**Mathematics**

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 6313–6324.
DOI: 10.3934/math.2020406
Received: 18 March 2020
Accepted: 15 July 2020
Published: 07 August 2020

*Research article*

# Role of shape operator in warped product submanifolds of nearly cosymplectic manifolds

**Rifaqat Ali[1,*], Nadia Alluhaibi[2], Khaled Mohamed Khedher[3], Fatemah Mofarreh[4,*] and Wan Ainun Mior Othman[5]**

[1] Department of Mathematics, College of Science and Arts Muhayil, King Khalid University, 9004 Abha, Saudi Arabia

[2] Department of Mathematics, Science and Arts College, Rabigh Campus, King Abdulaziz University, Jeddah, Saudi Arabia

[3] Department of Civil Engineering, College of Engineering, King Khalid University, Abha–61421, KSA, Department of Civil Engineering, ISET, Nabeul, DGET, Tunisia

[4] Mathematical Science Department Faculty of Science Princess Nourah bint Abdulrahman University, Riyadh 11546, Saudi Arabia

[5] Institute of Mathematical Sciences, Faculty of Science, University of Malaya, 50603 Kuala Lumpur, Malaysia

* **Correspondence:** Email: rrafat@kku.edu.sa; fyalmofarrah@pnu.edu.sa.

**Abstract:** In this paper, first, we find the integrability theorems for the invariant and slant distributions which appeared in the concept of semi-slant submanifolds. Utilizing these theorems, we prove that a semi-slant submanifold reduces to be a warped product semi-slant submanifold, provided some necessary and sufficient conditions concerning the shape operators. Also, it is shown that a few earlier results are exceptional cases of this paper results.

**Keywords:** warped product; characterizations; integrability conditions; shape operators
**Mathematics Subject Classification:** 58C40, 53C42, 35P15

## 1. Introduction

The warped product is used to form a different semi-Riemannian manifolds. Such construction is of benefit in *General Relativity*, black holes study and cosmological models. The product manifold metric, at this case, turns to non-degenerate. The warped product manifolds were inaugurated by Bishop and O'Neill [12] to extend the Riemannian product manifolds. This idea concerning warped product submanifolds was given by Chen [15, 16]. Extending such a concept, recently, Uddin et al. [22] studied the non existence case of warped product semi-slant submanifold in terms of $M = M_\vartheta \times_f M_T$ whith $M_\vartheta$ and $M_T$ are proper slant and invariant submanifolds of the nearly cosymplectic manifold. Using the warped product semi-slant submanifold of type $M = M_T \times_f M_\vartheta$, the geometric inequality is obtained in [22] which disclosed the connection between the second fundamental form, slant immersion and warping function. Similarly, in [2, 4] it has been classified the warped product submanifolds at Sasakian (cosymplectic) manifolds with pointwise slant embedding. Characterization theorems for warped product submanifold at Kenmotsu manifold having the fiber is the slant submanifold, were proved in [3, 5]. A number of authors extended the warped product submanifolds idea in almost contact manifolds [as in [1, 6–8, 16] and references therein]. Studying the previous articles, an important question is arisen. Why we choose nearly cosymplectic structure? Answer is that cosymplectic structure does not admits any type of warped product semi-slant submanifold [19], while a near cosymplectic manifold has a nontrivial warped product semi-slant submanifolds in the shape of $M = M_T \times_f M_\vartheta$ see [22, 24]. Therefore such a class force to us to study in a nearly cosymplectic manifold. At the first part of this present paper, we derive integrability theorems under some restrictions.

**Remark 1.1.** We will use the following abberiation throughout the paper: "WPSSS" for Warped product sem-slant submanifold, "WF" for warping function, "RM" for Riemannian manifold, "SSS" for semi-slant submanifold and "NCM" for nearly cosymplectic manifold $\widetilde{M}$.

More precisely, we prove the following finding for invariant distribution.

**Theorem 1.1.** *An invariant distribution $\mathcal{D} \oplus <\zeta>$ of a SSS $M$ in a NCM $\widetilde{M}$ is integrable if and only if the following equality holds*

$$2g(\nabla_{W_0} W_1, W_2) = \csc^2\vartheta\Big(g(h(W_0, \varphi W_1), FW_2) - 2g(h(W_0, W_1), FTW_2) + g(h(W_1, \varphi W_0), FW_1)\Big), \quad (1.1)$$

*for any $W_0, W_1 \in \Gamma(\mathcal{D} \oplus \zeta)$ and $W_2 \in \Gamma(\mathcal{D}^\vartheta)$.*

Similarly, for a slant distribution we get the next integrability result.

**Theorem 1.2.** *The slant distribution $\mathcal{D}^\vartheta$ of a SSS $M$ in a NCM $\widetilde{M}$ is integrable if and only if*

$$\begin{aligned}\sin^2\vartheta g(\nabla_{W_2} W_3, W_1) =& \frac{1}{2}\Big\{g(h(W_1, W_2), FTW_3) + g(h(W_1, W_3), FTW_2) \\ & - g(h(\varphi W_1, W_2), FW_3) - g(h(\varphi W_1, W_3), FW_2) + \eta(W_1)g(\tilde{\nabla}_{W_3}\zeta, W_2)\Big\}, \quad (1.2)\end{aligned}$$

*for any $W_1 \in \Gamma(\mathcal{D} \oplus \zeta)$ and $W_2, W_3 \in \Gamma(\mathcal{D}^\vartheta)$.*

The results Theorem 1.1 and 1.2 will required for proof of main theorem of this paper. Next, we provide the characterizations of a class of SSS for being the class of WPSSS in a NCM by using the result of Hiepko [18]. Hence, we give the characterization theorem for a WPSSS of a NCM which is an important result of this study.

**Theorem 1.3.** *A semi-slant submanifold $M$ of a NCM $\widetilde{M}$ having the integrable distributions $\mathcal{D}$ and $\mathcal{D}^{\vartheta}$, is locally a WPSSS of the type $M = M_T \times_f M_{\vartheta}$ if and only if*

$$(i)\ A_{FW_2}\varphi W_1 = (W_1\lambda)W_2, \quad and \quad (ii)\ A_{FTW_2}W_1 = \frac{1}{3}\cos^2\vartheta(W_1\lambda)W_2, \tag{1.3}$$

*for all $W_1 \in \Gamma(\mathcal{D}\oplus\zeta)$ and $W_2 \in \Gamma(\mathcal{D}^{\vartheta})$. For all $W_3 \in \Gamma(\mathcal{D}^{\vartheta})$, it is also satisfied that $W_3\lambda = 0$, for a positive function $\lambda = \ln f$ on $M_T$.*

As a direct result of Theorem 1.3 in a sense of Papaghiuc [21] which shown that the class of semi-slant submanifold is generality of class of CR-submanifold with slant angle $\vartheta = \frac{\pi}{2}$. Therefore, we substitute $\vartheta = \frac{\pi}{2}$ in (1.3), then we get following result.

**Corollary 1.1.** *A CR-submanifold $M$ of NCM $\widetilde{M}$ having the integrable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$, is locally the CR-warped product of type $M = M_T \times_f M_{\perp}$ if and only if*

$$(i)\ A_{\varphi W_2}W_1 = (\varphi W_1\lambda)W_2, \tag{1.4}$$

*for all $W_1 \in \Gamma(\mathcal{D}\oplus\zeta)$ and $W_2 \in \Gamma(\mathcal{D}^{\perp})$. For all $W_3 \in \Gamma(\mathcal{D}^{\perp})$, it is also satisfied that $W\lambda = 0$, for the positive function $\lambda = \ln f$ on $M_T$.*

**Remark 1.2.** Interestingly to notice that Corollary 1.1 coincides with Theorem 3.1 in [24] and hence Theorem 1.3 is generalized Theorem 3.1 in [24]. Also, we give immediately consequences of our results.

Now, we give another interesting theorem

**Theorem 1.4.** *A mixed totally geodesic WPSSS $M = M_T \times_f M_{\vartheta}$ in a NCM $\widetilde{M}$ is a usual Riemannian product manifold of $M_T$ and $M_{\vartheta}$.*

The paper is organized as follows: In section 2, we highlight some preliminaries and formulas which are useful for our literature. In section 3, we give the definition of semi-slant submanifolds and provide the proofs of integrability theorems. In section 4, we define warped product manifolds and give the proof of characterization theorem.

## 2. Preliminaries

The odd-dimensional $C^{\infty}$-manifold $(\widetilde{M}, g)$ associated to almost contact structure $(\varphi, \zeta, \eta)$ is referred to as the *almost contact metric manifold* fulfilling coming properties:

$$\varphi^2 = -I + \eta\otimes\zeta,\ \ \eta(\zeta) = 1,\ \ \varphi(\zeta) = 0,\ \ \eta\circ\varphi = 0, \tag{2.1}$$

$$g(\varphi W_1, \varphi W_2) = g(W_1, W_2) - \eta(W_1)\eta(W_2),\ \ \eta(W_1) = g(W_1, \zeta), \tag{2.2}$$

$\forall\ W_1, W_2 \in \Gamma(T\widetilde{M})$ (see, for instance [10, 11]). A *cosymplectic manifold* [17, 22, 23] regarding Riemannian connection is contained the almost contact metric manifold which satisfied the next equation

$$(\widetilde{\nabla}_{W_1}\varphi)W_1 = 0, \tag{2.3}$$

It follows for a nearly cosymplectic manifold

$$(\widetilde{\nabla}_{W_1}\varphi)W_2 + (\widetilde{\nabla}_{W_2}\varphi)W_1 = 0, \tag{2.4}$$

for all vector fields $W_1, W_2$ are tangent to $\widetilde{M}$. The *Gauss* and *Weingarten* formulas which specifying the relation between Levi-Civitas connections $\nabla$ on a submanifold $M$ and $\widetilde{\nabla}$ on ambient manifold $\widetilde{M}$ are given by (for more detail see [16])

$$\widetilde{\nabla}_{W_1}W_2 = \nabla_{W_1}W_2 + h(W_1, W_2) \tag{2.5}$$

$$\widetilde{\nabla}_{W_1}\xi = -A_\xi W_1 + \nabla^{\perp}_{W_1}\xi, \tag{2.6}$$

for every $W_1,\ W_2 \in \Gamma(TM)$ and $\xi \in \Gamma(T^{\perp}M)$, in which $h$ and $A_\xi$ have this next relation

$$g(h(W_1, W_2), \xi) = g(A_\xi W_1, W_2) \tag{2.7}$$

The next relation is that

$$\varphi W_1 = TW_1 + FW_1, \tag{2.8}$$

in which $FW_1$ and $TW_1$are normal and tangential elements of $\varphi W_1$, respectively. If $M$ is invariant and anti-invariant then $FW_1$ as well as $TW_1$ are zero, in the same order. Similarly, we have

$$\varphi\xi = t\xi + f\xi \tag{2.9}$$

where $t\xi$ (resp. $f\xi$) are tangential (resp. normal) components of $\varphi\xi$. The covariant derivative of the endomorphism $\varphi$ is explained by

$$(\widetilde{\nabla}_{W_1}\varphi)W_2 = \widetilde{\nabla}_{W_1}\varphi W_2 - \varphi\widetilde{\nabla}_{W_1}W_2, \ \ \forall W_1, W_2 \in \Gamma(T\widetilde{M}). \tag{2.10}$$

In case the tangential and normal elements of $(\widetilde{\nabla}_{W_1}\varphi)W_2$ using $\mathcal{P}_{W_1}W_2$ and $\mathcal{Q}_{W_1}W_2$, for a nearly cosymplectic manifold, it is satisfied that

$$(i)\ \ \mathcal{P}_{W_1}W_2 + \mathcal{P}_{W_2}W_1 = 0, \ \ (ii)\ \ \mathcal{Q}_{W_1}W_2 + \mathcal{Q}_{W_2}W_1 = 0, \tag{2.11}$$

where $W_1, W_2$ are tangential to $\widetilde{M}$. For more details on properties of $\mathcal{P}$ and $\mathcal{Q}$, see [22].

There is a motivating class of submanifolds presented as slant submanifolds class. For any not zero vector $W_1$ tangential to $M$ about $p$, in which $W_1$ is not proportional to $\zeta_p$, $0 \leq \vartheta(W_1) \leq \pi/2$ is referred to the angle between $\varphi W_1$ and $T_pM$ which is named as Wirtinger angle. If $\vartheta(W_1)$ is constant for any $W_1 \in T_pM - <\zeta>$ at point $p \in M$, therefore $M$ is referred to as the slant submanifold [13] and $\vartheta$ is then slant angle of $M$. The following necessary and sufficient condition is an important for this paper which known as characterization slant submanifold and was proved in [13], a submanifold $M$ is slant if and only if the equality holds

$$T^2 = \lambda(-I + \eta\otimes\zeta), \tag{2.12}$$

for a constant $\lambda \in [0, 1]$ in which $\lambda = \cos^2\vartheta$, where $T$ is an endomorphism defined in (2.8). The following alliances are resulted from Eq (2.12).

$$g(TW_1, TW_2) = \cos^2\vartheta\Big\{g(W_1, W_2) - \eta(W_1)\eta(W_1)\Big\} \tag{2.13}$$

$$g(FW_1, FW_2) = \sin^2\vartheta\Big\{g(W_1, W_2) - \eta(W_1)\eta(W_2)\Big\}, \tag{2.14}$$

$\forall\ W_1, W_2 \in \Gamma(TM)$. The following result which was derived in [9] is the necessary and sufficient condition $M$ to remain slant if

$$\begin{aligned}(a)\quad & tFW_1 = \sin^2\vartheta(-W_1 + \eta(W_1)\zeta)\\ & \textit{and}\\ (b)\quad & fFW_1 = -FTW_1,\end{aligned} \tag{2.15}$$

for any $W_1 \in \Gamma(TM)$.

## 3. Semi-slant submanifolds

A generality of CR-submanifold in a almost Hermitian manifold by utilizing slant distribution was described using N. Papaghiuc [21]. Such submanifolds are referred to as semi-slant submanifold. This is determined by Cabererizo [14] in an almost contact manifold, i.e.,

**Definition 3.1.** The Riemannian submanifold $M$ is called a semi-slant in $\widetilde{M}$, in case it is spanned by two perpendicular distributions $\mathcal{D}$ and $\mathcal{D}^\vartheta$ that is $TM = \mathcal{D}^\vartheta \oplus \mathcal{D} \oplus \langle\zeta\rangle$. For more classification see in [9, 22].

**Remark 3.1.** A semi-slant submanifold is referred to as a mixed totally geodesic if $h(W_1, W_2) = 0$, for any $W_1 \in \Gamma(\mathcal{D}^\vartheta)$ and $W_2 \in \Gamma(\mathcal{D})$.

Now, we construct the integrability conditions of the distributions involving in the meaning of a semi-slant submanifold in a nearly cosymplectic manifold.

**Proof of Theorem 1.1 and 1.2**

**Proof of Theorem 1.1** From the Lie bracket, we get

$$g([W_0, W_1], W_2) = g(\tilde{\nabla}_{W_0} W_1, W_2) - g(\tilde{\nabla}_{W_1} W_0, W_2).$$

From (2.2) and $\zeta$ is orthogonal to $\mathcal{D}^\vartheta$, we get

$$g([W_0, W_1], W_2) = g(\tilde{\nabla}_{W_0} W_1, W_2) - g(\varphi\tilde{\nabla}_{W_1} W_0, \varphi W_2).$$

Then by utilizing (2.8) and (2.11), we attain

$$\begin{aligned}g([W_0, W_1], W_2) =& g(\tilde{\nabla}_{W_0} W_1, W_2) + g(\tilde{\nabla}_{W_1} W_0, \varphi TW_2) + g((\tilde{\nabla}_{W_1}\varphi)W_0, FW_2)\\ & - (\tilde{\nabla}_{W_1}\varphi W_0, FW_2).\end{aligned}$$

Thus by (2.3), (2.8) as well as (2.5), we have

$$\begin{aligned}g([W_0, W_1], W_2) =& g(\tilde{\nabla}_{W_0} W_1, W_2) + g(\tilde{\nabla}_{W_1} W_0, T^2 W_2) + g(\tilde{\nabla}_{W_1} W_0, FTW_2)\\ & - g((\tilde{\nabla}_{W_0}\varphi)W_1, FW_2) - g(h(W_1, \varphi W_0), FW_2).\end{aligned}$$

Utilizing (2.13), (2.5) as well as (2.11), resulted in

$$\begin{aligned} g([W_0,W_1],W_2) =& g(\tilde{\nabla}_{W_0}W_1,W_2)-\cos^2\vartheta g(\tilde{\nabla}_{W_1}W_0,W_2)+g(h(W_0,W_1),FTW_2)\\ &-g(\tilde{\nabla}_{W_0}\varphi W_1,FW_2)-g(\tilde{\nabla}_{W_0}W_1,\varphi FW_2)-g(h(W_1,\varphi W_0),FZ). \end{aligned}$$

Using (2.9) and (2.5), then we derive

$$\begin{aligned} g([W_0,W_1],W_2) =& g(\tilde{\nabla}_{W_0}W_1,W_2)-\cos^2\vartheta g(\tilde{\nabla}_{W_1}W_0,W_2)+g(h(W_0,W_1),FTW_2)\\ &-g(h(W_0,\varphi W_1),FW_2)-g(\tilde{\nabla}_{W_0}W_1,tFW_2)-g(\tilde{\nabla}_{W_0}W_1,fFW_2)\\ &-g(h(W_1,\varphi W_0),FW_2). \end{aligned}$$

Finally, from (2.15), we achieve that

$$\begin{aligned} g([W_0,W_1],W_2) =& g(\tilde{\nabla}_{W_0}W_1,W_2)-\cos^2\vartheta g(\tilde{\nabla}_{W_1}W_0,W_2)+2g(h(W_0,W_1),FTW_2)\\ &+\sin^2\vartheta g(\tilde{\nabla}_{W_0}W_1,W_2)-g(h(W_0,\varphi W_1),FW_2)-g(h(W_1,\varphi W_0),FW_2). \end{aligned}$$

That is the result which we wanted.

**Corollary 3.1.** *The distribution $\mathcal{D}\oplus\zeta$ is the totally geodesic foliation of a SSS $M$ in NCM $\widetilde{M}$ if and only if*

$$g(h(W_0,W_1),FTW_2)=\frac{1}{2}\Big\{g(h(W_0,\varphi W_1),FW_2)+g(h(W_1,\varphi W_0),FW_2)\Big\},$$

*for all $W_0,W_1\in\Gamma(\mathcal{D}\oplus\zeta)$ and $W_2\in\Gamma(\mathcal{D}^{\vartheta})$.*

*Proof.* From the total geodesic folition defintion, for every $W_0,W_1\in\Gamma(\mathcal{D}\oplus\zeta)$, then $\nabla_{W_0}W_1\in\Gamma(\mathcal{D}\oplus\zeta)$. Putting this in the Eq (1.1), we get required result. □

**Proof of Theorem 1.2**

By the property of Lie bracket, we have

$$g([W_2,W_3],W_0)=g(\tilde{\nabla}_{W_2}W_3,W_0)-g(\tilde{\nabla}_{W_3}W_2,W_0).$$

By (2.2), we get

$$g([W_2,W_3],W_0)=g(\tilde{\nabla}_{W_2}W_3,W_0)-g(\varphi\tilde{\nabla}_{W_3}W_2,\varphi W_0)+\eta(W_0)g(\tilde{\nabla}_{W_3}\zeta,W_2).$$

Using (2.11), we obtain

$$g([W_2,W_3],W_0)=g(\tilde{\nabla}_{W_2}W_3,W_0)+g((\tilde{\nabla}_{W_3}\varphi)W_2,\varphi W_0)-g(\tilde{\nabla}_{W_3}\varphi W_2,\varphi W_0)+\eta(W_0)g(\tilde{\nabla}_{W_3}\zeta,W_2).$$

Thus from (2.3) and (2.8), then above equation take the form

$$\begin{aligned} g([W_2,W_3],W_0) =& g(\tilde{\nabla}_{W_2}W_3,W_0)-g((\tilde{\nabla}_{W_2}\varphi)W_3,\varphi W_0)-g(\tilde{\nabla}_{W_3}TW_2,\varphi W_0)\\ &-g(\tilde{\nabla}_{W_3}FW_2,\varphi W_0)+\eta(W_0)g(\tilde{\nabla}_{W_3}\zeta,W_2). \end{aligned}$$

Utilizing (2.2), (2.11) and (2.6), we achieve that

$$g([W_2,W_3],W_0) = 2g(\tilde{\nabla}_{W_2}W_3,W_0) - g(\tilde{\nabla}_{W_2}\varphi W_3,\varphi W_0) + g(\varphi\bar{\nabla}_{W_3}TW_2,W_0) + g(h(\varphi W_0,W_3),FW_2) + \eta(W_0)g(\tilde{\nabla}_{W_3}\zeta,W_2)$$

Utilizing (2.8), (2.2) and (2.11), we get

$$\begin{aligned} g([W_2,W_3],W_0) =& 2g(\tilde{\nabla}_{W_2}W_3,W_0) + g(h(\varphi W_0,W_3),FW_2) + g(\varphi\tilde{\nabla}_{W_2}TW_3,W_0) \\ &- g(\tilde{\nabla}_{W_2}FW_3,\varphi W_0) - g((\tilde{\nabla}_{W_3}\varphi)TW_2,W_0) + g(\tilde{\nabla}_{W_3}\varphi TW_2,W_0) \\ &+ \eta(W_0)g(\tilde{\nabla}_{W_3}\zeta,W_2). \end{aligned}$$

From (2.13), (2.6) (2.11) and (2.8), we obtain

$$\begin{aligned} g([W_2,W_3],W_0) =& 2g(\tilde{\nabla}_{W_2}W_3,W_0) + g(h(\varphi W_0,W_3),FW_2) + g(\tilde{\nabla}_{W_2}T^2W_3,W_0) \\ &+ g(\tilde{\nabla}_{W_2}FTW_3,W_0) + g(h(\varphi W_0,W_2),FW_3) - \cos^2\vartheta g(\tilde{\nabla}_{W_3}W_2,W_0) \\ &- g(A_{FTW_2}W_0,W_3) - g(\mathcal{P}_{W_2}TW_3,W_0) - g(\mathcal{P}_{W_3}TW_2,W_0) \\ &+ \eta(W_0)g(\tilde{\nabla}_{W_3}\zeta,W_2). \end{aligned}$$

Using (2.13) and (2.6), we derive

$$\begin{aligned} g([W_2,W_3],W_0) =& 2g(\tilde{\nabla}_{W_2}W_3,W_0) + g(h(\varphi W_0,W_3),FW_2) + g(h(\varphi W_0,W_2),FW_3) \\ &- g(A_{FTW_2}W_0,W_3) - g(A_{FTW_3}W_0,W_2) - \cos^2\vartheta g(\tilde{\nabla}_{W_3}W_2,W_0) \\ &- \cos^2\vartheta g(\tilde{\nabla}_{W_2}W_3,W_0)) + \eta(W_0)g(\tilde{\nabla}_{W_3}\zeta,W_2) \\ &- g(\mathcal{P}_{W_2}TW_3,W_0) - g(\mathcal{P}_{W_3}TW_2,W_0). \end{aligned} \tag{3.1}$$

Now we compute last two terms of above equation by using property $\mathcal{P}$ (see [13]) as follows

$$\begin{aligned} g(\mathcal{P}_{W_2}TW_3,W_0) =& -g(TW_3,\mathcal{P}_{W_2}W_0) = -(\varphi W_3,\mathcal{P}_{W_2}W_0) \\ =& g(W_3,\varphi\mathcal{P}_{W_2}W_0) \\ =& -g(W_3,\mathcal{P}_{W_2}\varphi W_0) = (\mathcal{P}_{W_2}W_3,\varphi W_0). \end{aligned}$$

Thus by the hypothesis and property of Lie Bracket in Eq (3.1), we arrive at

$$\begin{aligned} \sin^2\vartheta g([W_2,W_3],W_0) =& 2g(\tilde{\nabla}_{W_2}W_3,W_0) + g(h(\varphi W_0,W_3),FW_2) + g(h(\varphi W_0,W_2),FW_3) \\ &- g(A_{FTW_2}W_0,W_3) - g(A_{FTW_3}W_0,W_2) + \eta(W_0)g(\tilde{\nabla}_{W_3}\zeta,W_2) \\ &- 2\cos^2\vartheta g(\bar{\nabla}_{W_2}W_3,W_0)) - g(\mathcal{P}_{W_2}W_3 + \mathcal{P}_{W_3}W_2,\varphi W_0). \end{aligned}$$

Applying the structure equation (2.12), we obtain

$$\begin{aligned} \sin^\theta g([W_2,W_3],W_0) =& 2\sin^2\vartheta g(\tilde{\nabla}_{W_2}W_3,W_0) + g(h(\varphi W_0,W_3),FW_2) + g(h(\varphi W_0,W_2),FW_3) \\ &- g(A_{FTW_2}W_0,W_3) - g(A_{FTW_3}W_0,W_2) + \eta(W_0)g(\tilde{\nabla}_{W_3}\zeta,W_2). \end{aligned}$$

Hence, our assertion has got proven. The proof is completed.

An application of Theorem 1.2, we introduce

**Corollary 3.2.** *The slant distribution $\mathcal{D}^{\vartheta}$ of SSS $M$ in a NCM $\widetilde{M}$, is the totally geodesic foliation in $M$ if and only if*

$$g(h(\varphi W_0, W_3), FZ_2) + g(h(\varphi W_0, W_2), FW_3) = g(A_{FTW_2} W_0, W_3) + g(A_{FTW_3} W_0, W_2).$$

*for any $W_0 \in \Gamma(\mathcal{D} \oplus \zeta)$ and $W_2, W_3 \in \Gamma(\mathcal{D}^{\vartheta})$.*

*Proof.* From the definition totally geodesic folition, for every $W_2, W_3 \in \Gamma(\mathcal{D}^{\vartheta})$, then $\nabla_{W_2} W_3 \in \Gamma(\mathcal{D}^{\vartheta})$. Inserting this in the Eq (1.2), we get required result. □

## 4. Warped product submanifolds

Suppose $M = M_1 \times_f M_2$ is the smooth Riemannian manifold associating to warped product metric $g = g_1 + f^2 g_2$ in which $f$ is a WF on $M$. This idea was given in [12] and derived the formula;

$$\nabla_{W_2} W_0 = \nabla_{W_0} W_2 = (W_0 \ln f) W_2, \tag{4.1}$$

For all $W_0, W_1 \in \Gamma(TM_1)$ and $W_2, W_3 \in \Gamma(TM_2)$. The gradient of $\ln f$ is denoted by $\nabla \ln f$ and defined as:

$$g(\nabla \ln f, W_0) = W_0 \ln f.$$

**Remark 4.1.** The warped product manifold $M = M_1 \times_f M_2$ becomes *trivial* if $f$ is a constant function.

**Remark 4.2.** The base $M_1$ is totally geodesic and fiber $M_2$ is totally umbilical in WPM $M = M_1 \times_f M_2$.

The two kinds of the warped product submanifolds are described as the products between proper slant submanifolds and invariant submanifolds follows by Definition 3.1, and it can be expressed as:

(i) $M = M_{\vartheta} \times_f M_T$
(ii) $M = M_T \times_f M_{\vartheta}$,

where $M_{\vartheta}$ and $M_T$ are slant as well as invariant submanifolds, in the same order. For the first case, we recall the following finding which obtained by Uddin et al. [22] which stats that a WPSSS of type $M = M_{\vartheta} \times_f M_T$ does not exist in a NCM $\widetilde{M}$. Therefore, we shall consider non-trivial WPSSS in terms of $M = M_T \times_f M_{\vartheta}$ in a NCM $\widetilde{M}$. The following results were proved in [22] related to such type warped product semi-slant in nearly cosymplectic manifold.

**Lemma 4.1.** *[22] A WPSSS $M = M_T \times_f M_{\vartheta}$ in a NCM $\widetilde{M}$ has the coming relations*

*(i)* $\zeta \ln f = 0$,
*(ii)* $g(h(W_0, W_1), FW_2) = 0$,
*(iii)* $g(h(W_2, \varphi W_0), FW_2) = (W_0 \lambda) \|W_2\|^2$,
*(iv)* $g(\mathcal{P}_{W_0} W_2, TW_2) = 2g(h(W_0, W_2), FTW_2)$,

*for all $W_2 \in \Gamma(TM_{\vartheta})$ and $W_0, W_1 \in \Gamma(TM_T)$.*

There is a another interesting lemma.

**Lemma 4.2.** *[22] A WPSSS $M = M_T \times_f M_\vartheta$ in a NCM $\widetilde{M}$ satisfying the relation*

$$g(h(W_0, W_2), FTW_2) = -g(h(W_0, TW_2), FW_2), = \frac{1}{3}(W_0 \ln f)\cos^2\vartheta ||W_2||^2,$$

*for all $W_2 \in \Gamma(TM_\vartheta$ and $W_0 \in \Gamma(TM_T)$.*

Recalling the result of S. Hiepko [18], it is suitable to prove such characterization theorems for a WPSSS. The main result will be proved now.

**Proof of Theorem 1.3**

Let $M = M_T \times_f M_\vartheta$ is a WPSSS in a NCM $\widetilde{M}$ with $M_\vartheta$ and $M_T$ are proper slant and invariant submanifolds of $\widetilde{M}$. Therefore, the first part direct follows Lemmas 4.1 (iii) and Lemma 4.2 together Eq (2.7).

Conversely let $M$ is a SSS with condition (1.3) is satisfied. As we assumed that invariant distribution is integrable in hypothesis of theorem. From Theorem 1.1, we have necessary and sufficient condition of the integrability of $\mathcal{D}\oplus\zeta$ is

$$2\sin^2\vartheta g(\nabla_{W_0}W_1, W_2) = g(A_{FW_2}\varphi W_0, W_1) + g(A_{FW_2}\varphi W_1, W_0) - 2g(A_{FTW_2}W_0, W_1).$$

for $W_2 \in \Gamma(\mathcal{D}^\vartheta)$ and $W_0, W_1 \in \Gamma(\mathcal{D}\oplus\zeta)$. Thus relation (1.3)(i)-(ii) imply that

$$2\sin^2\vartheta g(\nabla_{W_0}W_1, W_2) = (W_0\lambda)g(W_2, W_1) + (W_1\lambda)g(W_0, W_2) - \frac{2}{3}\cos^2\vartheta(W_0\lambda)g(W_1, W_2).$$

which implies that

$$\sin^2\vartheta g(\nabla_{W_0}W_1, W_2) = 0.$$

As we have seen that $\mathcal{D}^\vartheta$ is proper slant then $\sin\vartheta \neq 0$, which means that $\nabla_{W_0}W_1 \in \Gamma(\mathcal{D}\oplus\zeta)$ for any $W_0, W_1 \in \Gamma(\mathcal{D}\oplus\zeta)$, it is concluded that the leaves of $\mathcal{D}\oplus\zeta$ are totally geodesic in $M$ into an immersion of $\widetilde{M}$. Also from Theorem 1.2, it can be seen that the slant distribution $\mathcal{D}^\vartheta$ is integrable if and only if the following equation holds

$$\begin{aligned}\sin^2\vartheta g(\nabla_{W_2}W_3, W_0) =&\frac{1}{2}\Big\{g(h(W_0, W_2), FTW_3) + g(h(W_0, W_3), FTW_2)\\ &- g(h(\varphi W_0, W_2), FW_3) - g(h(\varphi W_0, W_3), FW_2) + \eta(W_0)g(\tilde{\nabla}_{W_3}\zeta, W_2)\Big\}.\end{aligned}$$

Then above equation can be written by using (2.7).

$$\begin{aligned}2\sin^2\vartheta g(\nabla_{W_2}W_3, W_0) =&g(A_{FTW_2}W_0, W_3)) - g(A_{FW_2}\varphi W_0, W_3)\\ &+ g(A_{FTW_3}W_0, W_2) - g(A_{FW_3}\varphi W_0, W_2) + \eta(W_0)g(\tilde{\nabla}_{W_3}\zeta, W_2).\end{aligned}$$

Thus from (1.3)(i)-(ii), we obtain

$$2\sin^2\vartheta g(\nabla_{W_2}W_3, W_0) = \frac{2}{3}\cos^2\vartheta(W_0\lambda)g(W_2, W_3) - 2(W_0\lambda)g(W_3, W_2) + \eta(W_0)g(\tilde{\nabla}_{W_3}\zeta, W_2).$$

It implies that

$$\sin^2\vartheta g(\nabla_{W_2}W_3,W_0)=\Big(\frac{\cos^2\vartheta-3}{3}\Big)(W_0\lambda)g(W_2,W_3)+\frac{1}{2}\eta(W_0)g(\tilde{\nabla}_{W_3}\zeta,W_2). \tag{4.2}$$

Moreover, we also assumed that $\mathcal{D}^\vartheta$ to be integrable, thus we consider a integral manifold $M_\vartheta$ of $\mathcal{D}^\vartheta$, that is, $M_\vartheta$ is leaf of integrable distribution $\mathcal{D}^\vartheta$ and $h^\vartheta$ is the second fundamental form resulting from immersion $M_\vartheta$ in $M$. Therefore, equation (4.2) becomes

$$g(h^\vartheta(W_2W_3,W_0)=\frac{1}{3}\Big\{\cot^2\vartheta-3\csc^2\vartheta\Big\}(W_0\lambda)g(W_2,W_3)+\frac{1}{2}\csc^2\vartheta\eta(W_0)g(\tilde{\nabla}_{W_3}\zeta,W_2). \tag{4.3}$$

By interchanging $W_2$ and $W_3$ in above equation, we obtain

$$g(h^\vartheta(W_2W_3,W_0)=\frac{1}{3}\{\cot^2\vartheta-3\csc^2\vartheta\}(W_0\lambda)g(W_2,W_3)+\frac{1}{2}\csc^2\vartheta\eta(W_0)g(\tilde{\nabla}_{W_2}\zeta,W_3). \tag{4.4}$$

Using (4.3) and (4.4), we attain

$$\begin{aligned}2g(h^\vartheta(W_2W_3,W_0)=&\frac{2}{3}\Big\{\cot^2\vartheta-3\csc^2\vartheta\}(W_0\lambda)g(W_2,W_3)\\&+\frac{1}{2}\csc^2\vartheta\eta(W_0)\{g(\tilde{\nabla}_{W_2}\zeta,W_3)+g(\tilde{\nabla}_{W_3}\zeta,W_2)\Big\}.\end{aligned}$$

Thus from the definition of gradient and killing vector field $\xi$ for a nearly cosymplectic manifold [see (2.1) in [17]], we get

$$g(h^\theta(W_2W_3,W_0)=\frac{1}{3}\Big\{\cot^2\vartheta-3\csc^2\vartheta\Big\}g(W_2,W_3)g(\nabla\lambda,W_0),$$

which leads to

$$h^\vartheta(W_2,W_3)=\frac{1}{3}\Big\{\cot^2\vartheta-3\csc^2\vartheta\Big\}g(W_2,W_3)\nabla\lambda.$$

Hence,it is concluded that $M_\vartheta$ is totally umbilical in $M$ having the following mean curvature vector

$$H^\vartheta=\frac{1}{3}(\cot^2\vartheta-3\csc^2\vartheta)\nabla\lambda,$$

where $\nabla\lambda$ is the gradient of $\lambda$. However, by direct computations as we known that $Z(\lambda)=0$, we derive

$$g(\nabla^\vartheta_{W_2}\nabla\lambda,W_0)=-g(\nabla\lambda,\nabla^\vartheta_{W_0}W_2). \tag{4.5}$$

Furthermore, $\nabla\lambda\in\Gamma(TM_T)$ as $M_T$ is a totally geodesic in $M$ by Remark 4.2, consequently

$$\nabla^\vartheta_{W_0}W_2\in\Gamma(TM_\vartheta),$$

for any $W_0\in\Gamma(\mathcal{D}\oplus\zeta)$ and $W_2\in\Gamma(\mathcal{D}^\vartheta)$. By equation (4.5), we get

$$g(\nabla^\vartheta_{W_2}\nabla\lambda,W_0)=0.$$

This shows that the mean curvature vector $H^\theta$ of $M_\vartheta$ is parallel reciprocal to the normal connection $\nabla^\vartheta$ of $M_\vartheta$ in $M$. Therefore, the spherical condition is fulfilled, such that $M_\vartheta$ is an extrinsic sphere in $M$. Using the result of Hiepko (cf. [18]), $M$ is the non-trivial warped product submanifold of the form $M=M_T\times_f M_\vartheta$, in which $M_T$ and $M_\vartheta$ are the integral manifold of $\mathcal{D}\oplus\zeta$ and $\mathcal{D}^\vartheta$, in the same order. It completes this proof.

**Proof of Theorem 1.4**

Thus from Lemma 4.2, for all $W_2 \in \Gamma(TM_\vartheta$ and $W_1 \in \Gamma(TM_T)$, we get

$$g(h(W_1, W_2), FTW_2) = -g(h(W_1, TW_2), FW_2) + \frac{1}{3}(W_1 \ln f) \cos^2 \vartheta \|W_2\|^2.$$

As we assumed that $M$ is mixed totally geodesic submanifold, that is $h(W_1, W_2) = h(W_1, TW_2) = 0$, for all $W_2 \in \Gamma(\mathcal{D}^\vartheta)$ and $W_0 \in \Gamma(\mathcal{D} \oplus \zeta)$.

This implies $\cos^2 \vartheta (W_1 \ln f)\|W_2\|^2 = 0$. But $M$ is proper slant submanifold, then $\cos \vartheta \neq 0$, that is $(W_1 \ln f)\|W_2\|^2 = 0$. Hence $(W_1 \ln f) = 0$, i.e, the warping function $f$ is a constant on $M$ and the proof is completed via Remark 4.1.

**Acknowledgements**

The authors thank the referees for their valuable and constructive comments for modifying the presentation of this work. They are grateful to Akram Ali for giving idea for the work and his help for finalizing the manuscript. The authors extend their appreciation to the deanship of scientific research at King Khalid University for funding this work through research groups program under grant number R.G.P.2/56/40.

This research also was funded by the Deanship of Scientific Research at Princess Nourah bint Abdulrahman University through the Fast-track Research Funding Program.

**Conflict of interest**

There is no conflict of interest between the authors.

**References**

1. A. H. Alkhaldi, A. Ali, *Classification of warped product submanifolds in Kenmotsu space forms admitting gradient Ricci solitons*, Mathematics, **7** (2019), 112.
2. A. Ali, P. Laurian-Ioan, *Geometric classification of warped products isometrically immersed into Sasakian space forms*, Math. Nachr., **292** (2019), 234–251.
3. A. Ali, S. Uddin, A. H. Alkhaldi, *Characterization theorems on warped product semi-slant submanifolds in Kenmotsu manifolds*, Filomat, **33** (2019), 4033–4043.
4. A. Ali, C. Ozel, *Geometry of warped product pointwise semi-slant submanifolds of cosymplectic manifolds and its applications*, Int. J. Geom. Methods Mod. Phys., **14** (2017), 1750042.
5. A. Ali, P. Laurian-Ioan, *Geometry of warped product immersions of Kenmotsu space forms and its applications to slant immersions*, J. Geom. Phys., **114** (2017), 276–290.
6. A. Ali, S. Uddin, W. A. M. Othman, *Geometry of warped product pointwise semi-slant submanifolds of Kaheler manifolds*, Filomat, **31** (2017), 3771–3788.
7. A. Ali, W. A. M. Othman, C. Ozel, et al. *A geometric inequality for warped product pseudo-slant submanifolds of nearly Sasakian manifolds*, C. R. Acad. Bulgare Sci., **70** (2017), 175–182.

8. A. Ali, W. A. M. Othman, C. Ozel, *Some inequalities for warped product pseudo-slant submanifolds of nearly Kenmotsu manifolds*, J. Inequal. Appl., **2015**, 2015.

9. F. R. Al-Solamy, S. Uddin, *Warped product immersions with slant factor in Sasakian manifolds*, Public. Math. Debreec., **91** (2017), 331–348.

10. D. E. Blair, *Riemannian geometry of contact and symplectic manifold*, Progress in Mathematics vol. 203, Birkhauser. Boston Inc., Boston, 2010.

11. D. E. Blair, D. K. Showers, *Almost contact manifolds with Killing structures tensors II*, J. Differ. Geom., **9** (1974), 577–582.

12. R. L. Bishop, B. O'Neil, *Manifolds of negative curvature*, Trans. Amer. Math. Soc., **145** (1969), 1–9.

13. J. L. Cabrerizo, A. Cabrerizo, L. M. Fernandez, et al. *Slant submanifolds in Sasakian manifolds*, Glasgow Math. J., **42** (2000), 125–138.

14. J. L. Cabrerizo, A. Carriazo, L. M. Fernandez, et al. *Semi-slant submanifolds of a Sasakian manifold*, Geom. Dedicata, **78** (1999), 183–199.

15. B. Y. Chen, *Geometry of warped product CR-submanifold in Kaehler manifolds*, Monatsh. Math., **133** (2001), 177–195.

16. B. Y. Chen, *Differential geometry of warped product manifolds and submanifolds*, Singapore, World Scientific, 2017.

17. H. Endo, *On the curvature tensor of nearly cosymplectic manifolds of constant $\phi$-sectional curvature*, Ann. St. Univ. Iasi., **51** (2005), 439–554.

18. S. Hiepko, *Eine inner kennzeichungder verzerrten produckt*, Math. Ann., **244** (1979), 209–215.

19. K. A. Khan, V. A. Khan, S. Uddin, *Warped product submanifolds of cosymplectic manifolds*, Balkan J. Geom. Appl., **13** (2008), 55–65.

20. A. Lotta, *Slant submanifolds in contact geometry*, Bull. Math. Soc. Roumanie., **39** (1996), 183–198.

21. N. Papaghiuc, *Semi-slant submanifold of Kaehler manifold*, An. St. Univ. Al. I. Cuza. Iasi, **40** (1994), 55–61.

22. S. Uddin, A. Mustafa, B. R. Wong, et al. *A geometric inequality for warped product semi-slant submanifolds of nearly cosymplectic manifolds*, Rev. Un. Mat. Argentina., **55** (2014), 55–69

23. S. Uddin, C. Ozel, *A classification theorem on totally umbilical submanifolds in a cosymplectic manifold*, Hecetppe J. Math. Stat., **43** (2014), 635–640.

24. S. Uddin, S. H. Kon, M. A. Khan, et al. *Warped product semi-invariant submanifolds of nearly cosymplectic manifolds*, Math. Probl. Eng. 2011., (2011): Article ID 230374.