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Research article

Invariants in partition classes

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Abstract: With $p(n,k)$ denote the numerical value of the number of partitions of the natural number n on exactly k parts. Form an arithmetic progression of k natural numbers with an arbitrary first value $x_1 = p(j,k)$, and the difference $d = m \cdot LCM(1,2,\dots,k)$, where j and m an arbitrary natural numbers. Calculate all the values of $\{p(x_i,k)\}_{i=1,2,\dots,k}$ and make the alternating sum with the appropriate binomial coefficients $\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}p(j+i\cdot d,k)$. The last sum has a constant value equal to $(-1)^{k-1}\frac{d^{k-1}}{k!}$, regardless of the first selected member x_1 of the arithmetic progression. We call this sum the first partition invariant, and it exists in all classes. In addition to these values there are a whole number of other invariant values, but they exist only in some classes, and so forth.

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1. Introduction

Let n and k be two positive integers. Denote by $p(n,k)$ the number of partitions of the positive number n on exactly k parts. Then the partition class k is the sequence $p(1,k), p(2,k), \dots, p(n,k), \dots$ We already know, see [1], all these values can be divided into the highest $d_0 = LCM(1,2,\dots,k)$ sub sequences, each of which is calculated by the same polynomial.

Choose a sequence of k natural numbers such that: the first member is arbitrary, and the rest form an arithmetic progression with a difference $d = m \cdot d_0,\ \ m \in \mathbb{N}$, starting from the chosen first member. For example:

$$x_1 = j,\ \ x_2 = j + d,\ \ \dots,\ \ x_k = j + (k-1)\cdot d,\quad j \in \mathbb{N}. \tag{1.1}$$

The corresponding number of partitions of the class k for the elements of the previous arithmetic progression's values is:

$$p(x_1,k),\ \ p(x_2,k),\ \ \dots\ ,\ \ p(x_k,k). \tag{1.2}$$

If the values, which are calculated using the same polynomial, multiplied by the corresponding binomial coefficients, form the alternate sum, we notice that the sum always has a value which is independent of x_1, no matter how we form the sequence (1.1).

For the partition function of classes we already know the following results, see [1, 2] for some details:

i) The values of the partition function of classes is calculated with one quasi polynomial.

ii) For each class k the quasi polynomial consists of at most $LCM(1, 2, \dots, k)$ different polynomials, each of them consists of a strictly positive and an alternating part.

iii) All polynomials within one quasi polynomial $p(n, k)$ are of degree $k - 1$.

iv) All the coefficients with the highest degrees down to $\left[\frac{k}{2}\right]$ are equal for all polynomials (all of strictly positive) and all polynomials differ only in lower coefficients (alternating part).

v) The form of any polynomial $p(n, k)$ is:

$$p(n, k) = a_1 n^{k-1} + a_2 n^{k-2} + \cdots + a_k, \tag{1.3}$$

where the coefficients $a_1, a_2, \dots, a_k$ are calculated in the general form.

Let us forget for a moment that the coefficients $a_1, a_2, \dots$ are known in general form. Knowing that all values for partitions class of the sequence (1.1) are obtained by one polynomial $p(n, k)$, it is possible to determine all unknown coefficients in a completely different way from that given in papers [1, 2]. To determine k unknowns, a k equation is required. For this purpose, it is sufficient to know all the values of the sequence (1.2). To this end, we must form the system (1.4) and solve it. (For $k = 10$, see [3]).

$$\begin{aligned} a_1 \cdot x_1^{k-1} + a_2 \cdot x_1^{k-2} + \cdots + a_k &= p(x_1, k) \\ a_1 \cdot x_2^{k-1} + a_2 \cdot x_2^{k-2} + \cdots + a_k &= p(x_2, k) \\ \dots \quad \dots \quad \dots & \\ a_1 \cdot x_k^{k-1} + a_2 \cdot x_k^{k-2} + \cdots + a_k &= p(x_k, k) \end{aligned} \tag{1.4}$$

The system (1.4) can be solved by Cramer's Rule. For further analysis, we need to find the following determinants. We will start with the known Vandermonde determinant, see [4].

$$\Delta_m = \begin{vmatrix} x_1^{m-1} & x_1^{m-2} & \dots & 1 \\ x_2^{m-1} & x_2^{m-2} & \dots & 1 \\ \dots & \dots & \dots & \\ x_m^{m-1} & x_m^{m-2} & \dots & 1 \end{vmatrix} = \prod_{1 \le i < j \le m} \left(x_i - x_j\right), \quad m > 1. \tag{1.5}$$

When we remove the first column and an arbitrary row from the previous determinant we obtain the Vandermonde determinant of one order less. The following results are known, see [4] and are needed for further exposure. If we remove the second column and an arbitrary a-th row from the determinant (1.5) we get

$$\begin{vmatrix} x_1^{m-1} & x_1^{m-3} & \dots & x_1 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ x_{a-1}^{m-1} & x_{a-1}^{m-3} & \dots & x_{a-1} & 1 \\ x_{a+1}^{m-1} & x_{a+1}^{m-3} & \dots & x_{a+1} & 1 \\ \dots & \dots & \dots & \dots & \dots \\ x_m^{m-1} & x_m^{m-3} & \dots & x_m & 1 \end{vmatrix} = \left(\sum_{0 \le i \le m}^{i \ne a} x_i\right) \cdot \prod_{0 \le i < j \le m}^{i,j \ne a} \left(x_i - x_j\right) \tag{1.6}$$

If we remove the third column and an arbitrary a-th row from the determinant (1.5) we get

$$\begin{vmatrix} x_1^{m-1} & x_1^{m-2} & x_1^{m-4} & \dots & x_1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_{a-1}^{m-1} & x_{a-1}^{m-2} & x_{a-1}^{m-4} & \dots & x_{a-1} & 1 \\ x_{a+1}^{m-1} & x_{a+1}^{m-2} & x_{a+1}^{m-4} & \dots & x_{a+1} & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_m^{m-1} & x_m^{m-2} & x_m^{m-4} & \dots & x_m & 1 \end{vmatrix} = \sum_{1\le i<j\le m}^{i,j\neq a} x_i \cdot x_j \prod_{0\le i<j\le m}^{i,j\neq a} \left(x_i - x_j\right) \tag{1.7}$$

Generaly, if we remove the b-th column and an arbitrary a-th row from the determinant (1.5) we get

$$\Delta_m^{(a,b)} = \begin{vmatrix} x_1^{m-1} & x_1^{m-2} & \dots & x_1^{b+1} & x_1^{b-1} & \dots & x_1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ x_{a-1}^{m-1} & x_{a-1}^{m-2} & \dots & x_{a-1}^{b+1} & x_{a-1}^{b-1} & \dots & x_{a-1} & 1 \\ x_{a+1}^{m-1} & x_{a+1}^{m-2} & \dots & x_{a+1}^{b+1} & x_{a+1}^{b-1} & \dots & x_{a+1} & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ x_m^{m-1} & x_m^{m-2} & \dots & x_m^{b+1} & x_m^{b-1} & \dots & x_m & 1 \end{vmatrix} = \left(\sum_{1\le t_1<t_2\cdots<t_{b-1}\le m}^{t_1,\dots,t_{b-1}\neq a} x_{t_1} x_{t_2} \dots x_{t_{b-1}} \right) \prod_{0\le i<j\le m}^{i,j\neq a} \left(x_i - x_j\right).$$

The label $\Delta_m^{(a,b)}$ means that from Δ_m remove the a-th row and b-th column from the set of variables x_a.

2. Invariants of the partitions classes

2.1. The first partition invariant of classes

Theorem 1. Let m, j and k be three positive integers and

$$I_1(k, j, d) = \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} p(j + i \cdot d, k),$$

where $d = m \cdot LCM(1, 2, 3, \dots, k)$. Then $I_1(k, j, d) = (-1)^{k-1} \frac{d^{k-1}}{k!}$ and is independent of j. ($I_1(k, j, d)$ is the first partition invariant which exists in all classes.)

Proof. Among the values of the class k we choose the ones corresponding to the sequence (1.1), and they are given with the sequence (1.2). According to [2], all the elements in (1.2) can be calculated using the same polynomial $p(n, k)$ with degree $k - 1$. Elements of the following sequence:

$$q, q + d, \dots, q + (k - 1) \cdot d, \quad q \neq j,$$

are calculated with not necessarily the same polynomial as the previous one. Let the polynomial $p(n, k)$ have the form as in (1.3). To determine the coefficients $a_1, a_2, \dots, a_k$ it suffices to know the k values: $p(x_1, k), p(x_2, k), \dots, p(x_k, k)$ where $x_1 = j,\ x_2 = j + d, \dots,\ x_k = j + (k - 1)d$ are different numbers. Since $\Delta_k \neq 0$, system (1.4) always has a unique solution, because all the elements of the set $\{x_1, x_2, \dots, x_k\}$ are different from one another. According to Cramer's Rule, to determine the coefficient of the highest degree of the polynomial (1.3), which calculates the value of the number of partitions of class k, we have the following formula:

$$a_1 = \frac{p(x_1, k)\,\Delta_k^{(1,1)} - p(x_2, k)\,\Delta_k^{(2,1)} + \dots + (-1)^{k-1} p(x_k, k)\,\Delta_k^{(k,1)}}{\Delta_k} \tag{2.1}$$

Determinants $\Delta_k^{(a,1)}, (1 \le a \le k)$ are also Vandermonde and their values are equal to Δ_{k-1}. Let $\{x_i\}_{1\le i\le k}$ satisfy (1.1) then for $1 \le a \le k$ it holds that

$$\Delta_k^{(a,1)} = \Delta_{k-1} = \frac{\Delta_k}{\prod\limits_{i=1}^{a-1} (x_i - x_a) \prod\limits_{i=a+1}^{k} (x_a - x_i)} = \frac{\Delta_k}{(-1)^{a-1} (a-1)! d^{a-1} \cdot (-1)^{k-a} (k-a)! d^{k-a}} = \frac{(-1)^{k-1} \Delta_k}{(a-1)! (k-a)! d^{k-1}}.$$

Replacing in (2.1), after shortening with Δ_k we have

$$a_1 = (-1)^{k-1} \left(\frac{p(x_1,k)}{0! (k-1)! d^{k-1}} - \frac{p(x_2,k)}{1! (k-2)! d^{k-1}} + \ldots + (-1)^{k-1} \frac{p(x_k,k)}{(k-1)! 0! d^{k-1}} \right).$$

The coefficient a_1 is already defined in [2] where it is shown that $a_1 = \frac{1}{k!(k-1)!}$. Substituting into the previous equality and multiplying by $(-1)^{k-1}$, we obtain

$$\frac{(-1)^{k-1}}{k! (k-1)!} = \frac{p(x_1,k)}{0! (k-1)! d^{k-1}} - \frac{p(x_2,k)}{1! (k-2)! d^{k-1}} + \cdots + (-1)^{k-1} \frac{p(x_k,k)}{(k-1)! 0! d^{k-1}}.$$

Multiplying the last equality with $(k-1)! d^{k-1}$ we obtain

$$(-1)^{k-1} \frac{d^{k-1}}{k!} = \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} p(j + i \cdot d, k),$$

which was to be proved. As these values are equal to each observed number of objects (1.2) within a class, the sum is invariant for any observed class. □

All classes of the partition do not contain all the invariants we will list. This primarily refers to the classes from the beginning. Only the first invariant appears in all classes. The second invariant holds starting from the third class. The third invariant holds starting from the fifth class. Fourth, from the seventh class, etc. This coincides with the appearance of the common coefficients $\{a_k\}$ in quasi polynomials $p(n,k)$, $k \in \mathbb{N}$.

Theorem 2. Let m, j and k be three positive integers, $k \ge 3$ and

$$I_2(k,j,d) = \sum_{i=0}^{k-1} (-1)^{i+1} \left(j(k-1) + \left(\binom{k}{2} - i \right) d \right) \binom{k-1}{i} p(j + i \cdot d, k)$$

where $d = m \cdot LCM(2,3,\ldots,k)$. Then $I_2(k,j,d) = (-1)^k \frac{(k-3)d^{k-1}}{4(k-2)!}$ and is independent of j.

Remark. In the previous expression, we should not simplify as then the value for $k = 3$ cannot be obtained. However, the value for $k = 3$ exists and is equal to zero.

Proof. Analogously to Theorem 1, the fact that the sum does not depend on the parameter j is a consequence of the periodicity per modulo $LCM(2,3,\ldots,k)$ using the same polynomial to calculate the partition class values.

In [2] it is shown how the system of linear equations can determine the other unknown coefficient of the polynomials which are calculated values of the partition classes. This coefficient is obtained from Cramer's Rule on system (1.4) and a_2 is given by

$$a_2 = \frac{-p(x_1,k)\,\Delta_k^{(1,2)} + p(x_2,k)\,\Delta_k^{(2,2)} - \ldots (-1)^k\, p(x_k,k)\,\Delta_k^{(k,2)}}{\Delta_k}. \tag{2.2}$$

Considering (1.6), knowing that $\{x_i\}_{i=1,2,\ldots,k}$ is an arithmetic progression, determinants $\Delta_k^{(a,2)}$ can be written for $1 \le a \le k$ with

$$\Delta_k^{(a,2)} = \left(\sum_{1\le i\le k}^{i\neq a} x_i\right)\Delta_{k-1} = \frac{\left((k-1)\,j + \left(\binom{k}{2} - a + 1\right)d\right)\Delta_k}{\prod\limits_{i=1}^{a-1}(x_i - x_a)\prod\limits_{i=a+1}^{k}(x_a - x_i)}$$

$$= \frac{\left((k-1)\,j + \left(\binom{k}{2} - a + 1\right)d\right)\Delta_k}{(-1)^{a-1}(a-1)!d^{a-1}\cdot(-1)^{k-a}(k-a)!d^{k-a}}$$

$$= \frac{(-1)^{k-1}\left((k-1)\,j + \left(\binom{k}{2} - a + 1\right)d\right)\Delta_k}{(a-1)!\,(k-a)!d^{k-1}}.$$

Knowing the value of the coefficient $a_2 = \frac{k-3}{4(k-1)!(k-2)!}$ [2] and substituting in (2.2), and after multiplication with $(-1)^k(k-1)!d^{k-1}$ we obtain

$$(-1)^k\frac{k-3}{4(k-2)!}d^{k-1} = \left((k-1)\,j + \binom{k}{2}d\right)\binom{k-1}{0}p(j,k) -$$

$$\left((k-1)\,j + \left(\binom{k}{2} + 1\right)d\right)\binom{k-1}{1}p(j+d,k) + \ldots$$

$$= \sum_{i=0}^{k-1}(-1)^{i+1}\left(j(k-1) + \left(\binom{k}{2} - i\right)d\right)\binom{k-1}{i}p(j+i\cdot d,k).\square$$

2.2. The third partition invariant of classes

These invariants are in all classes starting from the fifth. For simplicity we denote them by

$$R(i,j,k,d) = \frac{1}{2}\left(\left((k-1)\,j + \left(\binom{k}{2} - i\right)d\right)^2 - (k-1)\,j^2\right.$$

$$\left. - \left(\frac{1}{6}k(k-1)(2k-1) - i^2\right)d^2 - 2d\cdot j\left(\binom{k}{2} - i\right)\right).$$

Theorem 3. Let m, j and k be three positive integers, $k \ge 5$ and

$$I_3(k,j,d) = \sum_{i=0}^{k}(-1)^i R(i,j,k,d)\binom{k-1}{i}p(j+i\cdot d,k),$$

where $d = m \cdot LCM(2, 3, \ldots, k)$. Then $I_3(k, j, d) = (-1)^{k-1} \frac{9k^3-58k^2+75k-2}{288(k-3)!} d^{k-1}$ and is independent of j.
Proof. For the third invariant we need the value of the third polynomial coefficient of $p(n, k)$, and it is shown [2] that this is

$$a_3 = \frac{9k^3 - 58k^2 + 75k - 2}{288(k-1)!(k-3)!}, \quad k \geq 5.$$

On the other hand, we have

$$a_3 = \frac{p(x_1, k)\Delta_k^{(1,3)} - p(x_2, k)\Delta_k^{(2,3)} + \cdots + (-1)^{k-1} p(x_k, k)\Delta_k^{(k,3)}}{\Delta_k} \tag{2.3}$$

From formula (1.7) we find $\Delta_k^{(a,3)}$. The required sum $\sum\limits_{1 \leq i < j \leq k} x_i \cdot x_j$ is convenient to calculate from the equality

$$\sum_{1 \leq i < j \leq k}^{i,j \neq a} x_i \cdot x_j = \frac{1}{2}\left(\left(\sum_{1 \leq i \leq k}^{i \neq a} x_i\right)^2 - \sum_{1 \leq i \leq k}^{i \neq a} x_i^2\right),$$

where the sequence $\{x_i\}$ satisfies (1.1). Then, we should determine the quotient which can be simplified by reducing the following:

$$\frac{\Delta_k^{(a,3)}}{\Delta_k} = \frac{R(a-1, j, k, d)}{\prod\limits_{i=1}^{a-1}(x_i - x_a)\prod\limits_{i=a+1}^{k}(x_a - x_i)}.$$

By multiplying (2.3) with $(-1)^{k-1}(k-1)!d^{k-1}$ and after shortening we obtain:

$$I_3(k, j, d) = \sum_{i=0}^{k-1} (-1)^i \cdot R(i, j, k, d)\binom{k-1}{i} p(j + i \cdot d, k).\Box$$

In every subsequent invariant, the proceedings become more complex. But, it is quite clear how further invariants can be calculated.

3. Consideration of special cases

For each partitions class k, $k \in \mathbb{N}$ we determine $d_0 = LCM(1, 2, 3, \ldots, k)$, and then form $d = m \cdot d_0$, $m \in \mathbb{N}$. In addition arbitrarily choose the natural number j and than form sequences (1.1) and (1.2). Finally, we form an appropriate sum which is for the first invariant:

$$\sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} p(j + i \cdot d, k) = \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} p(x_{i+1}, k), \, j \in \mathbb{N}. \tag{3.1}$$

Sum (3.1) has a constant value in each partitions class and can be nominated as the first partitions class invariant.

3.1. The first partitions class invariant

For $k = 1$, sum (3.1) has a constant value of 1.

For $k = 2$, $d_0 = 2$. If we choose some $m \in \mathbb{N}$ and set $d = 2m$, the sum (3.1) has the form: $p(j, 2) - p(j + d, 2), j \in \mathbb{N}$. According to [1], it is known that $p(n, 2) = \left[\frac{n}{2}\right]$. Distinguishing between even and odd numbers of j (j and $j + d$ have the same parity) and substituting into the sum, we obtain that the result, in both cases, is equal to $-\frac{d}{2} = -m$.

For $k = 3$, $d_0 = 6$. If we choose some $m \in \mathbb{N}$ and set $d = 6m$ the sum (3.1) has the form:

$$p(j, 3) - 2 \cdot p(j + d, 3) + p(j + 2d, 3), j \in \mathbb{N}. \tag{3.2}$$

According to [1], it is known that:

$$p(n, 3) = \frac{n^2 + \omega_i}{12}, \quad i = n \mod 6, \quad \omega_i \in \{0, -1, -4, 3, -4, -1\}. \tag{3.3}$$

By replacing (3.3) in relation (3.2) we get

$$\frac{j^2 + w_{i_1}}{12} - 2\frac{(j + d)^2 + w_{i_2}}{12} + \frac{(j + 2d)^2 + w_{i_3}}{12}.$$

Note that: $i_1 = j \ mod\ 6$, $i_2 = (j + d)\ \ mod\ 6$, $i_3 = (j + 2d)\ \ mod\ 6$ and $w_{i_1} = w_{i_2} = w_{i_3}$. Finally, we get the unique sum $6m^2$.

For $k = 4$, $d_0 = 12$. If we choose some $m \in \mathbb{N}$ and set $d = 12m$ the sum (3.1) has the form:

$$p(j, 4) - 3p(j + d, 4) + 3p(j + 2d, 4) - p(j + 3d, 4), j \in \mathbb{N}. \tag{3.4}$$

According to [1], it is known that:

$$p(n, 4) = \frac{1}{144}n^3 + \frac{1}{48}n^2 + \begin{cases} \frac{w_i}{144}, & n \text{ even}, \\ -\frac{1}{16}n + \frac{w_i}{144}, & n \text{ odd}, \end{cases} \quad i \equiv n \mod 12, \tag{3.5}$$

$$w_i \in \{0, 5, -20, -27, 32, -11, -36, 5, 16, -27, -4, -11\}.$$

Similar to case $k = 3$, by distinguishing the even and odd j and replacing (3.5) in relation (3.4) we obtain that the corresponding sums in both cases are equal to: $-72m^3$. (Note that: $i_1 = j \ mod\ 12$, $i_2 = (j + d)\ mod\ 12$, $i_3 = (j + 2d)\ mod\ 12$, $i_4 = (j + 3d)\ mod\ 12$ and $w_{i_1} = w_{i_2} = w_{i_3} = w_{i_4}$.)

The number of invariants increases, when the class number increases. Starting with class three, another invariant can be observed.

3.2. The second partitions class invariant

Form in the same way as in the previous section: d_0, d and the sequences (1.1) and (1.2) as well as the sum:

$$\sum_{i=0}^{k-1} (-1)^i \left(j(k - 1) + \left(\binom{k}{2} - i \right) d \right) \binom{k - 1}{i} p(j + i \cdot d, k).$$

Previous sum has a constant value in each partitions class (starting from third class) and can be nominated as the second partitions class invariant.

For $k = 3$, $d_0 = 6$. If we choose some $m \in \mathbb{N}$ and set $d = 6m$ the general form of the second invariant in the third class can be written as

$$(2j + 3d)\, p\,(j, 3) - 2\,(2j + 2d)\, p\,(j + d, 3) + (2j + d)\, p\,(j + 2d, 3)\,, \; j \in \mathbb{N}$$

The values $p\,(j, 3)\,, p\,(j + d, 3)$ and $p\,(j + 2d, 3)$ are calculated using the same polynomial (3.3). Using (3.3) in the last equality we have

$$(2j + 3d)\,\frac{j^2 + w_{i_1}}{6} - 2\,(2j + 2d)\,\frac{(j + d)^2 + w_{i_2}}{6} + (2j + d)\,\frac{(j + 2d)^2 + w_{i_3}}{6}$$

Note that: $i_1 = j \; mod \; 6$, $i_2 = (j + d)\, mod \; 6$, $i_3 = (j + 2d)\, mod \; 6$ and $w_{i_1} = w_{i_2} = w_{i_3}$. The last equality is identical to zero.

For $k = 4$, $d_0 = 12$. If we choose some $m \in \mathbb{N}$ and set $d = 12m$ the general form of the second invariant in the fourth class can be written as

$$(3j + 6d)\, p\,(j, 4) - 3\,(3j + 5d)\, p\,(j + d, 4) + 3\,(3j + 4d)\, p\,(j + 2d, 4) - (3j + 3d)\, p\,(j + 3d, 4)\,. \quad (3.6)$$

The last equations can be verified in an analogous manner, by using the same form of the known polynomial for the fourth class given in (3.5). Note that: $i_1 = j \; mod \; 12$, $i_2 = (j + d)\, mod \; 12$, $i_3 = (j + 2d)\, mod \; 12$, $i_4 = (j + 3d)\, mod \; 12$ and $w_{i_1} = w_{i_2} = w_{i_3} = w_{i_4}$. By distinguishing the even and odd j and replacing (3.5) in relation (3.6) we obtain that the corresponding sums in both cases are equal to: $-216m^3$.

3.3. The third partition invariants

Form in the same way as in the previous two section: d_0, d and the sequences (1.1) and (1.2) as well as the sum $I_3\,(k, j, d)$ (Theorem 3). For each class (starting from the fifth) $I_3\,(k, j, d)$ has constant values and can be nominated as the third partitions class invariant. It is known [1] that

$$p\,(n, 5) = \frac{1}{2880}n^4 + \frac{1}{288}n^3 + \frac{1}{288}n^2 + \begin{cases} -\frac{1}{24}n + \frac{w_i}{2880}, & n \text{ even}, \\ -\frac{1}{96}n + \frac{w_i}{2880}, & n \text{ odd}, \end{cases} \quad i \equiv n \bmod 60, \quad (3.7)$$

w_i are following numeric respectively:

$$\begin{aligned} &0, 9, 104, -351, -576, 905, -216, -351, -256, 9, 360, -31, -576, 9, 104, 225, \\ &-576, 329, -216, -351, 320, 9, -216, -31, -576, 585, 104, -351, -576, 329, 360, \\ &-351, -256, 9, -216, 545, -576, 9, 104, -351, 0, 329, -216, -351, -256, 585, \\ &-216, -31, -576, 9, 680, -351, -576, 329, -216, 225, -256, 9, -216, -31. \end{aligned}$$

For $k = 5$, $d_0 = 60$. If we choose some $m \in \mathbb{N}$ and set $d = 60m$ the invariant $I_3\,(k, j, d)$ can be written as:

$$\frac{1}{2}\left((4j+10d)^2-4j^2-20d\cdot j-30d^2\right)p(j,5)-\frac{1}{2}\left((4j+9d)^2-4j^2-18d\cdot j-29d^2\right)p(j+d,5)$$
$$+\frac{1}{2}\left((4j+8d)^2-4j^2-16d\cdot j-26d^2\right)p(j+2d,5)-\frac{1}{2}\left((4j+7d)^2-4j^2-14d\cdot j-21d^2\right)p(j+3d,5)$$
$$+\frac{1}{2}\left((4j+6d)^2-4j^2-12d\cdot j-14d^2\right)p(j+4d,5).$$

Substituting (3.7) into the previous formula by distinguishing between even and odd j, we obtain a unique value of $1080000m^4$.

Remark 1. From the Table 1, see [5], given at the end of the paper it is possible to check all of these explicitly with numerical values. For example:

1. Check the first invariant in the third class. Take $m = 2$, $j = 5$. The first invariant formula is

$$p(5,3)-2\cdot p(17,3)+p(27,3).$$

From the Table we find: $p(5,3) = 2$, $p(17,3) = 24$, $p(29,3) = 70$. By substitution we find $2-2\cdot 24+70 = 24(= 6m^2)$.

2. Check the second invariant in the forth class. Take $m = 1$, $j = 3$. The second invariant formula is

$$81p(3,4)-3\cdot 69p(15,4)+3\cdot 57p(27,4)-45p(39,4).$$

From the Table we find: $p(3,4) = 0$, $p(15,4) = 27$, $p(27,4) = 150$, $p(39,4) = 441$. By substitution we find:

$$81\cdot 0-3\cdot 69\cdot 27+3\cdot 57\cdot 150-45\cdot 441 = -216(= -216m^3).$$

3. Check the third invariant in the fifth class. Take $m = 1$, $j = 1$. The third invariant formula is

$$127806\cdot p(1,5)-380904\cdot p(61,5)$$
$$+419076\cdot p(121,5)-206664\cdot p(181,5)+40686\cdot p(241,5) = 1080000.$$

Using formulas from (3.7), we find that: $p(1,5) = 0$, $p(61,5) = 5608$, $p(121,5) = 80631$, $p(181,5) = 393369$ and $p(241,5) = 1220122$, and so by checking we are assured of the accuracy.

Remark 2. Obviously, $p(n,k)$ define values only for $n \geq k$. The invariants determine very precisely that values for $n < k$ should be taken as zero.

Table 1. Partition classes values.

d_0	1	2	6	12	60	60	420	840	2520	2520			
n/k	1	2	3	4	5	6	7	8	9	10	11	...	$p(n)$
1	1	0	0	0	0	0	0	0	0	0	0		1
2	1	1	0	0	0	0	0	0	0	0	0		2
3	1	1	1	0	0	0	0	0	0	0	0		3
4	1	2	1	1	0	0	0	0	0	0	0		5
5	1	2	2	1	1	0	0	0	0	0	0		7
6	1	3	3	2	1	1	0	0	0	0	0		11
7	1	3	4	3	2	1	1	0	0	0	0		15
8	1	4	5	5	3	2	1	1	0	0	0		22
9	1	4	7	6	5	3	2	1	1	0	0		30
10	1	5	8	9	7	5	3	2	1	1	0		42
11	1	5	10	11	10	7	5	3	2	1	1		56
12	1	6	12	15	13	11	7	5	3	2	1	...	77
13	1	6	14	18	18	14	11	7	5	3	2	...	101
14	1	7	16	23	23	20	15	11	7	5	3	...	135
15	1	7	19	27	30	26	21	15	11	7	5	...	176
16	1	8	21	34	37	35	28	22	15	11	7	...	231
17	1	8	24	39	47	44	38	29	22	15	11	...	297
18	1	9	27	47	57	58	49	40	30	22	15	...	385
19	1	9	30	54	70	71	65	52	41	30	22	...	490
20	1	10	33	64	84	90	82	70	54	42	30	...	627
21	1	10	37	72	101	110	105	89	73	55	43	...	792
22	1	11	40	84	119	136	131	116	94	75	56	...	1002
23	1	11	44	94	141	163	164	146	123	97	77	...	1255
24	1	12	48	108	164	199	201	186	157	128	100	...	1575
25	1	12	52	120	192	235	248	230	201	164	133	...	1958
26	1	13	56	136	221	282	300	288	252	212	171	...	2436
27	1	13	61	150	255	331	364	352	318	267	223	...	3010
28	1	14	65	169	291	391	436	434	393	340	282	...	3718
29	1	14	70	185	333	454	522	525	488	423	362	...	4565
30	1	15	75	206	377	532	618	638	598	530	453	...	5604
31	1	15	80	225	427	612	733	764	732	653	573	...	6842
32	1	16	85	249	480	709	860	919	887	807	709	...	8349
33	1	16	91	270	540	811	1009	1090	1076	984	884	...	10143
34	1	17	96	297	603	931	1175	1297	1291	1204	1084	...	12310
35	1	17	102	321	674	1057	1369	1527	1549	1455	1337	...	14883
36	1	18	108	351	748	1206	1579	1801	1845	1761	1626	...	17977
37	1	18	114	378	831	1360	1824	2104	2194	2112	1984	...	21637

4. Conclusions

In this paper, authors have demonstrated a new approach to partitions class invariants, as a way of proving the relevance and accuracy of all formulas given in [1, 2]. Also, it I can be considered to be another way to obtain some of the formulas in [2]. The quasi polynomials $p(n,k)$ needed to calculate the number of partitions of a number n to exactly k parts consists of at most $LCM(1,2,\ldots,k)$ different polynomials. The invariants claim that the more different polynomials in one quasi polynomial, the more invariable sizes connect them.

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Conflict of interest

Authors declare no conflicts of interest in this paper.

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