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Research article

Soft near-semirings

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Abstract: Soft set theory was introduced by Molodtsov to deal with uncertainty. In this manuscript, we commence with the notion of soft near-semirings based on soft set theory. Several associated concepts including soft subnear-semirings, idealistic soft near-semirings, soft near-semiring homomorphism and soft near-semiring anti-homomorphism are also introduced. Moreover, numerous related properties are discussed and illustrated by suitable examples. We also present the notion of chain condition along with its applications towards soft near-semirings. We conclude that applying different types of union operations on soft sets without further restrictions are possible as long as chain condition is satisfied. Few related characteristics of soft left (resp., right) near-semirings are also explored by using soft near-semiring homomorphism and soft near-semiring anti-homomorphism. Furthermore, we found that under soft near-semiring anti-homomorphism the image of soft right (left) near-semiring is a soft left (right) near-semiring. This study also bridging the link between classical near-semirings and soft sets theory.

Keywords: soft near-semirings; soft idealistic near-semirings; soft anti-homomorphism
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1. Introduction

Scientists in engineering, economics, social sciences and several other fields are dealing with the difficulties of modeling the data containing uncertainties. Many theories of mathematics like probability theory, fuzzy set theory [22], rough set theory [19] were introduced to handling uncertainties. But classical mathematics may not be efficiently worked due to having several inherent complications. Consequently, Molodtsov [17] presented the soft set theory, which is proven useful to deal with different types of uncertainties existing in given data. Furthermore, in [17, 18] the authors explored many applications in different fields of sciences. Currently, a lot of work has been done on the soft set theory and is developing speedily. Plenty of applications of soft sets have been explored towards data analysis, decision making theory etc. The application of soft sets towards data analysis under incomplete informations has been presented in [23]. The notion of fuzzy parameterized soft sets (FP-soft sets) along with its applications towards decision making theory was discussed in [5]. In [6], authors produced further applications of FP-soft sets towards the problems containing uncertainties. Recently, the application of soft sets towards multi-attribute decision-making theory is addressed in [11]. As well as concerned about theoretical aspects of soft sets, different operations on soft sets [4, 15] and mappings [16] have been introduced in literature. Subsequently, number of soft algebraic structures like soft groups [3], soft rings [1], soft semirings [8], soft nearrings [20] have been introduced and discussed in the literature.

On the other hand, number of applications of near-semirings have been explored in the literature, Krishna & Chatterjee introduced the application of near-semirings towards sequential machine [14]. Near-semirings is also recognized in the study of automata theory and computer languages, one can expect that soft near-semirings will be useful in the study of fuzzy (soft) automata and languages.

Near-semiring is the common generalization of nearrring and semiring. Near-semiring R is an algebraic structure equipped with two binary operations "+" and "$\cdot$" such that $(R, +)$ is a monoid, $(R, \cdot)$ is a semigroup, and both structures are linked through a single (left or right) distributive law with 0 as a one sided absorbing element. In addition, if $0 \in R$ such that $a + 0 = 0 + a = a$, $a \cdot 0 = 0 \cdot a = 0$, then we call R a zero-symmetric near-semiring (or seminearring). Near-semirings established naturally from the mappings on monoid to itself under component-wise addition and composition of mappings. Several algebraists have considered near-semirings in different aspects. Due to having strong relationships of near-semirings with different types of algebras, recently the authors [7] introduced the notion of balanced near-semirings. On the other hand, ideals have their own importance in algebra, particularly when we study rings with two binary operations. In general, ideals in a near-semiring do not match with the standard ideals of rings (resp., nearrings, semirings) and hence numerous results in rings (resp., nearrings, semirings) theory have no similarities in near-semirings using merely ideals. Thus, the classified notion of ideal i.e., S-ideal [2] has been introduced in seminearring (zero symmetric near-semirings) theory. Furthermore, special types of prime ideals of seminearrings have been explored in [12].

In this paper, we initiate the notion of soft near-semirings and apply a number of soft set operations for the sake of investigation. We present the concepts of soft subnear-semiring, soft ideals, idealistic soft near-semirings and provide numerous explanatory examples for every notion. We also introduce the notion of chain conditions among the soft sets and by applying it we explore that if the chain condition exists among soft sets then we can apply the union operation without imposing any further restriction. We also apply the concept of chain conditions to soft near-semirings. By utilizing the term g-soft subset [10], we investigate few results related with soft near-semirings. Moreover, the notions of soft near-semiring homomorphism and soft near-semiring anti-homomorphisms are also introduced along with suitable examples. We investigate that the soft anti-homomorphic image of soft right (left) near-semiring (resp., ideal) is a soft left (right) near-semiring (resp., ideal). We mainly emphasis on the algebraic properties of soft near-semirings. Moreover, near-semirings have a lot of applications towards computer languages, so one can expect that soft near-semirings will play a role in fuzzy (soft) languages.

2. Preliminaries

An algebraic system $(R, +, \cdot)$ is said to be a right (resp., left) near-semiring if R is a an additive monoid under addition, semigroup w.r.t multiplication, satisfying right (resp., left) distributive law $(a+b)\cdot c = a\cdot c + b\cdot c$ (resp., $c\cdot(a+b) = c\cdot a + c\cdot b$), $\forall\, a, b, c \in R$ and $0\cdot\ a = 0$, $\forall\, a \in R$. In addition, if $0\cdot\ a = a\cdot\ 0 = 0$ for all $a \in R$, then we call it a zero-symmetric near-semiring (or seminearrings). The additive semigroup $(R, +)$ of a near-semiring $(R, +, \cdot)$ is said to be an R-semigroup.

We call a subset S of a near-semiring $(R, +, \cdot)$ a subnear-semiring if it is closed under both the operations "+" and "·". In other words, a subsemigroup $(S, +)$ of a near-semiring $(R, +, \cdot)$ is a sub near-semiring, if $0 \in S$ and $SS \subseteq S$.

In [9] the authors defined the ideal of a seminearring (zero symmetric near-semiring) as the kernel of a seminearring homomorphism. Later, [2] J. Ahsan generalized the definition of ideal of seminearrings. Following [2], a subset I of a seminearring R is said to be a right (left) S-ideal if (i) $x + y \in I$, for all x, $y \in I$, (ii) $xr(rx) \in I$, for all $x \in I$ and $r \in R$.

A mapping $\phi : R \to R'$ is said to be a homomorphism between near-semirings R and R', if

$$\phi(a+b) = \phi(a) + \phi(b),\ \phi(ab) = \phi(a)\phi(b)$$

for all a, $b \in R$ and $\phi(0) = 0$. Anti-homomorphism between near-semirings R and R' is the map $\phi : R \to R'$ satisfying $\phi(a+b) = \phi(a) + \phi(b)$, $\phi(ab) = \phi(b)\phi(a)$ and $\phi(0) = 0$ for all $a, b \in R$.

For further concepts about near-semirings one may consult [9, 13, 21].

Throughout, by R we mean right near-semiring (or seminearring) unless otherwise specified and by an ideal of a near-semiring we mean a subset I of a near-semiring R which satisfies (i) $x + y \in I$, for all $x, y \in I$ and (ii) $xr(rx) \in I$, for all $x \in I$ and $r \in R$.

For the sake of completion, we present few useful terminologies from soft set theory. Here, U represent the initial universe set, E stands for the set of parameters, $\wp(U)$ is the power set of U and A is contained in E.

Definition 1. *[17]Let η be the map from $A \to \wp(U)$. Then, the pair (η, A) is said to be a soft set over U.*

Definition 2. *[15]If (β, B) and (γ, C) be the two soft sets over the same universe U, then we call (β, B) is a soft subset of (γ, C) i.e., $(\beta, B) \tilde{\subset} (\gamma, C)$, if; (i) $B \subset C$, (ii) For each $\varepsilon \in B$, $\beta(\varepsilon)$ and $\gamma(\varepsilon)$ are identical approximations. Similarly, if $(\alpha, A) \tilde{\subset} (\beta, B)$ and $(\beta, B) \tilde{\subset} (\alpha, A)$, then $(\alpha, A) = (\beta, B)$.*

Definition 3. *[15] Union of two soft sets (α, A) and (β, B) over U is the soft set (ϑ, C), where $C = A \cup B$, and for all $c \in C$ we have,*

$$\vartheta(c) = \begin{cases} \alpha(c), & \text{if } c \in A \setminus B, \\ \beta(c), & \text{if } c \in B \setminus A, \\ \alpha(c) \cup \beta(c), & \text{if } c \in A \cap B. \end{cases}$$

We denote it by $(\alpha, A) \tilde{\cup} (\beta, B) = (\vartheta, C)$.

Definition 4. *[10] Let (α, A), (β, B) be two soft sets over U. Then,*
(i) *(α, A) is a (generalized) g-soft subset of (β, B) written as $(\alpha, A) \prec (\beta, B)$, if for each $a \in A$ there exists $b \in B$ implies $\alpha(a) \subset \beta(b)$.*

(ii) *(α, A) and (β, B) are generalized g-soft equal i.e., $(\alpha, A) \underset{g}{=} (\beta, B)$, if (α, A) is a g-soft subset of (β, B) and (β, B) is a g-soft subset of (α, A).*

Definition 5. *[4] Let us consider a soft sets (α, A), (β, B) over U. Then,*
(i) *the extended intersection of both soft subsets over U is the soft set $(\boldsymbol{\theta}, C)$, where $C = A \cup B$, and for all $e \in C$, we have*

$$\boldsymbol{\theta}(e) = \begin{cases} \alpha(e), & \text{if } e \in A \setminus B, \\ \beta(e), & \text{if } e \in B \setminus A, \\ \alpha(e) \cup \beta(e), & \text{if } e \in A \cap B. \end{cases}$$

It is represented by $(\alpha, A) \underset{e}{\cap} (\beta, B) = (\boldsymbol{\theta}, C)$.
(ii) *the restricted union of (α, A) and (β, B), where $A \cap B \neq \emptyset$ over the same universe U is represented by $(\alpha, A) \cup_R (\beta, B)$, and is defined by $(\alpha, A) \cup_R (\beta, B) = (\gamma, C)$ where $C = A \cap B$ and for all $c \in C$, $\gamma(c) = \alpha(c) \cup \beta(c)$.*

Definition 6. *[8] Let $(\zeta_i, A_i)_{i\in I}$ be the family of non-empty soft sets over the same universe U. Then,*
(i) *their union $B = \cup_{i\in I} A_i$ and for all $x \in B$, $\beta(x) = \cup_{i\in I(x)} \zeta_i(x)$ where $I(x) = \{i \in I : x \in A_i\}$. We represent $\tilde{U}_{i\in I}(\zeta_i, A_i) = (\beta, B)$.*
(ii) *AND-soft set denoted by $\tilde{\wedge}_{i\in I}(\zeta_i, A_i)$ is the soft (Ψ, B) such that $B = \prod_{i\in I} A_i$, and $\Psi(x) = \cap_{i\in I} \zeta_i(x_i)$ for all $x = (x_i)_{i\in I} \in B$. Likewise, the OR-soft set operation represented by $\tilde{\vee}_{i\in I}(\zeta_i, A_i)$ is the soft set (α, B) such that $B = \prod_{i\in I} A_i$ and $\alpha(x) = \cup_{i\in I} \zeta_i(x_i)$ for all $x = (x_i)_{i\in I} \in B$.*

3. Soft near-semirings

Let R be a near-semiring, A the nonempty set, and $\mathfrak{R}$ is the binary relation between A and R i.e., the subset of $A \times R$. Then, we define the set-valued function $\eta : A \to \wp(R)$ by $\eta(x) = \{y \in R : (x, y) \in \mathfrak{R}\}$ for all $x \in A$. We call a pair (η, A) , the soft set over R and is established by the relation $\mathfrak{R}$. Throughout, g-soft represents the generalized soft set.

Definition 7. *Let (ζ, A) be a non-null soft set over a near-semiring R. Then, (ζ, A) is said to be a soft near-semiring over R, if $\zeta(x)$ is a subnear-semiring of R for all $x \in Supp(\zeta, A)$.*

Example 1. Consider a right near-semiring $R = \{0, a_1, a_2, a_3, a_4\}$ with operations tables given below.

$+$	0	a_1	a_2	a_3	a_4
0	0	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_2	a_4	a_4
a_2	a_2	a_2	a_2	a_4	a_4
a_3	a_3	a_4	a_4	a_3	a_4
a_4	a_4	a_4	a_4	a_4	a_4

$\cdot$	0	a_1	a_2	a_3	a_4
0	0	0	0	0	0
a_1	0	a_1	a_1	a_1	a_1
a_2	0	a_1	a_2	a_2	a_4
a_3	0	a_1	a_2	a_2	a_4
a_4	0	a_1	a_2	a_2	a_4

Now for $A = \{0, a_2, a_4\}$, we define a map $\zeta : A \to \wp(R)$ by $\zeta(x) = \{y \in R : x\mathfrak{R}y \Leftrightarrow xy \in \{0, a_2, a_4\}\}$, for all $x \in A$. Then, $\zeta(0) = R$, $\zeta(a_2) = \{0, a_2, a_3, a_4\}$, $\zeta(a_4) = \{0, a_2, a_3, a_4\}$ are subnear-semirings of R. Hence (ζ, A) is a soft (right) near-semiring.

Definition 8. *Let (ζ, A) and (η, B) be two soft near-semirings over R. Then, the soft near-semiring (η, B) is said to be a soft subnear-semiring of (ζ, A) if:*
(i) *$B \subset A$,*
(ii) *$\eta(x)$ is a subnear-semiring of $\zeta(x)$ for all $x \in Supp(\eta, B)$.*

Example 2. Let us consider $R = \{0, a_1, a_2, a_3, a_4\}$ be a (right) near-semiring whose operations tables are given in example1. Let $A = \{0, a_2, a_4\}$ and $B = \{0, a_4\}$. *We define a mapping* $\zeta : A \to \wp(R)$ and $\eta : B \to \wp(R)$ by $\zeta(x) = \{y \in R : x\mathfrak{R}y \Leftrightarrow xy \in \{0, a_2, a_4\}$ and $\eta(x) = \{y \in R : x\mathfrak{R}y \Leftrightarrow xy \in \{0, a_2, a_4\}$. *Then,* $\zeta(0) = R$, $\zeta(a_2) = \{0, a_2, a_3, a_4\}$, $\zeta(d) = \{0, a_2, a_3, a_4\}$ are (right) subnear-semirings of R. Also $\eta(0) = R$ and $\eta(a_4) = \{0, a_2, a_3, a_4\}$ are soft (right) subnear-semirings of $\zeta(0)$ and $\zeta(a_4)$ respectively. Hence (η, B) is a soft (right) subnear-semiring of $\zeta(x)$ for all $x \in Supp(\eta, B)$.

Exploiting definition4, we can define a generalized soft subnear-semiring i.e., g-soft subnear-semiring as follows.

Definition 9. *Let (ζ, A) and (η, B) be two soft near-semirings over R. Then, the soft near-semiring (η, B) is called a g-soft subnear-semiring of (ζ, A), if for each $b \in B$ there exist $a \in A$ such that $\eta(b)$ is a subnear-semiring of $\zeta(a)$.*

Example 3. In example*2,* we observe that $\eta(0) = R$ is a g-soft subnear-semiring of $\zeta(0)$. On the other hand, $\eta(a_4)$ is a g-soft subnear-semiring of soft near-semirings $\zeta(a_4)$ and $\zeta(0)$. Thus, (η, B) is a g-soft subnear-semiring of $\zeta(x)$.

Remark 1. Soft subnear-semiring of a soft near-semiring (ζ, A) implies$g$-soft subnear-semiring. But the converse doesn't exist, in general.

Hereafter, we will continue with the usual operations because we have noticed that each result of soft subnear-semirings is also valid for g-soft subnear-semirings.

Theorem 1. *Let (ζ, A) and (η, B) be the two soft (right) near-semirings over R. Then, $(\zeta, A) \tilde{\wedge} (\eta, B)$ is a soft (right) near-semiring over R if it is non-null.*

Proof. Let $(\zeta, A) \tilde{\wedge} (\eta, B) = (\Psi, C)$, where $C = A \times B$ and $\Psi(x, y) = \zeta(x) \cap \eta(y)$ for all $(x, y) \in C$ [15]. By assumption, (Ψ, C) is a non-null soft set over R, if $(x, y) \in Supp\,(\Psi, C)$ then $\Psi(x, y) = \zeta(x) \cap \eta(y) \neq \emptyset$. $\zeta(x)$ and $\eta(y)$ are subnear-semirings of R implies $\Psi(x, y)$ is a subnear-semiring of R for all $(x, y) \in Supp(\Psi, C)$ and thus $(\Psi, C) = (\zeta, A) \tilde{\wedge} (\eta, B)$ is a soft near- semiring over R. □

Evidently, if (η, B) is a soft subnear-semiring of (ζ, A), then $Supp(\eta, B) \subset Supp(\zeta, A)$.

Example 4. Let $R = \{0, a_1, a_2\}$ be a (right) near-semiring with the operations tables given below.

$+$	0	a_1	a_2
0	0	a_1	a_2
a_1	a_1	a_1	a_2
a_2	a_2	a_2	a_2

$\cdot$	0	a_1	a_2
0	0	a_1	a_2
a_1	a_1	a_1	a_2
a_2	a_2	a_1	a_2

Now for $A = \{0, a_1\}$, we define the mapping $\zeta : A \to \wp(R)$ by $\zeta(x) = \{y \in R : x\Re y \Leftrightarrow x + y = \{0, a_1\}\}$, and for $B = \{0, a_2\}$ let $\eta : B \to \wp(R)$ defined by $\eta(x) = \{y \in R : x\Re y \Leftrightarrow xy \in \{0, a_2\}\}$ respectively. Now $\zeta(0) = \{0, a_1\}$, $\zeta(a_1) = \{0, a_1\}$, hence (ζ, A) is a soft (right) near-semiring over near-semiring R. Also, $\eta(0) = \{0, a_2\}$ and $\eta(a_2) = \{0, a_2\}$, and thus (η, B) is a soft near-semiring. Taking $A \times B = \{(0, 0), (0, a_2), (a_1, 0), (a_1, b)\}$ and following the definition of $(\zeta, A)\ \tilde{\wedge}(\eta, B)$, we have $(0, 0) \in A \times B$, and $\Psi(0, 0) = \zeta(0) \cap \eta(0) = \{0\}$, also for $(0, a_2) \in A \times B$, $\Psi(0, a_2) = \zeta(0) \cap \eta(a_2) = \{0\}$; $\Psi(a_1, 0) = \zeta(a_1) \cap \eta(0) = \{0\}$ are all soft near-semirings over the near-semiring R. Consequently, $(\zeta, A)\ \tilde{\wedge}\ (\eta, B)$ is a soft near-semiring.

Theorem 2. *Let (ζ, A) and (η, A) be the two soft near-semirings over the near-semiring R. Then,*
(1) $(\zeta, A)\tilde{\sqcap}\ (\eta, A)$ *is a soft near-semiring over R if it is non-null.*
(2) *If $\eta(x) \subset \zeta(x)$ for all $x \in A$, then (η, A) is a soft subnear-semiring of (ζ, A).*
(3) $(\zeta, A)\tilde{\sqcap}\ (\eta, A)$ *is a soft subnear-semiring of both (ζ, A) and (η, A), if it is non-null.*

Proof. (1) We know that $(\zeta, A)\ \tilde{\sqcap}\ (\eta, A) = (\xi, A)$, where $\xi(x) = \zeta(x) \cap \eta(x)$ for all $x \in A$ [8]. Let (ξ, A) be a non-null soft set over R. If $x \in Supp\ (\xi, A)$ then $\xi(x) = \zeta(x) \cap \eta(x) \neq \emptyset$. Since, the sets $\zeta(x)$ and $\eta(x)$ are both non-empty and subnear-semirings of R implies $\xi(x)$ is a subnear-semiring of R for all $x \in Supp(\xi, A)$, and thus $(\xi, A) = (\zeta, A)\ \tilde{\sqcap}\ (\eta, B)$ is a soft near-semiring over R, as required.

(2) Immediate.

(3) Follows from (1) & (2). □

Proposition 1. *Let $(\zeta_i, A_i)_{i\in I}$ be the family of non-empty soft near-semirings over a near-semiring R. Then,*

(1) $\tilde{\wedge}_{i\in I}(\zeta_i, A_i)$ *is a soft near-semiring over R if it is non-null.*

(2) $\tilde{\sqcap}_{i\in I}(\zeta_i, A_i)$ *is a soft near-semiring over R if it is non-null.*

Proof. (1) Following definition 6(*ii*), we assume (Ψ, B) is non-null. By hypothesis, $x = (x_i)_{i\in I} \in Supp(\Psi, B)$. Then, $\Psi(x) = \cap_{i\in I}\zeta_i(x_i) \neq \emptyset$. Since the non-empty set $\zeta_i(x_i)$ is a subnear-semiring of R which implies (ζ_i, A_i) is a soft near-semiring over R for each $i \in I$ and hence $\Psi(x)$ is a subnear-semiring of R for each $x \in Supp(\Psi, B)$. It follows that $\tilde{\wedge}_{i\in I}\ (\zeta_i, A_i) = (\Psi, B)$ is a soft near-semiring over R.

(2) Since we know that $\tilde{\sqcap}_{i\in I}(\zeta_i, A_i) = (\xi, B)$, where $B = \cap_{i\in I}A_i$, and $\xi(x) = \cap_{i\in I}\zeta_i(x)$ for all $x \in B$. We assume (ξ, B) is non-null. If $x \in Supp(\xi, B)$ then $\xi(x) = \cap_{i\in I}\zeta_i(x) \neq \emptyset$. Thus, for each $i \in I$, the non-empty set $\zeta_i(x)$ is a sub near-semiring of R, as (ζ_i, A_i) is a soft near-semiring over R. So $\xi(x)$ is a subnear-semiring of R for all $x \in Supp(\xi, B)$, and hence $\tilde{\sqcap}_{i\in I}(\zeta_i, A_i) = (\xi, B)$ is a soft near-semiring over R. □

Theorem 3. *Let $(\zeta_i, A_i)_{i\in I}$ be the family of soft near-semiring over the near-semiring R. If the family $\{A_i : i \in I\}$ is a pairwise disjoint, then $\tilde{\cup}_{i\in I}(\zeta_i, A_i)$ is a soft near-semiring over the near-semiring R.*

Proof. Let $\tilde{\cup}_{i\in I}(\zeta_i, A_i) = (\vartheta, B)$. Consider $B = \cup_{i\in I}A_i$ and for all $x \in B$, $\vartheta(x) = \cup_{i\in I(x)}\zeta_i(x)$ where $I(x) = \{i \in I : x \in A_i\}$. First note that (ϑ, B) is non-null, since $Supp\ (\vartheta, B) = \cup_{i\in I}Supp(\zeta_i, A_i) \neq \emptyset$. Consider $x \in Supp(\vartheta, B)$, then $\vartheta(x) = \cup_{i\in I(x)}\ \zeta_i(x) \neq \emptyset$. Thus, we have $\zeta_{i_0}(x) \neq \emptyset$, for some $i_0 \in I(x)$. Since $\{A_i : i \in I\}$ are pairwise disjoint so i_0 is the unique and hence $\eta(x)$ coincides with $\zeta_{i_0}(x)$. (ζ_{i_0}, A_{i_0}) being a soft near-semiring over R which implies that the nonempty set $\zeta_{i_0}(x)$ is a subnear-semiring of R. It *follows that $\eta(x) = \zeta_{i_0}(x)$ is a subnear-semiring of R for all $x \in Supp(\vartheta, B)$. Hence $\tilde{\cup}_{i\in I}(\zeta_i, A_i) = (\vartheta, B)$ is a soft near-semiring over R.* □

While considering the non-disjoint family $\{A_i : i \in I\}$, the above theorem doesn't hold true.

Example 5. Let $R = \{0, a_1, a_2, a_3, a_4\}$ be a (right) near-semiring with the operations tables given below.

$+$	0	a_1	a_2	a_3	a_4
0	0	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_2	a_4	a_4
a_2	a_2	a_2	a_2	a_4	a_4
a_3	a_3	a_4	a_4	a_3	a_4
a_4	a_4	a_4	a_4	a_4	a_4

$\cdot$	0	a_1	a_2	a_3	a_4
0	0	0	0	0	0
a_1	0	a_1	a_1	a_1	a_1
a_2	0	a_2	a_2	a_2	a_4
a_3	0	a_3	a_3	a_3	a_3
a_4	0	a_4	a_4	a_4	a_4

Now for $A = \{0, a_1\}$, let $\zeta : A \to \wp(R)$ be a set-valued function defined by $\zeta(x) = \{y \in R : x\mathfrak{R}y \Leftrightarrow x + y \in \{0, a_1\}$ for all $x \in A$. Then, $\zeta(0) = \{0, a_1\}$, $\zeta(a_1) = \{0, a_1\}$, are subnear-semirings of R and hence (ζ, A) is a soft near-semiring over R. Similarly, for $B = \{0, a_3\}$, let $\eta : B \to \wp(R)$ be the function defined by $\eta(x) = \{y \in R : x\mathfrak{R}y \Leftrightarrow x + y \in \{0, a_3\}$ for all $x \in B$. Then, $\eta(0) = \{0, a_3\}$ and $\eta(c) = \{0, a_3\}$ are (right) subnear-semirings of R so (η, B) is a soft near-semiring over R. We see that A and B are not disjoint, we have $A \cap B = \{0\}$, and $\gamma(0) = \zeta(0) \cup \eta(0) = \{0, a_1, a_3\}$ is not a subnear-semiring of R for $a_1 + a_3 = a_4 \notin \gamma(0)$. Hence $(\zeta, A) \tilde{\cup} (\eta, B) = (\gamma, C)$ is not a soft (right) near-semiring over R if $A \cap B \neq \emptyset$.

To overcome the condition of "pairwise disjoint family" in theorem3, we introduce the concept of chain condition in definition10, and for illustration we also provide an example6.

Definition 10. *Let $(\alpha_i, A_i)_{i\in I}$ be the family of soft near-semirings over the near-semiring R. We say that the chain among the soft near-semirings exists if for each $x \in A_i$, there exists a chain $\alpha_1(x) \subseteq \alpha_2(x) \subseteq ...$ which terminates. OR if there exist a chain among soft near-semirings $(\alpha_1, A_1) \subseteq (\alpha_2, A_2) \subseteq (\alpha_3, A_3)$ ... which terminates.*

We illustrate definition10 by providing example.

Example 6. Let $R = \{0, a_1, a_2\}$ be a (right) near-semiring with the operations tables given below.

$+$	0	a_1	a_2
0	0	a_1	a_2
a_1	a_1	a_1	a_2
a_2	a_2	a_2	a_2

$\cdot$	0	a_1	a_2
0	0	a_1	a_2
a_1	a_1	a_1	a_2
a_2	a_2	a_1	a_2

For $A = \{0, a_1\}$, let $\zeta : A \to \wp(R)$ be the set-valued function defined by $\zeta(x) = \{0\} \cup \{y \in R : x\mathfrak{R}y \Leftrightarrow xy \in \{a_2\}\}$ for all $x \in A$, then $\zeta(0) = \{0, a_2\}$ and $\zeta(a_1) = \{0, a_2\}$ are subnear-semirings of R and thus (ζ, A) is a soft near-semiring. Again consider $B = \{0, a_1, a_2\}$, and $\eta : B \to \wp(R)$ be a set-valued function defined by $\eta(x) = \{y \in R : x\mathfrak{R}y \Leftrightarrow x + y \in \{0, a_1, a_2\}\}$, then $\eta(0) = R$, $\eta(a_1) = R$, and $\eta(a_2) = R$, hence (η, B) is a soft near-semiring over R. Since $\zeta(0) \subseteq \eta(0)$ and $\zeta(a_1) \subseteq \eta(a_1)$ hence $(\zeta, A) \subseteq (\eta, B)$. As $A \cap B = \{0, a_1\} \neq \emptyset$, and $\vartheta(0) = \zeta(0) \cup \eta(0) = \eta(0) = \{0, a_2\}$, $\vartheta(a_1) = \zeta(a_1) \cup \eta(a_1) = R$, and $\vartheta(a_2) = R$ are all soft near-semirings. Thus, $(\zeta, A) \tilde{\cup} (\eta, B) = (\vartheta, C)$ where $C = A \cup B$ is a soft near-semiring.

Referring example 5, in which we have seen that if the chain condition on soft sets doesn't exist then the union of two soft near-semirings is not a soft near-semiring. Similarly, we can use freely the operation of restricted union for the family of non-null soft sets if they obey the chain condition, we demonstrate it in theorem 4.

Definition 11. *Let $(\alpha_i, A_i)_{i\in I}$ be a non-empty family of soft sets over a common universe U with $\cap_{i\in I}A_i \neq \emptyset$. Then, the restricted union of these soft sets is denoted by $\cup_{\mathcal{R}i\in I}(\alpha_i, A_i)_{i\in I}$ defined to be the soft set (β, B) such that $B = \cap_{i\in I}A_i$ and for all $x \in B$, $\beta(x) = \cup_{i\in I(x)}\alpha_i(x)$ where $I(x) = \{i \in I : x \in A_i\}$. In this case, we write $\cup_{\mathcal{R}i\in I}(\alpha_i, A_i)_{i\in I} = (\beta, B)$.*

Theorem 4. *Let $(\alpha_i, A_i)_{i\in I}$ be the family of soft near-semiring over R with $\cap_{i\in I}A_i \neq \emptyset$ satisfying the chain condition. Then, $\cup_{\mathcal{R}i\in I}(\alpha_i, A_i)_{i\in I}$ s a soft near-semiring over the near-semiring R.*

Proof. By definition11, we have $\cup_{\mathcal{R}i\in I}(\alpha_i, A_i)_{i\in I} = (\beta, B)$, where $B = \cap_{i\in I}A_i \neq \emptyset$ and for all $x \in B, \beta(x)$ $= \cup_{i\in I(x)}\alpha_i(x)$. Since (β, B) is non-null, $Supp\ (\beta, B) = \cup_{i\in I}Supp(\alpha_i, A_i)_{i\in I} \neq \emptyset$. Let $x \in Supp(\beta, B)$, then $\beta(x) = \cup_{i\in I}\alpha_i(x) \neq \emptyset$. Thus, $\alpha_{i_0}(x) \neq \emptyset$; for some $i_0 \in I(x)$. By assumption $B = \cap_{i\in I}A_i \neq \emptyset$, let $x \in B$ such that $\beta(x) = \cup_{i\in I(x)}\alpha_i(x)$. Let i_0 be the supremum for $i \in I(x)$ such that $\beta(x)$ coincides with $\alpha_{i_0}(x)$. Also (α_{i_0}, A_{i_0}) is a soft near-semiring over R, which implies that the non-empty set $\alpha_{i_0}(x)$ is a subnear-semiring of R. Thus, $\beta(x) = \alpha_{i_0}(x)$ is a subnear-semiring of R for all $x \in Supp(\beta, B)$. Hence $\cup_{\mathcal{R}i\in I}(\alpha_i, A_i)_{i\in I} = (\beta, B)$ is a soft near-semiring over R. □

It is notable that if the chain among different soft sets over the same universe U exists then the operation of union as well as restricted union can be freely apply to soft sets (resp., soft near-semirings).

3.1. Soft ideals and idealistic soft near-semirings

Definition 12. *Let (ζ, A) be a soft near-semiring over a near-semiring R. A non-null soft set (γ, I) over R is said to be a soft left (resp. right) ideal of (ζ, A), and is represented by $(\gamma, I)\ \tilde{\lhd}_l\ (\zeta, A)$ (resp. $(\gamma, I)\ \tilde{\lhd}_R\ (\zeta, A)$), if it satisfies.*

(i) $I \subset A$.

(ii) $\gamma(x)\ \tilde{\lhd}_l\ \zeta(x)$ (resp. $\gamma(x)\ \tilde{\lhd}_R\ \zeta(x)$) for all $x \in (\gamma, I)$.

If (γ, I) is both soft left and soft right ideal of (ζ, A), then we call (γ, I) is the soft ideal of (ζ, A) and is mentioned by $(\gamma, I)\ \tilde{\lhd}\ (\zeta, A)$.

Example 7. Let $R = \{0, a_1, a_2, a_3, a_4\}$ be a (right) near-semiring with the operations tables given in example*1*. Let $A = \{0, a_2, a_4\}$, and $\zeta : A \longrightarrow \wp(R)$ is defined by

$$\zeta(x) = \{y \in R : x\mathfrak{R}y \Leftrightarrow xy \in \{0, a_2, a_4\}\} \text{ for all } x \in A.$$

Clearly, $\zeta(0) = R$, $\zeta(a_2) = \{0, a_2, a_3, a_4\}$ and $\zeta(a_4) = \{0, a_2, a_3, a_4\}$ are (right) subnear-semirings over R, hence (ζ, A) be a soft near-semiring. Let (γ, I) be the function $\gamma : I \longrightarrow \wp(R)$ defined as $\gamma(x) = \{y \in R : x\mathfrak{R}y \Leftrightarrow xy \in \{0, a_1\}\}$ where $I = \{a_2, a_4\} \subset A$. Then, $\gamma(a_2) = \{0, a_1\}\ \tilde{\lhd}\ \zeta(a_1) = \{0, a_2, a_3, a_4\}$, $\gamma(a_4) = \{0, a_1\}\ \tilde{\lhd}\ \zeta(a_4) = \{0, a_2, a_3, a_4\}$, thus (γ, I) is a soft (right) ideal of (ζ, A). Hence $(\gamma, I)\ \tilde{\lhd}\ (\zeta, A)$.

Proposition 2. *Let (γ_1, I_1) and (γ_2, I_2) be soft ideals of a soft near-semiring (ζ, A) over R. Then $(\gamma_1, I_1)\ \tilde{\sqcap}\ (\gamma_2, I_2)$ is a soft ideal of (ζ, A) if it is non-null.*

Proof. Assume that $(\gamma_1, I_1) \mathrel{\tilde{\lhd}} (\zeta, A)$ and $(\gamma_2, I_2) \mathrel{\tilde{\lhd}} (\zeta, A)$. We can write $(\gamma_1, I_1) \mathrel{\tilde{\sqcap}} (\gamma_2, I_2) = (\gamma, I)$, where $I = I_1 \cap I_2$ and $\gamma(x) = \gamma_1(x) \cap \gamma_2(x)$ for all $x \in I$. We have $I \subset A$, assume that (γ, I) is non-null. If $x \in Supp(\gamma, I)$, then $\gamma(x) = \gamma_1(x) \cap \gamma_2(x) \neq \emptyset$. Since $(\gamma_1, I_1) \mathrel{\tilde{\lhd}} (\zeta, A)$ and $(\gamma_2, I_2) \mathrel{\tilde{\lhd}} (\zeta, A)$, we deduce that non-empty sets $\gamma_1(x)$ and $\gamma_2(x)$ are the ideals of $\zeta(x)$. It follows that $\gamma(x) \mathrel{\tilde{\lhd}} \zeta(x)$ for all $x \in Supp(\gamma, I)$. Therefore $(\gamma_1, I_1) \cap (\gamma_2, I_2) = (\gamma, I)$ is a soft ideal of (ζ, A). Thus $(\gamma_1, I_1) \mathrel{\tilde{\sqcap}} (\gamma_2, I_2) = (\gamma, I)$ is a soft ideal of (ζ, A). □

Example 8. Refer to the near-semiring described in the tables of example*1*. Let $A = \{0, a_2, a_4\}$, and $\zeta : A \longrightarrow \wp(R)$ be the set-valued function defined by $\zeta(x) = \{y \in R : x\mathfrak{R}y \Leftrightarrow xy \in \{0, a_2, a_4\}$ for all $x \in A$. Then $\zeta(0) = R$, $\zeta(a_2) = \{0, a_2, a_3, a_4\}$ and $\zeta(a_4) = \{0, a_2, a_3, a_4\}$ are subnear-semirings over R, hence (ζ, A) be a soft near-semiring. Let (γ_1, I_1) be a soft set over R where $I_1 = \{a_2, a_4\} \subset A = \{0, a_2, a_4\}$ such that $\gamma_1(x) = \{y \in R : x\mathfrak{R}y \Leftrightarrow xy \in \{0, a_2\}\}$, then $\gamma_1(a_2) = \{0, a\} \mathrel{\tilde{\lhd}} \zeta(a_2) = \{0, a_2, a_3, a_4\}$, $\gamma_1(a_4) = \{0, a_1\} \mathrel{\tilde{\lhd}} \zeta(a_4) = \{0, a_2, a_3, a_4\}$, thus (γ_1, I_1) is a soft ideal of (ζ, A) i.e., $(\gamma_1, I_1) \mathrel{\tilde{\lhd}} (\zeta, A)$. Again consider (γ_2, I_2) where $I_2 = \{0, a_2\} \subset A = \{0, a_2, a_4\}$, and $\gamma_2(x) = \{y \in R : xy \in \{0, a_1\}\}$, $\gamma_2(0) = \{0, a_1, a_2, a_3, a_4\} \mathrel{\tilde{\lhd}} \zeta(0) = R$, $\gamma_2(a_2) = \{0, a_1\} \mathrel{\tilde{\lhd}} \zeta(a_2) = \{0, a_2, a_3, a_4\}$, thus (γ_2, I_2) is also soft ideal of (ζ, A) i.e., $(\gamma_2, I_2) \mathrel{\tilde{\lhd}} (\zeta, A)$. Let $(\gamma_1, I_1) \mathrel{\tilde{\sqcap}} (\gamma_2, I_2) = (\gamma, I)$ where $I = I_1 \cap I_2 = \{a_2\}$. For every $x \in I$ we have $\gamma(x) = \gamma_1(x) \cap \gamma_2(x)$ that is $\gamma(a_2) = \gamma_1(a_2) \cap \gamma_2(a_2) = \{0, a_1\} \mathrel{\tilde{\lhd}} \zeta(a_2) = \{0, a_2, a_3, a_4\}$, hence $(\gamma_1, I_1) \mathrel{\tilde{\sqcap}} (\gamma_2, I_2)$ is a soft (right) ideal of (ζ, A).

Theorem 5. *Let (γ_1, I) and (γ_2, J) be the soft ideals of a soft near-semiring (ζ, A) over R. If I and J are disjoint then $(\gamma_1, I) \mathrel{\tilde{\cup}} (\gamma_2, J)$ is a soft ideal of (ζ, A).*

Proof. Consider $(\gamma_1, I) \mathrel{\tilde{\lhd}} (\zeta, A)$, $(\gamma_2, J) \mathrel{\tilde{\lhd}} (\zeta, A)$ and $(\gamma_1, I) \mathrel{\tilde{\cup}} (\gamma_2, J) = (\gamma, U)$, where $U = I \cup J$. Since $U \subset A$, we assume that I and J are disjoint, i.e., $I \cap J = \emptyset$, then for every $x \in Supp(\gamma, U)$, we have either $x \in I\backslash J$ or $x \in J\backslash I$. If $x \in I\backslash J$ then $\gamma(x) = \gamma_1(x) \neq \emptyset$ is an ideal of $\zeta(x)$ as (γ_1, I) is an ideal of (ζ, A). Again, if $x \in J\backslash I$ then $\gamma(x) = \gamma_2(x) \neq \emptyset$ is an ideal of $\zeta(x)$ because (γ_2, J) is an ideal of (ζ, A). Thus, $\gamma(x)$ is an ideal of $\zeta(x)$ for all $x \in Supp(\gamma, U)$. Hence $(\gamma_1, I) \mathrel{\tilde{\cup}} (\gamma_2, J) = (\gamma, U)$ is a soft ideal of (ζ, A). □

For illustration of theorem 5, we provide the below example.

Example 9. We refer near-semiring described in example*1*. Let $\zeta : A \longrightarrow \wp(R)$ be a set-valued function defined by $\zeta(x) = \{y \in R : x\mathfrak{R}y \Leftrightarrow xy \in \{0, a, b, d\}\}$, where $A = \{0, a, b, c, d\}$. Clearly, $\zeta(0) = R$, $\zeta(a) = R, \zeta(b) = R, \zeta(c) = R, \zeta(d) = R$ are the soft near-semirings over the near-semiring R. Let (γ_1, I_1) be the soft set over near-semiring R, where $I_1 = \{b, d\} \subset A$, $\gamma_1(x) = \{y \in R : x\mathfrak{R}y \Leftrightarrow xy \in \{0, a\}\}$, $\gamma_1(b) = \{0, a\} \mathrel{\tilde{\lhd}} \zeta(b) = R$, $\gamma_1(d) = \{0, a\} \mathrel{\tilde{\lhd}} \zeta(d)$. Hence $(\gamma_1, I_1) \mathrel{\tilde{\lhd}} (\zeta, A)$. Again consider (γ_2, I_2) be soft set over R, where $I_2 = \{0, c\} \subset A$, $\gamma_2(x) = \{y \in N : x\mathfrak{R}y \Leftrightarrow xy \in \{0, a, b, c, d\}\}$, then $\gamma_2(0) = \{0, a, b, c, d\} \mathrel{\tilde{\lhd}} \zeta(0)$, $\gamma_2(c) = \{0, a, b, c, d\} \mathrel{\tilde{\lhd}} \zeta(c)$. Consider $I_1 \cup I_2 = \{b, d\} \cup \{0, c\} = \{0, b, c, d\}$, now $\gamma_1(0) \cup \gamma_2(0) = \gamma_2(0) \mathrel{\tilde{\lhd}} \zeta(0)$, similarly $\gamma_1(b) \cup \gamma_2(b) = \gamma_1(b) \mathrel{\tilde{\lhd}} \zeta(b)$, so $\gamma_1(c) \cup \gamma_2(c) = \gamma_2(c) \mathrel{\tilde{\lhd}} \zeta(c)$ and $\gamma_1(d) \cup \gamma_2(d) = \gamma_1(d) \mathrel{\tilde{\lhd}} \zeta(d)$. Hence $(\gamma_1, I_1) \mathrel{\tilde{\cup}} (\gamma_2, I_2)$ is a soft ideal of (ζ, A) when I_1 and I_2 are disjoint.

In general, theorem 5 doesn't hold true if the ideals I and J are not pairwise disjoint as we have observed in example 5 in the case of soft near-semirings. On the other hand, if there exists chain condition between soft ideals then again $(\gamma_1, I) \mathrel{\tilde{\cup}} (\gamma_2, J)$ is a soft ideal of a soft near-semiring, we have already seen in the case of soft near-semirings in example 6.

Theorem 6. *Let $\tilde{\wedge}_{i\in I}(\zeta_i, A_i)$ be a soft near-semiring over the near-semiring R and $(\gamma_i, A_i)_{i\in I}$ be a non-empty family of soft ideals of $\tilde{\wedge}_{i\in I}(\zeta_i, A_i)$. Then, we have the followings.*
(1) *If both $\tilde{\wedge}_{i\in I}(\zeta_i, A_i)$ and $\tilde{\wedge}_{i\in I}(\gamma_i, A_i)$ are non-null then $\tilde{\wedge}_{i\in I}(\gamma_i, A_i)$ is a soft ideal of the soft near-semiring $\tilde{\wedge}_{i\in I}(\zeta_i, A_i)$.*
(2) *If $\{A_i : i \in I\}$ are pairwise disjoint, i.e., $i \neq j$, $A_i \cap A_j = \emptyset$ then $\tilde{\cup}_{i\in I}(\gamma_i, A_i)$ is a soft ideal of $\tilde{\wedge}_{i\in I}(\zeta_i, A_i)$.*

Proof. (1) We may write $\tilde{\wedge}_{i\in I}(\zeta_i, A_i) = (\Psi, B)$, where $B = \prod_{i\in I} A_i$ and $\Psi(x) = \cap_{i\in I}\zeta_i(x_i)$ for all $x = (x_i)_{i\in I} \in B$. Similarly, let $\tilde{\wedge}_{i\in I}(\gamma_i, A_i) = (\gamma, C)$, where $C = \prod_{i\in I} A_i$, and $\gamma(x) = \cap_{i\in I}\gamma_i(x_i)$ for all $x = (x_i)_{i\in I} \in C$. Suppose that (Ψ, B) and (γ, C) are non-null. Following proposition1(1), (Ψ, B) is a soft near-semiring over R. To prove (γ, C) is a soft ideal of the soft near-semiring (Ψ, B), note that $C \subset B$. Moreover, if $x = (x_i)_{i\in I} \in Supp(\gamma, C)$, then $\gamma(x) = \cap_{i\in I}\gamma_i(x_i) \neq \emptyset$. Since (γ_i, A_i) is a soft ideal of (Ψ, A) for all $i \in I$, we deduce that the nonempty set $\gamma_i(x_i)$ is an ideal of $\Psi(x_i)$ for all $i \in I$. It follows that $\gamma(x) = \cap_{i\in I}\gamma_i(x_i)$ is an ideal of $\Psi(x) = \cap_{i\in I}\zeta(x_i)$ for all $x = (x_i)_{i\in I} \in Supp\ (\gamma, C)$. Hence we conclude that $\tilde{\wedge}_{i\in I}(\gamma_i, A_i) = (\gamma, C)$ is a soft ideal of the soft near-semiring $(\Psi, B) = \tilde{\wedge}_{i\in I}(\zeta_i, A_i)$.

(2) The proof is similar to theorem5. □

Definition 13. *Let (ζ, A) be a non-null soft set over R. Then, we call (ζ, A) an idealistic soft near-semiring over R if $\zeta(x)$ is an ideal of R for every $x \in Supp\ (\zeta, A)$.*

Example 10. Let $R = \{0, a_1, a_2, a_3, a_4\}$ be a near-semiring under the operations defined in example *1*. Consider a soft near-semiring (ζ, A) described in example *1* which is not an idealistic soft near-semiring because $\zeta(a_2)$ and $\zeta(a_4)$ are not ideals of R. On the other hand, if we take $B = \{0, a_4\}$, and redefine $\zeta : B \to \wp(R)$ such that $\zeta(x) = \{y \in R : x\mathfrak{R}y \Leftrightarrow xy \in \{0\}\}$ for all $x \in B$. Then, $\zeta(0) = R$, $\zeta(a_4) = \{0\}$ are subnear-semirings of R and hence (ζ, B) is a soft near-semiring over near-semiring R. Also $\zeta(0)$ and $\zeta(a_4)$ are also ideals of R, hence (ζ, B) is an idealistic soft near-semiring over R.

Proposition 3. *Let (ζ, A) be a soft set over a near semiring R and let $B \subset A$. If (ζ, A) is an idealistic soft near-semiring over R then (ζ, B) is also an idealistic soft near-semiring, whenever it is non-null.*

Proof. Immediate. □

In general, the converse of proposition 3 is not true.

Example 11. Following near-semiring described in example *1*. Let $\zeta : A \longrightarrow \wp(R)$ be a map defined by $\zeta(x) = \{y \in R : x\mathfrak{R}y \Leftrightarrow xy \in \{0, a_1, a_2, a_4\}\}$, where $A = \{a_1, a_2\}$. Evidently, $\zeta(a_1) = R$, $\zeta(a_2) = R$ are ideals over R the soft near-semirings over the near-semiring R. Hence (ζ, A) is idealistic soft near-semiring over R.

Theorem 7. *Let (ζ, A) and (η, B) be idealistic soft near semirings over R. Then, $(\zeta, A) \tilde{\sqcap} (\eta, B)$ is an idealistic soft near-semiring over R, if it is non-null.*

Proof. Since $(\zeta, A) \tilde{\sqcap} (\eta, B) = (\xi, C)$, where $C = A \cap B$ and $\xi(x) = \zeta(x) \cap \eta(x)$ for all $x \in C$. Suppose (ξ, C) is a non-null soft set over a near-semiring R. If $x \in Supp(\xi, C)$, then $\xi(x) = \zeta(x) \cap \eta(x) \neq \emptyset$, and thus $\zeta(x)$ and $\eta(x)$ are both ideals of R. So $\xi(x)$ is an ideal of R for all $x \in Supp(\xi, C)$. Hence $(\zeta, A) \tilde{\sqcap} (\eta, B) = (\xi, C)$ is an idealistic soft near-semiring over R. □

Theorem 8. *Let* (ζ, A) *and* (η, B) *be idealistic soft near-semirings over a near-semiring R. If* $A \cap B = \emptyset$, *then the union* $(\zeta, A) \tilde{\cup} (\eta, B)$ *is an idealistic soft near-semiring over R.*

Proof. By definition3, we can write $(\zeta, A) \tilde{\cup} (\eta, B) = (\vartheta, C)$, where $C = A \cup B$ and for every $x \in C$,

$$\vartheta(x) = \begin{cases} \zeta(x), & \text{if } x \in A \setminus B, \\ \eta(x), & \text{if } x \in B \setminus A, \\ \zeta(x) \cup \eta(x), & \text{if } x \in A \cap B. \end{cases}$$

We assume that $A \cap B = \emptyset$. Then for every $x \in Supp(\vartheta, C)$, we have either $x \in A \backslash B$ or $x \in B \backslash A$. If $x \in A \backslash B$, then $\vartheta(x) = \zeta(x)$ is an ideal of R, since (ζ, A) is an idealistic soft near-semiring over R. If $x \in B \backslash A$, then $\vartheta(x) = \eta(x)$ is an ideal of R, since (η, B) is an idealistic soft semiring over R. Consequently, $\vartheta(x)$ is an ideal of R for all $x \in Supp(\vartheta, C)$. Thus $(\vartheta, C) = (\zeta, A) \tilde{\cup} (\eta, B)$ is an idealistic soft near-semiring over R. □

In general, theorem 8 doesn't hold true if A and B are not disjoint. But if idealistic soft near-semirings obeying chain condition stated in definition10 then $(\zeta, A) \tilde{\cup} (\eta, B)$ is an idealistic soft near-semiring over R.

Theorem 9. *Let* (ζ, A) *and* (η, B) *be the two idealistic soft near-semirings over near-semiring R. Then, a non-null* $(\zeta, A) \tilde{\wedge} (\eta, B)$ *is an idealistic soft near-semiring over R.*

Proof. By definition 6 (ii), we have $(\Psi, C) = (\zeta, A) \tilde{\wedge} (\eta, B)$, where $C = A \times B$ and $\Psi(x, y) = \zeta(x) \cap \eta(y)$ for all $x, y \in C$. Let (Ψ, C) is a non-null soft set over R. If $(x, y) \in Supp(\Psi, C)$, then $\Psi(x, y) = \zeta(x) \cap \eta(y) \neq \emptyset$. Since (ζ, A) and (η, B) be idealistic soft near-semirings over a near-semiring R which implies that the nonempty sets $\zeta(x)$ and $\eta(y)$ are both ideals of R. Hence (Ψ, C) is an ideal of R for all $(x, y) \in$ Supp (Ψ, C), so $(\Psi, C) = (\zeta, A) \tilde{\wedge} (\eta, B)$ is an idealistic soft near-semiring over R. □

Definition 14. *Let* (ζ, A) *and* (η, B) *be the two soft near-semirings over near-semirings* R_1 *and* R_2, *respectively. Suppose* $f : R_1 \to R_2$ *and* $g : A \to B$ *be the two mappings. Then, the pair* (f, g) *is said to be a soft near-semiring homomorphism if:*
(i) f *is a near-semiring epimorphism.*
(ii) g *is a surjective.*
(iii) $f(\zeta(x)) = \eta(g(x))$ *for each* $x \in A$.

Refer to the above definition, we say that (ζ, A) is soft homomorphic to(η, B), if the soft near-semiring homomorphism exists between (ζ, A) and (η, B). Similarly, we call (f, g) a soft near-semiring isomorphism, if f is an isomorphism of near-semirings and g is a bijective. (ζ, A) is soft isomorphic to (η, B) which is denoted by $(\zeta, A) \simeq (\eta, B)$.

Example 12. Let $R = \{0, a_1, a_2\}$ be a right near-ring in the below tables.

$+$	0	a_2	a_2
0	0	a_1	a_2
a_1	a_1	a_1	a_2
a_2	a_2	a_2	a_2

$\cdot$	0	a_1	a_2
0	0	a_1	a_2
a_1	a_1	a_1	a_2
a_2	a_2	a_1	a_2

Let us define a map $f : R \to R$ as $f(x) = x$ which is a near-semiring epimorphism. Assume $A = B = \{0, a_1, a_2\}$, and $g : A \to B$ with $g(0) = 0$, $g(a_1) = a_1$ and $g(a_2) = a_2$ is a surjective mapping. Let $\zeta : A \to \wp(R)$ by $\zeta(x) = \{0\} \cup \{y \in R : x\mathfrak{R}y \Leftrightarrow xy = x\}$ for all $x \in A$. Clearly, $\zeta(0) = \{0\}$, $\zeta(a_1) = \{0, a_1\}$, $\zeta(a_2) = \{0, a_2\}$ are subnear-semirings of R, hence (ζ, A) is a soft near-semiring over near-semiring R. A map $\eta : B \to \wp(R)$ is defined by $\eta(x) = \{0\} \cup \{y \in R : x\mathfrak{R}y \Leftrightarrow xy = x\}$. Then, $\eta(0) = \{0\}$, $\eta(a_1) = \{0, a_1\}$, $\eta(a_2) = \{0, a_2\}$ are soft near-semirings over R. We observe that$f(\zeta(0)) = f(0) = \{0\}$ and $\eta(g(0)) = \eta(0) = \{0\}$, so $f(\zeta(0)) = \eta(g(0))$. On the other hand, $f(\zeta(a)) = \{0, a_1\}$ and $\eta(g(a_1)) = \eta(a_1) = \{0, a_1\}$, which implies $f(\zeta(a_1)) = \eta(g(a_1))$. Similarly, $f(\zeta(a_2)) = f(a_2) = \{0, a_2\}$ and $\eta(g(a_2)) = \eta(a_2) = \{0, a_2\}$, so $f(\zeta(a_2)) = \eta(g(a_2))$. Hence $f(\zeta(x)) = \eta(g(x))$ for all $x \in A$.

In example 12, f is an isomorphism of near-semirings and g is a bijective mapping so (f, g) is a soft near-semiring isomorphism. Hence (ζ, A) is soft isomorphic to (η, B) i.e., $(\zeta, A) \simeq (\eta, B)$.

Proposition 4. *Let $f : R \to S$ be a near-semiring epimorphism between near-semirings R and S. If (ζ, A) is an idealistic soft near-semiring over R then $(f(\zeta), A)$ is an idealistic soft near-semiring over R.*

Proof. The set $(f(\zeta), A)$ is a non-null soft set over R as (ζ, A) is an idealistic soft near-semiring over R. Thus, for any $x \in Supp(f(\zeta), A)$, $f(\zeta)(x) = f(\zeta(x)) \neq \emptyset$, implies that the non-empty set $\zeta(x)$ is an ideal of R. Hence, under a surjective homomorphism, $f(\zeta(x))$ is an ideal of S which implies $f(\zeta(x))$ is an ideal of B for all $x \in Supp(f(\zeta), A)$. Hence $(f(\zeta), A)$ is an idealistic soft near-semiring over R. □

3.2. Soft anti-homomorphism of near-semirings

We introduce the concept of soft near-semirings anti-homomorphism and also apply it to soft near-semirings. We investigate that image of soft right (left) near-semiring under near-semiring anti-homomorphism is a soft left (right) near-semiring.

Definition 15. *Let (η, A) be the soft right near-semiring over right near-semiring R_1 and (ζ, B) be the soft left near-semiring over the left near-semiring R_2. Suppose $f : R_1 \to R_2$ and $g : A \to B$ be the mappings. The pair (f, g) is said to be a soft near-semiring anti-homomorphism if,*
(i) f is a near-semirings anti-epimorphism.
(ii) g is onto mapping.
(iii) $f(\eta(x)) = \zeta(g(x))$ for each $x \in A$.

If f is a near-semiring anti-isomorphism and g is a bijective mapping then we call (f, g) a soft near-semiring anti-isomorphism. In other words, we say that (ζ, A) is soft anti-isomorphic to (η, B) i.e., $(\eta, A) \simeq (\zeta, B)$.

Theorem 10. *The image of soft right (left) near-semiring is a soft left (right) near-semiring under the mapping of soft near-semiring anti-homomorphism.*

We provide example in the support of the above theorem.

Example 13. Consider a soft right near-semiring (η, B) defined in example*2*, and let $R = \{0, a_1, a_2, a_3, a_4\}$ be a left near-semiring with the operations tables given below.

+	0	a_1	a_2	a_3	a_4
0	0	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_2	a_4	a_4
a_2	a_2	a_2	a_2	a_4	a_4
a_3	a_3	a_4	a_4	a_3	a_4
a_4	a_4	a_4	a_4	a_4	a_4

$\cdot$	0	a_1	a_2	a_3	a_4
0	0	0	0	0	0
a_1	0	a_1	a_1	a_1	a_1
b	0	a_1	a_2	a_2	a_2
a_3	0	a_1	a_2	a_2	a_2
d	0	a_1	a_4	a_4	a_4

And for $B = \{0, a_2, a_4\}$, let $\zeta : B \to \wp(R)$ be a set valued function defined by $\zeta(x) = \{y \in R : y\Re x \Leftrightarrow yx \in \{0, a_2, a_4\}\}$ for all $x \in B$. Then, $\zeta(0) = R$, $\zeta(a_2) = \{0, a_2, a_3, a_4\}$, $\zeta(a_4) = \{0, a_2, a_3, a_4\}$ are left subnear-semirings of R. Thus (ζ, B) is a soft left near-semiring. Following example*2,* $\eta(0) = R$, $\eta(a_2) = \{0, a_2, a_3, a_4\}$, $\eta(a_4) = \{0, a_2, a_3, a_4\}$ are sub near-semirings of R. Thus (η, A) is a soft right near-semiring. Let us consider a map $f : R \to R$ defined by $f(x) = x$ is a anti-epimorphism of near-semirings, which is easy to verify that $f(a_2 \cdot a_4) = f(a_4) \cdot f(a_2)$ implies $d = d$. On the other hand, we see that $f(a_2 \cdot a_4) \neq f(a_2) \cdot f(a_4)$ since $a_4 \neq a_2$, thus f is purely an anti-homomorphism. And $g : A \to B$ defined by $g(x) = x$ is surjective mapping. Finally, it is easy to see that $f(\eta(x)) = \zeta(g(x))$ for all $x \in A$. Since f is an anti-isomorphism of near-semirings and g is a bijective mapping, hence soft right near-semiring (η, A) is anti-isomorphic to soft left near-semiring (ζ, B).

Remark 2. *Soft anti-homomorphic image of soft right* (*left*) *ideal of near-semiring is a soft left* (*right*) *ideal.*

4. Conclusions

In this manuscript, we have introduced and discussed the algebraic structure of near-semirings on the ground of soft set theory. We initiate the concept of a soft near-semiring with major focused on its algebraic properties. Numerous associated concepts are also introduced such as soft subnear-semirings, soft ideals, idealistic soft near-semirings, soft near-semiring homomorphisms and soft near-semiring anti-homomorphisms. The above newly established soft algebraic structures are further investigated by applying different operations of soft set theory. In this pursuance, the notion of chain condition is presented and applied to soft near-semirings (resp., soft ideals). Finally, we explored through soft anti-homomorphism map that the image of soft right (left) near-semiring is a soft left (right) near-semiring. Based on the obtained results, we can apply the soft set theory to some other substructures of near-semirings that would be useful in the characterization of near-semirings. We can further discuss both balanced near-semirings and affine near-semirings on the ground of soft sets. Near-semiring is used in theory of automata and other computer languages, our presented work may be useful in studying fuzzy automata.

Conflict of interest

The authors declare that they have no conflict of interest.

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