Research article

Interval-valued Choquet integral for set-valued mappings: definitions, integral representations and primitive characteristics

Zengtai Gong[1,*], Xuyang Kou[1] and Ting Xie[2]

[1] College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

[2] College of Social Development and Public Administration, Northwest Normal University, Lanzhou 730070, China

* **Correspondence:** Email: zt-gong@163.com; Tel: +869317971430.

Abstract: In this paper, a new kind of real-valued major Choquet integral, real-valued minor Choquet integral and interval-valued Choquet integrals for set-valued functions is introduced and investigated. The representations of the Choquet integral of set-valued functions with respect to a fuzzy measure are given. In particular, we focus on the case of the distorted Lebesgue measure as a fuzzy measure. Furthermore, the characteristics of the primitive of Choquet integral for set-valued functions are given as Radon-Nikodym property in some sense.

1. Introduction

The Choquet integral with respect to a fuzzy measure was proposed by Murofushi and Sugeno [13]. It was introduced by Choquet [4] in potential theory with the concept of capacity in 1950s. So far many studies have been devoted to a discrete case because their wide application in decision-making. In fact, there are many applications of the Choquet integral as an aggregation function, it has been used for utility theory in the field of economic theory [3, 6], fuzzy reasoning [21], pattern recognition, information fusion and data mining [6, 14], in particular, for multi-criteria decision making [7], and also as a psychological model of subjective preference [22], and so on. As Sugeno's description in [19], due to a wide range of applications for Choquet integral in decision problems, so far many studies have been devoted to a discrete case. However, the analytic properties of the Choquet integrals of set-valued functions with respect to fuzzy measure have not been fully discussed, including the properties of the primitives functions of Choquet integrals, the representation of Choquet integrals and the derivative of integral primitive functions in some sense, and so on. Indeed, integrals of set-valued function had been

studied by Aumann [1]. Jang et al. [10] defined Choquet integrals of set-valued functions by using the measurable selection functions, that is by the Aumann's approach, and it has been discussed and generalized by Gong, Jang, Pap et al. [8, 10, 11, 16, 18]. In 2018, Paternain et al. [17] proposed a new approach for the interval-valued Choquet integral that takes into account every possible permutation fitting to the considered ordinal structure of data. Wu et al. [9] defined real-valued Choquet integrals for set-valued functions by

$$(C) \int_A F(x)d\mu = \int_0^\infty \mu(F_\alpha \cap A)d\alpha, \tag{1}$$

where F is a measurable set-valued function, $F_\alpha = \{x | F(x) \cap [\alpha, \infty] \neq \emptyset\}, \alpha \geq 0$. Specially, it could be computed by the Choquet integral of a real-valued function, that is $(C) \int_A F(x)d\mu = (c) \int_A f(x)d\mu$, where $f(x) = \sup F(x), x \in X$.

It is well known that set-valued functions $F(t) = [0, e^t]$ and $G(t) = [t, e^t]$ defined on a compact convex set $[0, 1]$ are quite different, but the integral values are equal by Eq (1) since their $\sup F(x)$ and $\sup G(x)$ are equal completely. Based on this consideration, we propose a new kind of real-valued major Choquet integral, real-valued minor Choquet integral and interval-valued Choquet integrals for set-valued functions, and investigate some properties, such as the representations of the Choquet integral of set-valued functions with respect to a fuzzy measure, the characteristics of the primitive of Choquet integral for set-valued functions as a Radon-Nikodym property in some sense, and so on. It shows that these results we obtained in this paper are consistent with the results of [10, 11] for a compact-convex-set-valued function, and we notice that the method proposed in this paper is more simply than the calculations in [10, 11].

This paper is organized as follows. Section 2 presents some concepts on fuzzy measures and Choquet integral of set-valued functions. Also, the main definition of Choquet integral of set-valued functions in this paper is given. Section 3 defines major and minor Choquet integrals for set-valued function and shows the basic transformation theorem. Section 4 shows the representation theorem of Choquet integrals for set-valued functions as a Radon-Nikodym properties. Section 5 discusses some properties of the primitives functions of Choquet integrals. Section 6 introduces a simple representation of Choquet integral for set-valued functions with respect to a distorted probability measure. Section 7 concludes this paper.

2. Preliminaries and Definitions

This section introduces some notations and basic concepts on fuzzy measures, Choquet integrals, set-valued functions and distorted fuzzy measure. Also, the main definition of Choquet integral of set-valued functions in this paper is given.

Let X be a nonempty set and $\mathcal{A}$ a $\sigma-$algebra on X. A fuzzy measure on X is a set function $\mu : \mathcal{A} \longrightarrow [0, \infty]$ satisfying the following conditions [23]:

(1) $\mu(\phi) = 0$;

(2) $A \in X, B \in X, A \subset B$ implies $\mu(A) \leq \mu(B)$;

(3) In X, if $A_1 \subset A_2 \cdots \subset A_n \cdots$ and $\bigcup_0^\infty A_n \in \mathcal{A}$, then $\mu(\bigcup_{n=1}^\infty A_n) = \lim_{n \to \infty} \mu(A_n)$;

(4) In X, if $A_1 \supset A_2 \cdots \supset A_n \supset \cdots, A_n \in \mathcal{A}$, and there exist a n_0, such that $\mu(A_{n_0}) < \infty$, then $\mu(\bigcap_{n=1}^\infty A_n) = \lim_{n \to \infty} \mu(A_n)$.

If μ is a fuzzy measure, then $(X, \mathcal{A}, \mu)$ is said to be a fuzzy measure space. $R^+ = [0, \infty]$ denotes the set of extended nonnegative real numbers. $P(R^+)\backslash\{\emptyset\}$ denotes the class of all the nonempty subsets of R^+.

Remark 2.1. *Let* $m : R^+ \longrightarrow R^+$ *be a continuous and increasing function and* $m(0) = 0$. *A fuzzy measure* $\mu = \mu_m$, *a distorted Lebesgue measure, is defined by* $\mu_m(\cdot) = m(\lambda(\cdot))$. *Similarly, let* $m : [0, 1] \longrightarrow [0, 1]$ *and* $m(0) = 0$. *A fuzzy measure* $\mu = \mu_m = m \circ p$, *a distorted Probability measure, is defined by* $\mu_m(\cdot) = m(p(\cdot))$.

Note that μ_m is induced from the Lebesgue measure λ a monotone transformation where $\mu_m([a, b]) = m(\lambda([a, b])) = m(b - a)$.

Definition 2.1. *[4] Let* $(X, \mathcal{A}, \mu)$ *be a fuzzy measure space,* $f : X \longrightarrow R^+$ *be a measurable real-valued function. Then the Choquet integral of* f *on* A *is defined as*

$$(c) \int_A f d\mu = \int_0^\infty \mu(f_\alpha \cap A) d\alpha,$$

where $f_\alpha = \{x | f(x) \geq \alpha\}$, $\alpha \geq 0$, *and the right part is the Lebesgue integral.*

Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space. A set-valued mapping $F : X \longrightarrow P(R^+)\backslash\{\emptyset\}$ is said to be measurable if its graph is measurable, i.e., $\{(w, r) \in X \times R^+ : r \in F(w)\} \in \mathcal{A} \times \mathfrak{B}$, where $\mathfrak{B}$ is the Borel algebra of R^+ (refer to [5]).

Definition 2.2. *[9] Let* $(X, \mathcal{A}, \mu)$ *be a fuzzy measure space,* $F : X \longrightarrow P(R^+)\backslash\{\emptyset\}$ *be a measurable set-valued function. Then the real-valued Choquet integral of* F *on* A *is defined as*

$$(C) \int_A F d\mu = \int_0^\infty \mu(F_\alpha \cap A) d\alpha,$$

where $F_\alpha = \{x | F(x) \cap [\alpha, \infty] \neq \emptyset\}, \alpha \geq 0$, *and the right part is the Lebesgue integral.*

Remark 2.2. *(i) Instead of* $(C) \int_X F d\mu$, *we will write* $(C) \int F d\mu$;
(ii) If F *is a Choquet integrably bounded set-valued function, then* F *is Choquet integrable;*
(iii) A set-valued function $F : X \longrightarrow R^+$ *is said to be measurable if*

$$F^{-1}(A) = \{x \in X | F(x) \cap A \neq \emptyset\} \in \mathcal{A}$$

for every $A \in \mathcal{A}(R^+)$, *where* $\mathcal{A}(R^+)$ *is the Borel algebra of* R^+.

Note that if $F(x)$ degenerates into a real-valued function $f(x)$, $F_\alpha \cap A = \{x | f(x) \geq \alpha\}(\alpha \geq 0)$, then the above mentioned definition of the Choquet integral of set-valued functions is consistent with Definition 2.1. Furthermore, unless otherwise stated, in this paper, $(C) \int, (c) \int, \int$ respectively denote the Choquet integral of set-valued functions, real-valued functions and the Lebesgue integral.

Definition 2.3. *Let* $(X, \mathcal{A}, \mu)$ *be a fuzzy measure space,* $F : X \longrightarrow P(R^+)\backslash\{\emptyset\}$ *be a measurable set-valued function. Then major Choquet integral, minor Choquet integral, Choquet integrals of* F *on* A *are defined respectively by*

$$(C) \overline{\int_A} F d\mu = \int_0^\infty \mu(F_\alpha \cap A) d\alpha,$$

$$(C) \underline{\int}_A F d\mu = \int_0^\infty \mu(A \setminus \{x | F(x) \cap [0, \alpha) \neq \emptyset\}) d\alpha,$$

and

$$(C) \int_A F d\mu = [(C) \underline{\int}_A F d\mu, (C) \overline{\int}_A F d\mu],$$

where $F_\alpha = \{x | F(x) \cap [\alpha, \infty] \neq \emptyset\}, \alpha \geq 0$. right part is the Lebesgue integral.

Remark 2.3. *When $F(x)$ degenerates into a nonnegative real-valued function $f(x)$, then $F_\alpha \cap A = \{x | f(x) \geq \alpha\}(\alpha \geq 0)$ and $A \setminus \{x | F(x) \cap [0, \alpha) \neq \emptyset\} = \{x | F(x) \cap [\alpha, +\infty] \neq \emptyset\}$. Thus, Definition 3.2 is consistent with the classical Choquet integral of real-valued functions.*

Lemma 2.1. *[9] Let $F : X \longrightarrow P(R^+) \setminus \{\emptyset\}$ be a measurable set-valued function, $E \in \mathcal{A}$. Then*
   (1) $(C) \int_E F d\mu = \int_0^\infty \mu(F_{\tilde{\alpha}} \cap E) d\alpha$, where $F_{\tilde{\alpha}} = \{x | F(x) \cap (\alpha, \infty] \neq \emptyset\}, \alpha \geq 0$;
   (2) $(C) \int_E F d\mu = (c) \int_E g d\mu$, where $g(x) = \sup F(x), x \in E$.

Theorem 2.1. *Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space, $F : X \longrightarrow P(R^+) \setminus \{\emptyset\}$ be a measurable set-valued function, $A \in \mathcal{A}$, $f^+(x) = \sup F(x), f^-(x) = \inf F(x)$. Then*

$$(C) \overline{\int}_A F d\mu = (c) \int_A f^+ d\mu,$$

$$(C) \underline{\int}_A F d\mu = (c) \int_A f^- d\mu,$$

and

$$(C) \int_A F d\mu = [(c) \int_A f^- d\mu, (c) \int_A f^+ d\mu].$$

Proof. Since $(C) \overline{\int}_A F d\mu = \int_0^\infty \mu(F_\alpha \cap A) d\alpha$, then by Lemma 2.1 (2), we have

$$(C) \overline{\int}_A F d\mu = (c) \int_A f^+ d\mu.$$

Let $F_\alpha = \{x | F(x) \cap [\alpha, \infty] \neq \emptyset\}, f_\alpha^- = \{x | f^-(x) \cap [\alpha, +\infty] \neq \emptyset\}$, then $(f_\alpha^-)^c = \{x | f^-(x) \cap [0, \alpha) \neq \emptyset\}$. It is easy to prove that $(f_\alpha^-)^c = \{x | F(x) \cap [0, \alpha) \neq \emptyset\}$. In fact, for any $t \in (f_\alpha^-)^c$, $\inf F(t) < \alpha$, there exists a $y \in F(t)$ such that $y < \alpha$. It means that $F(t) \cap [0, \alpha) \neq \emptyset$. That is to say, $t \in \{x | F(x) \cap [0, \alpha) \neq \emptyset\}$. Therefore, $(f_\alpha^-)^c \subseteq \{x | F(x) \cap [0, \alpha) \neq \emptyset\}$.

Conversely, for any $t \in \{x | F(x) \cap [0, \alpha) \neq \emptyset\}$, $F(t) \cap [0, \alpha) \neq \emptyset$, there exists a $y \in F(t)$ such that $y < \alpha$. It follows that $\inf F(t) < \alpha$, i.e. $f^-(t) < \alpha$. That is, $\{x | F(x) \cap [0, \alpha) \neq \emptyset\} \subseteq (f_\alpha^-)^c$. Thus $(f_\alpha^-)^c = \{x | F(x) \cap [0, \alpha) \neq \emptyset\}$. We obtain

$$(c) \int_A f^- d\mu = \int_0^\infty \mu(f_\alpha^- \cap A) d\alpha = \int_0^\infty \mu(A \setminus (f_\alpha^-)^c) d\alpha$$
$$= \int_0^\infty \mu(A \setminus \{x | F(x) \cap [0, \alpha) \neq \emptyset\}) d\alpha = (C) \underline{\int}_A F d\mu.$$

Naturally,

$$(C) \int_A F d\mu = [(c) \int_A f^- d\mu, (c) \int_A f^+ d\mu].$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Example 2.1. *Let* $([0, 1], \mathcal{A}, \mu)$ *be a fuzzy measure space,* $F(x) = [0, e^x]$ *and* $G(x) = [x, e^x]$ *be set-valued functions on* $[0, 1]$. *Then* $f^-(x) = \inf F(x) = 0, f^+(x) = \sup F(x) = e^x, g^-(x) = \inf G(x) = x, g^+(x) = \sup G(x) = e^x$. *For* $f^+(x) = e^x$, *we have*

$$f_\alpha^+ = \{x | e^x \geq \alpha\} = \begin{cases} [0, 1], & 0 \leq \alpha \leq 1, \\ [\ln \alpha, 1] & 1 < \alpha \leq e, \\ \emptyset, & \alpha > e. \end{cases}$$

It follows by Definition 2.1,

$$(c) \int_{[0,1]} f^+(x)d\mu = \int_0^{+\infty} \mu(f_\alpha^+ \cap [0, 1])d\alpha = \int_0^1 \mu([0, 1])d\alpha + \int_1^e \mu([\ln \alpha, 1])d\alpha$$

$$= \mu([0, 1]) + \int_1^e \mu([\ln \alpha, 1])d\alpha.$$

Obviously, $(c) \int_{[0,1]} f^-(x)d\mu = 0$.
Thus, by the Theorem 2.1,

$$(C) \int_{[0,1]} F(x)d\mu = [0, \mu([0, 1]) + \int_1^e \mu([\ln \alpha, 1])d\alpha].$$

For $g^-(x) = \inf G(x) = x$, *we have*

$$g_\alpha^- = \{x | x \geq \alpha\} = \begin{cases} [\alpha, 1], & 0 \leq \alpha \leq 1, \\ \emptyset, & \alpha > 1. \end{cases}$$

It follows by Definition 2.1,

$$(c) \int_{[0,1]} g^-(x)d\mu = \int_0^{+\infty} \mu(g_\alpha^- \cap [0, 1])d\alpha = \int_0^1 \mu([\alpha, 1])d\alpha.$$

Thus, by the Theorem 2.1,

$$(C) \int_{[0,1]} G(x)d\mu = [\int_0^1 \mu([\alpha, 1])d\alpha, \mu([0, 1]) + \int_1^e \mu([\ln \alpha, 1])d\alpha].$$

However, if we define Choquet integral of F on A according to Definition 2.2, and by Lemma 2.1, the Choquet integrations of F and G equal to $\mu([0, 1]) + \int_1^e \mu([\ln \alpha, 1])d\alpha$, *although the interval-valued functions F and G are different on* $[0, 1]$.

Remark 2.4. *Theorem 2.1 shows that Definition 2.3 is consistent with the results of Aumann-type Choquet integral in [11] for interval-valued functions, but we notice that the method proposed in this paper is simpler than the calculations in [11].*

3. The representation and properties for the Choquet integral of the set-valued functions

Let μ be a distorted fuzzy measure and consider $\mu([\tau, t])$ for a closed interval $[\tau, t]$, then $\mu([\tau, t])$ is increasing for t and decreasing for τ. Throughout the paper, let $m(t)$ and $f(t)$ be continuously differentiable functions. Let $\mu([\tau, t])$ be differentiable with respect to τ on $[0, t]$ for every $t > 0$. We require the regularity condition that $\mu(\{t\}) = 0$ holds for every $t \geq 0$. Let $\mu'([\tau, t])$ denote $(\partial / \partial \tau) \mu([\tau, t])$, if μ is a distorted Lebesgue measure μ_m, then $\mu'([\tau, t]) = -m'(t - \tau)$. For set-valued function $F : X \longrightarrow P(R^+) \backslash \{\emptyset\}$. For convenience, $f^+(x) = \sup F(x)$ and $f^-(x) = \inf F(x)$ respectively denotes major function and minor function of $F(x)$. Furthermore, the definitions of major Choquet integral, minor Choquet integral and interval-valued Choquet integral for set-valued function, play important roles in discussing the problems concerning the representation theorem.

Definition 3.1. *Let $F : X \longrightarrow P(R^+) \backslash \{\emptyset\}$ be a strictly monotone increasing set-valued function, if major function $f^+(x) = \sup F(x)$ and minor function $f^-(x) = \inf F(x)$ are strict monotone increasing.*

Lemma 3.1. *[19] Let $f(t)$ be nonnegative measurable and strictly increasing function. Then the Choquet integral of $f(t)$ with respect to μ on $[0, t]$ is represented as*

$$(c) \int_{[0,t]} f(\tau) d\mu(\tau) = - \int_0^t \mu'([\tau, t]) f(\tau) d\tau,$$

In particular, for $\mu = \mu_m$, then

$$(c) \int_{[0,t]} f(\tau) d\mu_m(\tau) = \int_0^t m'(t - \tau) f(\tau) d\tau.$$

Theorem 3.1. *Let $F : X \longrightarrow P(R^+) \backslash \{\emptyset\}$ be a measurable and strictly increasing set-valued function, $f^+(x)$ and $f^-(x)$ be continuous differentiable real-valued functions, where $f^-(x) = \inf F(x), f^+(x) = \sup F(x)$. Then the Choquet integral of $F(t)$ with respect to μ on $[0, t]$ is represented as*

$$(C) \overline{\int}_{[0,t]} F(\tau) d\mu = - \int_0^t \mu'([\tau, t]) f^+(\tau) d\tau,$$

$$(C) \underline{\int}_{[0,t]} F(\tau) d\mu = - \int_0^t \mu'([\tau, t]) f^-(\tau) d\tau,$$

and

$$(C) \int_{[0,t]} F d\mu = [- \int_0^t \mu'([\tau, t]) f^-(\tau) d\tau, - \int_0^t \mu'([\tau, t]) f^+(\tau) d\tau].$$

In particular, for $\mu = \mu_m$, then

$$(C) \overline{\int}_{[0,t]} F(\tau) d\mu_m = \int_0^t m'(t - \tau) f^+(\tau) d\tau,$$

$$(C) \underline{\int}_{[0,t]} F(\tau) d\mu_m = \int_0^t m'(t - \tau) f^-(\tau) d\tau,$$

and

$$(C) \int_{[0,t]} F d\mu_m = [\int_0^t m'(t - \tau) f^-(\tau) d\tau, \int_0^t m'(t - \tau) f^+(\tau) d\tau].$$

Proof. For $f^+(x) = \sup F(x)$. By Theorem 2.1, we have

$$(C)\overline{\int}_{[0,t]} F d\mu = (c)\int_{[0,t]} f^+ d\mu,$$

and

$$(c)\int_{[0,t]} f^+ d\mu = \int_0^\infty \mu(\{x|f^+ \geq \alpha\} \cap [0,t])d\alpha$$

$$= \int_0^{f^+(0)} \mu([0,t])d\alpha + \int_{f^+(0)}^{f^+(t)} \mu([(f^+(\alpha))^{-1},t])d\alpha$$

$$= \mu([0,t])f^+(0) + \int_{f^+(0)}^{f^+(t)} \mu([(f^+(\alpha))^{-1},t])d\alpha.$$

On the other hand,

$$\int_{f^+(0)}^{f^+(t)} \mu([(f^+(\alpha))^{-1},t])d\alpha = \int_{[0,t]} \mu([\tau,t])(f^+(\tau))'d\tau$$

$$= u([\tau,t])f^+(\tau)|_0^t - \int_{[0,t]} \mu'([\tau,t])f^+(\tau)d\tau$$

$$= \mu(\{t\})f^+(t) - \mu([0,t])f^+(0) - \int_{[0,t]} \mu'([\tau,t])f^+(\tau)d\tau,$$

where $\alpha = f^+(\tau)$, and $(f^+(\tau))'d\tau = d\alpha$. Note that $\mu(\{t\}) = 0$, $f^+(0) = 0$.
That is

$$(C)\overline{\int}_{[0,t]} F(\tau)d\mu = -\int_0^t \mu'([\tau,t])f^+(\tau)d\tau.$$

Especially, $\mu = \mu_m$, then $-\mu'([\tau,t]) = m'(t-\tau)$, we obtain

$$(C)\overline{\int}_{[0,t]} F(\tau)d\mu_m = \int_0^t m'(t-\tau)f^+(\tau)d\tau.$$

Similarly, we can also obtain

$$(C)\underline{\int}_{[0,t]} F(\tau)d\mu_m = \int_0^t m'(t-\tau)f^-(\tau)d\tau.$$

Since $F(x) = [f^-(x), f^+(x)]$, thus

$$(C)\int_{[0,t]} F d\mu = (c)\int_{[0,t]} [f^-, f^+]d\mu = [(c)\int_{[0,t]} f^- d\mu, (c)\int_{[0,t]} f^- d\mu]$$

$$= [-\int_0^t \mu'([\tau,t])f^-(\tau)d\tau, -\int_0^t \mu'([\tau,t])f^+(\tau)d\tau].$$

Note that $f^- = \inf F(x)$ or $f^+ = \sup F(x)$ is a constant function M. By Theorem 1, we have

$$(C)\overline{\int}_{[0,t]} F(\tau)d\mu = (c)\int_{[0,t]} Md\mu = M \cdot \mu([0,t]),$$

$$(C)\underline{\int}_{[0,t]} F(\tau)d\mu = (c)\int_{[0,t]} Md\mu = M \cdot \mu([0,t]).$$

Hence, f^- and f^+ is a constant function, Theorem 3.1 still holds. Therefore, we have the following corollary. □

Theorem 3.2. *Let $F : X \longrightarrow P(R^+)\backslash\{\emptyset\}$ be a measurable and increasing set-valued functions, $f^+(x)$ and $f^-(x)$ be continuous differentiable real-valued function, where $f^-(x) = \inf F(x)$, $f^+(x) = \sup F(x)$. Then*

$$(C)\overline{\int}_{[0,t]} F(\tau)d\mu = -\int_0^t \mu'([\tau,t])f^+(\tau)d\tau,$$

$$(C)\underline{\int}_{[0,t]} F(\tau)d\mu = -\int_0^t \mu'([\tau,t])f^-(\tau)d\tau,$$

and

$$(C)\int_{[0,t]} Fd\mu = [-\int_0^t \mu'([\tau,t])f^-(\tau)d\tau, -\int_0^t \mu'([\tau,t])f^+(\tau)d\tau].$$

In particular, for $\mu = \mu_m$, then

$$(C)\overline{\int}_{[0,t]} F(\tau)d\mu_m = \int_0^t m'(t-\tau)f^+(\tau)d\tau,$$

$$(C)\underline{\int}_{[0,t]} F(\tau)d\mu_m = \int_0^t m'(t-\tau)f^-(\tau)d\tau,$$

and

$$(C)\int_{[0,t]} Fd\mu_m = [\int_0^t m'(t-\tau)f^-(\tau)d\tau, \int_0^t m'(t-\tau)f^+(\tau)d\tau].$$

Example 3.1. *Let $F(x) = [x, e^x]$ be a set-valued function, $\mu = \mu_m$ be a distorted Lebesgue measure, and $m(x) = x^2$. Then $f^+(x) = \sup F(x) = e^x$, $f^-(x) = \inf F(x) = x$, $m'(t-\tau) = 2(t-\tau)$. By Theorem 3.2, we have*

$$(C)\overline{\int}_{[0,t]} F(\tau)d\mu = \int_0^t m'(t-\tau)f^+(\tau)d\mu = \int_0^t 2(t-\tau)e^\tau d\tau = 2(e^t - t - 1),$$

$$(C)\underline{\int}_{[0,t]} F(\tau)d\mu = \int_0^t m'(t-\tau)f^-(\tau)d\tau = \int_0^t 2(t-\tau)\tau d\tau = \frac{1}{3}t^3.$$

Hence

$$(C)\int_{[0,t]} Fd\mu = [\frac{1}{3}t^3, 2(e^t - t - 1)].$$

Especially, when $t = 1$, we could obtain the following statement.

$$(C)\int_{[0,1]} Fd\mu = [\frac{1}{3}, 2(e-2)].$$

Example 3.2. *Let $F(x) = [0, e^x]$ and $G(x) = [x, e^x]$ be two set-valued functions on $[0, 1]$, $\mu = \mu_m$ be a distorted Lebesgue measure, and $m(x) = x^2$. Then $f^-(x) = \inf F(x) = 0$, $f^+(x) = \sup F(x) = e^x$, $g^-(x) = \inf G(x) = x$, $g^+(x) = \sup G(x) = e^x$. By Example 2.1,*

$$(C) \int_{[0,1]} F(x)d\mu = [0, \mu([0, 1]) + \int_1^e \mu([\ln \alpha, 1])d\alpha].$$

$$(C) \int_{[0,1]} G(x)d\mu = [\int_0^1 \mu([\alpha, 1])d\alpha, \mu([0, 1]) + \int_1^e \mu([\ln \alpha, 1])d\alpha].$$

Since $\mu(\cdot) = \mu_m(\cdot) = m(\lambda(\cdot))$ is a distorted Lebesgue measure, and the distorted function $m(x) = x^2$, then

$$\mu([0, 1]) + \int_1^e \mu([\ln \alpha, 1])d\alpha = 1 + \int_0^1 (1 - \ln \alpha)^2 d\alpha = 1 + 2e - 5 = 2(e - 2),$$

$$\int_0^1 \mu([\alpha, 1])d\alpha = \int_0^1 (1 - \alpha)^2 d\alpha = \frac{1}{3}.$$

Therefore,

$$(C) \int_{[0,1]} Fd\mu_m = [\frac{1}{3}, 2(e - 2)].$$

Obviously, the result coincides with Example 3.1.

4. Radon-Nikodym properties of the Choquet integral for the set-valued functions

As Sugeno's description in article [19], due to a wide range of applications for Choquet integral in decision problems, so far many studies have been devoted to a discrete case. However, the analytic properties of the Choquet integrals of set-valued functions with respect to fuzzy measure have not been fully discussed, including the properties of the primitives functions of Choquet integrals, the representation of Choquet integrals and the derivative of integral primitive functions in some sense, and so on. It will be discussed in this section by using the definitions of real-valued major Choquet integral, real-valued minor Choquet integral and interval-valued Choquet integrals for set-valued functions.

Given two continuous increasing and nonnegative real-valued functions $g_1(x)$, $g_2(x)$ with $g_1(0) = g_2(0) = 0$, and a fuzzy measure μ_m. How to define a continuous increasing and differentiable set-valued function $F(x)$, such that

$$g_1(t) = (C) \underline{\int}_{[0,t]} F(\tau)d\mu_m(\tau), \ g_2(t) = (C) \overline{\int}_{[0,t]} F(\tau)d\mu_m(\tau).$$

Definition 4.1. *[19] For a real-valued function $f(t)$ on $[a, b]$, we call $\mathcal{F}(s) = \int_0^{+\infty} e^{-st} f(t)dt$ the Laplace transformation of $f(t)$ if $\int_0^{+\infty} e^{-st} f(t)dt$ with respect to s is convergent. We denote its Laplace transformation as $\mathcal{F}(s) = L[f(t)]$ and the inverse Laplace transformation as $f(t) = L^{-1}[\mathcal{F}(s)]$.*

Theorem 4.1. *Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space, $g_1(t)$ and $g_2(t)$ be nonnegative continuous real-valued functions. $F : [0, +\infty) \longrightarrow P(R^+)\backslash\{\emptyset\}$ be a measurable set-valued function with*

$g_1(t) = (C)\underline{\int}_{[0,t]} F(\tau)d\mu, g_2(t) = (C)\overline{\int}_{[0,t]} F(\tau)d\mu,$ $f^+(x)$ *and* $f^-(x)$ *be strictly monotone increasing and continuously differentiable, where* $f^-(x) = \inf F(x), f^+(x) = \sup F(x)$. *Then*

$$G_1(s) = sM(s)\mathcal{F}_1(s), \ g_1(t) = L^{-1}[sM(s)\mathcal{F}_1(s)],$$

$$G_2(s) = sM(s)\mathcal{F}_2(s), \ g_2(t) = L^{-1}[sM(s)\mathcal{F}_2(s)],$$

where $G_1(s) = L[g_1(t)], G_2(s) = L[g_2(t)], \ M(s) = L[m(t)], \mathcal{F}_1(s) = L[f^-(t)], \mathcal{F}_2(s) = L[f^+(t)]$

Proof. By Theorem 3.1, we have

$$(C)\overline{\int}_{[0,t]} F(\tau)d\mu = -\int_0^t \mu'([\tau,t])f^+(\tau)d\tau,$$

$$(C)\underline{\int}_{[0,t]} F(\tau)d\mu = -\int_0^t \mu'([\tau,t])f^-(\tau)d\tau,$$

That is,

$$g_1(t) = -\int_0^t \mu'([\tau,t])f^-(\tau)d\tau, \ g_2(t) = -\int_0^t \mu'([\tau,t])f^+(\tau)d\tau.$$

Thus, we obtain

$$G_2(s) = \int_0^{+\infty} e^{-st}g_2(t)dt = \int_0^{+\infty} e^{-st}\int_0^t m'(t-\tau)f^+(\tau)d\tau dt.$$

On the other hand,

$$\int_0^{+\infty} e^{-st}\int_0^t m'(t-\tau)f^+(\tau)d\tau dt$$

$$= \int_0^{+\infty}\int_\tau^{+\infty} e^{-st}m'(t-\tau)f^+(\tau)dt d\tau$$

$$= \int_0^{+\infty} f^+(\tau)[e^{-st}m(t-\tau)|_\tau^{+\infty} - \int_\tau^{+\infty} m(t-\tau)de^{-st}]d\tau$$

$$= s\int_0^{+\infty} f^+(\tau)[e^{-s\tau}\int_0^{+\infty} e^{-s(t-\tau)}m(t-\tau)d(t-\tau)]d\tau$$

$$= s\int_0^{+\infty} f^+(\tau)e^{-s\tau}M(s)d\tau$$

$$= sM(s)\int_0^{+\infty} e^{-s\tau}f^+(\tau)d\tau$$

$$= sM(s)\mathcal{F}_2(s).$$

Therefore, $G_2(s) = sM(s)\mathcal{F}_2(s)$. Correspondingly, $g_2(t) = L^{-1}[sM(s)\mathcal{F}_2(s)]$. Similarly, we can also obtain

$$G_1(s) = sM(s)\mathcal{F}_1(s), \ g_1(t) = L^{-1}[sM(s)\mathcal{F}_1(s)].$$

The proof is complete. □

Example 4.1. *Let* $F(t) = [\frac{1}{2}t, t]$, *distorted function* $m(t) = t^n$, *then* $f^-(t) = \inf F(t) = \frac{1}{2}t$, $f^+(t) = \sup F(t) = t$. *By Laplace transformation, we obtain*

$$\mathcal{M}(s) = \int_0^{+\infty} t^n e^{-st} dt = n!/s^{n+1},$$

$$\mathcal{F}_1(s) = \int_0^{+\infty} \frac{1}{2} t e^{-st} dt = 1/2s^2, \quad \mathcal{F}_2(s) = \int_0^{+\infty} t e^{-st} dt = 1/s^2.$$

According to Theorem 4.1, we have

$$\mathcal{G}_2(s) = s\mathcal{M}(s)\mathcal{F}_2(s) = n!/s^{n+2},$$

Thus

$$g_2(t) = L^{-1}[s\mathcal{M}(s)\mathcal{F}_2(s)] = t^{n+1}/(n+1).$$

By the same way, we have

$$g_1(t) = L^{-1}[s\mathcal{M}(s)\mathcal{F}_1(s)] = t^{n+1}/2(n+1).$$

Theorem 4.2. *Let* $(X, \mathcal{A}, \mu)$ *be a fuzzy measure space,* $g_1(t)$ *and* $g_2(t)$ *be nonnegative continuous real-valued functions. Then there exists* $F : [0, +\infty) \longrightarrow P(R^+)\backslash\{\emptyset\}$ *being a measurable set-valued function, such that*

$$g_1(t) = (C)\underline{\int}_{[0,t]} F(\tau)d\mu, \quad g_2(t) = (C)\overline{\int}_{[0,t]} F(\tau)d\mu,$$

furthermore

$$[g_1(t), g_2(t)] = (C)\int_{[0,t]} F(\tau)d\mu.$$

Proof. Make $f^+(t) = L^{-1}[\mathcal{G}_2(s)/s\mathcal{M}(s)]$. By Theorem 3.2, we have

$$(C)\overline{\int}_{[0,t]} F(\tau)d\mu = \int_0^t m'(t-\tau)f^+(\tau)d\tau.$$

We just need to prove that f^+ is nonnegative continuous and differentiable increasing, and satisfying

$$g_2(t) = \int_0^t m'(t-\tau)f^+(\tau)d\tau.$$

Note that $F(t) = [f^-(t), f^+(t)]$. In fact, since $g_1(t)$ is nonnegative continuous and differentiable on $[0, +\infty)$, so $f^+(t)$ is also nonnegative continuous and differentiable. It is easy to calculate that

$$\int_0^t m'(t-\tau)f^+(\tau)d\tau = L^{-1}[L[\int_0^t m'(t-\tau)f^+(\tau)d\tau]]$$

and

$$L[\int_0^t m'(t-\tau)f^+(\tau)d\tau] = \int_0^{+\infty} e^{-st} \int_0^t m'(t-\tau)f^+(\tau)d\tau dt$$

$$= \int_0^{+\infty} \int_\tau^{+\infty} e^{-st} m'(t-\tau) f^+(\tau) dt d\tau$$

$$= \int_0^{+\infty} f^+(\tau) [e^{-st} m(t-\tau)|_\tau^{+\infty} - \int_\tau^{+\infty} m(t-\tau) de^{-st}] d\tau$$

$$= s \int_0^{+\infty} f^+(\tau) [e^{-s\tau} \int_0^{+\infty} e^{-s(t-\tau)} m(t-\tau) d(t-\tau)] d\tau$$

$$= s \int_0^{+\infty} f^+(\tau) e^{-s\tau} \mathcal{M}(s) d\tau$$

$$= s \mathcal{M}(s) \int_0^{+\infty} e^{-s\tau} f^+(\tau) d\tau$$

$$= s \mathcal{M}(s) \mathcal{F}_2(s)$$

$$= s \mathcal{M}(s) \cdot \mathcal{G}_2(s) / s \mathcal{M}(s)$$

$$= \mathcal{G}_2(s).$$

Therefore,

$$g_2(t) = \int_0^t m'(t-\tau) f^+(\tau) d\tau.$$

Similarly, make $f^-(t) = L^{-1}[\mathcal{G}_1(s)/s\mathcal{M}(s)]$, we can also obtain

$$g_1(t) = \int_0^t m'(t-\tau) f^-(\tau) d\tau.$$

Then by Theorem 3.2, we get

$$[g_1(t), g_2(t)] = (C) \int_{[0,t]} F(\tau) d\mu.$$

where $\mathcal{G}_1(s) = L[g_1(t)], \mathcal{G}_2(s) = L[g_2(t)], \mathcal{M}(s) = L[m(t)]. \mathcal{F}_1(s) = L[f^-(t)], \mathcal{F}_2(s) = L[f^+(t)]$. The proof is complete. □

Example 4.2. *Let $g_1(t) = t^2, g_2(t) = e^{2t} - 1$ and distorted function $m(t) = t + \frac{1}{2}at^2$, $a > 0$, by Laplace transformation, we have*

$$\mathcal{M}(s) = \int_0^{+\infty} (t + \frac{1}{2}at^2) e^{-st} dt = \frac{1}{s^2} + \frac{a}{s^3},$$

$$\mathcal{G}_1(s) = \int_0^{+\infty} t^2 e^{-st} dt = 2/s^3, \mathcal{G}_2(s) = \int_0^{+\infty} (e^{2t} - 1) e^{-st} dt = \frac{a}{s(s-a)},$$

Hence, by Theorem 4.2, there exists $F(t) = [f^-(t), f^+(t)]$, and

$$f^-(t) = L^{-1}[\mathcal{G}_1(s)/s\mathcal{M}(s)] = \frac{2}{a}(1 - e^{-at}),$$

$$f^+(t) = L^{-1}[\mathcal{G}_2(s)/s\mathcal{M}(s)] = \frac{1}{2}(e^{2t} + e^{-2t}) = \cosh(2t).$$

5. The primitive function characterization of the Choquet integral for set-valued functions

Definition 5.1. *[12, 15] Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space.*

(1) μ is said to be weakly null-additive if for any $A, B \in \mathcal{A}$, $\mu(A) = \mu(B) = 0$ implies $\mu(A \cup B) = 0$;

(2) μ is said to have strong order continuity if for any $A_n \subset \mathcal{A}, A \in \mathcal{A}$ with $\mu(A) = 0$, $A_n \downarrow A$ implies $\mu(A_n) \longrightarrow 0$;

(3) μ is said to have pseudo-metric generating property if for any $\varepsilon > 0$, there is $\delta > 0$, such that $\mu(A \cup B) < \varepsilon$, whenever $A, B \in \mathcal{A}$, and $\mu(A) \vee \mu(B) < \delta$;

(4) μ is said to have property (S) if for any $A_n \subset \mathcal{A}$ with $\mu(A_n) \longrightarrow 0$, there exists a subsequence A_{n_i} of A_n such that $\mu(\overline{\lim_{i \to \infty}}(A_{n_i})) = 0$.

Definition 5.2. *Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space, and v be a set-valued function, denoted by $v(A) = [v^-(A), v^+(A)], \forall A \in \mathcal{A}$. Then*

(1) v is said to be weakly null-additive if for any $A, B \in \mathcal{A}$, $v(A) = v(B) = 0$ (i.e., $v^-(A) = v^-(B) = 0, v^+(A) = v^+(B) = 0$) implies $v(A \cup B) = 0$ (i.e., $v^-(A \cup B) = 0, v^+(A \cup B) = 0$);

(2) v is said to have strong order continuity if for any $A_n \subset \mathcal{A}, A \in \mathcal{A}$ with $v(A) = 0$ (i.e., $v^-(A) = 0, v^+(A) = 0$), $A_n \downarrow A$ implies $v(A_n) \longrightarrow 0$ (i.e., $v^-(A_n) \longrightarrow 0, v^+(A_n) \longrightarrow 0$);

(3) v is said to have pseudo-metric generating property if for any $\varepsilon > 0$, there is $\delta > 0$, such that $v(A \cup B) < \varepsilon$ (i.e., $v^-(A \cup B) < \varepsilon, v^+(A \cup B) < \varepsilon$), whenever $A, B \in \mathcal{A}$, and $v(A) \vee v(B) < \delta$ (i.e., $v^-(A) \vee v^-(B) < \delta, v^+(A) \vee v^+(B) < \delta$);

(4) v is said to have property (S) if for any $A_n \subset \mathcal{A}$ with $v(A_n) \longrightarrow 0$ (i.e., $v^-(A_n) \longrightarrow 0, v^+(A_n) \longrightarrow 0$), there exists a subsequence A_{n_i} of A_n such that $v(\overline{\lim_{i \to \infty}}(A_{n_i})) = 0$ (i.e., $v^-(\overline{\lim_{i \to \infty}}(A_{n_i})) = 0, v^+(\overline{\lim_{i \to \infty}}(A_{n_i})) = 0$).

In the following, let F be a nonnegative measurable set-valued function. For any $A \in \mathcal{A}$, the set-valued function v is defined as $v(A) = [v^-(A), v^+(A)]$, where $v^-(A) = (C)\underline{\int}_A F d\mu$, $v^+(A) = (C)\overline{\int}_A F d\mu$.

Theorem 5.1. *If μ is weakly null-additive, then $v(A) = [v^-(A), v^+(A)]$ is also weakly null-additive.*

Proof. For any $A, B \in \mathcal{A}$ with $v^+(A) = v^+(B) = 0$, let $F_\alpha = \{x | F(x) \cap [\alpha, \infty] \neq \emptyset\}, \alpha \geq 0$, from the definitions of $v^+(A)$ and the major Choquet integral, we have

$$v^+(A) = \int_0^\infty \mu(F_\alpha \cap A) d\alpha = \int_0^\infty \mu(F_\alpha \cap B) d\alpha = v^+(B) = 0,$$

in the case of the Lebesgue measure, $\mu(F_\alpha \cap A) = 0$ and $\mu(F_\alpha \cap B) = 0$ almost everywhere for $\alpha \in [0, +\infty)$. By the monotonicity of the Lebesgue measure, we have

$$\mu(F_\alpha \cap A) = \mu(F_\alpha \cap B) = 0 \Rightarrow \mu[(F_\alpha \cap A) \cup (F_\alpha \cap B)] = 0.$$

Thus

$$v^+(A \cup B) = (C)\overline{\int}_{A \cup B} F d\mu = \int_0^\infty \mu(F_\alpha \cap (A \cup B)) d\alpha$$
$$= \int_0^\infty \mu[(F_\alpha \cap A) \cup (F_\alpha \cap B)] d\alpha = 0.$$

By the same way, we can also obtain

$$v^-(A \cup B) = (C)\underline{\int}_{A \cup B} F d\mu = 0.$$

Therefore, $v(A \cup B) = [(C)\underline{\int}_{A \cup B} F d\mu, (C)\overline{\int}_{A \cup B} F d\mu] = 0$, so $v(A)$ is weakly null-additive. The proof is complete. □

Theorem 5.2. *Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space, F be a measurable set-valued function and Choquet integrable on X. For $v(A) = [v^-(A), v^+(A)], v^-(A) = (C)\underline{\int}_A F d\mu, v^+(A) = (C)\overline{\int}_A F d\mu$, we have*
1. *If $\mu(A) = 0$, then $v(A) = 0$;*
2. *If $\{A_n\} \subset \mathcal{A}, \mu(\{A_n\}) \longrightarrow 0$, then $v(\{A_n\}) \longrightarrow 0$;*
3. *v is a fuzzy measure.*

Proof. (1) Obviously.

(2) By $\{A_n\} \subset \mathcal{A}, \mu(\{A_n\}) \longrightarrow 0$, hence $\mu(F_\alpha \cap A_n) \leq \mu(F_\alpha \cap X)$, then

$$\mu(F_\alpha \cap A_n) \longrightarrow 0.$$

Since $F(x)$ is Choquet integrable, thus

$$v^+(X) = (C)\overline{\int} F d\mu = \int_0^\infty \mu(F_\alpha \cap X) d\alpha < \infty.$$

By the dominated convergence theorem of the Lebesgue integral, we obtain

$$v^+(A_n) = (C)\overline{\int}_{A_n} F d\mu = \int_0^\infty \mu(F_\alpha \cap A_n) d\alpha \longrightarrow 0,$$

That is $v^+(\{A_n\}) \longrightarrow 0$.

Similarly, we can also obtain $v^-(\{A_n\}) = (C)\underline{\int}_{A_n} F d\mu \longrightarrow 0$, therefore, $v(\{A_n\}) \longrightarrow 0$.

(3) v is a fuzzy measure, we just need to prove that v satisfies the following conditions:
i) $v(\emptyset) = 0$.
ii) For any $A, B \in \mathcal{A}, A \subset B$, since $\mu(F_\alpha \cap A) \leq \mu(F_\alpha \cap B)$. Thus, we have

$$v^+(A) = (C)\overline{\int}_A F d\mu = \int_0^\infty \mu(F_\alpha \cap A) d\alpha$$
$$\leq (C)\overline{\int}_B F d\mu = \int_0^\infty \mu(F_\alpha \cap B) d\alpha = v^+(B).$$

Similarly, we can also obtain

$$v^-(A) = (C)\underline{\int}_A F d\mu \leq (C)\underline{\int}_B F d\mu = v^-(B).$$

Hence, $v(A) \leq v(B)$.

iii) For $A_1 \subset A_2 \cdots \subset A_n \cdots , A_n \in \mathcal{A}$, Since $F(x)$ is Choquet integrable, i.e. $v^+(A_n) \leq v^+(X) < \infty$, and $v^+(A_n)$ is monotone increasing, hence $\lim_{n\to\infty} v^+(A_n)$ exists.
On the other hand,

$$v^+(A_n) = (C)\overline{\int}_{A_n} Fd\mu = \int_0^\infty \mu(F_\alpha \cap A_n)d\alpha, \ \mu(F_\alpha \cap A_n) \leq \mu(F_\alpha \cap X).$$

By the Lebesgue integral dominated convergence theorem, we have

$$\lim_{n\to\infty} v^+(A_n) = \int_0^\infty \lim_{n\to\infty} \mu(F_\alpha \cap A_n)d\alpha = \int_0^\infty \mu(F_\alpha \cap \bigcup_{n=1}^\infty A_n)d\alpha = v^+(\bigcup_{n=1}^\infty A_n).$$

That is $v^+(\bigcup_{n=1}^\infty A_n) = \lim_{n\to\infty} v^+(A_n)$.
Similarly, we can also obtain $v^-(\bigcup_{n=1}^\infty A_n) = \lim_{n\to\infty} v^-(A_n)$. Thus

$$v(\bigcup_{n=1}^\infty A_n) = \lim_{n\to\infty} v(A_n).$$

iv) For $A_1 \supset A_2 \cdots \supset A_n \supset \cdots , A_n \in \mathcal{A}$, Since $F(x)$ is Choquet integrable, i.e. $v^+(A_n) \leq v^+(X) < \infty$, and $v^+(A_n)$ is monotone decreasing, hence $\lim_{n\to\infty} v^+(A_n)$ exists.
On the other hand,

$$v^+(A_n) = (C)\overline{\int}_{A_n} Fd\mu = \int_0^\infty \mu(F_\alpha \cap A_n)d\alpha, \ \mu(F_\alpha \cap A_n) \leq \mu(F_\alpha \cap X).$$

By the Lebesgue integral dominated convergence theorem, we have

$$\lim_{n\to\infty} v^+(A_n) = \int_0^\infty \lim_{n\to\infty} \mu(F_\alpha \cap A_n)d\alpha = \int_0^\infty \mu(F_\alpha \cap \bigcap_{n=1}^\infty A_n)d\alpha = v^+(\bigcap_{n=1}^\infty A_n).$$

Similarly, we can also obtain

$$v^-(\bigcap_{n=1}^\infty A_n) = \lim_{n\to\infty} v^-(A_n).$$

Therefore, $v(\bigcap_{n=1}^\infty A_n) = \lim_{n\to\infty} v(A_n)$.
The proof is complete. □

Theorem 5.3. *Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space, F be a measurable set-valued function and Choquet integrable on X, $v(A) = [v^-(A), v^+(A)], v^-(A) = (C)\underline{\int}_A Fd\mu, v^+(A) = (C)\overline{\int}_A Fd\mu$. If μ has strong order continuity, then v also has strong order continuity.*

Proof. For any $A \in \mathcal{A}, \{A_n\} \subset \mathcal{A}, A_n \downarrow A$, and $v(A) = 0$, denotes $F_\alpha = \{x|F(x) \cap [\alpha, \infty] \neq \emptyset\}$, i.e.

$$v^+(A) = (C)\overline{\int}_A Fd\mu = \int_0^\infty \mu(F_\alpha \cap A)d\alpha = 0.$$

Thus, in the case of the Lebesgue measures, $\mu(F_\alpha \cap A) = 0$ almost everywhere for $\alpha \in [0, +\infty)$. By μ is strong order continuity, and $\mu(A) = 0$, hence, $\mu(F_\alpha \cap A_n) \downarrow 0$.

On the other hand, since $F(x)$ is Choquet integrable, we have

$$(C)\overline{\int}_{A_n} Fd\mu = \int_0^\infty \mu(F_\alpha \cap A_n)d\alpha \le \int_0^\infty \mu(F_\alpha \cap X)d\alpha = (C)\overline{\int} Fd\mu < \infty.$$

By the Lebesgue integral dominated convergence theorem, we have

$$v^+(A_n) = (C)\overline{\int}_{A_n} Fd\mu = \int_0^\infty \mu(F_\alpha \cap A_n)d\alpha \longrightarrow 0.$$

By the same way, we can also obtain

$$v^-(A_n) = (C)\underline{\int}_{A_n} Fd\mu \longrightarrow 0.$$

Therefore, $v(A_n) = [(C)\underline{\int}_{A_n} Fd\mu, (C)\overline{\int}_{A_n} Fd\mu] \to 0$, so v has strong order continuity. The proof is complete. □

Theorem 5.4. *Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space, F be a measurable set-valued function and Choquet integrable on X, $v(A) = [v^-(A), v^+(A)]$, $v^-(A) = (C)\underline{\int}_A Fd\mu, v^+(A) = (C)\overline{\int}_A Fd\mu$. If μ has pseudo-metric generating property, then v also has pseudo-metric generating property.*

Proof. Let $F_\alpha = \{x | F(x) \cap [\alpha, \infty] \ne \emptyset\}$. Since $F(x)$ is Choquet integrable, then

$$v^+(X) = (C)\overline{\int} Fd\mu = \int_0^\infty \mu(F_\alpha \cap X)d\alpha < \infty.$$

Thus, for every $\varepsilon > 0$, there exist $a, b(0 < a < b)$ such that

$$\int_0^a \mu(F_\alpha)d\alpha + \int_b^\infty \mu(F_\alpha)d\alpha < \frac{\varepsilon}{2}.$$

Becuase μ has pseudo-metric generating property, then there exist $\delta > 0$, for any $A, B \in \mathcal{A}$ and $\mu(A) \vee \mu(B) < \delta$ such that

$$\mu(A \cup B) < \frac{\varepsilon}{2(b-a)},$$

for $\varepsilon > 0$, let $\delta_1 = a\delta$, by $v(A) \vee v(B) < \delta_1$(i.e. $v^+(A) \vee v^+(B) < \delta_1$), we have

$$\int_0^a \mu(F_\alpha \cap A)d\alpha \vee \int_0^a \mu(F_\alpha \cap B)d\alpha < \delta_1.$$

On the other hand, μ has pseudo-metric generating property, by the monotonicity of μ, for any $\alpha \in [a, \infty)$, we obtain $\mu(F_\alpha \cap A) \vee \mu(F_\alpha \cap B) < \delta$, then for any $\alpha \in [a, \infty)$, we have

$$\mu(F_\alpha \cap (A \cup B)) < \frac{\varepsilon}{2(b-a)},$$

i.e.

$$\int_a^b \mu(F_\alpha \cap (A \cup B))d\alpha < \frac{\varepsilon}{2}.$$

Hence

$$v^+(A \cup B) = (C)\overline{\int}_{A \cup B} F d\mu$$

$$= \int_0^a \mu(F_\alpha \cap (A \cup B))d\alpha + \int_a^b \mu(F_\alpha \cap (A \cup B))d\alpha + \int_b^\infty \mu(F_\alpha \cap (A \cup B))d\alpha$$

$$\leq \int_0^a \mu(F_\alpha)d\alpha + \int_b^\infty \mu(F_\alpha)d\alpha + \int_a^b \mu(F_\alpha \cap (A \cup B))d\alpha$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Similarly, we can also obtain

$$v^-(A \cup B) = (C)\underline{\int}_{A \cup B} F d\mu < \varepsilon.$$

Thus, $v(A \cup B) = [(C)\underline{\int}_{A \cup B} F d\mu, (C)\overline{\int}_{A \cup B} F d\mu] < \varepsilon$, so μ has pseudo-metric generating property. The proof is complete. □

Theorem 5.5. *Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space, F be a measurable set-valued function and Choquet integrable on X. $v(A) = [v^-(A), v^+(A)], v^-(A) = (C)\underline{\int}_A F d\mu, v^+(A) = (C)\overline{\int}_A F d\mu$, if μ has property (S), then v also has property (S).*

Proof. Let $\{A_n\} \subset \mathcal{A}, \mu(\{A_n\}) \longrightarrow 0, F_\alpha = \{x|F(x) \cap [\alpha, \infty] \neq \emptyset\}$, then $\mu(F_\alpha \cap A_n) \longrightarrow 0$,
i.e.

$$v^+(A_n) = (C)\overline{\int}_{A_n} F d\mu = \int_0^\infty \mu(F_\alpha \cap A_n)d\alpha \longrightarrow 0.$$

Hence, for any $\alpha \geq 0$, by the Lebesgue integral theory, measurable function sequence $\{\mu(F_\alpha \cap A_n)\}_n \longrightarrow 0$. According to Riesz Theorem from real analysis, there exists subsequence $\{\mu(F_\alpha \cap A_n\}_k$, such that

$$\{\mu(F_\alpha \cap A_n\}_k \longrightarrow 0(k \to \infty).$$

By the monotony of μ, we obtain $\{\mu(F_\alpha \cap A_n\}_k \longrightarrow 0$. We suppose that $\mu(F_\alpha \cap A_n) \longrightarrow 0(n \to 0)$, then $\mu(F_{\frac{1}{2}} \cap A_n) \longrightarrow 0(n \to 0)$ if $\alpha = \frac{1}{2}$. By the property (S) of μ, so there exists a subsequence $A_{n_i}^{(1)}$ of A_n such that

$$\mu(\overline{\lim_{i \to \infty}}(F_{\frac{1}{2}} \cap A_{n_i})) = 0,$$

there must be $\mu(F_{\frac{1}{2}} \cap A_{n_i}^{(1)}) \longrightarrow 0(n \to 0)$. Hence, there exists a subsequence $A_{n_i}^{(2)}$ of $A_{n_i}^{(1)}$, we have

$$\mu(\overline{\lim_{i \to \infty}}(F_{\frac{1}{2^2}} \cap A_{n_i}^{(2)})) = 0.$$

Repeating this procedure, we can obtain a subsequence $A_{n_i}^{(j)}$ of An such that

$$\{A_{n_i}^{(j)}\}_i \supset \{A_{n_i}^{(j+1)}\}_i, \ \mu(\overline{\lim_{i \to \infty}}(F_{\frac{1}{2^k}} \cap A_{n_i}^{(j)})) = 0.$$

Let $A_{n_i} = A_{n_i}^{(j)}$, for any $\alpha > 0$, there exists a with $\frac{1}{2^a} < \alpha$, and

$$\bigcap_{i=1}^{\infty}\bigcup_{i=j}^{\infty}(F_\alpha \cap A_{n_i}) = \bigcap_{i=a}^{\infty}\bigcup_{i=j}^{\infty}(F_\alpha \cap A_{n_i}) \subset \bigcap_{i=a}^{\infty}\bigcup_{i=j}^{\infty}(F_\alpha \cap A_{n_i}^j) \subset \bigcap_{i=a}^{\infty}\bigcup_{i=j}^{\infty}(F_\alpha \cap A_{n_i}^a)$$

$$\subset \bigcap_{i=a}^{\infty}\bigcup_{i=j}^{\infty}(F_{\frac{1}{2^a}} \cap A_{n_i}^1) = \bigcap_{i=1}^{\infty}\bigcup_{i=j}^{\infty}(F_{\frac{1}{2^a}} \cap A_{n_i}^a) = \overline{\lim_{i\to\infty}}(F_{\frac{1}{2^a}} \cap A_{n_i}^{m_0}).$$

for $\alpha > 0$, it follows that

$$0 \le \mu(\overline{\lim_{i\to\infty}}(F_\alpha \cap A_{n_i})) \le \mu(\overline{\lim_{i\to\infty}}(F_{\frac{1}{2^{m_0}}} \cap A_{n_i}^{m_0})) = 0.$$

That is

$$v^+(\overline{\lim_{i\to\infty}}(A_{n_i})) = \int_0^\infty \mu(\overline{\lim_{i\to\infty}}(F_\alpha \cap A_{n_i}))d\alpha = 0.$$

Similarly, we can also obtain $v^-(\overline{\lim_{i\to\infty}}(A_{n_i})) = 0$. Therefore, $v(\overline{\lim_{i\to\infty}}(A_{n_i})) = 0$, so μ also has property (S). The proof is complete. □

6. The Choquet integral of the set-valued functions under distorted fuzzy measures

The properties of the Choquet integrals with respect to fuzzy measures have been studied extensively. This section we discussed the representation of the Choquet integral for set-valued functions with respect to a distorted fuzzy measure by probability measure p [2].

Definition 6.1. *[20] Let $f(x)$ be a continuous function. We say that $f(x)$ does not have plateaus when $A_\alpha = \{x|f(x) = \alpha\}$ is finite for all α.*

Lemma 6.1. *[20] Let $X = [0,1]$, $f(x)$ be a continuous function without plateaus. Let μ be a distorted probability measure $\mu_m = m \circ p$ with $m(x)$ a continuous strictly monotonic function with $m(1) = 1$ and p a probability such that $p(\{x\}) = 0$ for all x. Then there exists a monotone increasing function $f^* : X \longrightarrow X$ such that*

$$(c)\int f(x)dm \circ p = (c)\int f^*(x)dm \circ p.$$

Theorem 6.1. *Let $X = [0,1]$, $F : X \longrightarrow P(X)\backslash\{\emptyset\}$ be measurable set-valued function, f^- and f^+ be continuous differentiable and not have plateaus, $\mu_m = m \circ p$ is a distorted fuzzy measure by probability measure, and satisfying: (i) $m(x)$ is a continuous strictly monotonic function with $m(1) = 1$; (ii) $p(\{x\}) = 0$ for all x. Then, there exists a monotone increasing set-valued function $G : X \longrightarrow P(X)\backslash\{\emptyset\}$, such that*

$$(C)\int F(x)dm \circ p = (C)\int G(x)dm \circ p.$$

Proof. By Theorem 3.2, we have

$$(C)\overline{\int} Fdm \circ p = (c)\int f^+dm \circ p,$$

$$(C)\underline{\int} Fdm \circ p = (c)\int f^- dm \circ p.$$

Since $f^+(x)$ is a continuous differentiable and not have plateaus, according to Lemma 6.1, there exists a monotone increasing function $g^+ : X \longrightarrow X$ such that

$$(c)\int f^+ dm \circ p = (c)\int g^+ dm \circ p.$$

Similarly, we also obtain exists a monotone increasing function $g^- : X \longrightarrow X$, such that

$$(c)\int f^- dm \circ p = (c)\int g^- dm \circ p.$$

On the other hand, due to $F(x) = [f^-(x), f^+(x)]$, we have

$$(C)\int Fdm \circ p = [(c)\underline{\int} f^- dm \circ p, (c)\overline{\int} f^+ dm \circ p].$$

Hence, there exists a monotone increasing set-valued function $G(x) = [g^-(x), g^+(x)]$ with $g^+(x) = \sup G(x)$ and $g^-(x) = \inf G(x)$, such that

$$(C)\overline{\int} G(x)dm \circ p = (c)\int g^+(x)dm \circ p,$$

$$(C)\underline{\int} G(x)dm \circ p = (c)\int g^-(x)dm \circ p.$$

Therefore

$$(C)\int F(x)dm \circ p = (C)\int G(x)dm \circ p.$$

The proof is complete. □

7. Conclusion

The aim of this paper is attempt to discuss the representation of Choquet integral of set-valued functions with respect to fuzzy measures and the characteristic of its primitive. We firstly define and discuss real-valued major Choquet integral, real-valued minor Choquet integral and interval-valued Choquet integrals for set-valued functions with respect to fuzzy measures, which achieved the domination of the Choquet integral of set-valued functions with respect to fuzzy measures. Meanwhile, the calculation of the Choquet integral for set-valued function is investigated, and we have shown a basic representation theorem of Choquet integral for set-valued function as a Radon-Nikodym property in some sense. On the other hand, the characteristics of the primitive of the Choquet integral for set-valued functions are investigated. At the same time, we discussed the representation for Choquet integral of set-valued functions with respect to a distorted fuzzy measure by probability measure which is a classically fuzzy measure. Our results improve and generalize the corresponding results of [9]. In the future research, we shall discuss some applications for Choquet integral of set-valued functions with respect to fuzzy measures.

Acknowledgments

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. R. J. Aumann, *Integrals of set-valued functions*, J. Math. Anal. Appl., **12** (1965), 1–12.

2. P. Billingsley, *Probability and Measure*, Wiley, 1994.

3. D. Candeloro, R. Mesiar, A. R. Sambucini, *A special class of fuzzy measures: Choquet integral and applications*, Fuzzy Sets and Systems, **355** (2019), 83–99.

4. G. Choquet, *Theory of capacities*, J. Annals det Institut Fourier, **5** (1955), 131–295

5. S. J. Cho, B. S. Lee, G. M. Lee, et al. *Fuzzy integrals for set-valued mappings*, Fuzzy Sets and Systems, **117** (2001), 333–337.

6. M. Grabisch, T. Murofushi, M, Sugeno, *Fuzzy Measures and Integrals: Theory and Applications*, Physica-verlag, Heidelberg, 2000.

7. M. Grabisch, C. Labreuche, *A decade of applications of the Choquet and Sugeno integrals in multi-criteria decision aid*, Ann. Oper. Res., **175** (2010), 247–286.

8. Z. T. Gong, Z. Q. Wei, *Choquet integral of set-valued functions with respect to multisubmeasures*, Journal of Shandong University (Natural Science) (in Chinese), **50** (2015), 63–71.

9. Y. Huang, C. X. Wu, *Real-valued Choquet integral for set-valued mappings*, Int. J. Approx. Reason., **55** (2014), 683–688.

10. L. C. Jang, J. S. Kwon, *On the representation of Choquet of set-valued functions and null set*, Fuzzy Sets and Systems, **112** (2000), 233–239.

11. L. C. Jang, B. M. Kil, Y. K. Kim, et al. *Some properties of Choquet integrals of set-valued functions*, Fuzzy Sets and Systems, **91** (1997), 95–98.

12. J. Li, *Order continuity of monotone set function and convergence of measurable functions sequence*, Appl. Math. Comput., **135** (2003), 211–218.

13. T. Murofushi, M. Sugeno, *An integral of fuzzy measures and the Choquet integral as integral with respect to a fuzzy measure*, Fuzzy Sets and Systems, **29** (1989), 201–227.

14. F. Y. Meng, Q. Zhang, *Induced continuous Choquet integral operators and their application to group decision making*, Comput. Ind. Eng., **68** (2014), 42–53.

15. Y. Ouyang, J. Li, *Some properties of monotone set functions defined by Choquet integral*, Journal of Southeast University (in Chinese), **19** (2003), 424–427.

16. E. Pap, A. Iosif, A. Gavrilut, *Integrability of an interval-valued multifunction with respect to an interval-valued set multifunction*, Iranian Journal of Fuzzy Systems, **15** (2018), 47–63.

17. D. Paternain, L. De Miguel, G. Ochoa, et al, *The interval-Valued Choquet integral based on admissible permutations*, IEEE T. Fuzzy Syst., **27** (2019), 1638–1647.

18. C. Stamate, A. Croitoru, *The general Pettis-Sugeno integral of vector multifunctions relative to a vector fuzzy multimeasure*, Fuzzy Sets and Systems, **327** (2017), 123–136.

19. M. Sugeno, *A note on derivatives of functions with respect to fuzzy measures*, Fuzzy Sets and Systems, **222** (2013), 1–17.

20. V. Torra, Y. Narukawa, *Numerical integration for the Choquet integral*, Inform. Fusion, **31** (2016), 137–145.

21. V. Torra, Y. Narukawa, *The interpretation of fuzzy integrals and their application to fuzzy systems*, Int. J. Approx. Reason., **41** (2006), 43–58.

22. A. Tversky, D. Kahneman, *Advances in prospect theory: cumulative representation of uncertainty*, J. Risk Uncertain, **5** (1992), 297–323.

23. C. X. Wu, M. Ma, *The Foundament of Fuzzy Analysis*, National Defense Industry Press (in Chinese), Beijing, 1991.