right)\right\}, \tag{3.9}$$

and

$$\left|\frac{1}{6}\left[f(b_1)+f(b_2)+4f\left(\frac{b_1+b_2}{2}\right)\right]-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx\right|$$
$$\leq\left(\frac{b_2-b_1}{4}\right)\left\{\frac{5}{9}|f'(b_2)|+\eta\left(|f'(b_1)|,|f'(b_2)|\right)\left[\int_0^1 h\left(\frac{1+t}{2}\right)\left|\frac{2}{3}-t\right|dt\right.\right.$$
$$\left.\left.+\int_0^1 h\left(\frac{t}{2}\right)\left|\frac{1}{3}-t\right|dt-\frac{211}{162}\mu(b_1-b_2)^2\right]\right\}. \tag{3.10}$$

Remark 5. If we take $h(t) = t$ and $\mu = 0$ then inequalities (3.7)–(3.10) reduce to inequalities (17) and (18) in [33].

Theorem 3.2. *Let $f : J \subset [0, 1) \to \mathbb{R}$ be a differentiable mapping on $\mathring{J}$ with $f'' \in L^1[b_1, b_2]$, where $b_1, b_2 \in J$ and $b_1 < b_2$. If $|f''|$ is genarilized strongly modified h-convex on $[b_1, b_2]$, then*

$$\left|f\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx\right|\leq\frac{(b_2-b_1)^2}{16}\left\{\left[\frac{1}{3}|f''(b_1)|+\int_0^1 t^2h(t)\right.\right.$$
$$\left.\eta\left(\left|f''\left(\frac{b_1+b_2}{2}\right)\right|,|f''(b_1)|\right)dt-\frac{1}{20}\mu\left(\frac{b_1+b_2}{2}-b_1\right)^2\right]+\left[\frac{1}{3}\left|f''\left(\frac{b_1+b_2}{2}\right)\right|\right.$$
$$\left.\left.+\int_0^1(t-1)^2h(t)\eta\left(|f''(b_2)|,\left|f''\left(\frac{b_1+b_2}{2}\right)\right|\right)dt\ -\frac{1}{20}\mu\left(a_2-\frac{b_1+b_2}{2}\right)\right]\right\}. \tag{3.11}$$

Proof. From Lemma (2.12), we have

$$\left|f\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx\right|$$
$$\leq\frac{(b_2-b_1)^2}{16}\left[\int_0^1 t^2\left|f''\left(t\frac{b_1+b_2}{2}+(1-t)\,b_1\right)\right|dt+\int_0^1(t-1)^2\left|f''\left(tb_2+(1-t)\,\frac{b_1+b_2}{2}\right)\right|dt\right].$$

Since $|f''|$ is generalized strongly modified h convex function, so

$$\left|f\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx\right|$$
$$\leq\frac{(b_2-b_1)^2}{16}\left[\int_0^1 t^2\left(|f''(b_1)|\;+h(t)\eta\left(\left|f''\left(\frac{b_1+b_2}{2}\right)\right|,|f''(b_1)|\right)-\mu(t)\,(1-t)\left(\frac{b_1+b_2}{2}-b_1\right)^2\right)dt\right.$$
$$\left.+\int_0^1(t-1)^2\left(\left|f''\left(\frac{b_1+b_2}{2}\right)\right|+h(t)\eta\left(|f''(b_2)|,\left|f''\left(\frac{b_1+b_2}{2}\right)\right|\right)\;-\mu(t)\,(1-t)\left(b_2-\frac{b_1+b_2}{2}\right)^2\right)dt\right]$$
$$=\frac{(b_2-b_1)^2}{16}\left[\frac{1}{3}\,|f''(b_1)|+\int_0^1 t^2h(t)\eta\left(\left|f''\left(\frac{b_1+b_2}{2}\right)\right|,|f''(b_1)|\right)dt\right.$$
$$-\mu\left(\frac{b_1+b_2-2b_1}{2}\right)^2\int_0^1 t^3\,(1-t)\,dt+\frac{1}{3}\left|f''\left(\frac{b_1+b_2}{2}\right)\right|+\int_0^1(t-1)^2$$
$$\left.\times h(t)\eta\left(|f''(b_2)|,\left|f''\left(\frac{b_1+b_2}{2}\right)\right|\right)dt+\mu\left(\frac{2b_2-b_1-b_2}{2}\right)^2\int_0^1(t-1)^3\,tdt\right].\tag{3.12}$$

And

$$\left|f\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx\right|$$
$$\leq\frac{(b_2-b_1)^2}{16}\left\{\left[\frac{1}{3}\,|f''(b_1)|+\int_0^1 t^2h(t)\;\;\eta\left(\left|f''\left(\frac{b_1+b_2}{2}\right)\right|,|f''(b_1)|\right)dt-\frac{1}{20}\mu\left(\frac{b_1+b_2}{2}-b_1\right)^2\right]\right.$$
$$\left.+\left[\frac{1}{3}\left|f''\left(\frac{b_1+b_2}{2}\right)\right|\;+\int_o^1(t-1)^2\,h(t)\eta\left(|f''(b_2)|,\left|f''\left(\frac{b_1+b_2}{2}\right)\right|\right)dt-\frac{1}{20}\mu\left(b_2-\frac{b_1+b_2}{2}\right)\right]\right\}.$$

This completes the proof. □

Remark 6. If we take $h(t)=t$ and $\mu=0$ then inequality (3.11) reduces to inequality (24) in [33].

Theorem 3.3. *Let $f:J\subset[0,1)\to\mathbb{R}$ be a differentiable mapping on $\mathring{J}$ with $f''\in L^1[b_1,b_2]$, where $b_1,b_2\in J$ and $b_1<b_2$.If $|f''|^q$ for $q\geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$ is generelized strongly modified h-convx on $[b_1,b_2]$, then*

$$\left|f\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx\right|$$
$$\leq\frac{(b_2-b_1)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left\{\left(\frac{1}{3}\left|f''(b_1)\right|^q+\int_0^1 t^2h(t)\eta\left(\left|f''\left(\frac{b_1+b_2}{2}\right)\right|^q,\left|f''(b_1)\right|^q\right)dt-\frac{1}{20}\mu\left(\frac{b_2-b_1}{2}\right)^2\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{3}\left|f''\left(\frac{b_1+b_2}{2}\right)\right|^q+\int_0^1(t-1)^2h(t)\ \eta\left(\left|f''(b_2)\right|^q,\left|f''\left(\frac{b_1+b_2}{2}\right)\right|^q\right)dt-\frac{1}{20}\mu\left(\frac{b_2-b_1}{2}\right)^2\right)^{\frac{1}{q}}\right\}. \tag{3.13}$$

Proof. Suppose that p≥ 1 , using Lemma (2.12) and power mean inequality, we have

$$\left|f\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx\right|$$
$$\leq\frac{(b_2-b_1)^2}{16}\left[\int_0^1 t^2\left|f''\left(t\frac{b_1+b_2}{2}+(1-t)b_1\right)\right|dt+\int_0^1(t-1)^2\left|f''\left(tb_2+(1-t)\frac{b_1+b_2}{2}\right)\right|dt\right]$$
$$\leq\frac{(b_2-b_1)^2}{16}\left(\int_0^1 t^2dt\right)^{\frac{1}{p}}\left(\int_0^1 t^2\left|f''\left(t\frac{b_1+b_2}{2}+(1-t)b_1\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b_2-b_1)^2}{16}\left(\int_0^1(t-1)^2dt\right)^{\frac{1}{p}}\left(\int_0^1(t-1)^2\left|f''\left(tb_2+(1-t)\frac{b_1+b_2}{2}\right)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $|f''|^q$ is generalized strongly modified h-convex, then we have

$$\int_0^1 t^2\left|f''\left(t\frac{b_1+b_2}{2}+(1-t)b_1\right)\right|^q dt$$
$$\leq\int_0^1 t^2f''\left|b_1\right|^q dt+\int_0^1 t^2h(t)\eta\left(\left|f''\left(\frac{b_1+b_2}{2}\right)\right|^q,\left|f''(b_1)\right|^q\right)dt-\int_0^1\mu(1-t)t^2t\left(\frac{b_1+b_2}{2}-b_1\right)^2dt$$
$$\leq\frac{1}{3}f''\left|b_1\right|^q+\int_0^1 t^2h(t)\eta\left(\left|f''\left(\frac{b_1+b_2}{2}\right)\right|^q,\left|f''(b_1)\right|^q\right)dt-\frac{1}{20}\mu\left(\frac{b_2-b_1}{2}\right)^2.$$

And

$$\int_0^1(t-1)^2\left|f''\left(tb_2+(1-t)\frac{b_1+b_2}{2}\right)\right|^q dt$$
$$\leq\int_0^1(t-1)^2\left|f''\left(\frac{b_1+b_2}{2}\right)\right|^q dt+\int_0^1(t-1)^2h(t)\eta\left(\left|f''(b_2)\right|^q,\left|f''\left(\frac{b_1+b_2}{2}\right)\right|^q\right)dt$$
$$-\int_0^1\mu(t-1)^2t(1-t)\left(b_2-\frac{b_1+b_2}{2}\right)^2dt$$
$$\leq\frac{1}{3}\left|f''\left(\frac{b_1+b_2}{2}\right)\right|^q+\int_0^1(t-1)^2h(t)\eta\left(\left|f''(b_2)\right|^q,\left|f''\left(\frac{b_1+b_2}{2}\right)\right|^q\right)dt-\frac{1}{20}\mu\left(\frac{b_2-b_1}{2}\right)^2.$$

After simplification, we have

$$\left|f\left(\frac{b_1+b_2}{2}\right)-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx\right|$$
$$\leq\frac{(b_2-b_1)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left\{\left(\frac{1}{3}|f''(b_1)|^q+\int_0^1 t^2h(t)\eta\left(\left|f''\left(\frac{b_1+b_2}{2}\right)\right|^q,|f''(b_1)|^q\right)dt-\frac{1}{20}\mu\left(\frac{b_2-b_1}{2}\right)^2\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{3}\left|f''\left(\frac{b_1+b_2}{2}\right)\right|^q+\int_0^1(t-1)^2h(t)\eta\left(|f''(b_2)|^q,\left|f''\left(\frac{b_1+b_2}{2}\right)\right|^q\right)dt-\frac{1}{20}\mu\left(\frac{b_2-b_1}{2}\right)^2\right)^{\frac{1}{q}}\right\}.$$

Which completes the proof. □

Remark 7. If we take $h(t)=t$ and $\mu=0$, then inequality (3.13) reduces to inequality (25) in [33].

Theorem 3.4. *Let $f:\mathring{J}\subset R\to R$ be a n-times differentiable generalized strongly modified h-convex, function on $\mathring{J}$ where $b_1,b_2\in\mathring{J}$ with $b_1<b_2$ and $f'\in L^1[b_1,b_2]$. If $\left|f'\right|^p$ is generalized strongly modified h-convex, function with $\mu\geq 1$, then for $n\geq 2$ and $p\geq 1$, we have*

$$\left|\frac{f(b_1)+f(b_2)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx-\Sigma_{k=2}^{n-1}\frac{(k-1)(b_2-b_1)^k}{2(k+1)!}f^{(k)}(b_1)\right|$$
$$\leq\frac{(b_2-b_1)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{p}}\left[\frac{n-1}{n+1}\left|f^{(n)}(b_2)\right|^p+\int_0^1 h(t)t^{n-1}(n-2t)dt\eta\left(\left|f^{(n)}(b_1)\right|^p,|f^n(b_2)|^p\right)\right.$$
$$\left.-\mu\frac{(n-1)}{(n+1)(n+3)}(x-y)^2\right].\tag{3.14}$$

Proof. **Case-i:** Since it is known that $\left|f'\right|$ is generalized strongly modified h-convex function, then using the property of modules, and Lemma (2.13), we have following inequality for $p=1$

$$\left|\frac{f(b_1)+f(b_2)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx-\Sigma_{k=2}^{n-1}\frac{(k-1)(b_2-b_1)^k}{2(k+1)!}f^{(k)}(b_1)\right|$$
$$\leq\frac{(b_2-b_1)^n}{2n!}\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(tb_1+(1-t)b_2)\right|dt.\tag{3.15}$$

Using the definition of generalized strongly modified h-convex function, we have

$$\left|\frac{f(b_1)+f(b_2)}{2}-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}f(x)dx-\Sigma_{k=2}^{n-1}\frac{(k-1)(b_2-b_1)^k}{2(k+1)!}f^{(k)}(b_1)\right|$$
$$\leq\frac{(b_2-b_1)^n}{2!}\int_0^1(n-2t)\left[|f^n(b_2)|+h(t)\eta(|f^n(b_1)|,|f^n(b_2)|)-\mu(x-y)^2t(1-t)\right]dt$$
$$\leq\frac{(b_2-b_1)^n}{2n!}\left[|f^n(b_2)|\int_0^1 t^{n-1}(n-2t)dt+\eta(|f^n(b_1)|,|f^n(b_2)|)\int_0^1 h(t)t^{n-1}(n-2t)dt\right.$$
$$\left.-\mu(x-y)^2\int_0^1 t^n(1-t)(n-2t)dt\right].\tag{3.16}$$

As

$$\int_0^1 t^{n-1}(n-2t)dt = \frac{n-1}{n+1} \tag{3.17}$$

$$\int_0^1 (1-t)(n-2t)dt = \frac{n-1}{(n+1)(n+3)}. \tag{3.18}$$

Substituting (3.17) and (3.18) in (3.16), we have

$$\begin{aligned}
&\left|\frac{f(b_1)+f(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx - \Sigma_{k=2}^{n-1}\frac{(k-1)(b_2-b_1)^k}{2(k+1)!}f^{(k)}(b_1)\right| \\
&\leq \frac{(b_2-b_1)^n}{2n!}\left[\frac{n-1}{n+1}\left|f^{(n)}(b_2)\right| + \eta\left(|f^n(b_1)|, |f^n(b_2)|\right)\int_0^1 h(t)t^{n-1}(n-2t)dt\right. \\
&\left.-\mu\frac{n-1}{(n+1)(n+3)}(x+y)^2\right].
\end{aligned}$$

Case-ii For $p > 1$ applying Holder inequality, we have

$$\begin{aligned}
&\left|\frac{f(b_1)+f(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx - \Sigma_{k=2}^{n-1}\frac{(k-1)(b_2-b_1)^k}{2(k+1)!}f^{(k)}(b_1)\right| \\
&\leq \frac{(b_2-b_1)^n}{2n!}\left[\int_0^1 t^{n-1}(n-2t)dt\right]^{1-\frac{1}{p}}\left[\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(tb_1+(1-t)b_2)\right|^p dt\right]^{\frac{1}{p}}.
\end{aligned}$$

Using definition of generalized strongly modified h-convex function, we have

$$\begin{aligned}
&\left|\frac{f(b_1)+f(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx - \Sigma_{k=2}^{n-1}\frac{(k-1)(b_2-b_1)^k}{2(k+1)!}f^{(k)}(b_1)\right| \\
&\leq \frac{(b_2-b_1)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{p}}\left[\left|f^{(n)}(b_2)\right|^p\int_0^1 t^{n-1}(n-2t)dt + \eta\left(|f^n(b_1)|, |f^n(b_2)|\right)\right. \\
&\left.\int_0^1 h(t)t^{n-1}(n-2t)dt - \mu(x-y)^2\int_0^1 t^n(n-2t)(1-t)dt\right] \\
&\times\left|\frac{f(b_1)+f(b_2)}{2} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(x)dx - \Sigma_{k=2}^{n-1}\frac{(k-1)(b_2-b_1)^k}{2(k+1)!}f^{(k)}(b_1)\right| \\
&\leq \frac{(b_2-b_1)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{p}}\left[\frac{n-1}{n+1}\left|f^{(n)}(b_2)\right|^P + \int_0^1 h(t)t^{n-1}(n-2t)dt\eta\left(\left|f^{(n)}(b_1)\right|^p, |f^n(b_2)|^p\right)\right. \\
&\left.-\mu\frac{(n-1)}{(n+1)(n+3)}(x-y)^2\right].
\end{aligned}$$

Which completes the proof. □

Remark 8. If we take $h(t) = t$ then Theorem (3.13) reduces to Theorem (2.5) in [23].

Theorem 3.5. *Suppose that $f : [b_1, b_2] \to \mathbb{R}$ is a differentiable function, $g : [b_1, b_2] \to \mathbb{R}^+$ is a continuous function and symmetric about $\frac{b_1+b_2}{2}$ and $|f'|$ is a generalized strongly modified h-convex function. Then*

$$\left|\frac{f(b_1)+f(b_2)}{2}\int_{b_1}^{b_2} g(x)dx - \int_{b_1}^{b_2} f(x)g(x)dx\right|$$
$$\leq \frac{b_2-b_1}{4}\left[2|f'(b_2)| + K\eta(|f'(b_1)|, |f'(b_2)|) - \frac{\mu}{2}(1-t^2)(b_1-b_2)^2\right]\int_0^1 \int_{\frac{1+t}{2}b_1+\frac{1-t}{2}b_2}^{\frac{1-t}{2}b_1+\frac{1+t}{2}b_2} g(u)dudt \tag{3.19}$$

where $k = max_{t\in[0,1]}|g(t)|$ and $g(t) = h\left(\frac{1-t}{2}\right) + h\left(\frac{1+t}{2}\right)$.

Proof. From Lemma (2.14) and the fact that $|f'|$ is generalized strongly modified h-convex, we have

$$\left|\frac{f(b_1)+f(b_2)}{2}\int_{b_1}^{b_2} g(x)dx - \int_{b_1}^{b_2} f(x)g(x)dx\right|$$
$$\leq \frac{b_2-b_1}{4}\int_0^1 \int_{\frac{1+t}{2}b_1+\frac{1-t}{2}b_2}^{\frac{1-t}{2}b_1+\frac{1+t}{2}b_2} g(u)\left[\left|f'\left(\frac{1+t}{2}b_1+\frac{1-t}{2}b_2\right)\right| + \left|f'\left(\frac{1-t}{2}b_1+\frac{1+t}{2}b_2\right)\right|\right]dudt$$
$$\leq \frac{b_2-b_1}{4}\int_0^1 \int_{\frac{1+t}{2}b_1+\frac{1-t}{2}b_2}^{\frac{1-t}{2}b_1+\frac{1+t}{2}b_2} g(u)\left[2|f'(b_2)| + h\left(\frac{1+t}{2}\right)\eta(|f'(b_1)|, |f'(b_2)|)\right.$$
$$\left.-2\mu\left(\frac{1+t}{2}\right)\left(\frac{1-t}{2}\right)(b_1-b_2)^2 + h\left(\frac{1-t}{2}\right)\eta(|f'(b_1)|, |f'(b_2)|)\right]dudt$$
$$= \frac{b_2-b_1}{4}\int_0^1 \int_{\frac{1+t}{2}b_1+\frac{1-t}{2}b_2}^{\frac{1-t}{2}b_1+\frac{1+t}{2}b_2} g(u)\left(2|f'(b_2)| + \left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]\eta(|f'(b_1)|, |f'(b_2)|)\right.$$
$$\left.-\frac{\mu}{2}(1-t^2)(b_1-b_2)^2\right)dudt$$
$$= \frac{b_2-b_1}{4}\left(2|f'(b_2)| + \left[h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right]\eta(|f'(b_1)|, |f'(b_2)|) - \frac{\mu}{2}(1-t^2)(b_1-b_2)^2\right)$$
$$\int_0^1 \int_{\frac{1+t}{2}b_1+\frac{1-t}{2}b_2}^{\frac{1-t}{2}b_1+\frac{1+t}{2}b_2} g(u)dudt$$
$$\leq \frac{b_2-b_1}{4}\left[2|f'(b_2)| + K\eta(|f'(b_1)|, |f'(b_2)|) - \frac{\mu}{2}(1-t^2)(b_1-b_2)^2\right]\int_0^1 \int_{\frac{1+t}{2}b_1+\frac{1-t}{2}b_2}^{\frac{1-t}{2}b_1+\frac{1+t}{2}b_2} g(u)dudt$$

where $k = max_{t\in[0,1]}|g(t)|$ and $g(t) = h\left(\frac{1-t}{2}\right) + h\left(\frac{1+t}{2}\right)$. □

Remark 9. If we take $\mu = 0, h(t) = t^s$ then Theorem (3.15) reduces to Theorem (3) in [25].

Corollary 4. *In theorem (3.15) if we choose $h(t) = t$ then $k = 1$ and $\mu = 0$, we have the inequality of the theorem (2) (Gordji, Dragomir and Delaver).*

$$\left|\frac{f(b_1)+f(b_2)}{2}\int_{b_1}^{b_2} g(x)dx - \int_{b_1}^{b_2} f(x)g(x)dx\right|$$
$$\leq \frac{b_2-b_1}{4}\left[2|f'(b_2)| + \eta(|f'(b_1)|, |f'(b_2)|)\right]\int_0^1 \int_{\frac{1+t}{2}b_1+\frac{1-t}{2}b_2}^{\frac{1-t}{2}b_1+\frac{1+t}{2}b_2} g(u)dudt. \tag{3.20}$$

Corollary 5. *In corollary (3.17) if we choose $g = 1, \eta(x, y) = x - y$, we have the following inequality*

$$\left|\frac{f(b_1) + f(b_2)}{2} - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} f(x)dx\right| \leq \frac{b_2 - b_1}{8}\left[f'(b_1) + f'(b_2)\right]. \tag{3.21}$$

for convex functions that is equivalent to Theorem (1.2) in [1].

4. Application to means

For two positive numbers $b_1 > 0$ and $b_2 > 0$, define

$$\begin{cases} A(b_1, b_2) = \frac{b_1+b_2}{2}, \\ G(b_1, b_2) = \sqrt{b_1 b_2}, \\ H(b_1, b_2) = \frac{2b_1 b_2}{b_1+b_2}, \\ L(b_1, b_2) = \begin{cases} \left[\frac{b_2^{s+1}-b_1^{s+1}}{(s+1)(b_2-b_1)}\right]^{\frac{1}{s}}, & b_1 \neq b_2 \\ b_1, & b_1 = b_2, \end{cases} \\ I(b_1, b_2) = \begin{cases} \frac{1}{e}\left(\frac{b_2^{b_2}}{b_1^{b_1}}\right)^{\frac{1}{b_2-b_1}}, & b_1 \neq b_2 \\ b_1, & b_1 = b_2 \end{cases} \\ H_{w,s}(b_1, b_2) = \begin{cases} \left[\frac{b_1^s + w(b_1 b_2)^{\frac{s}{2}} + b_2^s}{w+2}\right]^{\frac{1}{s}}, & s \neq 0 \\ \sqrt{b_1 b_2}, & s = 0 \end{cases} \end{cases} \tag{4.1}$$

for $0 \leq w \leq \infty$. These means are respectively called the arithematic, geometric, harmonic, generalized logarithmic, identric and Heronian means of two positive numbers b_1 and b_2.

Applying Theorem (3.1) to $f(x) = x^s$ for $s \neq 0$ and $x > 0$ result in the following inequalities for means.

Theorem 4.1. *Let $b_1 > 0$, $b_2 > 0$, $q \geq 1$, either $s > 1$ and $(s-1)q \geq 1$ or $s < 0$. Then*

$$\begin{aligned} &\left|A(\alpha b_1^s, \beta b_2^s) + \frac{2-\alpha-\beta}{2}A^s(b_1, b_2) - L^s(b_1, b_2)\right| \\ &\leq \left(\frac{b_2 - b_1}{8}\right)(2)^{\frac{1}{q}}\left\{\left(1 - 2\alpha + 2\alpha^2\right)^{1-\frac{1}{q}}\left[\frac{1}{2}\left(1 - 2\alpha + 2\alpha^2\right)\left|sb_2^{s-1}\right|^q\right.\right. \\ &+ \int_0^1 |1 - \alpha - t|\left(h\left(\frac{1+t}{2}\right)\eta\left(\left|sb_1^{s-1}\right|^q, \left|sb_2^{s-1}\right|^q\right)\right)dt \\ &\left.- \frac{\mu}{4}(b_1 - b_2)^2\left(\frac{-2\alpha^4 + 8\alpha^3 - 8\alpha + 5}{12}\right)\right]^{\frac{1}{q}} + \left(1 - 2\beta + 2\beta^2\right)^{1-\frac{1}{q}} \\ &\times\left[\frac{1}{2}\left(1 - 2\beta + 2\beta^2\right)\left|sb_2^{s-1}\right|^q + \int_0^1 |\beta - t|\, h(\frac{t}{2})\eta\left(\left|sb_1^{s-1}\right|^q, \left|sb_2^{s-1}\right|^q\right)dt\right. \\ &\left.\left.- \frac{\mu}{4}(b_1 - b_2)^2\left(\frac{-2\beta^4 + 8\beta^3 - 8\beta + 5}{12}\right)\right]^{\frac{1}{q}}\right\}. \end{aligned} \tag{4.2}$$

Taking $f(x) = lnx$ for $x > 0$ in Theorem (3.1) results in the following inequality for means.

Theorem 4.2. *For $b_1 > 0$, $b_2 > 0$, $b_1 \neq b_2$ and $q \geq 1$, we have*

$$\begin{aligned}
&\left|\frac{lnG^2(b_1^{\alpha}, b_2^{\beta})}{2} + \frac{2-\alpha-\beta}{2} lnA(b_1, b_2) - lnI(b_1, b_2)\right| \\
&\leq \left(\frac{b_2 - b_1}{8}\right)(2)^{\frac{1}{q}} \left\{\left(1 - 2\alpha + 2\alpha^2\right)^{1-\frac{1}{q}} \left[\frac{1}{2}\left(1 - 2\alpha + 2\alpha^2\right)\left|\frac{1}{b_2}\right|^q\right.\right. \\
&+ \int_0^1 |1 - \alpha - t| \left(h\left(\frac{1+t}{2}\right)\eta\left(\left|\frac{1}{b_1}\right|^q, \left|\frac{1}{b_2}\right|^q\right)\right) dt \\
&\left. - \frac{\mu}{4}(b_1 - b_2)^2\left(\frac{-2\alpha^4 + 8\alpha^3 - 8\alpha + 5}{12}\right)\right]^{\frac{1}{q}} + \left(1 - 2\beta + 2\beta^2\right)^{1-\frac{1}{q}} \\
&\times \left[\frac{1}{2}\left(1 - 2\beta + 2\beta^2\right)\left|\frac{1}{b_2}\right|^q + \int_0^1 |\beta - t|\, h(\frac{t}{2})\eta\left(\left|\frac{1}{b_1}\right|^q, \left|\frac{1}{b_2}\right|^q\right) dt\right. \\
&\left.\left. - \frac{\mu}{4}(b_1 - b_2)^2\left(\frac{-2\beta^4 + 8\beta^3 - 8\beta + 5}{12}\right)\right]^{\frac{1}{q}}\right\}. \qquad (4.3)
\end{aligned}$$

Finally, we can establish an inequality for the Heronian mean as follows.

Theorem 4.3. *For $b_2 > b_1 > 0$, $b_1 \neq b_2$ $w \geq 0$ and $s \geq 4$ or $0 \neq s < 1$, we have*

$$\begin{aligned}
&\left|\frac{H^s_{w,s}(b_1, b_2)}{H(b_1^s, b_2^s)} + H^{\frac{s}{2}+1}_{w,(\frac{s}{2}+1)}\left(\frac{b_2}{b_1} + \frac{b_1}{b_2}, 1\right) - H^s_{w,s}\left(\frac{L(b_1^2, b_2^2)}{G^2(b_1, b_2)}, 1\right)\right| \\
&\leq \frac{(b_2 - b_1)A(b_1, b_2)}{2G^2(b_1, b_2)} \left\{\frac{\frac{1}{2}|s|}{w+2}\left(G^{2(s-1)}\left(b_2, \frac{1}{b_1}\right) + \frac{w}{2}G^{s-\frac{1}{2}}\left(b_2, \frac{1}{b_1}\right)\right)\right. \\
&+\eta\left(\frac{|s|}{w+2}\left(G^{2(s-1)}\left(b_1, \frac{1}{b_2}\right) + \frac{w}{2}G^{s-\frac{1}{2}}\left(b_1, \frac{1}{b_2}\right)\right), \frac{|s|}{w+2}\left(G^{2(s-1)}\left(b_2, \frac{1}{b_1}\right) + \frac{w}{2}G^{s-\frac{1}{2}}\left(b_2, \frac{1}{b_1}\right)\right)\right) \\
&\left.\times \int_0^1 \left[h\left(\frac{1+t}{2}\right) + h(\frac{t}{2})\right]\left|\frac{1}{2} - t\right| dt - (\frac{5}{64})\mu\left(\frac{(b_1 - b_2)A(b_1, b_2)}{G^2(b_1, b_2)}\right)^2\right\}. \qquad (4.4)
\end{aligned}$$

Proof. Let $f(x) = \frac{x^s + wx^{\frac{s}{2}} + 1}{w+2}$ for $x > 0$ and $s \notin (1, 4)$. Then

$$f'(x) = \frac{s}{w+2}\left(x^{s-1} + \frac{w}{2}x^{\frac{s}{2}-1}\right).$$

By corollary (3.6) it follows that

$$\left|\frac{1}{2}\left[\frac{f\left(\frac{b_2}{b_1}\right)+f\left(\frac{b_1}{b_2}\right)}{2}+f\left(\frac{\frac{b_2}{b_1}+\frac{b_1}{b_2}}{2}\right)\right]-\frac{1}{\frac{b_2}{b_1}-\frac{b_1}{b_2}}\int_{\frac{b_1}{b_2}}^{\frac{b_2}{b_1}} f(x)dx\right|$$
$$=\left|\frac{1}{2}\left\{\frac{1}{2}\left[\frac{b_2^s+w(b_1b_2)^{\frac{s}{2}}+b_1^s}{b_1^s(w+2)}+\frac{b_1^s+w(b_1b_2)^{\frac{s}{2}}+b_2^s}{b_2^s(w+2)}\right]+\frac{\left(\frac{b_2}{b_1}+\frac{b_1}{b_2}\right)^s+w\left(\frac{b_2}{b_1}+\frac{b_1}{b_2}\right)^{\frac{s}{2}}+1}{w+2}\right\}\right.$$
$$\left.-\frac{1}{w+2}\left[\frac{\left(\frac{b_2}{b_1}\right)^{s+1}-\left(\frac{b_1}{b_2}\right)^{s+1}}{(s+1)\left(\frac{b_2}{b_1}-\frac{b_1}{b_2}\right)}+w\frac{\left(\frac{b_2}{b_1}\right)^{\frac{s}{2}+1}-\left(\frac{b_1}{b_2}\right)^{\frac{s}{2}+1}}{\left(\frac{s}{2}+1\right)\left(\frac{b_2}{b_1}-\frac{b_1}{b_2}\right)}+1\right]\right|$$
$$=\left|\frac{H^s_{w,s}(b_1,b_2)}{H(b_1^s,b_2^s)}+H^{\frac{s}{2}+1}_{w,(\frac{s}{2}+1)}\left(\frac{b_2}{b_1}+\frac{b_1}{b_2},1\right)-H^s_{w,s}\left(\frac{L(b_1^2,b_2^2)}{G^2(b_1,b_2)},1\right)\right|. \tag{4.5}$$

On the other hand, we have

$$\frac{\frac{b_2}{b_1}-\frac{b_1}{b_2}}{4}\left\{\frac{1}{2}\left|f'\left(\frac{b_2}{b_1}\right)\right|+\eta\left(\left|f'\left(\frac{b_1}{b_2}\right)\right|,\left|f'\left(\frac{b_2}{b_1}\right)\right|\right)\int_0^1\left[h\left(\frac{1+t}{2}\right)+h(\frac{t}{2})\right]\left|\frac{1}{2}-t\right|dt-\frac{\mu}{2}\left(\frac{b_1}{b_2}-\frac{b_2}{b_1}\right)^2(\frac{5}{128})\right\}$$
$$=\frac{b_2^2-b_1^2}{4b_1b_2}\left\{\frac{1}{2}\left|\frac{s}{w+2}\left(\left(\frac{b_2}{b_1}\right)^{s-1}+\frac{w}{2}\left(\frac{b_2}{b_1}\right)^{\frac{s}{2}-1}\right)\right|\right.$$
$$+\eta\left(\left|\frac{s}{w+2}\left(\left(\frac{b_1}{b_2}\right)^{s-1}+\frac{w}{2}\left(\frac{b_1}{b_2}\right)^{\frac{s}{2}-1}\right)\right|,\left|\frac{s}{w+2}\left(\left(\frac{b_2}{b_1}\right)^{s-1}+\frac{w}{2}\left(\frac{b_2}{b_1}\right)^{\frac{s}{2}-1}\right)\right|\right)$$
$$\left.\times\int_0^1\left[h\left(\frac{1+t}{2}\right)+h(\frac{t}{2})\right]\left|\frac{1}{2}-t\right|dt-\frac{\mu}{2}\left(\frac{b_1^2-b_2^2}{b_1b_2}\right)^2\left(\frac{5}{128}\right)\right\}$$
$$=\frac{(b_2-b_1)A(b_1,b_2)}{2G^2(b_1,b_2)}\left\{\frac{\frac{1}{2}|s|}{w+2}\left(G^{2(s-1)}\left(b_2,\frac{1}{b_1}\right)+\frac{w}{2}G^{s-\frac{1}{2}}\left(b_2,\frac{1}{b_1}\right)\right)\right.$$
$$+\eta\left(\frac{|s|}{w+2}\left(G^{2(s-1)}\left(b_1,\frac{1}{b_2}\right)+\frac{w}{2}G^{s-\frac{1}{2}}\left(b_1,\frac{1}{b_2}\right)\right),\frac{|s|}{w+2}\left(G^{2(s-1)}\left(b_2,\frac{1}{b_1}\right)+\frac{w}{2}G^{s-\frac{1}{2}}\left(b_2,\frac{1}{b_1}\right)\right)\right)$$
$$\left.\times\int_0^1\left[h\left(\frac{1+t}{2}\right)+h(\frac{t}{2})\right]\left|\frac{1}{2}-t\right|dt-(\frac{5}{64})\mu\left(\frac{(b_1-b_2)A(b_1,b_2)}{G^2(b_1,b_2)}\right)^2\right\}. \tag{4.6}$$

Obviously (4.5) and (4.6) yield (4.4). □

5. Conclusions

Fractional differential and integral equations play increasingly important roles in the modeling of engineering and science problems. It has been established fact that, in many situations, these models provide more suitable results than analogous models with integer derivatives. Fractional integral inequality results when $0 < q < 1$ can be developed when the nonlinear term is increasing and satisfies a one sided Lipschitz condition. Using the integral inequality result and the computation of the solution of the linear fractional equation of variable coefficients, Gronwall inequality results can be established. In the present report, we developed the fractional integral inequalities for more broader class of convex functions named as generalized strongly modified h-convex functions, we also established some applications of derived inequalities to means. Our results extend and generalize many existing results, for example [1, 23, 33–35].

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Conflict of interest

The authors declare that no competing interests exist.

References

1. M. E. Gordji, S. S. Dragomir, M. R. Delavar, *An inequality Related to ϕ-convex functions*, Int. J. Nonlinear Anal. Appl., **6** (2015), 26–32.

2. Z. Meng, G. Li, D. Yang, et al. *A new directional stability transformation method of chaos control for first order reliability analysis*, Struct. Multidiscip. Optim., **55** (2017), 601–612.

3. Z. Meng, Z. Zhang, H. Zhou, *A novel experimental data-driven exponential convex model for reliability assessment with uncertain-but-bounded parameters*, Appl. Math. Modell., **77** (2020), 773–787.

4. G. Farid, W. Nazeer, M. S. Saleem, et al. *Bounds of Riemann-Liouville fractional integrals in general form via convex functions and their applications*, Mathematics, **6** (2018), 248.

5. M. K. Wang, Y. M. Chu, W. Zhang, *Monotonicity and inequalities involving zero-balanced hypergeometric function*, Math. Inequal. Appl., **22** (2019), 601–617.

6. Y. M. Chu, M. A. Khan, T. Ali, et al. *Inequalities for a-fractional differentiable functions*, J. Inequal. Appl., **2017** (2017), 1–12.

7. Y. C. Kwun, M. S. Saleem, M. Ghafoor, et al. *Hermite-Hadamard-type inequalities for functions whose derivatives are convex via fractional integrals*, J. Inequal. Appl., **2019** (2019), 1–16.

8. D. Ucar, V. F. Hatipoglu, A. Akincali, *Fractional integral inequalities on time scales*, Open J. Math. Sci., **2** (2018), 361–370.

9. Y. C. Kwun, G. Farid, W. Nazeer, et al. *Generalized riemann-liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of hadamard inequalities*, IEEE access, **6** (2018), 64946–64953.
10. S. Kermausuor, *Simpson's type inequalities for strongly (s, m)-convex functions in the second sense and applications*, Open J. Math. Sci., **3** (2019), 74–83.
11. I. A. Baloch, S. S. Dragomir, *New inequalities based on harmonic log-convex functions*, Open J. Math. Anal., **3** (2019), 103–105.
12. H. Bai, M. S. Saleem, W. Nazeer, et al. *Hermite-Hadamard-and Jensen-type inequalities for interval nonconvex function*, J. Math., **2020** (2020), 1–6.
13. W. Iqbal, K. M. Awan, A. U. Rehman, et al. *An extension of Petrovic's inequality for (h-) convex ((h-) concave) functions in plane*, Open J. Math. Sci., **3** (2019), 398–403.
14. S. Zhao, S. I. Butt, W. Nazeer, et al. *Some Hermite-Jensen-Mercer type inequalities for k-Caputo-fractional derivatives and related results*, Adv. Differ. Equ., **2020** (2020), 1–17.
15. M. A. Khan, S. Begum, Y. Khurshid, et al. *Ostrowski type inequalities involving conformable fractional integrals*, J. Inequal. Appl., **2018** (2018), 1–14.
16. M. A. Khan, Y. Khurshid, T. S. Du, et al. *Generalization of Hermite-Hadamard type inequalities via conformable fractional integrals*, J. Funct. Space., **2018** (2018), 1–12.
17. X. M. Zhang, Y. M. Chu, X. H. Zhang, *The Hermite-Hadamard type inequality of GA-convex functions and its application*, J. Inequal. Appl., **2010** (2010), 1–11.
18. B. S. Mordukhovich, N. M. Nam, *An easy path to convex analysis and applications*, Morgan Claypool, **6** (2014), 1–218.
19. S. Varosanec, *On h-convexity*, J. Math. Anal. Appl., **326** (2007), 303–311.
20. M. Noor, K. Noor, U. Awan, *Hermite-Hadamard type inequalities for modified h-convex functions*, Trans. J. Math. Mechanics, **6** (2014), 2014.
21. M. E. Gordji, M. R. Delavar, M. D. Sen, *On ψ-convex functions*, J. Math. Inequal., **10** (2016), 173–183.
22. N. Merentes, K. Nikodem, *Remarks on strongly convex functions*, Aequationes Math., **80** (2010), 193–199.
23. M. U. Awan, M. A. Noor, *On Strongly Generalized Convex Functions*, Filomat, **31** (2017), 5783–5790.
24. T. Zhao, M. S. Saleem, W. Nazeer, et al. *On generalized strongly modified h-convex functions*, J. Inequal. Appl., **2020** (2020), 1–12.
25. M. V. Cortez, Y. C. R. Oliveros, *An inequalities s-ϕ-convex*. DoI: 10.18576/Aninequalitiess-fi-convex(verde).
26. B. de Finetti, *Sulla stratificazioni convesse*, Ann. Math. Pura. Appl., **30** (1949), 173–183.
27. O. L. Mangasarian, *Pseudo-convex functions*, J. Soc. Ind. Appl. Math., **3** (1965), 281–290.
28. B. T. Polyak, *Existence theorems and convergence of minimizing sequences in extremum problems with restrictions*, Soviet Math. Dokl., **7** (1966), 287–290.

29. X. M. Yang, *E-convex sets, E-convex functions and E-convex programming*, J. Optim. Theory Appl., **109** (2001), 699–704.
30. I. Hsu, R. G. Kuller, *Convexity of vector-valued functions*, Proc. Amer. Math. Soc., **46** (1974), 363–366.
31. B. Y. Xi, F. Qi, *Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means*, J. Funct. Space. Appl., **2012** (2012), 1–14.
32. M. E. Ozdernir, C. Yildiz, A. C. Akdemir, et al. *On some inequalities for s-convex functions and applications*, J. Inequal. Appl., **2013** (2013), 333.
33. Y. C. Kwun, M. S. Saleem, M. Ghafoor, *Hermite-Hadamardd-type inequalities for functions whose derivatives are η-convex via fractional integrals*, J. Inequal. Appl., **2019** (2019), 1–16.
34. H. Kadakal, M. Kadakal, I. Iscan, *New type integral inequalities for three times differentiable preinvex and prequasiinvex functions*, Open J. Math. Anal., **2** (2018), 33–46.
35. S. Mehmood, G. Farid, K. A. Khan, et al. *New fractional Hadamard and Fejer-Hadamard inequalities associated with exponentially (h, m)-convex functions*, Eng. Appl. Sci. Lett., **3** (2020), 45–55.