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Research article

On some refinements for inequalities involving zero-balanced hypergeometric function

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Abstract: In the article, we present an elegant double inequality for the ratio of the zero-balanced hypergeometric functions, which improve and refine some previously known results and also give a positive answer the question by proposed by Ismail.

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1. Introduction

Let $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \cdots$. Then the Gaussian hypergeometric function $F(a, b; c; r)$ [1–3] is defined by

$$F(a,b;c;r) = {}_2F_1(a,b;c;r) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} r^n \quad (|r| < 1), \tag{1.1}$$

where $(a)_0 = 1$, $(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ for $n \in \mathbb{N} = \{1, 2, \cdots\}$ is the Pochhammer symbol and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the classical Euler gamma function [4, 5]. If $c = a + b$, then $F(a, b; a + b; r)$ is said to be zero-balanced. In particular, the complete elliptic integral $\mathcal{K}(r)$ [6, 7] and generalized complete elliptic integral $\mathcal{K}_a(r)$ $(r \in (0, 1), a \in (0, 1/2])$ [8] of the first kind are the special cases of the Gaussian hypergeometric function $F(a, b; c; r)$. Indeed, $\mathcal{K}(r)$ and $\mathcal{K}_a(r)$ can be expressed by

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(1/2, 1/2; 1; r^2\right)$$

and

$$\mathcal{K}_a(r) = \frac{\pi}{2} F(a, 1-a; 1; r^2), \quad r \in (0,1). \tag{1.2}$$

In 2016, Takeuchi [9] introduced the complete p-elliptic integral $K_p(r)$ of the first kind in terms of the Gaussian hypergeometric functions as follows

$$K_p(r) = \frac{\pi_p}{2} F\left(1/p; 1-1/p; 1; r^p\right),$$

where π_p is given by

$$\frac{\pi_p}{2} = \frac{1}{p} B(1/p, 1-1/p) = \frac{\pi}{p\sin(\pi/p)}$$

and

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is the Beta function.

Recently, the Gaussian hypergeometric function and its special cases have attracted the attention of many researchers [10–25] due to they have wide applications in pure and applied mathematics [26–40].

Anderson et al. [41] proved that the double inequality

$$\frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} > \frac{1}{1+r} \tag{1.3}$$

holds for all $r \in (0,1)$.

Motivated by inequality (1.3), many researchers provided its improvements, variants, refinements and generalizations. For example, Alzer and Richards [42] proved that the double inequality

$$\frac{1}{1+\sigma(a)r} < \frac{\mathcal{K}_a(r)}{\mathcal{K}_a(\sqrt{r})} < \frac{1}{1+\tau_a r} \tag{1.4}$$

holds for all $a \in (0, 1/2]$ and $r \in (0,1)$ with the best possible factors $\sigma(a) = a(1-a)$ and $\tau_a = 0$.

Recently, Yin et al. [43] generalized inequality (1.4) to the case of complete p-elliptic integral of the first kind $\mathcal{K}_p(r)$ and a sharp improvement of (1.4) was presented by Zhao et al. in [8]. Ismail [42, p. 1669] asked whether the inequality (1.4) can be extended to the zero-balanced hypergeometric function. Inspired by this question, Richards [44] proved the following result which is to answer the question by Ismail from another point of view.

Theorem A. *(See [44]) Let $a, b > 0$. Then the double inequality*

$$\frac{1}{(1+r)^{\lambda(a,b)}} < \frac{F(a,b;a+b;r^2)}{F(a,b;a+b;r)} < \frac{1}{(1+r)^{\mu(a,b)}}$$

holds for all $r \in (0,1)$ with the best possible exponents $\lambda(a,b) = ab/(a+b)$ and $\mu(a,b) = 0$.

The main purpose of this paper is to generalize the inequality (1.4) to the case of zero-balanced hypergeometric function and also make some refinements of (1.4) and Theorem A under certain restriction of a, b, which gives an affirmative answer to the question by Ismail. Our main result is the following Theorem 1.1.

Throughout this paper, we mainly focus on the parameters satisfying $a + b \geq 6ab/5$ for $a, b > 0$. By the symmetry, our parameters might be only consider as $0 < a < b$ and $a + b \geq 6ab/5$, which is equivalent to

$$\left\{(a,b)|\ 0 < a \leq \frac{5}{6},\ b > 0\right\} \cup \left\{(a,b)|\ \frac{5}{6} < a \leq \frac{5}{3},\ a < b \leq \frac{5a}{6a-5}\right\}. \tag{1.5}$$

For convenience, we denote $\sigma(a, b)$ and $\tau(a, b)$ by σ and τ simply if no risk for confusion, where

$$\sigma(a,b) = \frac{ab}{a+b} \quad \text{and} \quad \tau(a,b) = \frac{ab(a+b-ab+1)}{2(a+b)(a+b+1)}.$$

Theorem 1.1. *Let $a, b > 0$ with $a + b \geq 6ab/5$. Then the double inequality*

$$\frac{1+\tau r^2+\alpha r^3}{1+\sigma r} < \frac{F(a,b;a+b;r^2)}{F(a,b;a+b;r)} < \frac{1+\tau r^2+\beta r^3}{1+\sigma r} \tag{1.6}$$

holds for all $r \in (0, 1)$ if and only if $\alpha \leq \alpha_0$ and $\beta \geq \beta_0$, where

$$\alpha_0 = \alpha_0(a,b) = -\frac{ab(a+1)(b+1)\left[a(2+a)+(1-a)(2+a)b+(1-a)b^2\right]}{3(a+b)^2(a+b+1)(a+b+2)},$$
$$\beta_0 = \beta_0(a,b) = \frac{ab(a+1)(b+1)}{2(a+b)(1+a+b)}.$$

Remark 1.2. For later use, we need discuss about the sign of α_0 and $\alpha_0 + \tau$ for $a, b > 0$ with $a + b \geq 6ab/5$.

- Let $\widehat{\alpha}(a,b) = a(2+a)+(1-a)(2+a)b+(1-a)b^2$. Then it is easy to see that $\widehat{\alpha}(a,b) > 0$ for $0 < a \leq 1$ and $b > 0$. Moreover, it follows from (1.5) that

$$\widehat{\alpha}(a,b) \geq \widehat{\alpha}\left(a, \frac{5a}{6a-5}\right) = \frac{3a^2[(2a-3)^2+1]}{2(6a-5)^2} > 0$$

  for $1 < a \leq 5/3$ and $a < b \leq 5a/(6a-5)$. This yields $\alpha_0 < 0$.

- By calculations, we obtain

$$\begin{aligned}\alpha_0+\tau = \frac{ab}{6(a+b)^2(1+a+b)(2+a+b)}\Big[&a(1+a)(2+a) \\ &+\frac{1}{125}[(5-3a)(50+330a+223a^2)+294a^3]b \\ &+(1+a)(3-2a+2a^2)b^2+(2a-1)(a-1)b^3\Big].\end{aligned}$$

  From the above expression, we clearly see that $\alpha_0 + \tau > 0$ for $0 < a \leq 1/2$ (or $1 \leq a \leq 5/3$) and $b > 0$, and $\alpha_0 + \tau < 0$ for $1/2 < a \leq 5/6$ and sufficiently large $b > 0$.

As mentioned in (1.2), if $b = 1 - a$, then $F(a, b; a + b; r^2)/F(a, b; a + b; r)$ reduces to the ratio of generalized complete elliptic integral of the first kind $\mathcal{K}_a(r)/\mathcal{K}_a(\sqrt{r})$ and $a(1 - a) \leq 1/4$ for $0 < a < 1$. That is to say Theorem 1.1 in [8] can be derived from our Theorem 1.1 as a corollary.

Corollary 1.3. *Let $a \in (0, 1/2]$. Then the double inequality*

$$\frac{1 + \tau(a)r^2 + \lambda(a)r^3}{1 + \sigma(a)r} < \frac{\mathcal{K}_a(r)}{\mathcal{K}_a(\sqrt{r})} < \frac{1 + \tau(a)r^2 + \mu(a)r^3}{1 + \sigma(a)r} \tag{1.7}$$

holds for all $r \in (0, 1)$, where $\sigma(a) = \sigma(a, 1 - a), \tau(a) = \tau(a, 1 - a)$ and $\lambda(a) = \alpha_0(a, 1 - a)$, $\mu(a) = \beta_0(a, 1 - a)$ are defined in the literature [8].

Remark 1.4. Corollary 1.3 gives an affirmative answer to the question by Ismail. Moreover, the bounds for inequality (1.7) are better than (1.4). Indeed, from $\lambda(a) < 0$, $\sigma(a) = \tau(a) + \mu(a)$ and $\tau(a) + \lambda(a) = a(1 - a)[(1 - 2a)(6 + 13a + 33a^2) + 2a^3(25 + 4a)]/36 > 0$ for $a \in (0, 1/2]$, we obtain

$$\begin{aligned}\frac{1 + \tau(a)r^2 + \lambda(a)r^3}{1 + \sigma(a)r} &= \frac{1}{1 + \sigma(a)r} + \frac{r^2}{1 + \sigma(a)r}\Big[\tau(a) + \lambda(a)r\Big]\\ &> \frac{1}{1 + \sigma(a)r} + \frac{r^2}{1 + \sigma(a)r}\Big[\tau(a) + \lambda(a)\Big] > \frac{1}{1 + \sigma(a)r}\end{aligned}$$

and

$$\frac{1 + \tau(a)r^2 + \mu(a)r^3}{1 + \sigma(a)r} < \frac{1 + \tau(a)r + \mu(a)r}{1 + \sigma(a)r} = 1$$

for $r \in (0, 1)$ and $a \in (0, 1/2]$.

Remark 1.5. Under the assumption of Theorem 1.1, the upper bound of (1.6) is better than that in Theorem A due to

$$\frac{1 + \tau r^2 + \beta r^3}{1 + \sigma r} < \frac{1 + \tau r + \beta r}{1 + \sigma r} = 1$$

from $\sigma = \tau + \beta_0$. On the other hand, in order to compare the lower bound of (1.6) and Theorem A, it suffices to take into account the sign of $f(r) := (1 + \tau r^2 + \alpha_0 r^3)(1 + r)^{\sigma} - (1 + \sigma r)$.

Differentiation yields

$$\begin{aligned}f'(r) &= (1 + r)^{\sigma-1}\Big[\sigma + 2\tau r + (3\alpha_0 + 2\tau + \sigma\tau)r^2 + \alpha_0(3 + \sigma)r^3\Big] - \sigma,\\ f''(r) &= (1 + r)^{\sigma-2}\hat{f}(r),\end{aligned} \tag{1.8}$$

where

$$\hat{f}(r) = 2\tau - \sigma + \sigma^2 + 2(3\alpha_0 + 2\tau + 2\sigma\tau)r + (2 + \sigma)(6\alpha_0 + \tau + \sigma\tau)r^2 + \alpha_0(\sigma + 2)(\sigma + 3)r^3.$$

It follows from

$$2\tau - \sigma + \sigma^2 = \frac{a^2b^2}{(a + b)^2(1 + a + b)}, \quad 3\alpha_0 + 2\tau + 2\sigma\tau = \frac{a^2b^2(5 + 3a + 3b + ab)}{(a + b)^2(1 + a + b)(2 + a + b)}$$

and $\alpha_0 < 0$ that $\hat{f}(r)$ can be regarded as the special polynomial defined in Lemma 2.1. We can verify but miss the details that $\hat{f}(1) < 0$ for $a, b > 0$ with $a + b \geq 6ab/5$. This in conjunction with Lemma 2.1 implies that there exists $r^* \in (0, 1)$ such that $\hat{f}(r) > 0$ for $r \in (0, r^*)$ and $\hat{f}(r) < 0$ for $r \in (r^*, 1)$. Combining this with (1.8) and $f(0) = f'(0) = 0$, we conclude that $f(r) > 0$ for $r \in (0, r^*]$ and $f(r)$ is strictly concave on $(r^*, 1)$.

For $a, b > 0$ with $a + b \geq 6ab/5$, we have the following two conclusion:

- if $f(1) \geq 0$, then $f(r) > \min\{f(r^*), f(1)\} \geq 0$ for $r \in (r^*, 1)$. This yields the lower bound of (1.6) is better than that in Theorem A for $r \in (0, 1)$ and we refer to see the domain of a, b illustrated in Figure 1;
- if $f(1) < 0$, then there exists $r' \in (r^*, 1)$ such that the lower bound of (1.6) is better than that in Theorem A for $r \in (0, r')$.

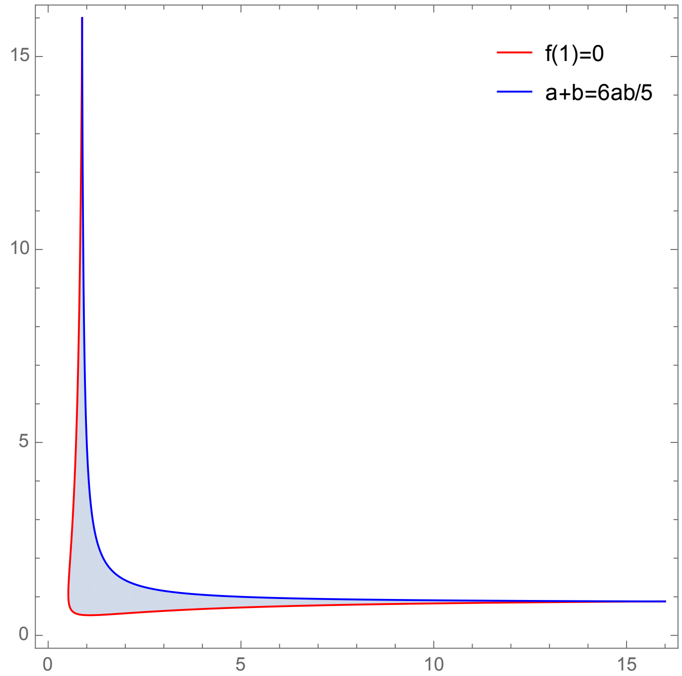

Figure 1. Visualized the domain $\{(a, b) | \, a, b > 0, a + b \geq 6ab/5, f(1) \geq 0\}$.

2. Lemmas

In this section, we introduce some notations and present some technical lemmas, which will be used in proving our main result.

Let $a, b > 0$ and

$$\Delta_n := \Delta_n(a, b) = \frac{(a)_n (b)_n}{n!(a + b)_n}.$$

Then we clearly see from (1.1) that $F(a, b; a + b; r)$ can be expressed simply as

$$F(a, b; a + b; r) = \sum_{n=0}^{\infty} \Delta_n r^n. \tag{2.1}$$

It is easy to verify that Δ_n satisfies the recurrence relation

$$\frac{\Delta_{n+1}}{\Delta_n} = \frac{(n + a)(n + b)}{(a + b + n)(n + 1)} \tag{2.2}$$

and also Δ_n is strictly decreasing for $n \geq 0$ if $a + b \geq ab$.

The following lemma provides a simple criterion to determine the sign of a class of special polynomial.

Lemma 2.1. *(See [45, Lemma 7) Let $n, m \in \mathbb{N} \cup \{0\}$ with $n > m$ and $P_n(t)$ be the polynomial of degree n defined by*

$$P_n(t) = \sum_{i=0}^{m} a_i t^i - \sum_{i=m+1}^{n} a_i t^i,$$

where $a_m, a_n > 0$ and $a_i \geq 0$ for $0 \leq i \leq n-1$ with $i \neq m$. Then there exist $t_0 \in (0, \infty)$ such that $P_n(t_0) = 0$ and $P_n(t) > 0$ for $t \in (0, t_0)$ and $P_n(t) < 0$ for $t \in (t_0, \infty)$.

Lemma 2.2. *(1) For $a, b > 0$, then $n\Delta_n(a, b)$ is strictly increasing for $n \geq 1$. In particular,*

$$\frac{\Delta_n(a, b)}{\Delta_m(a, b)} < \frac{m}{n} \tag{2.3}$$

for $m > n \geq 1$.

(2) For $a, b > 0$ with $a + b \geq 6ab/5$, then Δ_{n+1}/Δ_n is strictly increasing for $n \geq 1$.

(3) For $a, b > 0$ with $a + b \geq ab$, then $\Delta_{n+1}/\Delta_n > \Delta_{2n+2}/\Delta_{2n}$ holds for $n \geq 0$.

Proof. (1) From the recurrence relation (2.2) of Δ_n , we clearly see that

$$\frac{(n+1)\Delta_{n+1}}{n\Delta_n} = \frac{n+1}{n} \cdot \frac{(a+n)(b+n)}{(a+b+n)(n+1)} = 1 + \frac{ab}{n(a+b+n)} > 1$$

for $a, b > 0$ and $n \geq 1$. This yields the monotonicity of $n\Delta_n(a, b)$ with respect to n and inequality (2.3) follows directly from the monotonicity of $n\Delta_n(a, b)$.

(2) Taking the differentiation of (2.2) with respect to n yields

$$\frac{\partial(\Delta_{n+1}/\Delta_n)}{\partial n} = \frac{\delta_1(n; a, b)}{(n+1)^2(a+b+n)^2}, \tag{2.4}$$

where

$$\delta_1(n; a, b) = a^2 + a(1-a)b + (1-a)b^2 + 2(a+b-ab)n + n^2.$$

- If $0 < a \leq 1$ and $b > 0$, then we clearly see that $\delta_1(n; a, b) > 0$ for $n \geq 1$.
- If $1 < a \leq 5/3$, then it follows from (1.5) that

$$\begin{aligned}
\delta_1(n; a, b) \geq \delta(1; a, b) &= (a+1)^2 - (a-1)(a+2)b - (a-1)b^2 \\
&\geq (a+1)^2 - (a-1)(a+2) \cdot \frac{5a}{6a-5} - (a-1)\left(\frac{5a}{6a-5}\right)^2 \\
&= \frac{4 + 3(a-1)\left[4 + (a-1)(10 + (a-1)^2 + a^2)\right]}{(6a-5)^2} > 0
\end{aligned}$$

for $n \geq 1$.

Therefore, the proof is completed from (2.4) and $\delta_1(n; a, b) > 0$.

(3) From (2.2) we clearly see that

$$\frac{\Delta_{n+1}}{\Delta_n} - \frac{\Delta_{2n+2}}{\Delta_{2n}} = \frac{(n+a)(n+b)}{(a+b+n)(n+1)} - \frac{(2n+a)(2n+b)(2n+1+a)(2n+1+b)}{(a+b+2n)(2n+1)(a+b+2n+1)(2n+2)}$$

$$= \frac{ab\delta_2(n;a,b)}{2(n+1)(2n+1)(a+b+n)(a+b+2n)(1+a+b+2n)}, \tag{2.5}$$

where

$$\delta_2(n;a,b) = (a+b)(1+a+b-ab) + [3(1+2a+2b)+a+b-ab]n + 4(3+a+b)n^2 + 8n^3.$$

Combining this with $a+b \geq ab$ and (2.5) yields the desired result. □

Lemma 2.3. *Let $a, b > 0$ with $a+b \geq 6ab/5$ and $\rho_n(a,b) = (1+\sigma)\Delta_{n+2} - 2(\alpha_0+\tau)\Delta_{2n+2} - 2\Delta_{2n+4}$. Then $\rho_n(a,b) > 0$ for $n \geq 2$.*

Proof. Let $\hat{\rho}_n(a,b) = (1+\sigma)\Delta_{n+2} - 2\Delta_{2n+4}$. Then Remark 1.2 makes us to know that the sign of $\alpha_0+\tau$ can not be determined. We divide into two cases to complete the proof by mathematical induction.

Case 1 $\alpha_0+\tau \leq 0$. It suffices to show that $\hat{\rho}_n(a,b) > 0$ for $n \geq 0$.

From the definition of Δ_n, we compute that

$$\hat{\rho}_0(a,b) = \frac{a^2(a+1)b^2(b+1)\eta(a,b)}{12(a+b)^2(1+a+b)(2+a+b)(3+a+b)}, \tag{2.6}$$

where

$$\eta(a,b) = 36 + 17a + a^2 + (17+2a-a^2)b + (1-a)b^2.$$

It follows from $0 < a \leq 1$ that $\eta(a,b) > 0$. For $a > 1$, we clearly see that $\eta(a,b) > 0$ from (1.5) and Lemma 2.1 together with

$$\eta\left(a, \frac{5a}{6a-5}\right) = \frac{6\left[24 + 212(a-1) + (a-1)^2(338+104a+a^2)\right]}{(6a-5)^2} > 0.$$

This in conjunction with (2.6) yields $\hat{\rho}_0(a,b) > 0$.

We assume that $\hat{\rho}_k(a,b) > 0$, namely, $(1+\sigma)\Delta_{k+2} > 2\Delta_{2k+4}$ for $k \geq 0$. By the induction hypothesis, it follows from Lemma 2.2(3) and $n = k+2$ that

$$\begin{aligned}\hat{\rho}_{k+1}(a,b) &= (1+\sigma)\Delta_{k+2}\cdot\frac{\Delta_{k+3}}{\Delta_{k+2}} - 2\Delta_{2k+6} > 2\Delta_{2k+4}\cdot\frac{\Delta_{k+3}}{\Delta_{k+2}} - 2\Delta_{2k+6}\\ &= 2\Delta_{2k+4}\left(\frac{\Delta_{k+3}}{\Delta_{k+2}} - \frac{\Delta_{2k+6}}{\Delta_{2k+4}}\right) > 0.\end{aligned}$$

This completes the proof of Case 1.

Case 2 $\alpha_0+\tau > 0$.

By simple calculations, $\rho_2(a,b)$ can be simplified as

$$\rho_2(a,b) = \frac{ab\prod_{i=0}^{3}(a+i)\prod_{j=0}^{3}(b+j)}{60480(a+b)\prod_{i=0}^{2}(a+b+i)\prod_{j=0}^{7}(a+b+j)}\sum_{k=0}^{7}\epsilon_k(a)b^k, \tag{2.7}$$

where

$$\epsilon_0(a) = 2a(1+a)(2+a)(823200 + 465754a + 87277a^2 + 5546a^3 + 23a^4),$$

$$\epsilon_1(a) = 3292800 + 10782032a + 13070784a^2 + 7204178a^3 + 2041437a^4 + 313859a^5 + 25539a^6 + 891a^7,$$
$$\epsilon_2(a) = 6802216 + 13070784a + 8341952a^2 + 1941253a^3 + 2474a^4 - 67079a^5 - 9902a^6 - 458a^7,$$
$$\epsilon_3(a) = 4790032 + 7204178a + 1941253a^2 - 896662a^3 - 577922a^4 - 121105a^5 - 11511a^6 - 423a^7,$$
$$\epsilon_4(a) = 1477354 + 2041437a + 2474a^2 - 577922a^3 - 227680a^4 - 36779a^5 - 2588a^6 - 56a^7,$$
$$\epsilon_5(a) = 207922 + 313859a - 67079a^2 - 121105a^3 - 36779a^4 - 4330a^5 - 168a^6,$$
$$\epsilon_6(a) = 11230 + 25539a - 9902a^2 - 11511a^3 - 2588a^4 - 168a^5,$$
$$\epsilon_7(a) = (1 - a)(46 + 937a + 479a^2 + 56a^3).$$

Obviously, it follows that $\epsilon_0(a) > 0, \epsilon_1(a) > 0$ for $a > 0$ and $\epsilon_7(a) \geq 0$ for $0 < a \leq 1$ and $\epsilon_7(a) < 0$ for $1 < a \leq 5/3$. Moreover, $\epsilon_k(a)$ $(k = 2, 3, \cdots, 6)$ is the special polynomial defined in Lemma 2.1. It follows from Lemma 2.1 and

$$\epsilon_2(5/3) = \frac{130507213472}{2187}, \quad \epsilon_3(5/3) = \frac{952529672}{81}, \quad \epsilon_4(5/3) = -\frac{166616512}{2187},$$
$$\epsilon_5(5/3) = -\frac{87250384}{243}, \quad \epsilon_6(5/3) = -\frac{1326560}{27}$$

that $\epsilon_2(a) > 0, \epsilon_3(a) > 0$ for $0 < a \leq 5/3$ and there exits $a_k \in (0, 5/3)$ $(k = 4, 5, 6)$ such that $\epsilon_k(a) > 0$ for $0 < a < a_k$ and $\epsilon_k(a) < 0$ for $a_k < a \leq 5/3$. The roots a_k can be computed numerically as follows $a_4 = 1.65774\cdots > a_5 = 1.41613\cdots > a_6 = 1.22785\cdots > 1$. So

- for $0 < a \leq 1$, then it follows that $\sum_{k=0}^{7} \epsilon_k(a)b^k > 0$;
- for $1 < a \leq 5/3$, then we can consider the following intervals $(1, a_6]$, $(a_6, a_5]$, $(a_5, a_4]$, $(a_4, 5/3]$ and $\sum_{k=0}^{7} \epsilon_k(a)b^k$ can be regarded as the special polynomial defined in Lemma 2.1 on each such interval, which yields $\sum_{k=0}^{7} \epsilon_k(a)b^k > 0$ following from (1.5) and Lemma 2.1 together with

$$\sum_{k=0}^{7} \epsilon_k(a)\left(\frac{5a}{6a-5}\right)^k = \frac{72a^2}{(6a-5)^7}\Big[48\left(295386449(a-1)^2 + 15384558(a^2-1) + 1526700a^2\right) + (a-1)^3(12215090768 + 17108173200a + 8769780982a^2 + 3825548386a^3(a-1) + 10871964835a^4 + 18178058803a^5 + 6741509385a^6 + 1047476909a^7 + 64085484a^8 + 772338a^9)\Big] > 0.$$

This in conjunction with (2.7) implies $\rho_2(a, b) > 0$.

Suppose that $\rho_k(a, b) > 0$ for $k \geq 2$. We now prove that $\rho_{k+1}(a, b) > 0$.

The induction hypothesis enables us to know that

$$(1 + \sigma)\Delta_{k+2} > 2(\alpha_0 + \tau)\Delta_{2k+2} + 2\Delta_{2k+4}. \tag{2.8}$$

It follows from Lemma 2.2(2) and (3) with $n = k + 2$ that

$$\frac{\Delta_{k+3}}{\Delta_{k+2}} - \frac{\Delta_{2k+4}}{\Delta_{2k+2}} > \frac{\Delta_{k+3}}{\Delta_{k+2}} - \frac{\Delta_{2k+6}}{\Delta_{2k+4}} > 0.$$

This in conjunction with (2.8) yields

$$\begin{aligned}\rho_{k+1}(a,b) &= (1+\sigma)\Delta_{k+2}\cdot\frac{\Delta_{k+3}}{\Delta_{k+2}} - 2(\alpha_0+\tau)\Delta_{2k+4} - 2\Delta_{2k+6}\\ &> [2(\alpha_0+\tau)\Delta_{2k+2}+2\Delta_{2k+4}]\cdot\frac{\Delta_{k+3}}{\Delta_{k+2}} - 2(\alpha_0+\tau)\Delta_{2k+4} - 2\Delta_{2k+6}\\ &= 2\Delta_{2k+4}\left[(\alpha_0+\tau)\frac{\Delta_{2k+2}}{\Delta_{2k+4}}\left(\frac{\Delta_{k+3}}{\Delta_{k+2}}-\frac{\Delta_{2k+4}}{\Delta_{2k+2}}\right)+\left(\frac{\Delta_{k+3}}{\Delta_{k+2}}-\frac{\Delta_{2k+6}}{\Delta_{2k+4}}\right)\right] > 0.\end{aligned}$$

□

Lemma 2.4. *Let* $a, b > 0$ *with* $a + b \geq 6ab/5$ *and*

$$\begin{aligned}A_n &= \Delta_{n+2} - \alpha_0\Delta_{2n+1} - \tau\Delta_{2n+2} - \Delta_{2n+4},\\ B_n &= \sigma\Delta_{n+2} - \alpha_0\Delta_{2n+2} - \tau\Delta_{2n+3} - \Delta_{2n+5}.\end{aligned}$$

Then (i) $A_n > 0$*; (ii)* $A_n + B_n > 0$ *for* $n \geq 0$.

Proof. (i) By Lemma 2.2(2) and Bernoulli's inequality we know that

$$\begin{aligned}\frac{\Delta_{n+2}}{\Delta_{2n+1}} &= \frac{\Delta_{n+2}}{\Delta_{n+3}}\cdot\frac{\Delta_{n+3}}{\Delta_{n+4}}\cdots\frac{\Delta_{2n}}{\Delta_{2n+1}} \geq \left(\frac{\Delta_{2n}}{\Delta_{2n+1}}\right)^{n-1} = \left[1+\frac{a+b-ab+2n}{(a+2n)(b+2n)}\right]^{n-1}\\ &\geq 1+\frac{(n-1)(a+b-ab+2n)}{(a+2n)(b+2n)}\end{aligned}$$

for $n \geq 2$. This in conjunction with (2.2) and the monotonicity of Δ_n with respect to n gives

$$\begin{aligned}\frac{A_n}{\Delta_{2n+1}} &= \frac{\Delta_{n+2}}{\Delta_{2n+1}} - \alpha_0 - \tau\frac{\Delta_{2n+2}}{\Delta_{2n+1}} - \frac{\Delta_{2n+4}}{\Delta_{2n+1}} \geq \frac{\Delta_{n+2}}{\Delta_{2n+1}} - \alpha_0 - (\tau+1)\frac{\Delta_{2n+2}}{\Delta_{2n+1}}\\ &\geq 1+\frac{(n-1)(a+b-ab+2n)}{(a+2n)(b+2n)} - \alpha_0 - \frac{(\tau+1)(a+2n+1)(b+2n+1)}{(a+b+2n+1)(2n+2)}\\ &\geq 1+\frac{(n-1)(a+b-ab+2n)}{(a+b+2n+1)(2n+2)} - \alpha_0 - \frac{(\tau+1)(a+2n+1)(b+2n+1)}{(a+b+2n+1)(2n+2)}\\ &= \frac{\delta_3(n;a,b)}{12(a+b)^2(1+a+b)(2+a+b)(1+n)(1+a+b+2n)},\end{aligned} \tag{2.9}$$

where $\delta_3(n;a,b) = \zeta_0(a,b) + \zeta_1(a,b)n + \zeta_2(a,b)n^2$ and $\zeta_0(a,b) = \sum_{k=0}^{5}\varepsilon_k^0(a)b^k$, $\zeta_1(a,b) = \sum_{k=0}^{5}\varepsilon_k^1(a)b^k$, $\zeta_2(a,b) = \sum_{k=0}^{4}\varepsilon_k^2(a)b^k$, and

$$\begin{aligned}&\varepsilon_0^0(a) = 6a^2(1+a)(2+a), \quad \varepsilon_1^0(a) = a(24+56a+29a^2+4a^3+a^4),\\ &\varepsilon_2^0(a) = 2(6+28a+17a^2-4a^3-2a^4), \quad \varepsilon_3^0(a) = 18+29a-8a^2-22a^3-10a^4-a^5,\\ &\varepsilon_4^0(a) = 2(3+2a-2a^2-5a^3-a^4), \quad \varepsilon_5^0(a) = a(1-a)(1+a),\\ &\varepsilon_0^1(a) = 6a^3(1+a)(2+a), \quad \varepsilon_1^1(a) = 2a^2(18+28a+3a^2-4a^3),\\ &\varepsilon_2^1(a) = 2a(18+20a-12a^2-8a^3+3a^4), \quad \varepsilon_3^1(a) = 2(6+28a-12a^2-26a^3-3a^4-2a^5),\\ &\varepsilon_4^1(a) = 2(9+3a-8a^2-3a^3-4a^4), \quad \varepsilon_5^1(a) = 2(1-a)(3-a+2a^2),\end{aligned}$$

$$\varepsilon_0^2(a) = 12a^2(1+a)(2+a), \quad \varepsilon_1^2(a) = 4a[12+25a+a^2(9-a)],$$
$$\varepsilon_2^2(a) = 4[6+25a+a^2(3a^2-a+6)], \quad \varepsilon_3^2(a) = \frac{4}{9}(3+a+a^2)\,[7+2(5-3a)(2+3a)]\,,$$
$$\varepsilon_4^2(a) = \frac{4}{125}\left[(5-3a)(75+20a+87a^2)+11a^3\right].$$

From the above expressions, we clearly see that $\zeta_2(a,b) > 0$ for $0 < a \le 5/3$ and $b > 0$. Moreover,

$$\begin{aligned}
&2\zeta_2(a,b) + \zeta_1(a,b)\\
&\quad = 6a^2(1+a)(2+a)(4+a) + 2a[48+118a+46a^2+a^2(2-a)+4a^2(4-a^2)]b\\
&\qquad + 2[24+118a+12a^2+16a^2(2-a)+4a^4+3a^5]b^2\\
&\qquad\quad + 2(42+64a-16a^2-26a^3-11a^4-2a^5)b^3 + 2(21-a+4a^2-11a^3-4a^4)b^4\\
&\qquad\qquad + 2(1-a)(3-a+2a^2)b^5,
\end{aligned}$$

which gives $2\zeta_2(a,b) + \zeta_1(a,b) > 0$ for $0 < a \le 1$. By the same argument as in Case 2 of Lemma 2.3, $2\zeta_2(a,b)+\zeta_1(a,b)$ can be regarded as the special polynomial of b defined in Lemma 2.1 for $1 < a \le 5/3$, which implies $2\zeta_2(a,b) + \zeta_1(a,b) > 0$ from Lemma 2.1 and

$$\begin{aligned}
2\zeta_2\left(a, \frac{5a}{6a-5}\right) + \zeta_1\left(a, \frac{5a}{6a-5}\right) = \frac{12a^4}{(6a-5)^5}\Big[&23168(a-1) + 2940a\\
&+ (a-1)^2(32043+6028a+20528a^2+14472a^3+768a^4)\Big] > 0.
\end{aligned}$$

From the above discussion, we clearly see that for $n \ge 2$,

$$\begin{aligned}
\delta_3(n;a,b) &\ge n\,[2\zeta_2(a,b)+\zeta_1(a,b)] + \zeta_0(a,b) \ge 2\,[2\zeta_2(a,b)+\zeta_1(a,b)] + \zeta_0(a,b)\\
&= 6a^2(1+a)(2+a)(9+2a) + 3a[72+176a+75a^2+5a^2(4-a^2)]b\\
&\quad + 6[18+88a+11a^2+12a^2(2-a)+2a^4+2a^5]b^2\\
&\qquad + 3(62+95a-24a^2-42a^3-18a^4-3a^5)b^3 + 6(15+2a^2-9a^3-3a^4)b^4\\
&\qquad\qquad + 3(1-a)(4-a+3a^2)b^5 := \hat{\zeta}(a,b).
\end{aligned} \tag{2.10}$$

Similarly, it follows from (2.10) that $\hat{\zeta}(a,b) > 0$ for $0 < a \le 1$ and $\hat{\zeta}(a,b) > 0$ for $1 < a \le 5/3$ from Lemma 2.1 and

$$\begin{aligned}
\hat{\zeta}\left(a, \frac{5a}{6a-5}\right) = \frac{108a^4}{(6a-5)^5}\Big[652 + (a-1)\Big(&1098(a^3-1) + 5202a\\
&+ 2908a^2(a^2-1) + 338a^3 + 11a^4 + 189a^5\Big)\Big] > 0.
\end{aligned}$$

This in conjunction with (2.9) and (2.10) implies that $A_n > 0$ for $n \ge 2$.

It remains to verify the sign of A_0 and A_1. We only give the details of calculations for A_0 and similar for A_1.

By the definition of Δ_n, A_0 can be rewritten as

$$A_0 = \frac{a(1+a)b(1+b)\hat{\zeta}_0(a,b)}{8(a+b)^3(1+a+b)^2(2+a+b)(3+a+b)},$$

where

$$\begin{aligned}\hat{\zeta}_0(a,b) &= 2a^2(1+a)(2+a)(3+a) + a[24+70a+51a^2+6a^3+a^3(2-a)]b \\ &\quad + [2(6-a)+2a^2+a(18+13a+a^2)(4-a^2)]b^2 + (22+51a-14a^2-24a^3-3a^4)b^3 \\ &\quad + (12+8a-13a^2-3a^3)b^4 + (1-a)(2+a)b^5.\end{aligned}$$

The same argument as in (2.10) enables us to know that $\hat{\zeta}_0(a,b) > 0$ for $0 < a \le 1$ and $\hat{\zeta}_0(a,b) > 0$ for $1 < a \le 5/3$ from Lemma 2.1 and

$$\begin{aligned}\hat{\zeta}_0\left(a, \frac{5a}{6a-5}\right) = \frac{12a^4}{(6a-5)^5}\Big[834 + (a-1)\Big(334 + 2134a + 2301a^2(a-1) \\ + 1722a^3 + 2952a^4 + 306a^5\Big)\Big] > 0.\end{aligned}$$

(ii) We first verify the sign of $A_0 + B_0$ and $A_1 + B_1$. Since the method is similar as above, we only give some necessary expression for $A_0 + B_0$ and then similar for $A_1 + B_1$.

By calculations, we obtain

$$A_0 + B_0 = \frac{a(1+a)b(1+b)\hat{\zeta}_1(a,b)}{120(a+b)^3(1+a+b)^2(2+a+b)(3+a+b)(4+a+b)},$$

where

$$\begin{aligned}\hat{\zeta}_1(a,b) &= 6a^2(1+a)(2+a)(3+a)(4+a) + a(288+2580a+3836a^2+1919a^3+358a^4+19a^5)b \\ &\quad + (2+a)(3+a)[24+410a+431a^2+2a^2(2-a)(18+7a)]b^2 \\ &\quad + (300+3836a+2877a^2-666a^3-869a^4-187a^5-11a^6)b^3 \\ &\quad + (210+1919a+379a^2-869a^3-346a^4-33a^5)b^4 \\ &\quad + (60+358a-78a^2-187a^3-33a^4)b^5 + (1-a)(2+a)(3+11a)b^6.\end{aligned}$$

We can compute easily that $\hat{\zeta}_1(a,b) > 0$ for $0 < a \le 1$ and $\hat{\zeta}_1(a,b) > 0$ for $1 < a \le 5/3$ from Lemma 2.1 and

$$\begin{aligned}\hat{\zeta}_1\left(a, \frac{5a}{6a-5}\right) = \frac{36a^4}{(6a-5)^6}\Big[59520 + (a-1)\Big(32020 + 164020a + 367670a^2 \\ + a^3(a-1)(1030750 + 709949a) + 42672a^5 + 179016a^6 + 7446a^7\Big)\Big] > 0.\end{aligned}$$

For $n \ge 2$, it follows from $\alpha_0 < 0$ and Lemma 2.3 together with the monotonicity of Δ_n with respect to n that

$$\begin{aligned}A_n + B_n &= (1+\sigma)\Delta_{n+2} - \alpha_0(\Delta_{2n+1} + \Delta_{2n+2}) - \tau(\Delta_{2n+2} + \Delta_{2n+3}) - (\Delta_{2n+4} + \Delta_{2n+5}) \\ &\ge (1+\sigma)\Delta_{n+2} - 2(\alpha_0 + \tau)\Delta_{2n+2} - 2\Delta_{2n+4} \\ &= \rho_n(a,b) > 0.\end{aligned}$$

□

Lemma 2.5. *Let* $a, b > 0$ *with* $a + b \geq ab$ *and*

$$\begin{aligned} C_n &= \sigma\Delta_{n+1} - \beta_0\Delta_{2n} - \tau\Delta_{2n+1} - \Delta_{2n+3}, \\ D_n &= \Delta_{n+2} - \beta_0\Delta_{2n+1} - \tau\Delta_{2n+2} - \Delta_{2n+4}. \end{aligned}$$

Then (i) $C_n < 0$*; (ii)* $C_n + D_n < 0$ *for* $n \geq 0$.

Proof. (i) By calculations, we obtain

$$C_0 = -\frac{a(1+a)b(1+b)\left[(a+b)(4+3a+3b) + 2(a+b+3)(a+b-ab)\right]}{6(a+b)^2(1+a+b)(2+a+b)} < 0. \tag{2.11}$$

The relation $\sigma = \beta_0 + \tau$ allows us to rewrite C_n as

$$\begin{aligned} C_n &= (\beta_0 + \tau)\Delta_{n+1} - \beta_0\Delta_{2n} - \tau\Delta_{2n+1} - \Delta_{2n+3} \\ &= \beta_0\left(\Delta_{n+1} - \frac{2n}{n+1}\Delta_{2n}\right) + \tau\left(\Delta_{n+1} - \frac{2n+1}{n+1}\Delta_{2n+1}\right) + \hat{C}_n, \end{aligned}$$

where

$$\hat{C}_n = \frac{n-1}{n+1}\beta_0\Delta_{2n} + \frac{n}{n+1}\tau\Delta_{2n+1} - \Delta_{2n+3}.$$

From Lemma 2.2(1) it suffices to show $\hat{C}_n < 0$ for $n \geq 1$.

For $n \geq 1$, Lemma 2.2(1) and $\tau, \beta_0 > 0$ together with the monotonicity of Δ_n with respect to n lead to

$$\begin{aligned} \hat{C}_n &< \left(\frac{n-1}{n+1}\beta_0 + \frac{n}{n+1}\tau\right)\Delta_{2n} - \Delta_{2n+3} \\ &< \Delta_{2n+3}\left[\left(\frac{n-1}{n+1}\beta_0 + \frac{n}{n+1}\tau\right)\frac{2n+3}{2n} - 1\right] := \Delta_{2n+3}\hat{C}_n^*. \end{aligned} \tag{2.12}$$

Simplifying $\hat{C}_n^*$ gives rise to

$$\begin{aligned} \hat{C}_n^* = -\frac{1}{4(a+b)(1+a+b)n(1+n)}\Big[&3a(1+a)b(1+b) \\ &+ 2\Big(2(a+b+1)(a+b-ab) + a^2b^2\Big)n + 4(1+a+b)(a+b-ab)n^2\Big] < 0. \end{aligned}$$

This in conjunction with (2.11) and (2.12) completes the proof of (i).

(ii) For $n \geq 0$, it follows from Lemma 2.2(1) and $\sigma = \tau + \beta_0$ together with the monotonicity of Δ_n with respect to n that

$$\begin{aligned} C_n + D_n &= \sigma\Delta_{n+1} + \Delta_{n+2} - \beta_0(\Delta_{2n} + \Delta_{2n+1}) - \tau(\Delta_{2n+1} + \Delta_{2n+2}) - (\Delta_{2n+3} + \Delta_{2n+4}) \\ &< \sigma\Delta_{n+1} + \Delta_{n+2} - 2(\tau + \beta_0)\Delta_{2n+2} - 2\Delta_{2n+4} \\ &= \sigma(\Delta_{n+1} - 2\Delta_{2n+2}) + \Delta_{n+2} - 2\Delta_{2n+4} < 0. \end{aligned}$$

This completes the proof. □

3. Proof of Theorem 1.1

Proof. Define

$$\varphi_{a,b}(r) = (1+\sigma r)F(a,b;a+b;r^2) - (1+\tau r^2+\alpha_0 r^3)F(a,b;a+b;r)$$

and

$$\phi_{a,b}(r) = (1+\sigma r)F(a,b;a+b;r^2) - (1+\tau r^2+\beta_0 r^3)F(a,b;a+b;r).$$

In order to prove the inequalities (1.6) is valid, it suffices to show $\varphi_{a,b}(r) > 0$ and $\phi_{a,b}(r) < 0$ for $r \in (0,1)$.

From (2.1), we can rewrite $\varphi_{a,b}(r)$ and $\phi_{a,b}(r)$, in terms of power series, as

$$\begin{aligned}\varphi_{a,b}(r) &= (1+\sigma r)\sum_{n=0}^{\infty}\Delta_n r^{2n} - (1+\tau r^2+\alpha_0 r^3)\sum_{n=0}^{\infty}\Delta_n r^n \\ &= r^4\left[\sum_{n=0}^{\infty}(A_n+B_n r)r^{2n}\right], \qquad (3.1)\\ \phi_{a,b}(r) &= (1+\sigma r)\sum_{n=0}^{\infty}\Delta_n r^{2n} - (1+\tau r^2+\beta_0 r^3)\sum_{n=0}^{\infty}\Delta_n r^n \\ &= r^3\left[\sum_{n=0}^{\infty}(C_n+D_n r)r^{2n}\right], \qquad (3.2)\end{aligned}$$

where A_n, B_n and C_n, D_n are defined in Lemma 2.4 and Lemma 2.5, respectively.

From (3.1) and (3.2), we only need to prove that $A_n + B_n r > 0$ and $C_n + D_n r < 0$ for $r \in (0,1)$ and $n \geq 0$.

Case 1 $A_n + B_n r > 0$.

- If $B_n \geq 0$, then it follows from Lemma 2.4*(i)* that $A_n + B_n r \geq A_n > 0$ for $r \in (0,1)$;
- If $B_n < 0$, then Lemma 2.4*(ii)* enables us to know that $A_n + B_n r > A_n + B_n > 0$ for $r \in (0,1)$.

Case 2 $C_n + D_n r < 0$.

- If $D_n \leq 0$, then it follows from Lemma 2.5*(i)* that $C_n + D_n r \leq C_n < 0$ for $r \in (0,1)$;
- If $D_n > 0$, then from Lemma 2.5*(ii)* we clearly see that $C_n + D_n r < C_n + D_n < 0$ for $r \in (0,1)$.

We are now in a position to prove α_0 and β_0 are the best possible constants.

Let

$$\Phi_{a,b}(r) = \frac{(1+\sigma r)F(a,b;a+b;r^2) - (1+\tau r^2)F(a,b;a+b;r)}{r^3 F(a,b;a+b;r)}. \qquad (3.3)$$

Then we clearly see that

$$\begin{cases}\lim\limits_{r\to 0^+}\Phi_{a,b}(r) = -\dfrac{ab(a+1)(b+1)\left[a(2+a)+(1-a)(2+a)b+(1-a)b^2\right]}{3(a+b)^2(a+b+1)(a+b+2)} = \alpha_0, \\ \lim\limits_{r\to 1^-}\Phi_{a,b}(r) = \dfrac{ab(a+1)(b+1)}{2(a+b)(1+a+b)} = \beta_0.\end{cases} \qquad (3.4)$$

For $\alpha_0 < c < \beta_0$, then it follows from (3.4) that there exist sufficiently small $r_1, r_2 \in (0,1)$ such that $\Phi_{a,b}(r) < c$ for $r \in (0, r_1)$ and $\Phi_{a,b}(r) > c$ for $r \in (1-r_2, 1)$. In other words,

$$\frac{F(a,b;a+b;r^2)}{F(a,b;a+b;r)} < \frac{1+\tau r^2 + c r^3}{1+\sigma r, \quad r \in (0, r_1)}$$

and

$$\frac{F(a,b;a+b;r^2)}{F(a,b;a+b;r)} > \frac{1+\tau r^2 + c r^3}{1+\sigma r}, \quad r \in (1-r_2, 1).$$

This completes the proof of Theorem 1.1. □

Remark 3.1. In the proof of Theorem A [44], the parameters a and b are required to meet the condition that $a + b > ab$, although this is still more relaxed than our parameter's condition $a + b \geq 6ab/5$ in Theorem 1.1.

It is natural to ask that can our parameter's condition be relaxed to $a + b > ab$? From Lemma 2.5, we clearly see that the upper bound in Theorem 1.1 still holds for $a + b \geq ab$ but the lower bound is not true. The reason is that $A_0(a, b) < 0$ for $3/2 \leq a \leq 3$ and $a + b = ab$. More precisely,

$$A_0\left(a, \frac{a}{a-1}\right) = -\frac{(1-a^2)(2a-1)[(3-a)(2a-3)(43a^2+72a-72)+5a^4]}{648(a^2+a-1)^2(a^2+2a-2)(a^2+3a-3)} < 0$$

for $3/2 \leq a \leq 3$. By the continuity, it follows from (3.1) that there exists a sufficiently small $r_{a,b} \in (0,1)$ such that $\varphi_{a,b}(r) < 0$ for $r \in (0, r_{a,b})$ when a and b lie in a very narrow strip near the curve segment $\{(a,b)|\ 3/2 \leq a \leq 3, a+b = ab\}$. We focus on the condition that $a + b \geq 6ab/5$ just to make the calculation simpler. Of course, one can give some refinements for our parameter condition.

To this end, numerical experiment results allow us to pose the following conjecture.

Conjecture. Let $a, b > 0$ with $a + b \geq 6ab/5$ and $\Phi_{a,b}(r)$ be defined as in (3.3). Then $\Phi_{a,b}(r)$ is strictly increasing from $(0, 1)$ onto (α_0, β_0).

4. Discussions

This paper deals with the zero-balanced hypergeometric function ${}_2F_1(a, b; a + b; r)$. In this study, we present an elegant double inequality for ${}_2F_1(a, b; a + b; r^2)/{}_2F_1(a, b; a + b; r)$, which gives some refinements for some previously known results and also answers to the question by Ismail in the affirmative.

5. Conclusions

In this paper, we have established a sharp double inequality involving the ratio of zero-balanced hypergeometric function ${}_2F_1(a, b; a + b; r^2)/{}_2F_1(a, b; a + b; r)$. More precisely, the double inequality

$$\frac{1+\tau r^2+\alpha_0 r^3}{1+\sigma r} < \frac{F(a,b;a+b;r^2)}{F(a,b;a+b;r)} < \frac{1+\tau r^2+\beta_0 r^3}{1+\sigma r}$$

holds for all $r \in (0, 1)$ with the best possible constants α_0 and β_0, where

$$\sigma = \frac{ab}{a+b}, \quad \tau = \frac{ab(a+b-ab+1)}{2(a+b)(a+b+1)},$$

$$\alpha_0 = -\frac{ab(a+1)(b+1)\left[a(2+a)+(1-a)(2+a)b+(1-a)b^2\right]}{3(a+b)^2(a+b+1)(a+b+2)},$$
$$\beta_0 = \frac{ab(a+1)(b+1)}{2(a+b)(1+a+b)}.$$

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Conflict of interest

The authors declare that they have no competing interests.

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