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Research article

Fractional generalized Hadamard and Fejér-Hadamard inequalities for m-convex functions

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Abstract: The objective of this paper is to present the fractional Hadamard and Fejér-Hadamard inequalities in generalized forms. By employing a generalized fractional integral operator containing extended generalized Mittag-Leffler function involving a monotone increasing function, we generalize the well known fractional Hadamard and Fejér-Hadamard inequalities for m-convex functions. Also we study the error bounds of these generalized inequalities. In connection with some published results from presented inequalities are obtained.

Keywords: m-convex function; fractional integral inequalities; generalized fractional integral operator; extended generalized Mittag-Leffler function
Mathematics Subject Classification: 26A51, 26A33, 33E12

1. Introduction and preliminary results

The theory of inequalities of convex functions is part of the general subject of convexity since a convex function is one whose epigraph is a convex set. Nonetheless it is a theory important per se, which touches almost all branches of mathematics. Probably, the first topic who make necessary the encounter with this theory is the graphical analysis. With this occasion we learn on the second derivative test of convexity, a powerful tool in recognizing convexity. Then comes the problem of finding the extremal values of functions of several variables and the use of Hessian as a higher dimensional generalization of the second derivative. Passing to optimization problems in infinite dimensional spaces is the next step, but despite the technical sophistication in handling such problems, the basic ideas are pretty similar with those underlying the one variable case.

The objective of this paper is to present the fractional Hadamard and Fejér-Hadamard inequalities in generalized forms. We start from the integral operators containing generalized Mittag-Leffler function defined by Prabhaker in [25].

Definition 1.1. Let σ, τ, ρ be positive real numbers and $\omega \in \mathbb{R}$. Then the generalized fractional integral operators containing Mittag-Leffler function $\epsilon^{\rho}_{\sigma,\tau,\omega,a^+}f$ and $\epsilon^{\rho}_{\sigma,\tau,\omega,b_-}f$ for a real valued continuous function f are defined by:

$$\left(\epsilon^{\rho}_{\sigma,\tau,\omega,a^+}f\right)(x) = \int_a^x (x-t)^{\tau-1}E^{\rho}_{\sigma,\tau}(\omega(x-t)^{\sigma})f(t)dt, \tag{1.1}$$

$$\left(\epsilon^{\rho}_{\sigma,\tau,\omega,b_-}f\right)(x) = \int_x^b (t-x)^{\tau-1}E^{\rho}_{\sigma,\tau}(\omega(t-x)^{\sigma})f(t)dt, \tag{1.2}$$

where the function $E^{\rho}_{\sigma,\tau}(t)$ is the generalized Mittag-Leffler function; $E^{\rho}_{\sigma,\tau}(t) = \sum_{n=0}^{\infty} \frac{(\rho)_n t^n}{\Gamma(\sigma n+\tau)n!}$ and $(\rho)_n = \frac{\Gamma(\rho+n)}{\Gamma(\rho)}$.

Fractional integral operators associated with generalized Mittag-Leffler function play a vital role in fractional calculus. Different fractional integral operators have different types of properties and these integral operators may be singular or non-singular depending upon their kernels. For example, the global Riemann Liouville integral is a singular integral operator but the singularity is integrable. Some new models [2, 7] have been designed due to the non-singularity of their defining integrals. Fractional integral operators are useful in the generalization of classical mathematical concepts. Fractional integral operators are very fruitful in obtaining fascinating and glorious results, for example fractional order systems and fractional differential equations are used in physical and mathematical phenomena. Many inequalities like Hadamard are studied in the context of fractional calculus operators, see [1, 6, 8, 20, 28].

After the existence of Prabhaker fractional integral operators, the researchers began to think in this direction and consequently they further generalized and extended these operators in different ways for instance see [14, 18, 23, 29], and references therein. By using the Mittag-Leffler function these fractional integral operators are generalized by many authors. In [27] Salim and Faraj defined the following fractional integral operators involving an extended Mittag-Leffler function in the kernel.

Definition 1.2. Let $\sigma, \tau, k, \delta, \rho$ be positive real numbers and $\omega \in \mathbb{R}$. Then the generalized fractional integral operators containing Mittag-Leffler function $\epsilon^{\rho,r,k}_{\sigma,\tau,\delta,\omega,a^+}f$ and $\epsilon^{\rho,r,k}_{\sigma,\tau,\delta,\omega,b_-}f$ for a real valued continuous function f are defined by:

$$\left(\epsilon^{\rho,r,k}_{\sigma,\tau,\delta,\omega,a^+}f\right)(x) = \int_a^x (x-t)^{\tau-1}E^{\rho,r,k}_{\sigma,\tau,\delta}(\omega(x-t)^{\sigma})f(t)dt, \tag{1.3}$$

$$\left(\epsilon^{\rho,r,k}_{\sigma,\tau,\delta,\omega,b_-}f\right)(x) = \int_x^b (t-x)^{\tau-1}E^{\rho,r,k}_{\sigma,\tau,\delta}(\omega(t-x)^{\sigma})f(t)dt, \tag{1.4}$$

where the function $E^{\rho,r,k}_{\sigma,\tau,\delta}(t)$ is the generalized Mittag-Leffler function; $E^{\rho,r,k}_{\sigma,\tau,\delta}(t) = \sum_{n=0}^{\infty} \frac{(\rho)_{kn} t^n}{\Gamma(\sigma n+\tau)(r)_{\delta n}}$ and $(\rho)_{kn} = \frac{\Gamma(\rho+kn)}{\Gamma(\rho)}$.

Further fractional integral operators containing an extended generalized Mittag-Leffler function in their kernels are defined as follows:

Definition 1.3. [18] Let $\omega, \tau, \delta, \rho, c \in \mathbb{C}$, $p, \sigma, r \geq 0$ and $0 < k \leq r + \sigma$. Let $f \in L_1[a, b]$ and $x \in [a, b]$. Then the generalized fractional integral operators $\epsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+} f$ and $\epsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-} f$ are defined by:

$$\left(\epsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+} f\right)(x; p) = \int_a^x (x-t)^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(x-t)^{\sigma}; p) f(t) dt, \tag{1.5}$$

$$\left(\epsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-} f\right)(x; p) = \int_x^b (t-x)^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(t-x)^{\sigma}; p) f(t) dt, \tag{1.6}$$

where

$$E^{\rho,r,k,c}_{\sigma,\tau,\delta}(t; p) = \sum_{n=0}^{\infty} \frac{\beta_p(\rho + nk, c - \rho)(c)_{nk} t^n}{\beta(\rho, c-\rho)\Gamma(\sigma n + \tau)(\delta)_{nr}}, \tag{1.7}$$

is the extended generalized Mittag-Leffler function.

Recently, Farid et al. defined a unified integral operator in [14] (see also [22]) as follows:

Definition 1.4. Let $f, g : [a, b] \to \mathbb{R}$, $0 < a < b$ be the functions such that f be positive and $f \in L_1[a, b]$ and g be a differentiable and strictly increasing. Also let $\frac{\phi}{x}$ be an increasing function on $[a, \infty)$ and $\omega, \tau, \delta, \rho, c \in \mathbb{C}$, $\Re(\tau), \Re(\delta) > 0$, $\Re(c) > \Re(\rho) > 0$ with $p \geq 0$, $\sigma, r > 0$ and $0 < k \leq r + \sigma$. Then for $x \in [a, b]$ the integral operators $({}_gF^{\phi,\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+} f)$ and $({}_gF^{\phi,\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-} f)$ are defined by:

$$({}_gF^{\phi,\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+} f)(x; p) = \int_a^x \frac{\phi(g(x) - g(t))}{g(x) - g(t)} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(x) - g(t))^{\sigma}; p) f(t) d(g(t)), \tag{1.8}$$

$$({}_gF^{\phi,\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-} f)(x; p) = \int_x^b \frac{\phi(g(t) - g(x))}{g(t) - g(x)} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(t) - g(x))^{\sigma}; p) f(t) d(g(t)). \tag{1.9}$$

The following definition of generalized fractional integral operator containing extended Mittag-Leffler function in the kernel can be extracted by setting $\phi(x) = x^{\tau}$ in Definition 1.4.

Definition 1.5. Let $f, g : [a, b] \to \mathbb{R}$, $0 < a < b$ be the functions such that f be positive and $f \in L_1[a, b]$ and g be a differentiable and strictly increasing. Also let $\omega, \tau, \delta, \rho, c \in \mathbb{C}$, $\Re(\tau), \Re(\delta) > 0$, $\Re(c) > \Re(\rho) > 0$ with $p \geq 0$, $\sigma, r > 0$ and $0 < k \leq r + \sigma$. Then for $x \in [a, b]$ the integral operators are defined by:

$$\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+} f\right)(x; p) = \int_a^x (g(x) - g(t))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(x) - g(t))^{\sigma}; p) f(t) d(g(t)), \tag{1.10}$$

$$\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-} f\right)(x; p) = \int_x^b (g(t) - g(x))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(t) - g(x))^{\sigma}; p) f(t) d(g(t)). \tag{1.11}$$

The following remark provides some connection of Definition 1.5 with already known operators:

Remark 1. (i) If we take $p = 0$ and $g(x) = x$ in equation (1.10), then it reduces to the fractional integral operators defined by Salim and Faraj in [27].
(ii) If we take $\delta = r = 1$ and $g(x) = x$ in (1.10), then it reduces to the fractional integral operators ${}_g\Upsilon^{\rho,1,k,c}_{\sigma,\tau,1,\omega,a^+}$ and ${}_g\Upsilon^{\rho,1,k,c}_{\sigma,\tau,1,\omega,b^-}$ containing generalized Mittag-Leffler function $E^{\rho,1,k,c}_{\sigma,\tau,1}(t;p)$ defined by Rahman *et al.* in [26].
(iii) If we set $p = 0, \delta = r = 1$ and $g(x) = x$ in (1.10), then it reduces to integral operators containing extended generalized Mittag-Leffler function introduced by Srivastava and Tomovski in [29].
(iv) If we take $p = 0, \delta = r = k = 1$ and $g(x) = x$, (1.10) reduces to integral operators defined by Prabhaker in [25] containing generalized Mittag-Leffler function.
(v) For $p = \omega = 0$ and $g(x) = x$ in (1.10), then generalized fractional integral operators ${}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}$ and ${}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,b^-}$ reduce to Riemann-Liouville fractional integral operators.

Our aim in this paper is to establish Hadamard and Fejér-Hadamard inequalities for generalized fractional integral operators containing extended generalized Mittag-Leffler function for a monotone increasing function via m-convex functions.

More than a hundred years ago, the mathematicians introduced the convexity and they established a lot of inequalities for the class of convex functions. The convex functions are playing a significant and a tremendous role in fractional calculus. Convexity has been widely employed in many branches of mathematics, for instance, in mathematical analysis, optimization theory, function theory, functional analysis and so on. Recently, many authors and researchers have given their attention to the generalizations, extensions, refinements of convex functions in multi-directions.

Definition 1.6. A function $f : [a,b] \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in [a,b]$ and $t \in [0,1]$.

The m-convex function is a close generalization of convex function and its concept was introduced by Toader [30].

Definition 1.7. A function $f : [0,b] \to \mathbb{R}$, $b > 0$ is said to be m-convex if for all $x, y \in [0,b]$ and $t \in [0,1]$

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

holds for $m \in [0,1]$.

If we take $m = 1$, we get the definition for convex function. An m-convex function need not be a convex function.

Example 1. *[24] A function $f : [0,\infty) \to \mathbb{R}$ defined by $f(x) = x^4 - 5x^3 + 9x^2 - 5x$ is $\frac{16}{17}$-convex but it is not m-convex for $m \in (\frac{16}{17}, 1]$.*

A lot of results and inequalities pertaining to convex, m-convex and related functions have been produced (see, [11–20] and references therein). Many fractional integral inequalities like Hadamard and Fejér-Hadamard are very important and researchers have produced their generalizations and refinements (see, [5] and references therein). Fractional inequalities have many applications, for instance, the most fruitful ones are used in establishing uniqueness of solutions of fractional boundary value problems and fractional partial differential equations. For instance the following Hadamard inequality is given in [21]:

Theorem 1.8. *Let $f : [0,\infty) \to \mathbb{R}$, be positive real function. Let $a, b \in [0,\infty)$ with $a < mb$ and $f \in L_1[a, mb]$. If f is m-convex on $[a, mb]$, then the following inequalities for the extended generalized fractional integrals hold:*

$$
\begin{aligned}
& f\left(\frac{a+mb}{2}\right)\left(\epsilon^{\omega',\rho,r,k,c}_{a^+,\sigma,\tau,\delta}1\right)(mb;p) \\
\leq\ & \frac{\left(\epsilon^{\omega',\rho,r,k,c}_{a^+,\sigma,\tau,\delta}f\right)(mb;p) + m^{\tau+1}\left(\epsilon^{m^\sigma\omega',\rho,r,k,c}_{b_-,\sigma,\tau,\delta}f\right)(\frac{a}{m};p)}{2} \\
\leq\ & \frac{m^{\tau+1}}{2}\left[\frac{f(a) - m^2 f(\frac{a}{m^2})}{mb-a}\left(\epsilon^{m^\sigma\omega',\rho,r,k,c}_{b_-,\sigma,\tau+1,\delta}1\right)(\frac{a}{m};p)\right. \\
& \left. + \left(f(b) + mf(\frac{a}{m^2})\right)\left(\epsilon^{m^\sigma\omega',\rho,r,k,c}_{b_-,\sigma,\tau,\delta}1\right)(\frac{a}{m};p)\right], \omega' = \frac{\omega}{(mb-a)^\sigma}.
\end{aligned}
$$

In the upcoming section we will derive the Hadamard inequality for m-convex functions by means of fractional integrals (1.10) and (1.11). This version of the Hadamard inequality gives at once the Hadamard inequalities quoted in Section 2. Further we will establish the Fejér-Hadamard inequality for these operators of m-convex functions which will provide the corresponding inequalities proved in [31]. Moreover in Section 3 by establishing two identities error estimations of the Hadamard and the Fejér-Hadamard inequalities are obtained.

2. Main results

Theorem 2.1. *Let $f, g : [a,b] \to \mathbb{R}$, $0 < a < b$, Range $(g) \subset [a,b]$, be the functions such that f be positive and $f \in L_1[a,b]$, g be differentiable and strictly increasing. If f be m-convex $m \in (0,1]$ and $g(a) < mg(b)$, then the following inequalities for fractional operators (1.10) and (1.11) hold:*

$$
\begin{aligned}
& f\left(\frac{g(a)+mg(b)}{2}\right)\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',a^+}1\right)(g^{-1}(mg(b));p) \\
\leq\ & \frac{\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',a^+}(f\circ g)\right)(g^{-1}(mg(b));p) + m^{\tau+1}\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,m^\sigma\omega',b_-}(f\circ g)\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)}{2} \\
\leq\ & \frac{m^{\tau+1}}{2}\left[\frac{f(g(a)) - m^2 f\left(\frac{g(a)}{m^2}\right)}{mg(b)-g(a)}\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau+1,\delta,m^\sigma\omega',b_-}1\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)\right. \\
& \left. + \left(f(g(b)) + mf\left(\frac{g(a)}{m^2}\right)\right)\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,m^\sigma\omega',b_-}1\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)\right], \omega' = \frac{\omega}{(mg(b)-g(a))^\sigma}.
\end{aligned}
$$

Proof. By definition of m-convex function f, we have

$$
2f\left(\frac{g(a)+mg(b)}{2}\right) \leq f\left(tg(a) + m(1-t)g(b)\right) + mf\left(tg(b) + (1-t)\frac{g(a)}{m}\right). \tag{2.1}
$$

Further from (2.1), one can obtain the following integral inequality:

$$
\begin{aligned}
&2f\left(\frac{g(a)+mg(b)}{2}\right)\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};p)dt \\
&\leq \int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};p)f(tg(a)+m(1-t)g(b))dt \\
&+m\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};p)f\left(tg(b)+(1-t)\frac{g(a)}{m}\right)dt.
\end{aligned} \tag{2.2}
$$

Setting $g(x) = tg(a) + m(1-t)g(b)$ and $g(y) = tg(b) + (1-t)\frac{g(a)}{m}$ in (2.2), we get the following inequality:

$$
\begin{aligned}
&2f\left(\frac{g(a)+mg(b)}{2}\right)\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',a^{+}}1\right)(g^{-1}(mg(b));p) \\
&\leq \left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',a^{+}}(f\circ g)\right)(g^{-1}(mg(b));p)+m^{\tau+1}\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,m^{\sigma}\omega',b_{-}}(f\circ g)\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right).
\end{aligned} \tag{2.3}
$$

Also by using the m-convexity of f, one can has

$$
\begin{aligned}
&f(tg(a)+m(1-t)g(b))+mf\left(tg(b)+(1-t)\frac{g(a)}{m}\right) \\
&\leq m\left(f(g(b))+mf\left(\frac{g(a)}{m^2}\right)\right)+\left(f(g(a))-m^2f\left(\frac{g(a)}{m^2}\right)\right)t.
\end{aligned} \tag{2.4}
$$

This leads to the following integral inequality:

$$
\begin{aligned}
&\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};p)f(tg(a)+m(1-t)g(b))dt \\
&+m\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};p)f\left(tg(b)+(1-t)\frac{g(a)}{m}\right)dt \\
&\leq m\left(f(g(b))+mf\left(\frac{g(a)}{m^2}\right)\right)\int_0^1 t^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};p)dt \\
&+\left(f(g(a))-m^2f\left(\frac{g(a)}{m^2}\right)\right)\int_0^1 t^{\tau}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};p)dt.
\end{aligned} \tag{2.5}
$$

Again by setting $g(x) = tg(a) + m(1-t)g(b)$, $g(y) = tg(b) + (1-t)\frac{g(a)}{m}$ in (2.5) and after calculation, we get

$$
\begin{aligned}
&\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',a^{+}}(f\circ g)\right)(g^{-1}(mg(b));p)+m^{\tau+1}\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,m^{\sigma}\omega',b_{-}}(f\circ g)\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right) \\
&\leq m^{\tau+1}\left(\frac{f(g(a))-m^2f\left(\frac{g(a)}{m^2}\right)}{mg(b)-g(a)}\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau+1,\delta,m^{\sigma}\omega',b_{-}}1\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)\right. \\
&\left.+\left(f(g(b))+mf\left(\frac{g(a)}{m^2}\right)\right)\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,m^{\sigma}\omega',b_{-}}1\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)\right).
\end{aligned} \tag{2.6}
$$

Combining (2.3) and (2.6) we get the desired result. □

Remark 2. • In Theorem 2.1, if we put $m = 1$, we get [31, Theorem 3.1]
• In Theorem 2.1, if we put $g = I$, we get [21, Theorem 3.1].

The following theorem gives the Fejér-Hadamard inequality for m-convex functions.

Theorem 2.2. *Let $f, g, h : [a, b] \to \mathbb{R}$, $0 < a < b$, Range (g), be the functions such that f be positive and $f \in L_1[a, b]$, g be a differentiable and strictly increasing and h be integrable and non-negative. If f is m-convex, $m \in (0, 1]$, $g(a) < mg(b)$ and $g(a) + mg(b) - mg(x) = g(x)$, then the following inequalities for fractional operator (1.11) hold:*

$$\begin{aligned}
&2f\left(\frac{g(a)+mg(b)}{2}\right)\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',mb_-}h\circ g\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)\\
&\leq (1+m)\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',b_-}(f\circ g)(h\circ g)\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)\\
&\leq \frac{f(g(a))-m^2f\left(\frac{g(a)}{m^2}\right)}{(g(b)-\frac{g(a)}{m})}\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',b_-}h\circ g\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)\\
&+m\left(f(g(b))+mf\left(\frac{g(a)}{m^2}\right)\right)\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',b_-}h\circ g\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right),\omega'=\frac{\omega}{(g(b)-\frac{g(a)}{m})^{\sigma}}.
\end{aligned}$$

Proof. Multiplying both sides of (2.1) by $2t^{\tau-1}h\left(tg(b)+(1-t)\frac{g(a)}{m}\right)E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};p)$ and integrating on $[0, 1]$, we get

$$\begin{aligned}
&2f\left(\frac{g(a)+mg(b)}{2}\right)\int_0^1 t^{\tau-1}h\left(tg(b)+(1-t)\frac{g(a)}{m}\right)E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};p)dt \qquad (2.7)\\
&\leq \int_0^1 t^{\tau-1}h\left(tg(b)+(1-t)\frac{g(a)}{m}\right)E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};p)f(tg(a)+m(1-t)g(b))dt\\
&+m\int_0^1 t^{\tau-1}h\left(tg(b)+(1-t)\frac{g(a)}{m}\right)E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};p)f\left(tg(b)+(1-t)\frac{g(a)}{m}\right)dt.
\end{aligned}$$

Setting $g(x) = tg(b)+(1-t)\frac{g(a)}{m}$ and also using $g(a)+mg(b)-mg(x) = g(x)$ the following inequality is obtained:

$$\begin{aligned}
&2f\left(\frac{g(a)+mg(b)}{2}\right)\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',b_-}h\circ g\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right) \qquad (2.8)\\
&\leq (1+m)\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',b_-}(f\circ g)(h\circ g)\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right).
\end{aligned}$$

Multiplying both sides of inequality (2.4) with $t^{\tau-1}h\left(tg(b)+(1-t)\frac{g(a)}{m}\right)E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega t^{\sigma};p)$ and integrating on $[0, 1]$, then setting $g(x) = tg(b) + (1-t)\frac{g(a)}{m}$ and also using $g(a) + mg(b) - mg(x) = g(x)$ we have

$$(1+m)\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',b_-}(f\circ g)(h\circ g)\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right) \tag{2.9}$$
$$\leq \frac{(f(g(a))-m^2 f\left(\frac{g(a)}{m^2}\right))}{g((b)-\frac{g(a)}{m})}\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',b_-}h\circ g\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right)$$
$$+m\left(f(g(b))+mf\left(\frac{g(a)}{m^2}\right)\right)\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega',b_-}h\circ g\right)\left(g^{-1}\left(\frac{g(a)}{m}\right);p\right).$$

Combining (2.8) and (2.9) we get the desired result. □

Remark 3.
- In Theorem 2.2, if we put $m=1$, then we get [31, Theorem 3.2].
- In Theorem 2.2, if we put $g=I$ and $p=0$, then we get results of [3].
- In Theorem 2.2, if we put $g=I$, then we get results of [4].

3. Error bounds

To find error estimates first we prove the following two lemmas.

Lemma 3.1. *Let $f,g:[a,mb]\to\mathbb{R}$, $0<a<mb$, Range $(g)\subset[a,mb]$ be the functions such that f be positive and $f\circ g\in L_1[a,mb]$ and g be a differentiable and strictly increasing. Also if $f(g(x))=f(g(a)+g(mb)-g(x))$, then the following equality for fractional operators (1.10) and (1.11) holds:*

$$\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}f\circ g\right)(mb;p)=\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}f\circ g\right)(a;p)$$
$$=\frac{\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}f\circ g\right)(mb;p)+\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}f\circ g\right)(a;p)}{2}. \tag{3.1}$$

Proof. By Definition 1.5 of the generalized fractional integral operator containing extended generalized Mittag-Leffler function, we have

$$\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}f\circ g\right)(mb;p)=\int_a^{mb}(g(mb)-g(x))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(mb)-g(x))^{\sigma};p)f(g(x))d(g(x)). \tag{3.2}$$

Replacing $g(x)$ by $g(a)+g(mb)-g(x)$ in (3.2) and using $f(g(x))=f(g(a)+g(mb)-g(x))$, we have

$$\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}f\circ g\right)(mb;p)=\int_a^{mb}(g(x)-g(a))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(x)-g(a))^{\sigma};p)f(g(x))d(g(x)).$$

This implies

$$\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}f\circ g\right)(mb;p)=\left({}_{g}\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}f\circ g\right)(a;p). \tag{3.3}$$

By adding (3.2) and (3.3), we get (3.1). □

Lemma 3.2. *Let $f, g, h : [a, mb] \to \mathbb{R}$, $0 < a < mb$, Range (g), Range $(h) \subset [a, mb]$ be the functions such that f be positive and $f \circ g$, $h \circ g \in L_1[a, mb]$, g be a differentiable and strictly increasing and h be non-negative and continuous. If $f' \circ g \in L_1[a, mb]$ and $h(g(t)) = h(g(a) + g(mb) - g(t))$, then the following equality for the generalized fractional integral operators (1.10) and (1.11) holds:*

$$\begin{aligned}
&\frac{f(g(a)) + f(g(mb))}{2}\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}h \circ g\right)(mb; p) + \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}h \circ g\right)(a; p)\right] \qquad (3.4)\\
&- \left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}(f \circ g)(h \circ g)\right)(mb; p) + \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}(f \circ g)(h \circ g)\right)(a; p)\right]\\
&= \int_a^{mb}\left[\int_a^t (g(mb) - g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(mb) - g(s))^{\sigma}; p)h(g(s))d(g(s))\right.\\
&\left.- \int_t^{mb}(g(s) - g(a))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(s) - g(a))^{\sigma}; p)h(g(s))d(g(s))\right]f'(g(t))d(g(t)).
\end{aligned}$$

Proof. To prove the lemma, we have

$$\begin{aligned}
&\int_a^{mb}\left[\int_a^t (g(mb) - g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(mb) - g(s))^{\sigma}; p)h(g(s))d(g(s))\right]f'(g(t))d(g(t))\\
&= f(g(mb))\int_a^{mb}(g(mb) - g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega(g(mb) - g(s))^{\sigma}; p\right)h(g(s))d(g(s))\\
&- \int_a^{mb}\left((g(mb) - g(t))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}\left(\omega(g(mb) - g(t))^{\sigma}; p\right)\right)f(g(t))h(g(t))d(g(t))\\
&= f(g(mb))\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}h \circ g\right)(mb; p) - \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}(f \circ g)(h \circ g)\right)(mb; p).
\end{aligned}$$

By using Lemma 3.1, we have

$$\begin{aligned}
&\int_a^{mb}\left[\int_a^t (g(mb) - g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(mb) - g(s))^{\sigma}; p)h(g(s))d(g(s))\right]f'(g(t))d(g(t))\\
&= \frac{f(g(mb))}{2}\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}h \circ g\right)(mb; p) + \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}h \circ g\right)(a; p)\right]\\
&\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}(f \circ g)(h \circ g)\right)(mb; p).
\end{aligned}$$

In the same way we have

$$\begin{aligned}
&\int_a^{mb}\left[-\int_t^{mb}(g(s) - g(a))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(s) - g(a))^{\sigma}; p)h(g(s))d(g(s))\right]f'(g(t))d(g(t))\\
&= \frac{f(g(a))}{2}\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}h \circ g\right)(mb; p) + \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}h \circ g\right)(a; p)\right]\\
&\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}(f \circ g)(h \circ g)\right)(a; p).
\end{aligned}$$

By adding (3.5) and (3.5), we get (3.4). □

By using Lemma 3.2, we prove the following theorem.

Theorem 3.3. *Let $f, g, h : [a, mb] \to \mathbb{R}$, $0 < a < mb$, Range (g), Range $(h) \subset [a, mb]$ be the functions such that f be positive and $(f \circ g)' \in L_1[a, mb]$, where g be a differentiable and strictly increasing and h be non-negative and continuous. Also let $h(g(t)) = h(g(a) + g(mb) - g(t))$ and $|(f \circ g)'|$ is m-convex on $[a, b]$. Then for $k < r + \Re(\sigma)$, the following inequality for fractional integral operators (1.10) and (1.11) holds:*

$$\left|\frac{f(g(a))+f(g(mb))}{2}\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}h\circ g\right)(mb;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}h\circ g\right)(a;p)\right]\right. \tag{3.5}$$
$$\left.-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}(f\circ g)(h\circ g)\right)(mb;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}(f\circ g)(h\circ g)\right)(a;p)\right]\right|$$
$$\leq \frac{\| h \|_\infty \, M(g(mb)-g(a))^{\tau+1}}{\tau(\tau+1)}(1-\Omega)\left[|f'(g(a))+mf'(g(b))|\right],$$

where $\| h \|_\infty = \sup_{t\in[a,mb]} |h(t)|$ *and*

$$\Omega = \frac{1}{\tau+2}\left[\left\{\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)^{\tau+2}\right\}+\left\{\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)^{\tau+2}\right\}\right]$$
$$-\frac{\tau+1}{\tau+2}\left[\left\{\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)^{\tau+2}\right\}+\left\{\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)^{\tau+2}\right\}\right]$$
$$-\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)^{\tau+1}\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)+\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)^{\tau+1}.$$

Proof. Using Lemma 3.2, we have

$$\left|\left(\frac{f(g(a))+f(g(mb))}{2}\right)\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}h\circ g\right)(mb;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}h\circ g\right)(a;p)\right]\right.$$
$$\left.-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}(f\circ g)(h\circ g)\right)(mb;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}(f\circ g)(h\circ g)\right)(a;p)\right]\right|$$
$$\leq \int_a^{mb}\left|\left[\int_a^t (g(mb)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(mb)-g(s))^{\sigma};p)h(g(s))d(g(s))\right.\right.$$
$$\left.\left.-\int_t^{mb}(g(s)-g(a))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(s)-g(a))^{\sigma};p)h(g(s))d(g(s))\right]\right||f'(g(t))|d(g(t)). \tag{3.6}$$

Using the m-convexity of $|(f \circ g)'|$ on $[a, b]$, we have

$$|f'(g(t))| \leq \frac{g(mb)-g(t)}{g(mb)-g(a)}|f'(g(a))| + m\frac{g(t)-g(a)}{g(mb)-g(a)}|f'(g(b))|, t \in [a, b]. \tag{3.7}$$

If we replace $g(s)$ by $g(a) + g(mb) - g(s)$ and using $h(g(s)) = h(g(a) + g(mb) - g(s))$, $t' = g^{-1}(g(a) + g(mb) - g(t))$, in second integral in the followings, we get

$$
\begin{aligned}
&\left|\int_a^t (g(mb)-g(s))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(mb)-g(s))^{\sigma};p)h(g(s))d(g(s))\right.\\
&\left.-\int_t^{mb} (g(s)-g(a))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(s)-g(a))^{\sigma};p)h(g(s))d(g(s))\right|\\
=&\left|-\int_t^{a} (g(mb)-g(s))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(mb)-g(s))^{\sigma};p)h(g(s))d(g(s))\right.\\
&\left.-\int_a^{t'} (g(mb)-g(s))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(mb)-g(s))^{\sigma};p)h(g(s))d(g(s))\right|\\
=&\left|\int_t^{t'} (g(mb)-g(s))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(mb)-g(s))^{\sigma};p)h(g(s))d(g(s))\right|
\end{aligned}
$$

$$
\leq \begin{cases} \int_t^{t'} |(g(mb)-g(s))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(mb)-g(s))^{\sigma};p)h(g(s))|d(g(s)), & t\in[a,\frac{a+mb}{2}] \\ \int_{t'}^{t} |(g(mb)-g(s))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(mb)-g(s))^{\sigma};p)h(g(s))|d(g(s)), & t\in[\frac{a+mb}{2},mb]. \end{cases} \tag{3.8}
$$

By (3.6)–(3.8) and using absolute convergence of extended Mittag-Leffler function, we have

$$
\begin{aligned}
&\left|\frac{f(g(a))+f(g(mb))}{2}\left(\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}h\circ g\right)(mb;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}h\circ g\right)(a;p)\right)\right.\\
&\left.-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}(f\circ g)(h\circ g)\right)(mb;p)+\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}(f\circ g)(h\circ g)\right)(a;p)\right]\right|\\
&\leq \int_a^{\frac{a+mb}{2}}\left(\int_a^{a+mb-t} |(g(mb)-g(s))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(mb)-g(s))^{\sigma};p)h(g(s))|d(g(s))\right)\\
&\times\left(\frac{g(mb)-g(t)}{g(mb)-g(a)}|f'(g(a))|+m\frac{g(t)-g(a)}{g(mb)-g(a)}|f'(g(b))|\right)d(g(t))\\
&+\int_{\frac{a+mb}{2}}^{mb}\left(\int_{a+mb-t}^{t} |(g(mb)-g(s))^{\tau-1} E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(mb)-g(s))^{\sigma};p)h(g(s))|d(g(s))\right)\\
&\times\left(\frac{g(mb)-g(t)}{g(mb)-g(a)}|f'(g(a))|+m\frac{g(t)-g(a)}{g(mb)-g(a)}|f'(g(b))|\right)d(g(t))\\
&\leq \frac{\parallel h\parallel_\infty M}{\tau(g(mb)-g(a))}\left[\int_a^{\frac{a+mb}{2}}((g(mb)-g(t))^{\tau}-(g(t)-g(a))^{\tau})(g(mb)-g(t))|f'(g(a))|d(g(t))\right.\\
&+m\int_a^{\frac{a+mb}{2}}((g(mb)-g(t))^{\tau}-(g(t)-g(a))^{\tau})m(g(t)-g(a))|f'(g(b))|d(g(t))\\
&+\int_{\frac{a+mb}{2}}^{mb}((g(t)-g(a))^{\tau}-(g(mb)-g(t))^{\tau})(g(mb)-g(t))|f'(g(a))|d(g(t))\\
&\left.+m\int_{\frac{a+mb}{2}}^{mb}((g(t)-g(a))^{\tau}-(g(mb)-g(t))^{\tau})m(g(t)-g(a))|f'(g(b))|d(g(t))\right].
\end{aligned} \tag{3.9}
$$

After some calculations, we get

$$\int_a^{\frac{a+mb}{2}} ((g(mb)-g(t))^\tau - (g(t)-g(a))^\tau)(g(mb)-g(t))d(g(t))$$
$$= \int_{\frac{a+mb}{2}}^{mb} ((g(t)-g(a))^\tau - (g(mb)-g(t))^\tau)(g(t)-g(a))d(g(t))$$
$$= \frac{(g(mb)-g(a))^{\tau+2}}{\tau+2} - \frac{\left(g(mb)-g(\frac{a+mb}{2})\right)^{\tau+2}}{\tau+2} - \frac{\left(g(\frac{a+mb}{2})-g(a)\right)^{\tau+1}}{\tau+1}$$
$$\left(g(mb)-g(\frac{a+mb}{2})\right) - \frac{\left(g(\frac{a+mb}{2})-g(a)\right)^{\tau+2}}{(\tau+1)(\tau+2)},$$

and

$$\int_a^{\frac{a+mb}{2}} ((g(mb)-g(t))^\tau - (g(t)-g(a))^\tau)(g(t)-g(a))d(g(t))$$
$$= \int_{\frac{a+mb}{2}}^{mb} ((g(t)-g(a))^\tau - (g(mb)-g(t))^\tau)(g(mb)-g(t))d(g(t))$$
$$= -\frac{\left(g(\frac{a+mb}{2})-g(a)\right)^{\tau+1}}{\tau+1}\left(g(mb)-g(\frac{a+mb}{2})\right) + \frac{(g(mb)-g(a))^{\tau+2}}{(\tau+1)(\tau+2)}$$
$$- \frac{\left(g(\frac{a+mb}{2})-g(a)\right)^{\tau+2}}{(\tau+1)(\tau+2)} - \frac{\left(g(mb)-g(\frac{a+mb}{2})\right)^{\tau+2}}{\tau+2}.$$

Using the above evaluations of integrals in (3.9), we get the required inequality (3.5). □

Remark 4.
- In Theorem 3.3, if we put $m = 1$, then we get [31, Theorem]
- In Theorem 3.3, if we put $g = I$ and $p = 0$, then we get [3, Theorem 2.3].
- In Theorem 3.3, if we put $g = I$, $p = 0$ and $m = 1$, then we get [19, Theorem 2.3].
- In Theorem 3.3, if we put $g = I$, then we get [4, Theorem].
- In Theorem 3.3, if we put $g = I$, $m = 1$, then we get [16, Theorem 2.3].
- In Theorem 3.3, for $\omega = p = 0$, $g = I$ and $h = 1$ along with $\tau = m = 1$, then we get [9, Theorem 2.2].
- In Theorem 3.3, if we put $\omega = p = 0$, $g = I$ and $h = 1$ with $m = 1$, then we get [28, Theorem 3].

Theorem 3.4. *Let $f, g, h : [a, mb] \to \mathbb{R}$, $0 < a < mb$, Range (g), Range $(h) \subset [a, mb]$ be the functions such that f be positive, $(f \circ g)' \in L_1[a, mb]$, g be a differentiable and strictly increasing and h be continuous. Also let $h(g(t)) = h(g(a) + g(mb) - g(t))$ and $|(f \circ g)'|^{q_1}$, $q_1 \geq 1$ is m-convex. Then for $k < r + \Re(\sigma)$, the following inequality for fractional integral operators (1.10) and (1.11) holds:*

$$\left|\left(\frac{f(g(a))+f(g(mb))}{2}\right)\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+} h\circ g\right)(mb;p) + \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-} h\circ g\right)(a;p)\right]\right.$$
$$\left.-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}(f\circ g)(h\circ g)\right)(mb;p) + \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}(f\circ g)(h\circ g)\right)(a;p)\right]\right| \tag{3.10}$$
$$\leq \frac{\|h\|_\infty M(g(mb)-g(a))^{\tau+1}}{\tau(\tau+1)}\left((1-\Psi)^{\frac{1}{p_1}}(1-\Omega)^{\frac{1}{q_1}}\right)\left(\frac{|f'(g(a))|^{q_1}+m|f'(g(b))|^{q_1}}{2}\right)^{\frac{1}{q_1}},$$

where $\|h\|_\infty = \sup_{t\in[a,mb]} |h(t)|$, $\frac{1}{p_1} + \frac{1}{q_1} = 1$,

$\Psi = \left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)^{\tau+1} + \left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)^{\tau+1}$ *and*

$$\Omega = \frac{1}{\tau+2}\left[\left\{\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)^{\tau+2}\right\} + \left\{\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)^{\tau+2}\right\}\right]$$
$$-\frac{\tau+1}{\tau+2}\left[\left\{\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)^{\tau+2}\right\} + \left\{\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)^{\tau+2}\right\}\right]$$
$$-\left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)^{\tau+1}\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right) + \left(\frac{g(\frac{a+mb}{2})-g(a)}{g(mb)-g(a)}\right)\left(\frac{g(mb)-g(\frac{a+mb}{2})}{g(mb)-g(a)}\right)^{\tau+1}.$$

Proof. Using Lemma 3.2, power mean inequality, (3.8) and m-convexity of $|(f\circ g)'|^{q_1}$ respectively, we have

$$\left|\left(\frac{f(g(a))+f(g(mb))}{2}\right)\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}h\circ g\right)(mb;p) + \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}h\circ g\right)(a;p)\right]\right.$$
$$\left.-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}(f\circ g)(h\circ g)\right)(mb;p) + \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb_-}(f\circ g)(h\circ g)\right)(a;p)\right]\right|$$
$$\leq\left[\int_a^{mb}\left|\int_t^{a+mb-t}(g(mb)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(mb)-g(s))^{\sigma};p)h(g(s))d(g(s))\right|d(g(t))\right]^{1-\frac{1}{q_1}} \quad (3.11)$$
$$\left[\int_a^{mb}\left|\int_t^{a+mb-t}(g(mb)-g(s))^{\tau-1}E^{\rho,r,k,c}_{\sigma,\tau,\delta}(\omega(g(mb)-g(s))^{\sigma};p)h(g(s))d(g(s))\right||f'(g(t))|^{q_1}\right]^{\frac{1}{q_1}}.$$

Since $|(f\circ g)'|^{q_1}$ is m-convex on $[a,b]$, we have

$$|f'(g(t))|^{q_1} \leq \frac{g(mb)-g(t)}{g(mb)-g(a)}|f'(g(a))|^{q_1} + m\frac{g(t)-g(a)}{g(mb)-g(a)}|f'(g(b))|^{q_1}. \quad (3.12)$$

Using $\parallel h \parallel_\infty = \sup_{t\in[a,mb]}|h(t)|$, and absolute convergence of extended Mittag-Leffler function, inequality (3.11) becomes

$$\left|\left(\frac{f(g(a))+f(g(mb))}{2}\right)\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}h\circ g\right)(mb;p) + \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb^-}h\circ g\right)(a;p)\right]\right.$$
$$\left.-\left[\left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,a^+}(f\circ g)(h\circ g)\right)(mb;p) + \left({}_g\Upsilon^{\rho,r,k,c}_{\sigma,\tau,\delta,\omega,mb^-}(f\circ g)(h\circ g)\right)(a;p)\right]\right|$$
$$\leq \parallel h\parallel_\infty^{1-\frac{1}{q_1}} M^{1-\frac{1}{q_1}}\left[\int_a^{\frac{a+mb}{2}}\left(\int_t^{a+mb-t}(g(mb)-g(s))^{\tau-1}d(g(s))\right)d(g(t))\right.$$
$$\left.+\int_{\frac{a+mb}{2}}^{b}\left(\int_{a+mb-t}^{t}(g(mb)-g(s))^{\tau-1}d(g(s))\right)d(g(t))\right]^{1-\frac{1}{q_1}}$$
$$\times \parallel h\parallel_\infty^{\frac{1}{q_1}} M^{\frac{1}{q_1}}\left[\int_a^{\frac{a+mb}{2}}\left(\int_t^{a+mb-t}(g(mb)-g(s))^{\tau-1}d(g(s))\right)\right.$$
$$\times\left(\frac{g(mb)-g(t)}{g(mb)-g(a)}|f'(g(a))|^{q_1} + m\frac{g(t)-g(a)}{g(mb)-g(a)}|f'(g(b))|^{q_1}\right)d(g(t))$$
$$+\int_{\frac{a+mb}{2}}^{b}\left(\int_{a+mb-t}^{t}(g(mb)-g(s))^{\tau-1}d(g(s))\right)$$
$$\left.\times\left(\frac{g(mb)-g(t)}{g(mb)-g(a)}|f'(g(a)))|^{q_1} + m\frac{g(t)-g(a)}{g(mb)-g(a)}|f'(g(b))|^{q_1}\right)d(g(t))\right]^{\frac{1}{q_1}}.$$

After integrating and simplifying above inequality, we get (3.10). □

Remark 5.
- In Theorem 3.4, if we put $m = 1$, then we get [31, Theorem].
- In Theorem 3.4, if we put $g = I$ and $p = 0$, then we get [3, Theorem 2.6].
- In Theorem 3.4, if we put $g = I$, $p = 0$ and $m = 1$, then we get [19, Theorem 2.6].
- In Theorem 3.4, if we put $g = I$, then we get [4, Theorem].
- In Theorem 3.4, if we put $g = I, m = 1$, then we get [16, Theorem 2.5].

4. Conclusions

This work provides the Hadamard and the Fejér-Hadamard inequalities for generalized extended fractional integral operators involving monotonically increasing function. These inequalities are obtained by using m-convex function which give results for convex function in particular. The presented results are generalizations of several fractional integral inequalities which are directly connected, consequently the well-known published results are quoted in remarks.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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