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Research article

Fixed points of Kannan maps in modular metric spaces

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Abstract: The notion of a modular metric on an arbitrary set and the corresponding modular spaces, generalizing classical modulars over linear spaces like Orlicz spaces, were recently introduced. In this paper we study the existence of fixed points for contractive and nonexpansive Kannan maps in the setting of modular metric spaces.These are related to the successive approximations of fixed points (via orbits) which converge to the fixed points in the modular sense, which is weaker than the metric convergence.

Keywords: fixed point; Kannan contraction mapping; Kannan nonexpansive mapping; modular metric spaces
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1. Introduction

Modular metric spaces were introduced in [4, 5]. Behind this new notion, there exists a physical interpretation of the modular. A modular on a set bases on a nonnegative (possibly infinite valued) "field of (generalized) velocities": to each time $\lambda > 0$ (the absoulute value of) an averge velocity $\omega_\lambda(\rho, \sigma)$ is associated in such that in order to cover the distance between points $\rho, \sigma \in \mathcal{M}$, it takes time λ to move from ρ to σ with velocity $\omega_\lambda(\rho, \sigma)$, while a metric on a set stands for non-negative finite distances between any two points of the set. The process of access to this notion of modular metric spaces is different. Actually we deal with these spaces as the nonlinear version of the classical modular spaces as introduced by Nakano [12] on vector spaces and modular function spaces introduced by Musielack [11] and Orlicz [13]. In [1, 2] the authors have defined and investigated the fixed point property in the context of modular metric space and introduced several results. For more on modular metric fixed point theory, the reader may consult the books [7–9]. Some recent work in modular metric space has been represented in [14, 15]. It is almost a century where several mathematicians have improved, extended and enriched the classical Banach contraction principle [1] in different directions along with variety of applications. In 1969, Kannan [6] proved that if X is complete, then a Kannan mapping has a fixed point. It is interesting that Kannan's theorem is independent of the Banach contraction principle [3].

In this research article, fixed point problem for Kannan mappings in the framework of modular metric spaces is investigated.

2. Basic Notation and Terminology

Let $\mathcal{M} \neq \emptyset$. Throughout this paper for a function $\omega : (0,\infty) \times \mathcal{M} \times \mathcal{M} \to [0,\infty]$, we will write

$$\omega_\lambda(\rho,\sigma) = \omega(\lambda,\rho,\sigma),$$

for all $\lambda > 0$ and $\rho, \sigma \in \mathcal{M}$.

Definition 1. *[4, 5] A function $\omega : (0,\infty) \times \mathcal{M} \times \mathcal{M} \to [0,\infty]$ is called a modular metric on $\mathcal{M}$ if following axioms hold:*

(i) $\rho = \sigma \Leftrightarrow \omega_\lambda(\rho,\sigma) = 0$, for all $\lambda > 0$;
(ii) $\omega_\lambda(\rho,\sigma) = \omega_\lambda(\sigma,\rho)$, for all $\lambda > 0$, and $\rho, \sigma \in \mathcal{M}$;
(iii) $\omega_{\lambda+\mu}(\rho,\sigma) \leq \omega_\lambda(\rho,\varsigma) + \omega_\mu(\varsigma,\sigma)$, for all $\lambda, \mu > 0$ and $\rho, \sigma, \varsigma \in \mathcal{M}$.
A modular metric ω on $\mathcal{M}$ is called regular if the following weaker version of (i) is satisfied

$$\rho = \sigma \text{ if and only if } \omega_\lambda(\rho,\sigma) = 0, \text{ for some } \lambda > 0.$$

Eventually, ω is called convex if for $\lambda, \mu > 0$ and $\rho, \sigma, \varsigma \in \mathcal{M}$, it satisfies

$$\omega_{\lambda+\mu}(\rho,\sigma) \leq \frac{\lambda}{\lambda+\mu}\omega_\lambda(\rho,\varsigma) + \frac{\mu}{\lambda+\mu}\omega_\mu(\varsigma,\sigma).$$

Throughout this work, we assume ω is regular.

Definition 2. *[4, 5] Let ω be a modular on $\mathcal{M}$. Fix $\rho_0 \in \mathcal{M}$. The two sets*

$$\mathcal{M}_\omega = \mathcal{M}_\omega(\rho_0) = \{\rho \in \mathcal{M} : \omega_\lambda(\rho,\rho_0) \to 0 \ \ as \ \ \lambda \to \infty\},$$

and

$$\mathcal{M}^*_\omega = \mathcal{M}^*_\omega(\rho_0) = \{\rho \in \mathcal{M} : \exists \lambda = \lambda(\rho) > 0 \text{ such that } \omega_\lambda(\rho,\rho_0) < \infty\}$$

are called modular spaces (around ρ_0).

It is obvious that $\mathcal{M}_\omega \subset \mathcal{M}^*_\omega$ but this involvement may be proper in general. It follows from [4, 5] that if ω is a modular on $\mathcal{M}$, then the modular space $\mathcal{M}_\omega$ can be equipped with a (nontrivial) metric, generated by ω and given by

$$d_\omega(\rho,\sigma) = \inf\{\lambda > 0 : \omega_\lambda(\rho,\sigma) \leq \lambda\},$$

for any $\rho, \sigma \in \mathcal{M}_\omega$. If ω is a convex modular on $\mathcal{M}$, according to [4,5] the two modular spaces coincide, i.e. $\mathcal{M}^*_\omega = \mathcal{M}_\omega$, and this common set can be endowed with the metric d^*_ω given by

$$d^*_\omega(\rho,\sigma) = \inf\{\lambda > 0 : \omega_\lambda(\rho,\sigma) \leq 1\},$$

for any $\rho, \sigma \in \mathcal{M}_\omega$. These distances will be called Luxemburg distances.

Following example presented by Abdou and Khamsi [1,2] is an important motivation of the concept modular metric spaces.

example 3. *Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \bigcup K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{N}_\infty$ we will denote the space of all extended measurable functions, i.e. all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By 1_A we denote the characteristic function of the set A. Let $\rho : \mathcal{N}_\infty \to [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if:*

(i) $\rho(0) = 0$;
(ii) ρ is monotone, i.e. $|f(\omega)| \leq |g(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{N}_\infty$;
(iii) ρ is orthogonally subadditive, i.e. $\rho(f 1_{A \cup B}) \leq \rho(f 1_A) + \rho(f 1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{N}$;
(iv) ρ has the Fatou property, i.e. $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{N}_\infty$;
(v) ρ is order continuous in $\mathcal{E}$, i.e. $g_n \in \mathcal{E}$ and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

Similarly, as in the case of measure spaces, we say that a set $A \in \Sigma$ is ρ-null if $\rho(g 1_A) = 0$ for every $g \in \mathcal{E}$. We say that a property holds ρ-almost everywhere if the exceptional set is ρ-null. As usual we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind we define

$$\mathcal{N}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{N}_\infty; |f(\omega)| < \infty \; \rho - a.e\},$$

where each $f \in \mathcal{N}(\Omega, \Sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal ρ-a.e. rather than an individual function. Where no confusion exists we will write $\mathcal{M}$ instead of $\mathcal{N}(\Omega, \Sigma, \mathcal{P}, \rho)$. Let ρ be a regular function pseudomodular.

(a) We say that ρ is a regular function semimodular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies $f = 0 \, \rho{-}a.e.$;
(b) We say that ρ is a regular function modular if $\rho(f) = 0$ implies $f = 0 \; \rho - a.e.$

The class of all nonzero regular convex function modulars defined on Ω will be denoted by $\mathfrak{R}$. Let us denote $\rho(f, E) = \rho(f 1_E)$ for $f \in \mathcal{N}$, $E \in \Sigma$. It is easy to prove that $\rho(f, E)$ is a function pseudomodular in the sense of Def.2.1.1 in [10] (more precisely, it is a function pseudomodular with the Fatou property). Therefore, we can use all results of the standard theory of modular function space as per the framework defined by Kozlowski in [10], see also Musielak [11] for the basics of the general modular theory. Let ρ be a convex function modular.

(a) The associated modular function space is the vector space $L_\rho(\Omega, \Sigma)$, or briefly L_ρ, defined by

$$L_\rho = \{f \in \mathcal{N}; \rho(\lambda f) \to 0 \; as \; \lambda \to 0\}.$$

(b) The following formula defines a norm in L_ρ (frequently called Luxemburg norm):

$$\|f\|_\rho = \inf\{\alpha > 0; \rho(f/\alpha) \leq 1\}.$$

A modular function spaces furnishes a wonderful example of a modular metric space. Indeed, let L_ρ be modular function space.

example 4. *Define the function ω by*

$$\omega_\lambda(f,g)=\rho\left(\frac{f-g}{\lambda}\right)$$

for all $\lambda>0$, and $f,g\in L_\rho$. Then ω is a modular metric on L_ρ. Note that ω is convex if and only if ρ is convex. Moreover we have

$$\|f-g\|_\rho=d^*_\omega(f,g),$$

for any $f,g\in L_\rho$.

For more examples readers can see [4,5]

Definition 5. *[1]*

(1). A sequence $\{\rho_n\}\subset\mathcal{M}_\omega$ is ω-convergent to $\rho\in\mathcal{M}_\omega$ if and only if $\omega_1(\rho_n,\rho)\to 0$.
(2). A sequence $\{\rho_n\}\subset\mathcal{M}_\omega$ is ω-Cauchy if $\omega_1(\rho_n,\rho_m)\to 0$ as $n,m\to\infty$.
(3). A set $K\subset\mathcal{M}_\omega$ is ω-closed if the limit of ω_1-convergent sequence of K always belongs to K.
(4). A set $K\subset\mathcal{M}_\omega$ is ω-bounded if

$$\delta_\omega=\sup\{\omega_1(\rho,\sigma);\rho,\sigma\in K\}<\infty.$$

(5). If any ω-Cauchy sequence in a subset K of $\mathcal{M}_\omega$ is a convergent sequence and its limit is in K, then K is called an ω-complete.
(6). The ρ-centered ω-ball of radius r is defined as

$$B_\omega(\rho,r)=\{\sigma\in\mathcal{M}_\omega;\ \omega_1(\rho,\sigma)\le r\},$$

for any $\rho\in\mathcal{M}_\omega$ and $r\ge 0$.

Let $(\mathcal{M},\omega)$ be a modular metric space. In the rest of this work, we assume that ω satisfies the Fatou property, i.e. if $\{\rho_n\}$ ω-converges to ρ and $\{\sigma_n\}$ ω-converges to σ, then we must have

$$\omega_1(\rho,\sigma)\le\liminf_{n\to\infty}\omega_1(\rho_n,\sigma_n),$$

for any $\rho\in\mathcal{M}_\omega$.

Definition 6. *Let $(\mathcal{M},\omega)$ be a modular metric space. We define an admissible subset of $\mathcal{M}_\omega$ as intersection of modular balls.*

Note that if ω satisfies the Fatou property, then the modular balls are ω-closed. Hence any admissible subset is ω-closed.

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3. Modular Kannan mappings in modular metric space

It is well-known that every Banach contractive mapping is a continuous function. In 1968, Kannan [6] was the first mathematician who found the answer and presented a fixed point result in the seting of metric space as following .

Theorem 7. *[6] Let $(\mathcal{M}, d)$ be a complete metric space and $\mathcal{J} : \mathcal{M} \to \mathcal{M}$ be a self-mapping satisfying*

$$d(\mathcal{J}(\rho), \mathcal{J}(\sigma)) \leq \alpha \left(d(\rho, \mathcal{J}(\rho)) + d(\sigma, \mathcal{J}(\sigma)) \right),$$

$\forall\ \rho, \sigma \in \mathcal{M}$ and $\alpha \in [0, \frac{1}{2})$. Then $\mathcal{J}$ has a unique fixed point $\varsigma \in \mathcal{M}$, and for any $\rho \in \mathcal{M}$ the sequence of itreaive $(\mathcal{J}^n(\rho))$ converges to ς.

Before we state our results, we introduce the defintion of Kannan mappings in modular metic spaces.

Definition 8. *Let K be a nonempty subset of $\mathcal{M}_\omega$. A mapping $\mathcal{J} : K \to K$ is called Kannan ω-Lipschitzian if $\exists\ \alpha \geq 0$ such that*

$$\omega_1(\mathcal{J}(\rho), \mathcal{J}(\sigma)) \leq \alpha \left(\omega_1(\rho, \mathcal{J}(\rho)) + \omega_1(\sigma, \mathcal{J}(\sigma)) \right),$$

$\forall\ \rho, \sigma \in K$. The mapping $\mathcal{J}$ is said to be:

(1). Kannan ω-contraction if $\alpha < 1/2$;
(2). Kannan ω-nonexpansive if $\alpha = 1/2$.
(3) $\varsigma \in K$ is said to be fixed point of $\mathcal{J}$ if $\mathcal{J}(\varsigma) = \varsigma$.

Note that all Kannan ω-Lipschitzian mappings have at most one fixed point due to the regularty of ω.

The following result discusses the existence of fixed point for kannan contraction maps in the setting of modular metric spaces.

Theorem 9. *Let $(\mathcal{M}, \omega)$ be a modular metric space. Assume that K is a nonempty ω-complete of $\mathcal{M}_\omega$. Let $\mathcal{J} : K \to K$ be a Kannan ω-contraction mapping. Let $\varsigma \in K$ be such that $\omega_1(\varsigma, \mathcal{J}(\varsigma)) < \infty$. Then $\{\mathcal{J}^n(\varsigma)\}$ ω-converges to some $\tau \in K$. Furthermore, we have $\omega_1(\tau, \mathcal{J}(\tau)) = \infty$ or $\omega_1(\tau, \mathcal{J}(\tau)) = 0$ (i.e., τ is the fixed point of $\mathcal{J}$)*

Proof. Let $\varsigma \in K$ such that $\omega_1(\varsigma, \mathcal{J}(\varsigma)) < +\infty$. Now we establish that $\{\mathcal{J}^n(\varsigma)\}$ is ω-convergent. As K is ω-complete, it suffices to prove that $\{\mathcal{J}^n(\varsigma)\}$ is ω-Cauchy. Since $\mathcal{J}$ is a Kannan ω-contraction mapping, so $\exists\ \alpha \in [0, 1/2)$ such that

$$\omega_1(\mathcal{J}(\rho), \mathcal{J}(\sigma)) \leq \alpha \left(\omega_1(\rho, \mathcal{J}(\rho)) + \omega_1(\sigma, \mathcal{J}(\sigma)) \right),$$

for any $\rho, \sigma \in K$. Set $k = \alpha/(1 - \alpha) < 1$. Furthermore

$$\omega_1(\mathcal{J}^n(\varsigma), \mathcal{J}^{n+1}(\varsigma)) \leq \alpha \left(\omega_1(\mathcal{J}^{n-1}(\varsigma), \mathcal{J}^n(\varsigma)) + \omega_1(\mathcal{J}^n(\varsigma), \mathcal{J}^{n+1}(\varsigma)) \right),$$

which implies

$$\begin{aligned}\omega_1(\mathcal{J}^n(\varsigma),\mathcal{J}^{n+1}(\varsigma)) &\le \frac{\alpha}{1-\alpha}\,\omega_1(\mathcal{J}^{n-1}(\varsigma),\mathcal{J}^n(\varsigma))\\ &= k\,\omega_1(\mathcal{J}^{n-1}(\varsigma),\mathcal{J}^n(\varsigma)),\end{aligned}$$

for any $n \ge 1$. Hence,

$$\omega_1(\mathcal{J}^n(\varsigma),\mathcal{J}^{n+1}(\varsigma)) \le k^n\,\omega_1(\varsigma,\mathcal{J}(\varsigma)),$$

for any $n \in \mathbb{N}$. As $\mathcal{J}$ is a Kannan ω-contraction mapping, so we get

$$\omega_1(\mathcal{J}^n(\varsigma),\mathcal{J}^{n+h}(\varsigma)) \le \alpha\left(\omega_1(\mathcal{J}^{n-1}(\varsigma),\mathcal{J}^n(\varsigma)) + \omega_1(\mathcal{J}^{n+h-1}(\varsigma),\mathcal{J}^{n+h}(\varsigma))\right),$$

which implies

$$\omega_1(\mathcal{J}^n(\varsigma),\mathcal{J}^{n+h}(\varsigma)) \le \alpha\left(k^{n-1} + k^{n+h-1}\right)\omega_1(\varsigma,\mathcal{J}(\varsigma)), \tag{NL}$$

for $n \ge 1$ and $h \in \mathbb{N}$. As $k < 1$ and $\omega_1(\varsigma,\mathcal{J}(\varsigma)) < +\infty$, we conclude that $\{\mathcal{J}^n(\varsigma)\}$ is ω-Cauchy, as claimed. Let $\tau \in K$ be the ω-limit of $\{\mathcal{J}^n(\varsigma)\}$. As K is ω-closed, we get $\tau \in K$. Suppose that $\omega_1(\tau,\mathcal{J}(\tau)) < +\infty$; then we will obtain that $\omega_1(\tau,\mathcal{J}(\tau)) = 0$. As

$$\begin{aligned}\omega_1(\mathcal{J}^n(\varsigma),\mathcal{J}(\tau))) &\le \alpha\Big(\omega_1(\mathcal{J}^{n-1}(\varsigma),\mathcal{J}^n(\varsigma)) + \omega_1(\tau,\mathcal{J}(\tau))\Big)\\ &\le \alpha\Big(k^{n-1}\,\omega_1(\varsigma,\mathcal{J}(\varsigma)) + \omega_1(\tau,\mathcal{J}(\tau))\Big),\end{aligned}$$

for any $n \ge 1$. By the use of Fatou's property, we obtain

$$\begin{aligned}\omega_1(\tau,\mathcal{J}(\tau))) &\le \liminf_{n\to\infty}\,\omega_1(\mathcal{J}^n(\varsigma),\mathcal{J}(\tau)))\\ &\le \alpha\,\omega_1(\tau,\mathcal{J}(\tau)).\end{aligned}$$

Since $\alpha < 1/2$, we conclude that $\omega_1(\tau,\mathcal{J}(\tau)) = 0$, i.e., τ is the fixed point of $\mathcal{J}$. □

The upcoming result is the analogue to Kannan's extention of the classical Banach contraction principle in modular metric space.

Corollary 10. *Let K be a nonempty ω-closed subset of $\mathcal{M}_\omega$. Let $\mathcal{J} : K \to K$ be a Kannan ω-contraction mapping such that $\omega_1(\rho,\mathcal{J}(\rho)) < +\infty$, for any $\rho \in K$. Then for any $\varsigma \in K$, $\{\mathcal{J}^n(\varsigma)\}$ ω-converges to the unique fixed point ς of $\mathcal{J}$. Furthermore, if α is the Kannan constant associated to $\mathcal{J}$, then we have*

$$\omega_1(\mathcal{J}^n(\varsigma),\tau) \le \alpha\left(\frac{\alpha}{1-\alpha}\right)^{n-1}\omega_1(\varsigma,\mathcal{J}(\varsigma)),$$

for any $\rho \in K$ and $n \ge 1$.

Proof. From Theorem 9, we can obtain the proof of first part directly. Using the inequality (NK) and since $k < 1$, we get

$$\liminf_{h\longrightarrow\infty}\omega_1(\mathcal{J}^n(\varsigma),\mathcal{J}^{n+h}(\varsigma)) \le \alpha\left(k^{n-1}\right)\omega_1(\varsigma,\mathcal{J}(\varsigma)), \tag{3.1}$$

Now, using the fatou's property, we have

$$\omega_1(\mathcal{J}^n(\varsigma),\tau) \le \alpha\,k^{n-1}\,\omega_1(\varsigma,\mathcal{J}(\varsigma)) = \alpha\left(\frac{\alpha}{1-\alpha}\right)^{n-1}\omega_1(\varsigma,\mathcal{J}(\varsigma)),$$

for any $n \ge 1$ and $\varsigma \in K$. □

Recall that an admissible subset of $\mathcal{M}_\omega$ is defined as an intersection of modular balls.

Definition 11. *We will say that:*

(i). if any decreasing sequence of nonempty ω-bounded admissible subsets in $\mathcal{M}_\omega$ have a nonempty intersection, then $\mathcal{M}_\omega$ is said to satisfy the property (R),

(ii). if for any nonempty ω-bounded admissible subset K with more than one point, there exists $\rho \in K$ such that

$$\omega_1(\rho, \sigma) < \delta_\omega(K) = \sup\{\omega_1(a, b);\ a, b \in K\},$$

for any $\sigma \in K$, then $\mathcal{M}_\omega$ is said to satisfy ω-quasi-normal property.

Following technical lemma is very useful in the proof of our theorem.

Lemma 12. *Suppose that $\mathcal{M}_\omega$ satisfy the both (R) property and the ω-quasi-normal property. Let K be a nonempty ω-bounded admissible subset of $\mathcal{M}_\omega$ and $\mathcal{J} : K \to K$ be a Kannan ω-nonexpansive mapping. Fix $r > 0$. Suppose that $A_r = \{\rho \in K;\ \omega_1(\rho, \mathcal{J}(\rho)) \le r\} \neq \emptyset$. Set*

$$K_r = \bigcap \{B_\omega(a, t); \mathcal{J}(A_r) \subset B_\omega(a, t)\} \cap K.$$

Then $K_r \neq \emptyset$, ω-closed admissible subset of K and

$$\mathcal{J}(K_r) \subset K_r \subset A_r \quad \textit{and} \quad \delta_\omega(K_r) \le r.$$

Proof. As $\mathcal{J}(A_r)$ is strictly contained in each balls and intersection of all balls contained in K_r. Thus $\mathcal{J}(A_r) \subset K_r$, and K_r is not empty. From definition of admissible set, we deduce that K_r is an admissible subset of K. Let us prove that $K_r \subset A_r$. Let $\rho \in K_r$. If $\omega_1(\rho, \mathcal{J}(\rho)) = 0$, then obviously we have $\rho \in A_r$. Otherwise, assume $\omega_1(\rho, \mathcal{J}(\rho)) > 0$. Set

$$s = \sup\ \{\omega_1(\mathcal{J}(\varsigma), \mathcal{J}(\rho));\ \varsigma \in A_r\}.$$

From the definition of s, we have $\mathcal{J}(A_r) \subset B_\omega(\mathcal{J}(\rho), s)$. Hence $K_r \subset B_\omega(\mathcal{J}(\rho), s)$, which implies $\omega_1(\rho, \mathcal{J}(\rho)) \le s$. Let $\varepsilon > 0$. Then $\exists\ \varsigma \in A_r$ such that $s - \varepsilon \le \omega_1(\mathcal{J}(\rho), \mathcal{J}(\varsigma))$. Hence

$$\begin{aligned}
\omega_1(\rho, \mathcal{J}(\rho)) - \varepsilon &\le s - \varepsilon \\
&\le \omega_1(\mathcal{J}(\rho), \mathcal{J}(\varsigma)) \\
&\le \frac{1}{2}\Big(\omega_1(\rho, \mathcal{J}(\rho)) + \omega_1(\varsigma, \mathcal{J}(\varsigma))\Big) \\
&\le \frac{1}{2}\Big(\omega_1(\rho, \mathcal{J}(\rho)) + r\Big).
\end{aligned}$$

As we are taking ε an arbitrarily positive number, so we get

$$\omega_1(\rho, \mathcal{J}(\rho)) \le \frac{1}{2}\Big(\omega_1(\rho, \mathcal{J}(\rho)) + r\Big),$$

which implies $\omega_1(\rho, \mathcal{J}(\rho)) \le r$, i.e., $\rho \in A_r$ as claimed. Since $\mathcal{J}(A_r) \subset K_r$, we get $\mathcal{J}(K_r) \subset \mathcal{J}(A_r) \subset K_r$, i.e., K_r is $\mathcal{J}$-invariant. Now we prove that $\delta_\omega(K_r) \le r$. First, we observe that

$$\omega_1(\mathcal{J}(\rho), \mathcal{J}(\sigma)) \le \frac{1}{2}\Big(\omega_1(\rho, \mathcal{J}(\rho)) + \omega_1(\varsigma, \mathcal{J}(\varsigma))\Big) \le r,$$

for any $\rho, \sigma \in A_r$. Fix $\rho \in A_r$. Then $\mathcal{J}(A_r) \subset B_\omega(\mathcal{J}(\rho), r)$. The definition of K_r implies $K_r \subset B_\omega(\mathcal{J}(\rho), r)$. Thus $\mathcal{J}(\rho) \in \bigcap_{\sigma \in K_r} B_\omega(\sigma, r)$, which implies $\mathcal{J}(A_r) \subset \bigcap_{\sigma \in K_r} B_\omega(\sigma, r)$. Again by the definition of K_r, we get $K_r \subset \bigcap_{\sigma \in K_r} B_\omega(\sigma, r)$. Therefore, we have $\omega_1(\sigma, \varsigma) \leq r$, for any $\sigma, \varsigma \in K_r$, i.e., $\delta_\omega(K_r) \leq r$. □

Now, we are able to state and prove our result for ω-nonexpansive Kannan maps on modular metric spaces.

Theorem 13. *Suppose that $\mathcal{M}_\omega$ satisfies both the (R) property and the ω-quasi-normal property. Let K be a nonempty ω-bounded admissible subset of $\mathcal{M}_\omega$ and $\mathcal{J} : K \to K$ is a Kannan ω-nonexpansive mapping. Then $\mathcal{J}$ has a fixed point.*

Proof. Set $r_0 = \inf \{\omega_1(\rho, \mathcal{J}(\rho));\ \rho \in K\}$ and $r_n = r_0 + 1/n$, for $n \geq 1$. By definition of r_0, the set $A_{r_n} = \{\rho \in K;\ \omega_1(\rho, \mathcal{J}(\rho)) \leq r_n\}$ is not empty, for any $n \geq 1$. Taking K_{r_n} defined in Lemma 12. It is simple to analyze that $\{K_{r_n}\}$ is a decreasing sequence of nonempty ω-bounded admissible subsets of K. The property (R) implies that $K_\infty = \bigcap_{n \geq 1} K_{r_n} \neq \emptyset$. Let $\rho \in K_\infty$. Then we have $\omega_1(\rho, \mathcal{J}(\rho)) \leq r_n$, for any $n \geq 1$. If we let $n \to \infty$, we get $\omega_1(\rho, \mathcal{J}(\rho)) \leq r_0$ which implies $\omega_1(\rho, \mathcal{J}(\rho)) = r_0$. Hence the set $A_{r_0} \neq \emptyset$. We claim that $r_0 = 0$. Otherwise, assume $r_0 > 0$ which implies that $\mathcal{J}$ fails to have a fixed point. Again consider the set K_{r_0} as defined in Lemma 12. Note that since $\mathcal{J}$ fails to have a fixed point and K_{r_0} is $\mathcal{J}$-invariant, then K_{r_0} has more than one point, i.e., $\delta_\omega(K_{r_0}) > 0$. It follows from the ω-quasi-normal property that there exists $\rho \in K_{r_0}$ such that

$$\omega(\rho, \sigma) < \delta_\omega(K_{r_0}) \leq r_0,$$

for any $\sigma \in K_{r_0}$. From Lemma 12, we know that $K_{r_0} \subset A_{r_0}$. From the definition of K_{r_0}, we have

$$\mathcal{J}(\rho) \in T(A_{r_0}) \subset K_{r_0}.$$

Hence Obviously this will imply

$$\omega_1(\rho, \mathcal{J}(\rho)) < \delta_\omega(K_{r_0}) \leq r_0,$$

which is a contradiction with the definition of r_0. Hence $r_0 = 0$ which implies that any point in K_∞ is a fixed point of $\mathcal{J}$, i.e., $\mathcal{J}$ has a fixed point in K. □

4. Conclusions

In this paper, we have introduced some notions to study the existence of fixed points for contractive and nonexpansive Kannan maps in the setting of modular metric spaces.Using the modular convergence sense , which is weaker than the metric convergence we have proved our results. The proved results generalized and improved some of the results of the literature.

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Conflict of interest

The author declares that they have no competing interests.

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