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Research article

On global well-posedness to 3D Navier-Stokes-Landau-Lifshitz equations

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Abstract: In this paper, we prove the global well-posedness of solutions for the Cauchy problem of three-dimensional incompressible Navier-Stokes-Landau-Lifshitz equations under the condition that $\|u_0\|_{H^{\frac{1}{2}}} + \|\nabla d_0\|_{H^{\frac{1}{2}+\varepsilon}}$ ($\varepsilon > 0$) is sufficiently small. This result can be seen as an improvement of the previous paper [20].

Keywords: Navier-Stokes-Landau-Lifshitz equations; local existence; global well-posedness
Mathematics Subject Classification: 35Q35, 35D35, 76D05.

1. Introduction

Consider the Cauchy problem of 3D Navier-Stokes-Landau-Lifshitz equation

$$\begin{cases} u_t + (u\cdot\nabla)u - \nu\Delta u + \nabla\cdot(\nabla d\odot\nabla d) = 0, \\ d_t + (u\cdot\nabla)d = \Delta d + |\nabla d|^2 d + d\times\Delta d, \\ \nabla\cdot u = 0, \quad |d| = 1, \\ u(x,0) = u_0(x), \quad d(x,0) = d_0(x). \end{cases} \tag{1.1}$$

where $u(x,t)$ describes the velocity, p represent the pressure and $d(x,t)$ stands for the magnetic moment respectively. The constant $\nu > 0$ means the shear viscosity coefficient of the fluid, and the symbol $\nabla d\odot\nabla d$ denotes a 3×3 matrix whose (i,j)th entry is given by $\partial_i d\cdot\partial_j d$ for $1\le i,j\le 3$. We note that if $d = 0$, system (1.1) reduces to be the classical Navier-stokes equations [9, 11, 13, 17, 22, 23], which have drawn much attention. Moreover, if $u = 0$ in system (1.1), we obtain the Landau-Lifshitz system [1–3]. In this paper, for the sake of simplicity, we set the coefficient $\nu\equiv 1$. and the operator $\Lambda^{2\delta}$ is defined through the Fourier transform (see [15]), namely

$$\Lambda^{2\delta} f(x) = (-\Delta)^{\delta} f(x) = \int_{\mathbb{R}^3} |x|^{2\delta}\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi,$$

and $\widehat{f}$ is the Fourier transform of f.

For the study on the weak solution to the incompressible Navier-Stokes-Landau-Lifshitz equations, we refer the reader to Wang and Guo [18, 19] and Guo and Liu [8]. By using the Faedo-Galerkin approximation and weak compactness theory, the authors studied the existence and uniqueness of the weak solution to system (1.1) in two-dimension and three-dimension. There are also some papers related to the strong solutions to Navier-Stokes-Landau-Lifshitz equations. In [4], by using energy methods and delicate estimates from harmonic analysis, Fan, Gao and Guo obtained some regularity criteria for the strong solutions in Besov and multiplier spaces; Supposed that $d_0 \in \dot{B}^{3/2}_{2,1}(\mathbb{R}^3)$ and $u_0 = (u_0^h, u_0^3) \in \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$ with $|d_0| = 1$ and $\nabla \cdot u_0 = 0$, by using the Fourier frequency localization and Bony's paraproduct decomposition, Zhai, Li and Yan [21] proved that there exists a unique global solution (u, d) with

$$\begin{cases} u \in C([0,\infty); \dot{B}^{1/2}_{2,1}) \bigcap \tilde{L}((0,\infty); \dot{B}^{1/2}_{2,1}) \bigcap L^1((0,\infty); \dot{B}^{5/2}_{2,1}), \\ d \in C([0,\infty); \dot{B}^{3/2}_{2,1}) \bigcap \tilde{L}([0,\infty); \dot{B}^{3/2}_{2,1}) \bigcap L^1([0,\infty); \dot{B}^{7/2}_{2,1}), \end{cases} \tag{1.2}$$

provided that the initial data satisfies

$$C\left\{\nu\left(\|u_0^h\|_{\dot{B}^{1/2}_{2,1}} + \|d_0\|_{\dot{B}^{3/2}_{2,1}}\right) + \left(\left(\|u_0^h\|_{\dot{B}^{1/2}_{2,1}} + \|d_0\|_{\dot{B}^{3/2}_{2,1}}\right)^{1/2}\left(\|u_0^3\|_{\dot{B}^{1/2}_{2,1}} + \nu\right)^{\frac{1}{2}}\right)\right\} \le \nu^2.$$

Recently, with the help of an energy method, Wei, Li and Yao [20] established the global well-posedness of strong solutions for system (1.1) provided that $\|u_0\|_{H^1} + \|d_0\|_{H^2}$ is sufficiently small. Moreover, by applying the Fourier splitting method, the authors also showed that the time decay rates of the higher-order spatial derivatives of the solutions.

Remark 1.1. When the term $d \times \Delta d$ is omitted, the system (1.1) reduces to the liquid crystals equations, which have been studied by many researchers, see for instance, [5–7, 12, 14, 16] and the reference therein.

In this paper, we first show the following local well-posedness result, which can be proved by using Banach fixed point theorem and the standard linearization argument. Since the proof is so standard, we omit it here.

Lemma 1.2 (Local well-posedness). *Let $(u_0, \nabla d_0) \in H^2(\mathbb{R}^3)$. Then, there exists a small time $\tilde{T} > 0$ and a unique strong solution $(u(x,t), \nabla d(x,t))$ to system (1.1) satisfying*

$$\begin{cases} (u, \nabla d) \in C([0,\tilde{T}]; H^2) \bigcap L^2(0,\tilde{T}; H^3), \\ (u_t, \nabla d_t) \in L^\infty(0,\tilde{T}; L^2) \bigcap L^2(0,\tilde{T}; H^1). \end{cases} \tag{1.3}$$

Now, we give the following theorem on the small initial data global well-posedness for system (1.1).

Theorem 1.3 (Small initial data global well-posedness). *Suppose $(u_0, d_0) \in H^s(\mathbb{R}^3) \times H^{s+1}(\mathbb{R}^3)$ with $s > 2$ and $\operatorname{div} u_0 = 0$. There exists a sufficiently small constant $K > 0$ and any $\varepsilon > 0$ such that if $\|u_0\|_{H^{\frac{1}{2}}} + \|\nabla d_0\|_{H^{\frac{1}{2}+\varepsilon}} \le K$, then there exists a unique global strong solution (u, d) and satisfy*

$$(u, \nabla d) \in C(0,T; H^s(\mathbb{R}^3)) \cap L^2(0,T; H^{s+1}(\mathbb{R}^3)).$$

Remark 1.4. It is worth pointing out that the constant ε in Theorem 1.3 can not reduced to 0 because of the Sobolev's embedding

$$\|d\|_{L^\infty} \le \|d\|_{L^6}^{\frac{2\varepsilon}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^{\frac{1}{1+2\varepsilon}} \le \|\Lambda d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^{\frac{1}{1+2\varepsilon}} \le \|\Lambda d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}.$$

Remark 1.5. Since one only need $\|u_0\|_{H^{\frac{1}{2}}} + \|\nabla d_0\|_{H^{\frac{1}{2}+\varepsilon}}$ is sufficiently small, this paper can be seen as an improvement of Wei, Li and Yao [20].

2. Proof of Theorem 1.3

Multiplying $(1.1)_1$ and $(1.1)_2$ by u and $\Delta d + |\nabla d|^2 d$, we obtain the fundamental energy estimate

$$\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2\right) + \|\nabla u\|_{L^2}^2 + \|\Delta d + |\nabla d|^2 d\|_{L^2}^2 = 0, \quad \forall t \ge 0. \tag{2.1}$$

Taking $\Lambda^{\frac{1}{2}}$ to $(1.1)_1$, taking $\Lambda^{\frac{3}{2}}$ to $(1.1)_2$, multiplying by $\Lambda^{\frac{1}{2}} u$ and $\Lambda^{\frac{3}{2}} d$ respectively, summing them up, we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\Lambda^{\frac{1}{2}} u\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}} d\|_{L^2}^2) + \|\Lambda^{\frac{3}{2}} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}} d\|_{L^2}^2 \\
=& -\int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}}(u\cdot\nabla u)\cdot\Lambda^{\frac{1}{2}} u dx + \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}}[\nabla\cdot(\nabla d\odot\nabla d)]\cdot\Lambda^{\frac{1}{2}} u dx \\
&- \int_{\mathbb{R}^3} \Lambda^{\frac{3}{2}}(u\cdot\nabla d)\cdot\Lambda^{\frac{3}{2}} d dx + \int_{\mathbb{R}^3} \Lambda^{\frac{3}{2}}(|\nabla d|^2 d)\cdot\Lambda^{\frac{3}{2}} d dx \\
&+ \int_{\mathbb{R}^3} \Lambda^{\frac{3}{2}}(d\times\Delta d)\cdot\Lambda^{\frac{3}{2}} d dx \\
=& I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{2.2}$$

By using the Kato-Ponce inequality [10], it yields that

$$|I_1| \le C\|\Lambda^{\frac{3}{2}} u\|_{L^2}\|\Lambda^{\frac{1}{2}} u\|_{L^6}\|u\|_{L^3} \le C\|\Lambda^{\frac{1}{2}} u\|_{L^2}\|\Lambda^{\frac{3}{2}} u\|_{L^2}^2, \tag{2.3}$$

$$|I_2| \le C\|\Lambda^{\frac{1}{2}} u\|_{L^6}\|\Lambda^{\frac{5}{2}} d\|_{L^2}\|\nabla d\|_{L^3} \le C\|\Lambda^{\frac{3}{2}} d\|_{L^2}(\|\Lambda^{\frac{3}{2}} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}} d\|_{L^2}^2), \tag{2.4}$$

and

$$\begin{aligned}
|I_3| \le& C\|\Lambda^{\frac{3}{2}} d\|_{L^6}\|\Lambda^{\frac{3}{2}}(u\cdot\nabla d)\|_{L^{\frac{6}{5}}} \\
\le& C\|\Lambda^{\frac{5}{2}} d\|_{L^2}(\|\Lambda^{\frac{3}{2}} u\|_{L^2}\|\nabla d\|_{L^3} + \|u\|_{L^3}\|\Lambda^{\frac{3}{2}}\nabla d\|_{L^2}) \\
\le& C(\|\Lambda^{\frac{3}{2}} d\|_{L^2} + \|\Lambda^{\frac{1}{2}} u\|_{L^2})(\|\Lambda^{\frac{3}{2}} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}} d\|_{L^2}^2).
\end{aligned} \tag{2.5}$$

Note that $|d| = 1$, we have $\|d\|_{L^\infty} \le C$ and $d\cdot d = 1$. Hence, $\Delta(d\cdot d) = 0$, which implies that

$$\nabla\cdot(d\nabla d) = |\nabla d|^2 + d\Delta d = 0.$$

Therefore, we have

$$d\cdot\Delta d = -|\nabla d|^2.$$

Using the above equality, we easily obtain

$$
\begin{aligned}
|I_4| \leq & C\|\Lambda^{\frac{5}{2}} d\|_{L^2} \|\Lambda^{\frac{1}{2}}(|\nabla d|^2 d)\|_{L^2} \\
\leq & C\|\Lambda^{\frac{5}{2}} d\|_{L^2} (\|\Lambda^{\frac{3}{2}} d\|_{L^6} \|\nabla d\|_{L^3} \|d\|_{L^\infty} + \|\Lambda^{\frac{1}{2}} d\|_{L^6} \|d \cdot \Delta d\|_{L^3}) \\
= & C\|\Lambda^{\frac{5}{2}} d\|_{L^2} (\|\Lambda^{\frac{3}{2}} d\|_{L^6} \|\nabla d\|_{L^3} \|d\|_{L^\infty} + \|\Lambda^{\frac{1}{2}} d\|_{L^6} \|d\|_{L^\infty} \|\Delta d\|_{L^3}) \\
\leq & C\|\Lambda^{\frac{3}{2}} d\|_{L^2} \|\Lambda^{\frac{5}{2}} d\|_{L^2}^2.
\end{aligned} \tag{2.6}
$$

Moreover, since

$$
\|d\|_{L^\infty} \leq \|d\|_{L^6}^{\frac{2\varepsilon}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^{\frac{1}{1+2\varepsilon}} \leq \|\Lambda d\|_{L^2}^{\frac{d\varepsilon}{1+d\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^{\frac{1}{1+d\varepsilon}} \leq \|\Lambda d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2},
$$

we easily obtain

$$
\begin{aligned}
|I_5| \leq & C\|\Lambda^{\frac{5}{2}} d\|_{L^2} \|\Lambda^{\frac{1}{2}}(d \times \Delta d)\|_{L^2} \\
\leq & C\|\Lambda^{\frac{5}{2}} d\|_{L^2} (\|\Lambda^{\frac{1}{2}} d\|_{L^6} \|\Delta d\|_{L^3} + \|d\|_{L^\infty} \|\Lambda^{\frac{5}{2}} d\|_{L^2}) \\
\leq & C\|\Lambda^{\frac{5}{2}} d\|_{L^2} [\|\Lambda^{\frac{1}{2}} d\|_{L^6} \|\Delta d\|_{L^3} + (\|\Lambda d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}) \|\Lambda^{\frac{5}{2}} d\|_{L^2}] \\
\leq & C(\|\Lambda d\|_{L^2} + \|\Lambda^{\frac{3}{2}} d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}) \|\Lambda^{\frac{5}{2}} d\|_{L^2}^2.
\end{aligned} \tag{2.7}
$$

It then follows from (2.2)–(2.7) that

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}(\|\Lambda^{\frac{1}{2}} u\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}} d\|_{L^2}^2) + \|\Lambda^{\frac{3}{2}} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}} d\|_{L^2}^2 \\
\leq & C(\|\Lambda^{\frac{1}{2}} u\|_{L^2} + \|\Lambda d\|_{L^2} + \|\Lambda^{\frac{3}{2}} d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2})(\|\Lambda^{\frac{3}{2}} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}} d\|_{L^2}^2).
\end{aligned} \tag{2.8}
$$

Taking $\Lambda^{\frac{3}{2}+\varepsilon}$ to $(1.1)_2$, multiplying by $\Lambda^{\frac{3}{2}+\varepsilon} d$, we deduce that

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2}^2 \\
= & -\int_{\mathbb{R}^3} \Lambda^{\frac{3}{2}+\varepsilon}(u \cdot \nabla d) \cdot \Lambda^{\frac{3}{2}+\varepsilon} d dx + \int_{\mathbb{R}^3} \Lambda^{\frac{3}{2}+\varepsilon}(|\nabla d|^2 d) \cdot \Lambda^{\frac{3}{2}+\varepsilon} d dx \\
& + \int_{\mathbb{R}^3} \Lambda^{\frac{3}{2}+\varepsilon}(d \times \Delta d) \cdot \Lambda^{\frac{3}{2}+\varepsilon} d dx \\
= & I_6 + I_7 + I_8.
\end{aligned} \tag{2.9}
$$

Using the Kato-Ponce inequality again, we derive that

$$
\begin{aligned}
|I_6| \leq & C\|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2} \|\Lambda^{\frac{1}{2}+\varepsilon}(u \cdot \nabla d)\|_{L^2} \\
\leq & C\|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2} \left(\|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^{\frac{6}{2\varepsilon+1}}} \|\nabla d\|_{L^{\frac{3}{1-\varepsilon}}} + \|u\|_{L^3} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^6}\right) \\
\leq & C\|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2} \left(\|\Lambda^{\frac{3}{2}} u\|_{L^2} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2} + \|\Lambda^{\frac{1}{2}} u\|_{L^2} \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2}\right) \\
\leq & C(\|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2} + \|\Lambda^{\frac{1}{2}} u\|_{L^2}) \left(\|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}} u\|_{L^2}^2\right),
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
|I_7| \leq & C\|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2} \|\Lambda^{\frac{1}{2}+\varepsilon}(|\nabla d|^2 d)\|_{L^2} \\
\leq & C\|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2} \left(\|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^6} \|\nabla d\|_{L^3} \|d\|_{L^\infty} + \|\Lambda^{\frac{1}{2}+\varepsilon} d\|_{L^6} \|d\|_{L^\infty} \|\Delta d\|_{L^3}\right) \\
\leq & C(\Lambda^{\frac{3}{2}} d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2})(\|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}} d\|_{L^2}^2),
\end{aligned} \tag{2.11}
$$

and

$$
\begin{aligned}
|I_8| \leq & C\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}\|\Lambda^{\frac{1}{2}+\varepsilon}(d\times\Delta d)\|_{L^2}\\
\leq & C\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}(\|\Lambda^{\frac{1}{2}+\varepsilon}d\|_{L^6}\|\Delta d\|_{L^3}+\|d\|_{L^\infty}\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2})\\
\leq & C\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}[\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}\|\Lambda^{\frac{5}{2}}d\|_{L^2}+(\|\Lambda d\|_{L^2}+\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2})\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}]\\
\leq & C(\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}+\|\Lambda d\|_{L^2})(\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}^2+\|\Lambda^{\frac{5}{2}}d\|_{L^2}^2),
\end{aligned} \tag{2.12}
$$

where we have used the facts that $d\cdot\Delta d=-|\nabla d|^2$ in (2.11) and $\|d\|_{L^\infty}\leq\|\Lambda d\|_{L^2}+\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}$ in (2.12), respectively. Combining (2.9)–(2.12) together gives

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}^2+\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}^2\\
\leq & C(\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}+\|\Lambda d\|_{L^2}+\|\Lambda^{\frac{3}{2}}d\|_{L^2}+\|\Lambda^{\frac{1}{2}}u\|_{L^2})\left(\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}^2+\|\Lambda^{\frac{5}{2}}d\|_{L^2}^2+\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2\right).
\end{aligned} \tag{2.13}
$$

Summing up (2.1), (2.8) and (2.13), we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2+\|\nabla d\|_{L^2}^2+\|\Lambda^{\frac{1}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}}d\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}^2)\\
&+\|\nabla u\|_{L^2}^2+\|\Delta d+|\nabla d|^2d\|_{L^2}^2+\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{5}{2}}d\|_{L^2}^2+\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}^2\\
\leq & C(\|\Lambda d\|_{L^2}+\|\Lambda^{\frac{1}{2}}u\|_{L^2}+\|\Lambda^{\frac{3}{2}}d\|_{L^2}+\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2})(\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{5}{2}}d\|_{L^2}^2+\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}^2).
\end{aligned} \tag{2.14}
$$

Taking K small enough such that

$$
\|\Lambda d\|_{L^2}+\|\Lambda^{\frac{1}{2}}u\|_{L^2}+\|\Lambda^{\frac{3}{2}}d\|_{L^2}+\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}<K<\frac{1}{2C}, \tag{2.15}
$$

then, $\|u\|_{L^2}^2+\|\nabla d\|_{L^2}^2+\|\Lambda^{\frac{1}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}}d\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}^2$ is decreasing. So, for any $0<T<\infty$, we have

$$
\begin{aligned}
&\frac{d}{dt}(\|u\|_{L^2}^2+\|\nabla d\|_{L^2}^2+\|\Lambda^{\frac{1}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{3}{2}}d\|_{L^2}^2+\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}^2)\\
&+\|\nabla u\|_{L^2}^2+\|\Delta d+|\nabla d|^2d\|_{L^2}^2+\|\Lambda^{\frac{3}{2}}u\|_{L^2}^2+\|\Lambda^{\frac{5}{2}}d\|_{L^2}^2+\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}^2\leq 0,
\end{aligned} \tag{2.16}
$$

which means

$$
\begin{cases}
u\in L^\infty(0,T;H^{\frac{1}{2}})\cap L^2(0,T;H^{\frac{3}{2}}),\\
\nabla d\in L^\infty(0,T;H^{\frac{1}{2}+\varepsilon})\cap L^2(0,T;H^{\frac{3}{2}+\varepsilon}).
\end{cases} \tag{2.17}
$$

By Lemma 1.2 and (2.17), we easily obtain the higher-order norm estimates for the solution, this complete the proof.

3. Conclusions

The three-dimensional incompressible Navier-Stokes-Landau-Lifshitz equations is an important hydrodynamics equations. The well-posedness and large time behavior of its solutions were studied by many authors. The latest result on the global well-posedness was studied by Wei, Li and Yao [20]. The author supposed that $\|u_0\|_{H^1}+\|d_0-\omega_0\|_{H^2}$ is sufficiently small, obtained the small initial data global well-posedness. In this paper, we improve the global well-posedness result in [20], only assume that $\|u_0\|_{H^{\frac{1}{2}}}+\|\nabla d_0\|_{H^{\frac{1}{2}+\varepsilon}}$ ($\varepsilon>0$) is sufficiently small, prove the global well-posedness for 3D Navier-Stokes-Landau-Lifshitz equations.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. F. Alouges, A. Soyeur, *On global weak solutions for Landau-Lifshitz equations: Existence and nonuniqueness*, Nonlinear Anal., **18** (1992), 1071–1084.
2. B. Guo, M. Hong, *The Landau-Lifshitz equation of the ferromagnetic spin chain and harmonic maps*, Calc. Var. Patial. Dif., **1** (1993), 311–334.
3. W. E, X. Wang, *Numerical methods for the Landau-Lifshitz equation*, SIAM J. Numer. Anal., **38** (2000), 1647–1665.
4. J. Fan, H. Gao, B. Guo, *Regularity criteria for the Navier-Stokes-Landau-Lifshitz system*, J. Math. Anal. Appl. **363** (2010), 29–37.
5. J. Fan, Y. Zhou, *Uniform local well-posedness for an Ericksen-Leslie's density-dependent parabolic-hyperbolic liquid crystals model*, Appl. Math. Lett., **74** (2017), 79–84.
6. E. Feireisl, E. Rocca, G. Schimperna, et al. *On a hyperbolic system arising in liquid crystals modeling*, J. Hyperbol. Differ. Eq., **15** (2018), 15–35.
7. D. Golovaty, P. Sternberg, R. Venkatraman, *A Ginzburg-Landau-type problem for highly anisotropic nematic liquid crystals*, SIAM J. Math. Anal., **51** (2019), 276–320.
8. B. Guo, F. Liu, *Weak and smooth solutions to incompressible Navier-Stokes-Landau-Lifshitz-Maxwell equations*, Front. Math. China, **14** (2019), 1133–1161.
9. T. Kato, *Strong L^p-solutions of the Navier-Stokes equations in $\mathbb{R}^m$ with applications to weak solutions*, Math. Z., **187** (1984), 471–480.
10. T. Kato, G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math., **41** (1988), 891–907.
11. H. Koch, D. Tataru, *Well-posedness for the Navier-Stokes equations*, Adv. Math., **157** (2001), 22–35.
12. F. Lin, J. Lin, C. Wang, *Liquid crystal flows in two dimensions*, Arch. Ration. Mech. Anal., **197** (2010), 297–336.
13. F. Planchon, *Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $\mathbb{R}^3$*, Ann. Inst. Henri Poincare, **13** (1996), 319–336.
14. M. Schonbek, Y. Shibata, *Global well-posedness and decay for a $\mathbb{Q}$ tensor model of incompressible nematic liquid crystals in $\mathbb{R}^N$*, J. Differ. Equations, **266** (2019), 3034–3065.

15. E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, USA, 1970.

16. I. W. Stewart, T. R. Faulkner, *The stability of nematic liquid crystals under crossed electric and magnetic fields*, Appl. Math. Lett., **13** (2000), 23–28.

17. R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

18. G. Wang, B. Guo, *Existence and uniqueness of the weak solution to the incompressible Navier-Stokes-Landau-Lifshitz model in 2-dimension*, Acta Math. Sci. Ser. B (Engl. Ed.), **37** (2017), 1361–1372.

19. G. Wang, B. Guo, *Global weak solution to the quantum Navier-Stokes-Landau-Lifshitz equations with density-dependent viscosity*, Discrete Contin. Dyn. Syst. Ser. B, **24** (2019), 6141–6166.

20. R. Wei, Y. Li, Z. Yao, *Decay rates of higher-order norms of solutions to the Navier-Stokes-Landau-Lifshitz system*, Appl. Math. Mech. (English Ed.), **39** (2018), 1499–1528.

21. X. Zhai, Y. Li, W. Yan, *Global solutions to the Navier-Stokes-Landau-Lifshitz system*, Math. Nachr., **289** (2016), 377–388.

22. C. Zhao, Y. Li, Z. Song, *Trajectory statistical solutions for the 3D Navier-Stokes equations: The trajectory attractor approach*, Nonlinear Anal. Real World Appl., **53** (2020), 103077.

23. X. Zhao, Y. Zhou, *Well-posedness and decay of solutions to 3D generalized Navier-Stokes equations*, Discrete Cont. Dyn. Syst. B, In press, doi: 10.3934/dcdsb.2020142.