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Research article

The existence of upper and lower solutions to second order random impulsive differential equation with boundary value problem

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Abstract: In this article, we consider the existence of upper and lower solutions to a second-order random impulsive differential equation. We first study the solution form of the corresponding linear impulsive system of the second-order random impulsive differential equation. Based on the form of the solution, we define the resolvent operator. Then, we prove that the fixed point of this operator is the solution to the equation. Finally, we construct the sum of two monotonic iterative sequences and prove that they are convergent. Thus, we conclude that the system has upper and lower solutions.

Keywords: random impulse differential equation; upper and lower solutions; monotonic iterative sequences; boundary value problem
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1. Introduction

Impulsive differential equations have many applications in engineering, science and finance. As a ubiquitous phenomenon, pulses exist in mechanical systems with impacts, optimal control models in economics, and the transfers of satellite orbit. It is difficult to model such phenomena using continuous models or discrete models [1, 2]. In the 1950s, an impulsive model was developed to describe such specific evolution of a dynamic system [1]. Impulsive differential systems describe the dynamic processes with discontinuous jump caused by sudden changes. A variety of impulsive systems were investigated in the literature [3–6, 23, 35, 38, 40].

The characteristics of impulsive differential equations have attracted the attention of scholars [31, 32]. In recent years, many scholars have studied the initial and boundary value problem of fixed impulse differential equations [16, 17, 22]. For example, the boundary value problem of impulsive equations have been examined in the literature [7–10, 15, 18, 24]. The existence and uniqueness of solutions to the following impulsive equation with boundary value problems have been investigated in the literature [11].

$$\begin{cases} -u'' = f(t,u,u'), & t \in J \setminus \{t_1,t_2,\cdots,t_m\}, \\ \Delta u(t_k) = I_k(u(t_k)), & k = 1,2,\cdots,m, \\ -\Delta u'(t_k) = N_k(u(t_k)), \\ au(0) - bu'(0) = 0, cu(1) + du'(1) = 0, \end{cases} \tag{1.1}$$

where $a > 0, b \geq 0, c > 0, d \geq 0$, $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$, $J = [0,1]$.

Some kinds of stochastic differential equation with fixed impulsive moments and Random impulsive differential equations also obtained considerable attention in the literature [12–14,20,21,25,26,36,37]. The Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays has been studied in [19]. The authors considered the following system

$$\begin{cases} \mathrm{d}(x(t)) = f(t,x_t)\mathrm{d}t + g(t,x_t)\mathrm{d}W(t), & t \geq 0, t \neq \xi_k, \\ x(\xi_k) = b_k(\tau_k)x(\xi_k^-), & k = 1,2,\cdots, \\ x_{t_0} = \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}, \end{cases} \tag{1.2}$$

where x_t is $\mathbb{R}^d$ -valued stochastic process such that $x_t \in \mathbb{R}^d$, $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$. Here, $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_k < \cdots < \lim_{k\to\infty} \xi_k = +\infty$, and $x(\xi_k^-) = \lim_{t\to\xi_k-0} x(t)$. Note that $\{N(t), t \geq 0\}$ is the simple counting process generated by ξ_k, and $\{W(t) : t \geq 0\}$ is a given m-dimensional Winer process.

The fixed point method [28] had been used to study the random impulsive differential equations. Niu et al. [27] used the fixed point method to address the existence and Hyers-Ulam stability for the following differential equation

$$\begin{cases} x''(t) = f(t,x(t)), & t \in J,\ t \neq \xi_k, \\ x(\xi_k) = b_k(\tau_k)x(\xi_k^-), & k = 1,2,\cdots, \\ x_0 = x_0, x'(0) = x_1. \end{cases} \tag{1.3}$$

Upper and lower solution method can be used to study fractional evolution equations [33] and impulsive differential equations [34].

To the best of our knowledge, the boundary value problem of second order random impulsive differential equation has not been studied using the upper and lower solution method in the literature. In this paper, we use the upper and lower solution method to study the following second order random impulsive differential equation with boundary value problem.

$$\begin{cases} -x''(t) = f(t,x(t),x'(t)), & t \in J', \\ x(\xi_k^+) = b_k(\tau_k)x(\xi_k^-), & k = 1,2,\cdots, \\ \alpha_0 x(0) - \alpha_1 x'(0) = x_0, \\ \beta_0 x(1) + \beta_1 x'(1) = x_0^*. \end{cases} \tag{1.4}$$

Where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping. $x(t)$ is a stochastic process taking values in the Euclidean space $(\mathbb{R}, \|\cdot\|)$. Then, we introduce τ_k to be a random variable defined from Ω to $E_k \triangleq (0, d_k)$, with $0 < d_k < 1$ for every $k \in \mathbb{N}^+$. We assume that τ_i and τ_j are independent of each other when $i \neq j$ for every $i, j \in \mathbb{N}^+$ and b_k: $E_k \to \mathbb{R}$ satisfies for every $k \in \mathbb{N}^+$, $b_k(\tau_k) \geq 0$. Set $\xi_k = \xi_{k-1} + \tau_k$. Obviously, $\{\xi_k\}$ is a process with independent increments and the impulsive moments ξ_k form a strictly increasing sequence, i.e. $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_k < \cdots < 1$. We hold the opinion that $x(\xi_k^-) = \lim_{t\to\xi_k-0} x(t)$, $x(\xi_k^+) = \lim_{t\to\xi_k+0} x(t)$. The convergence is under the meaning of the orbit. Since for a realization (sample) of random process, $\{\xi_k\}$ will become a series of fixed time points. Under that sense, so we can define the limit as we would in general. We suppose that $\{N(t) : t \geq 0\}$ is the simple counting process generated by ξ_k. Let $J = [0, 1]$, $\mathbb{R}^+ = (0, +\infty)$ and $J' \triangleq J \setminus \{\xi_1, \xi_2, \cdots\}$. Here, $\alpha_0, \alpha_1, \beta_0, \beta_1, x_0, x_0^*$ are constants satisfying $\alpha_0\alpha_1 \neq 0, \beta_0 \neq 0, \alpha_0, \alpha_1, \beta_0, \beta_1, x_0, x_0^* \geq 0$.

The rest of the paper is organised as follows: In section 2, we introduce some notations and preliminaries. In section 3, we use the upper and lower solution method to study the existence of solutions to the second order random impulsive differential equations. In section 4, we give an example to show the application of the main result. Finally, conclusions are presented.

2. Preliminaries

Suppose (Ω, Γ, P) is a probability space. Let $L^p(\Omega, \mathbb{R}^n)$ be the collection of all strongly measurable, *pth* integrable, and Γ_t-measurable with $\mathbb{R}^n$-valued random variables $x : \Omega \to \mathbb{R}^n$ and norm $L^p(\Omega, \mathbb{R}^n)$ for $p \geq 1$. Here, $\mathrm{E}(x) = \int_\Omega x \mathrm{d}\mathbb{P} < \infty$ is the expectation of x, and $L^p(\Omega, \mathbb{R}^n)$ is equipped with its natural norm $\|x\|_{L^p(\Omega,\mathbb{R}^n)} = \left(\int_\Omega \|x\|^p \mathrm{d}\mathbb{P}\right)^{\frac{1}{p}} = (\mathrm{E}\|x\|^p)^{\frac{1}{p}}$.

We introduce the space $PC = PC(J, L^2(\Omega, \mathbb{R}^n))$:=$\{u(t) \mid u(t) = u(t, \omega)$ is a strongly measurable, square integrable, random process from J into $L^2(\Omega, \mathbb{R}^n)$, and $u(t)$ is continuous when $t \in J'$ and left continuous when $t \in J \setminus J'\}$. We can prove that PC is a Banach space with norm

$$\| u \|_{PC} = \left(\sup_{t\in J} \mathrm{E} \| u(t) \|^2 \right)^{1/2}. \tag{2.1}$$

Then, we consider the space $PC^1 = PC^1(J, L^2(\Omega, \mathbb{R}^n))$:=$\{u(t) \mid u(t) = u(t, \omega)$ is a strongly measurable, square integrable, random process from J into $L^2(\Omega, \mathbb{R}^n)$, $u(t)$ is continuously differentiable when $t \in J'$ and left continuous when $t \in J \setminus J'$, $u'(\xi_k^-)$ and $u'(\xi_k^+)$ exist for $k = 1, 2, \cdots\}$. It is easy to see that PC^1 is also a Banach space with norm

$$\| u \|_{PC^1} = \max \left\{ \left(\sup_{t\in J} \mathrm{E} \| u(t) \|^2 \right)^{1/2}, \left(\sup_{t\in J} \mathrm{E} \| u'(t) \|^2 \right)^{1/2} \right\}. \tag{2.2}$$

The functions in PC^1 which satisfy the equation (1.4) are called the solutions of the equation (1.4).

For convince, in the rest of the paper we write $PC(J, L^2(\Omega, \mathbb{R}))$ as $PC(J, \mathbb{R})$, write $PC^1(J, L^2(\Omega, \mathbb{R}))$ as $PC^1(J, \mathbb{R})$. In this paper, the upper and lower solutions of equation (1.4) are studied in $PC^1(J, \mathbb{R})$ space.

Now we consider the equation (2.3)

$$\begin{cases} -u''(t) = f(t, h(t), h'(t)) - M[u(t) - h(t)], & t \in J', \\ u(\xi_k^+) = b_k(\tau_k)h(\xi_k^-), & k = 1, 2, \cdots, \\ \alpha_0 u(0) - \alpha_1 u'(0) = x_0, \\ \beta_0 u(1) + \beta_1 u'(1) = x_0^*, \end{cases} \tag{2.3}$$

where $h(t) \in PC^1(J, \mathbb{R})$ and M is a positive constant.

Lemma 2.1. Equation (2.3) has one solution, given by

$$u(t)=\sum_{k=0}^{\infty}\Big[C_1^k e^{\sqrt{M}t}+C_2^k e^{-\sqrt{M}t}+\hat{h}(t)\Big]I_{(\xi_k,\xi_{k+1}]}(t), \tag{2.4}$$

$$\begin{cases} C_1^k=C_1, & k=0,\\ C_1^k=\delta_k^- C_1-e^{-2\sqrt{M}\xi_1}\delta_k^- C_2+\sum_{n=1}^{k}\Delta_{n,k}^-[b_n(\tau_n)h(\xi_n)-\hat{h}(\xi_n)], & k=1,2,\cdots,\\ C_2^k=C_2, & k=0,\\ C_2^k=-e^{2\sqrt{M}\xi_1}\delta_k^+ C_1+\delta_k^+ C_2+\sum_{n=1}^{k}\Delta_{n,k}^+[b_n(\tau_n)h(\xi_n)-\hat{h}(\xi_n)], & k=1,2,\cdots, \end{cases} \tag{2.5}$$

and

$$\begin{aligned} \delta_k^- &= \frac{1}{2^k}\Bigg[1+\sum_{1\le i_1<j_1\le k}e^{-2\sqrt{M}(\xi_{j_1}-\xi_{i_1})}+\sum_{1\le i_1<j_1<i_2<j_2\le k}e^{-2\sqrt{M}[(\xi_{j_2}-\xi_{i_2})+(\xi_{j_1}-\xi_{i_1})]}\\ &\quad+\cdots+\sum_{1\le i_1<\cdots<j_n\le k\le j_n+1}e^{-2\sqrt{M}[(\xi_{j_n}-\xi_{i_n})+\cdots+(\xi_{j_1}-\xi_{i_1})]}\Bigg],\\ \delta_k^+ &= \frac{1}{2^k}\Bigg[1+\sum_{1\le i_1<j_1\le k}e^{2\sqrt{M}(\xi_{j_1}-\xi_{i_1})}+\sum_{1\le i_1<j_1<i_2<j_2\le k}e^{2\sqrt{M}[(\xi_{j_2}-\xi_{i_2})+(\xi_{j_1}-\xi_{i_1})]}\\ &\quad+\cdots+\sum_{1\le i_1<\cdots<j_n\le k\le j_n+1}e^{2\sqrt{M}[(\xi_{j_n}-\xi_{i_n})+\cdots+(\xi_{j_1}-\xi_{i_1})]}\Bigg], \end{aligned} \tag{2.6}$$

$$\begin{aligned} \Delta_{n,k}^- &= \frac{1}{2^{k-n+1}}e^{-\sqrt{M}\xi_n}\Bigg[1+\sum_{n\le i_1<j_1\le k}(-1)^{I_{\{n\}}(i_1)}e^{-2\sqrt{M}(\xi_{j_1}-\xi_{i_1})}\\ &\quad+\sum_{n\le i_1<j_1<i_2<j_2\le k}(-1)^{I_{\{n\}}(i_1)}e^{-2\sqrt{M}[(\xi_{j_2}-\xi_{i_2})+(\xi_{j_1}-\xi_{i_1})]}\\ &\quad+\cdots+\sum_{n\le i_1<\cdots<j_l\le k\le j_l+1}(-1)^{I_{\{n\}}(i_1)}e^{-2\sqrt{M}[(\xi_{j_l}-\xi_{i_l})+\cdots+(\xi_{j_1}-\xi_{i_1})]}\Bigg],\\ \Delta_{n,k}^+ &= \frac{1}{2^{k-n+1}}e^{\sqrt{M}\xi_n}\Bigg[1+\sum_{n\le i_1<j_1\le k}(-1)^{I_{\{n\}}(i_1)}e^{2\sqrt{M}(\xi_{j_1}-\xi_{i_1})}\\ &\quad+\sum_{n\le i_1<j_1<i_2<j_2\le k}(-1)^{I_{\{n\}}(i_1)}e^{2\sqrt{M}[(\xi_{j_2}-\xi_{i_2})+(\xi_{j_1}-\xi_{i_1})]}\\ &\quad+\cdots+\sum_{n\le i_1<\cdots<j_l\le k\le j_l+1}(-1)^{I_{\{n\}}(i_1)}e^{2\sqrt{M}[(\xi_{j_l}-\xi_{i_l})+\cdots+(\xi_{j_1}-\xi_{i_1})]}\Bigg], \end{aligned} \tag{2.7}$$

$$C_1=\frac{1}{|Q|}\begin{vmatrix} x_0 & \alpha_0+\sqrt{M}\alpha_1\\ \begin{matrix}x_0^*-[e^{\sqrt{M}}B^-(1)(\beta_0+\sqrt{M}\beta_1)\\ +e^{-\sqrt{M}}B^+(1)(\beta_0-\sqrt{M}\beta_1)\\ +\beta_0\hat{h}(1)+\beta_1\hat{h}'(1)]\end{matrix} & \begin{matrix}-e^{2\sqrt{M}\xi_1+\sqrt{M}}A^-(1)(\beta_0+\sqrt{M}\beta_1)\\ +e^{-\sqrt{M}}A^+(1)(\beta_0-\sqrt{M}\beta_1)\end{matrix} \end{vmatrix}, \tag{2.8}$$

$$C_2=\frac{1}{|Q|}\begin{vmatrix}\alpha_0-\sqrt{M}\alpha_1 & x_0\\ e^{\sqrt{M}}A^-(1)(\beta_0+\sqrt{M}\beta_1) & x_0^*-[e^{\sqrt{M}}B^-(1)(\beta_0+\sqrt{M}\beta_1)\\ -e^{2\sqrt{M}\xi_1-\sqrt{M}}A^+(1)(\beta_0-\sqrt{M}\beta_1) & +e^{-\sqrt{M}}B^+(1)(\beta_0-\sqrt{M}\beta_1)\\ & +\beta_0\hat{h}(1)+\beta_1\hat{h}'(1)]\end{vmatrix}, \tag{2.9}$$

$$|Q|=\begin{vmatrix}\alpha_0-\sqrt{M}\alpha_1 & \alpha_0+\sqrt{M}\alpha_1\\ e^{\sqrt{M}}A^-(1)(\beta_0+\sqrt{M}\beta_1) & -e^{2\sqrt{M}\xi_1+\sqrt{M}}A^-(1)(\beta_0+\sqrt{M}\beta_1)\\ -e^{2\sqrt{M}\xi_1-\sqrt{M}}A^+(1)(\beta_0-\sqrt{M}\beta_1) & +e^{-\sqrt{M}}A^+(1)(\beta_0-\sqrt{M}\beta_1)\end{vmatrix}. \tag{2.10}$$

Denote

$$\hat{h}(t)=-\frac{e^{-\sqrt{M}t}}{2\sqrt{M}}\int_0^t e^{\sqrt{M}s}\sigma(s)\mathrm{d}s+\frac{e^{\sqrt{M}t}}{2\sqrt{M}}\int_0^t e^{-\sqrt{M}s}\sigma(s)\mathrm{d}s, \tag{2.11}$$

$$\sigma(s)=f(s,h(s),h'(s))+Mh(s), \tag{2.12}$$

and the index function

$$I_A(t)=\begin{cases}1 & t\in A,\\ 0 & t\notin A,\end{cases} \tag{2.13}$$

$$I_{\{n\}}(i_1)=\begin{cases}1 & n=i_1,\\ 0 & n\neq i_1,\end{cases} \tag{2.14}$$

where

$$\begin{aligned}A^-(t)&=\sum_{k=0}^{\infty}\delta_k^- I_{(\xi_k,\xi_{k+1}]}(t),\\ A^+(t)&=\sum_{k=0}^{\infty}\delta_k^+ I_{(\xi_k,\xi_{k+1}]}(t),\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}B^-(t)&=\sum_{k=0}^{\infty}\sum_{n=1}^{k}\Delta_{n,k}^-[b_n(\tau_n)h(\xi_n)-\hat{h}(\xi_n)]I_{(\xi_k,\xi_{k+1}]}(t),\\ B^+(t)&=\sum_{k=0}^{\infty}\sum_{n=1}^{k}\Delta_{n,k}^+[b_n(\tau_n)h(\xi_n)-\hat{h}(\xi_n)]I_{(\xi_k,\xi_{k+1}]}(t).\end{aligned} \tag{2.16}$$

Proof. Suppose $\xi_1,\xi_2,\cdots$ is a sample orbit. Thus, when $t\in[0,\xi_1]$, the solution of the equation (2.3) is

$$u_0(t)=C_1e^{\sqrt{M}t}+C_2e^{-\sqrt{M}t}+\hat{h}(t). \tag{2.17}$$

When $t\in(\xi_1,\xi_2]$, we assume that the solution of the equation (2.3) is

$$u_1(t)=C_1^1e^{\sqrt{M}t}+C_2^1e^{-\sqrt{M}t}+\hat{h}(t). \tag{2.18}$$

Plug in the initial conditions

$$\begin{aligned}u_1(\xi_1^+)&=b_1(\tau_1)h(\xi_1),\\ u_1'(\xi_1^+)&=u_0'(\xi_1),\end{aligned} \tag{2.19}$$

we can get

$$C_1^1 = \frac{1}{2}e^{-\sqrt{M}\xi_1}[C_1 e^{\sqrt{M}\xi_1} - C_2 e^{-\sqrt{M}\xi_1} + b_1(\tau_1)h(\xi_1) - \hat{h}(\xi_1)],$$
$$C_2^1 = \frac{1}{2}e^{\sqrt{M}\xi_1}[-C_1 e^{\sqrt{M}\xi_1} + C_2 e^{-\sqrt{M}\xi_1} + b_1(\tau_1)h(\xi_1) - \hat{h}(\xi_1)]. \tag{2.20}$$

In the same way, we can get when $t \in (\xi_k, \xi_{k+1}]$,

$$C_1^k = \frac{1}{2}e^{-\sqrt{M}\xi_k}[C_1^{k-1} e^{\sqrt{M}\xi_k} - C_2^{k-1} e^{-\sqrt{M}\xi_k} + b_k(\tau_k)h(\xi_k) - \hat{h}(\xi_k)],$$
$$C_2^k = \frac{1}{2}e^{\sqrt{M}\xi_k}[-C_1^{k-1} e^{\sqrt{M}\xi_k} + C_2^{k-1} e^{-\sqrt{M}\xi_k} + b_k(\tau_k)h(\xi_k) - \hat{h}(\xi_k)]. \tag{2.21}$$

Based on the above discussion, mathematical induction can be obtained as

$$C_1^k = \delta_k^- C_1 - e^{-2\sqrt{M}\xi_1}\delta_k^- C_2 + \sum_{n=1}^{k} \Delta_{n,k}^-[b_n(\tau_n)h(\xi_n) - \hat{h}(\xi_n)],$$
$$C_2^k = -e^{2\sqrt{M}\xi_1}\delta_k^+ C_1 + \delta_k^+ C_2 + \sum_{n=1}^{k} \Delta_{n,k}^+[b_n(\tau_n)h(\xi_n) - \hat{h}(\xi_n)]. \tag{2.22}$$

Where $\delta_k^+, \delta_k^-, \Delta_k^+, \Delta_k^-$ are defined as the equations (2.6) and (2.7).

So, the solution of the equation (2.3) is

$$u(t) = \left[\sum_{k=0}^{\infty} C_1^k I_{(\xi_k,\xi_{k+1}]}(t)\right] e^{\sqrt{M}t} + \left[\sum_{k=0}^{\infty} C_2^k I_{(\xi_k,\xi_{k+1}]}(t)\right] e^{-\sqrt{M}t} + \hat{h}(t), \tag{2.23}$$

and

$$\sum_{k=0}^{\infty} C_1^k I_{(\xi_k,\xi_{k+1}]}(t) = \left[\sum_{k=0}^{\infty} \delta_k^- I_{(\xi_k,\xi_{k+1}]}(t)\right] C_1 - \left[\sum_{k=0}^{\infty} \delta_k^- I_{(\xi_k,\xi_{k+1}]}(t)\right] e^{-2\sqrt{M}\xi_1} C_2$$
$$+ \sum_{k=0}^{\infty}\sum_{n=1}^{k} \Delta_{n,k}^-[b_n(\tau_n)h(\xi_n) - \hat{h}(\xi_n)] I_{(\xi_k,\xi_{k+1}]}(t), \tag{2.24}$$

$$\sum_{k=0}^{\infty} C_2^k I_{(\xi_k,\xi_{k+1}]}(t) = -\left[\sum_{k=0}^{\infty} \delta_k^+ I_{(\xi_k,\xi_{k+1}]}(t)\right] e^{2\sqrt{M}\xi_1} C_1 + \left[\sum_{k=0}^{\infty} \delta_k^+ I_{(\xi_k,\xi_{k+1}]}(t)\right] C_2$$
$$+ \sum_{k=0}^{\infty}\sum_{n=1}^{k} \Delta_{n,k}^+[b_n(\tau_n)h(\xi_n) - \hat{h}(\xi_n)] I_{(\xi_k,\xi_{k+1}]}(t). \tag{2.25}$$

Therefore

$$u(t) = [A^-(t)C_1 - e^{-2\sqrt{M}\xi_1}A^-(t)C_2 + B^-(t)]e^{\sqrt{M}t}$$
$$+ [-e^{2\sqrt{M}\xi_1}A^+(t)C_1 + A^+(t)C_2 + B^+(t)]e^{-\sqrt{M}t} + \hat{h}(t), \tag{2.26}$$

$$u'(t) = \sqrt{M}[A^-(t)C_1 - e^{-2\sqrt{M}\xi_1}A^-(t)C_2 + B^-(t)]e^{\sqrt{M}t}$$
$$- \sqrt{M}[-e^{2\sqrt{M}\xi_1}A^+(t)C_1 + A^+(t)C_2 + B^+(t)]e^{-\sqrt{M}t} + \hat{h}'(t). \tag{2.27}$$

Substituting these equations into the boundary value conditions of the equation (2.3) yields and using the Cramer's rule, it follows from (2.26) and (2.27) that (2.5)–(2.16) hold.

Lemma 2.2. δ_k^- and δ_k^+ are uniformly bounded series.

Proof. We firstly consider δ_k^-,

$$e^{-2\sqrt{M}[(\xi_{j_m}-\xi_{i_m})+\cdots+(\xi_{j_1}-\xi_{i_1})]} \le 1. \tag{2.28}$$

So, for every $n \in \mathbb{N}^+$, we have

$$\sum_{1\le i_1<\cdots<j_n\le k\le j_n+1} e^{-2\sqrt{M}[(\xi_{j_n}-\xi_{i_n})+\cdots+(\xi_{j_1}-\xi_{i_1})]} \le \begin{pmatrix} k \\ 2n \end{pmatrix}, \tag{2.29}$$

$$\begin{aligned}\delta_k^- &= \frac{1}{2^k}\Bigg[1 + \sum_{1\le i<j\le k} e^{-2\sqrt{M}(\xi_j-\xi_i)} + \sum_{1\le i_1<j_1<i_2<j_2\le k} e^{-2\sqrt{M}[(\xi_{j_2}-\xi_{i_2})+(\xi_{j_1}-\xi_{i_1})]} \\ &\quad + \cdots + \sum_{1\le i_1<\cdots<j_n\le k\le j_n+1} e^{-2\sqrt{M}[(\xi_{j_n}-\xi_{i_n})+\cdots+(\xi_{j_1}-\xi_{i_1})]}\Bigg] \\ &\le \frac{1}{2^k}\left[1+\begin{pmatrix} k \\ 2 \end{pmatrix}+\begin{pmatrix} k \\ 4 \end{pmatrix}+\cdots+\begin{pmatrix} k \\ 2n \end{pmatrix}\right] = \frac{1}{2}.\end{aligned} \tag{2.30}$$

In the same way, we can prove that δ_k^+ is uniformly bounded and

$$\delta_k^+ \le \frac{1}{2}e^{2\sqrt{M}}. \tag{2.31}$$

Lemma 2.3. For every $n \in \mathbb{N}^+$, $\Delta_{n,k}^-$ and $\Delta_{n,k}^+$ are uniformly bounded series.

Proof. We can easily prove that

$$\begin{aligned}&-2^{n-1}\delta_k^- \le \Delta_{n,k}^- \le 2^{n-1}\delta_k^-, \\ &-2^{n-1}e^{\sqrt{M}}\delta_k^+ \le \Delta_{n,k}^+ \le 2^{n-1}e^{\sqrt{M}}\delta_k^+.\end{aligned} \tag{2.32}$$

So, we have

$$\begin{aligned}&|\Delta_{n,k}^-| \le 2^{n-1}\delta_k^- \le 2^{n-2}, \\ &|\Delta_{n,k}^+| \le 2^{n-1}e^{\sqrt{M}}\delta_k^+ \le 2^{n-2}e^{3\sqrt{M}},\end{aligned} \tag{2.33}$$

and we have proved the lemma.

Definition 2.1. Define the operator $\Lambda : PC^1[J,\mathbb{R}] \to PC^1[J,\mathbb{R}]$ such that

$$\Lambda h = \sum_{k=0}^{\infty}\left[C_1^k e^{\sqrt{M}t} + C_2^k e^{-\sqrt{M}t} + \hat{h}(t)\right] I_{(\xi_k,\xi_{k+1}]}(t). \tag{2.34}$$

Definition 2.2. $v_0(t) \in PC^1[J,\mathbb{R}]$ is called a lower solution of equation (1.4) if $v_0(t)$ satisfies the inequality group

$$\begin{cases} -v_0''(t) \le f(t, v_0(t), v_0'(t)), \\ v_0(\xi_k^+) \le b_k(\tau_k)v_0(\xi_k^-), \\ \alpha_0 v_0(0) - \alpha_1 v_0'(0) \le x_0, \\ \beta_0 v_0(1) + \beta_1 v_0'(1) \le x_0^*. \end{cases} \tag{2.35}$$

Definition 2.3. $\omega_0(t) \in PC^1[J,\mathbb{R}]$ is called an upper solution of equation (1.4) if $\omega_0(t)$ satisfies the inequality group

$$\begin{cases} -\omega_0''(t) \geq f(t,\omega_0(t),\omega_0'(t)), \\ \omega_0(\xi_k^+) \geq b_k(\tau_k)\omega_0(\xi_k^-), \\ \alpha_0\omega_0(0) - \alpha_1\omega_0'(0) \geq x_0, \\ \beta_0\omega_0(1) + \beta_1\omega_0'(1) \geq x_0^*. \end{cases} \tag{2.36}$$

Lemma 2.4. $h(t) \in PC^1[J,\mathbb{R}]$ is the solution of equation (1.4) if and only if $h(t) \in PC^1[J,\mathbb{R}]$ is the fix point of the operator Λ.

Proof. If $h(t)$ is the fix point of the operator Λ, it is to say that $h(t)$ satisfies the equation $\Lambda h(t) = h(t)$, then, in the equation (2.3), we have $u(t) = \Lambda h(t) = h(t)$, so, we can replace $u(t)$ with $h(t)$ in the equation (2.3), and we have

$$\begin{cases} -h''(t) = f(t,h(t),h'(t)), \\ h(\xi_k^+) = b_k(\tau_k)h(\xi_k^-), \\ \alpha_0 h(0) - \alpha_1 h'(0) = x_0, \\ \beta_0 h(1) + \beta_1 h'(1) = x_0^*, \end{cases} \tag{2.37}$$

and we have proved that $h(t)$ is a solution of the equation (1.4).

If $h(t)$ is the solution of the equation (1.4), then using the same method, we can easily know that it is also the fix point of the operator Λ, and we have proved the lemma.

Lemma 2.5.(The Arzela-Ascoli Theorem)([29]) The set $M \subset C^2[J,\mathbb{R}^n]$ is column compact tight if and only if

(i) The functions in the set M are uniformly bounded, that is to say, there exists a fixed constant K for all $u(t) \in M$, where $\|u(t)\| \leq K$.

(ii) Functions in the set M are equally continuous, that is to say, for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$ such that when $t_1, t_2 \in J$ and $\| t_1 - t_2 \| < \delta$ for all $u(t) \in M$, there is $\| u(t_1) - u(t_2) \| < \epsilon$.

Lemma 2.6.([30]) Suppose E is a semi-ordered Banach space. For $x_0, y_0 \in E$, $x_0 \leq y_0$, and $D = [x_0(t), y_0(t)]$, $A : D \to E$ is an operator. Assuming that the following conditions are satisfied

(i) A is an increasing operator,

(ii) x_0 is the lower solution of A and y_0 is the upper solution of A,

(iii) A is a continuous operator,

(iv) $A(D)$ is a relatively compact set of columns in E.

Then, A has a maximum fixed point and a minimum fixed point in D. Let x_0 and y_0 be the initial conditions. We then have the iteration sequences

$$x_n = Ax_{n-1}, y_n = Ay_{n-1}, n = 1, 2, \cdots. \tag{2.38}$$

Thus,

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots \leq y_n \leq \cdots \leq y_1 \leq y_0, \tag{2.39}$$

and

$$x_n \to x^*, y_n \to y^*. \tag{2.40}$$

3. Main result

(H_1) The equation (1.4) has the lower solution $v_0(t)$ and the upper solution $\omega_0(t)$ and they meet the inequality

$$v_0(t) \le \omega_0(t), \tag{3.1}$$

for any $t \in J$.
(H_2) There exists a constant $M > 0$, such that

$$f(t, x_1, y) - f(t, x_2, y) \ge -M(x_1 - x_2), \tag{3.2}$$

for any $t \in J$, $y \in PC^1(J, \mathbb{R})$ and $v_0(t) \le x_2(t) \le x_1(t) \le \omega_0(t)$.
(H_3) There exist constants B_1 and B_2 such that

$$\begin{cases} \sum_{k=0}^{\infty}\Big\{\sum_{n=1}^{k} \Delta^-_{n,k} b_n(\tau_n)\Big\} I_{(\xi_k,\xi_{k+1}]}(t) \le B_1, \\ \sum_{k=0}^{\infty}\Big\{\sum_{n=1}^{k} \Delta^+_{n,k} b_n(\tau_n)\Big\} I_{(\xi_k,\xi_{k+1}]}(t) \le B_2. \end{cases} \tag{3.3}$$

(H_4) There exists an increasing continuous function $\Theta(x)$ satisfies that $\frac{\Theta(x)}{x}$ is a decreasing function and

$$\mathrm{E}\|f(t, x, y)\|^2 \le \Theta(\|x\|_{PC^1}), \tag{3.4}$$

for every $t \in J$, $x_1, x_2 \in D = [v_0(t), \omega_0(t)]$, and $y \in PC^1(J, \mathbb{R})$.
(H_5) There exist a function $\Psi(t, x, y)$ and a constant K such that

(i) For each $t \in J$, the function $\Psi(t, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\Psi(t, 0, 0) = 0$. For every $x, y \in \mathbb{R}$, the function $\Psi(\cdot, x, y) : J \to \mathbb{R}$ is measurable;

(ii)

$$\mathrm{E} \parallel f(t, x_1, y_1) - f(t, x_2, y_2) \parallel^2 \le K\Psi(t, \mathrm{E}\|x_1 - x_2\|^2, \mathrm{E}\|x_1' - x_2'\|^2), \tag{3.5}$$

for every $t \in J$, $x(t) \in D = [v_0(t), \omega_0(t)]$.
(H_6) Define $P = 12\frac{e^{2\sqrt{M}}}{M}$, $P^* = 12e^{2\sqrt{M}} + \ln 3(m_1 + m_2)$, $m_1 = \sup_t \mathrm{E}\|C_1^k e^{\sqrt{M}t}\|^2$, and $m_2 = \sup_t \mathrm{E}\|C_2^k e^{-\sqrt{M}t}\|^2$.

Then we define the sequences v_n and ω_n as

$$\begin{aligned} v_n(t) = &\sum_{k=0}^{\infty}\Big[C_1^k(v_{n-1})e^{\sqrt{M}t} + C_2^k(v_{n-1})e^{-\sqrt{M}t}\Big] I_{(\xi_k,\xi_{k+1}]}(t) \\ &- \frac{e^{-\sqrt{M}t}}{2\sqrt{M}}\int_0^t e^{\sqrt{M}s}\Big[Mv_{n-1}(s) + f(s, v_{n-1}(s), v_{n-1}'(s))\Big]\mathrm{d}s \\ &+ \frac{e^{\sqrt{M}t}}{2\sqrt{M}}\int_0^t e^{-\sqrt{M}s}\Big[Mv_{n-1}(s) + f(s, v_{n-1}(s), v_{n-1}'(s))\Big]\mathrm{d}s, \end{aligned} \tag{3.6}$$

$$
\begin{aligned}
\omega_n(t) = &\sum_{k=0}^{\infty}\Big[C_1^k(\omega_{n-1})e^{\sqrt{M}t} + C_2^k(\omega_{n-1})e^{-\sqrt{M}t}\Big]I_{(\xi_k,\xi_{k+1}]}(t) \\
&- \frac{e^{-\sqrt{M}t}}{2\sqrt{M}}\int_0^t e^{\sqrt{M}s}\Big[M\omega_{n-1}(s) + f(s,\omega_{n-1}(s),\omega_{n-1}'(s))\Big]\mathrm{d}s \\
&+ \frac{e^{\sqrt{M}t}}{2\sqrt{M}}\int_0^t e^{-\sqrt{M}s}\Big[M\omega_{n-1}(s) + f(s,\omega_{n-1}(s),\omega_{n-1}'(s))\Big]\mathrm{d}s.
\end{aligned} \tag{3.7}
$$

Theorem 3.1.

If conditions $(H_1) \sim (H_6)$ are met, the equation (1.4) has the maximum solution $x^*(t)$ and the minimum solution $x_*(t)$ in $[v_0(t), \omega_0(t)] \bigcap PC^1[J,\mathbb{R}]$. And there exist $\omega_n(t) = \Lambda\omega_{n-1}(t)$ uniformly convergent to $x^*(t)$, $v_n(t) = \Lambda v_{n-1}(t)$ uniformly convergent to $x_*(t)$, $n = 1, 2, \cdots$. And if $x(t)$ is the solution of the equation (1.4), it satisfies

$$
\ln(\mathrm{E}\|x(t)\|^2) \leq P\int_0^1 \frac{\Theta(\mathrm{E}\|x(t)\|^2)}{\mathrm{E}\|x(t)\|^2}\mathrm{d}t + P^*. \tag{3.8}
$$

Proof. We will prove this theorem in five steps.
Step(1). We prove that $v_0(t)$ and $\omega_0(t)$ are the lower and upper solutions of the operator Λ, i.e., we should prove $v_0(t) \leq \Lambda v_0(t)$ and $\omega_0(t) \geq \Lambda\omega_0(t)$.

When there is no random impulsive, we set $v_1(t) = \Lambda v_0(t)$. Now, we only need to prove that $v_0(t) \leq v_1(t)$. Here, we use proof by contradiction. If it is not true, then there exist $t_0 \in J$ and $\varepsilon > 0$ such that

$$
\begin{cases} v_0(t_0) = v_1(t_0) + \varepsilon, \\ v_0(t) \leq v_1(t) + \varepsilon, \end{cases} \tag{3.9}
$$

for every $t \in J$. If $t_0 \in J \setminus (\{0\} \bigcup \{1\})$, it is easy to see that

$$
\begin{cases} v_0'(t_0) - v_1'(t_0) = 0, \\ v_0''(t_0) - v_1''(t_0) \leq 0. \end{cases} \tag{3.10}
$$

However,

$$
\begin{aligned}
&-v_1''(t) = f(t, v_0(t), v_0'(t)) - M[v_1(t) - v_0(t)] \geq -v_0''(t) - M[v_1(t) - v_0(t)], \\
&v_0''(t_0) - v_1''(t_0) \geq M[v_0(t_0) - v_1(t_0)] = M\varepsilon > 0,
\end{aligned} \tag{3.11}
$$

which is a contradiction to the inequality $v_0''(t_0) - v_1''(t_0) \leq 0$. Thus, our hypothesis does not work.

When $t_0 = 0$ or $t_0 = 1$, we assume that $t_0 = 0$. Therefore,

$$
\begin{aligned}
&m(t) = v_1(t) - v_0(t), \\
&\min_{t\in J}\{m(t)\} = m(0),
\end{aligned} \tag{3.12}
$$

assuming that $v_1(0) + \varepsilon = v_0(0)$, it is easy to see that

$$
v_1'(0) - v_0'(0) \geq 0. \tag{3.13}
$$

By the boundary value conditions, we can get

$$\begin{aligned} &\alpha_0 v_0(0) - \alpha_1 v_0'(0) \le x_0, \\ &\alpha_0 v_1(0) - \alpha_1 v_1'(0) = x_0. \end{aligned} \tag{3.14}$$

Thus, we have

$$\begin{aligned} &\alpha_1 v_1'(0) + \alpha_0 \varepsilon - \alpha_1 v_0'(0) \le 0, \\ &\alpha_1 [v_1'(0) - v_0'(0)] < 0, \end{aligned} \tag{3.15}$$

which is a contradiction to the hypothesis. Therefore, we have proved that $v(t) \le \Lambda v(t)$.

When the equation has the random pulses, we have

$$\begin{cases} -v_1''(t) = f(t, v_0(t), v_0'(t)) - M[v_1(t) - v_0(t)], \\ v_1(\xi_k^+) = b_k(\tau_k) v_0(\xi_k^-), \end{cases} \tag{3.16}$$

and $v_0(t)$ is the lower solution of the equation (1.4). Thus, according to the second inequality of (2.35), for every $t \in \{\xi_k\}_{k\in\mathbb{N}^+}$, we have

$$v_0(\xi_k^+) \le b_k(\tau_k) v_0(\xi_k^-) = v_1(\xi_k^+). \tag{3.17}$$

Thus, based on our discussion, we conclude that for every $t \in (\xi_k, \xi_{k+1}]$, $k = 1, 2, \cdots$,

$$v_1(t) \ge v_0(t). \tag{3.18}$$

Hence, we have proved that $v_1(t) \ge v_0(t)$ for every $t \in J$.

In the same way, we can prove that $\omega_0(t) \ge \Lambda\omega_0(t)$ for every $t \in J$.

Step(2). We prove that Λ is an increasing operator.

First of all, we take any $h_1(t)$ and $h_2(t)$, $h_1(t), h_2(t) \in PC^1[J, \mathbb{R}]$. Suppose $h_1(t) \ge h_2(t)$ for any $t \in J$. Then, we prove that $\Lambda h_1(t) \ge \Lambda h_2(t)$. Here, we use proof by contradiction. Let $h_1^*(t) = \Lambda h_1(t)$, $h_2^*(t) = \Lambda h_2(t)$. Then, we need to prove $h_1^*(t) \ge h_2^*(t)$.

When there is no random pulse, if the hypothesis is not true, then there must exist $t_0 \in J$ and $\varepsilon > 0$ such that $h_1^*(t_0) + \varepsilon = h_2^*(t_0)$ and $h_1^*(t) + \varepsilon \ge h_2^*(t)$ for every $t \in J$. If $t_0 \in J \setminus (\{0\} \bigcup \{1\})$, then we have

$$\begin{aligned} &h_1'^*(t_0) - h_2'^*(t_0) = 0, \\ &h_1''^*(t_0) - h_2''^*(t_0) \ge 0, \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} &-h_1''^*(t) = f(t, h_1(t), h_1'(t)) - M[h_1^*(t) - h_1(t)], \\ &-h_2''^*(t) = f(t, h_2(t), h_2'(t)) - M[h_2^*(t) - h_2(t)]. \end{aligned} \tag{3.20}$$

Thus

$$\begin{aligned} h_1''^*(t_0) - h_2''^*(t_0) &= f(t_0, h_2(t_0), h_2'(t_0)) - f(t_0, h_1(t_0), h_1'(t_0)) \\ &\quad + M[h_1^*(t_0) - h_2^*(t_0)] + M[h_2(t_0) - h_1(t_0)] \\ &\le M[h_1(t_0) - h_2(t_0)] + M[h_1^*(t_0) - h_2^*(t_0)] + M[h_2(t_0) - h_1(t_0)] \\ &\le -M\varepsilon < 0. \end{aligned} \tag{3.21}$$

Which is a contradiction. When $t_0 = 0$ or $t_0 = 1$, we assume that $t_0 = 0$ and $h_1^*(0) + \varepsilon = h_2^*(0)$. Then

$$h_1'^*(0) - h_2'^*(0) \geq 0. \tag{3.22}$$

Thus

$$\begin{aligned} \alpha_0 h_1^*(0) - \alpha_1 h_1'^*(0) = x_0, \\ \alpha_0 h_2^*(0) - \alpha_1 h_2'^*(0) = x_0. \end{aligned} \tag{3.23}$$

Take the difference of these equation yields

$$\alpha_1[h_1'^*(0) - h_2'^*(0)] + \alpha_0\varepsilon = 0. \tag{3.24}$$

That is to say $h_1'^*(0) - h_2'^*(0) < 0$, which is a contradiction.

When there exits random pulses, we have

$$\Lambda h_2(\xi_k^+) = h_2^*(\xi_k^+) = b_k(\tau_k)h_2(\xi_k^-) \leq b_k(\tau_k)h_1(\xi_k^-) = \Lambda h_1(\xi_k^+). \tag{3.25}$$

Then, for every $t \in (\xi_k, \xi_{k+1}]$, $k = 1, 2, \cdots$, we have

$$\Lambda h_1(t) \geq \Lambda h_2(t). \tag{3.26}$$

Thus, for every $t \in J$, the inequality $\Lambda h_1(t) \geq \Lambda h_2(t)$ is true.

Step(3). We prove that Λ is a continuous operator. That is to say, we should prove that for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that when $\| h_1(t) - h_2(t) \|_{PC^1} < \delta$, $\| \Lambda h_1(t) - \Lambda h_2(t) \|_{PC^1} < \varepsilon$.

We assume that

$$\Lambda h_1 = \sum_{k=0}^{\infty} \Big[C_1^k e^{\sqrt{M}t} + C_2^k e^{-\sqrt{M}t}\Big] I_{(\xi_k, \xi_{k+1}]}(t) + \hat{h}_1(t), \tag{3.27}$$

$$\Lambda h_2 = \sum_{k=0}^{\infty} \Big[\widetilde{C}_1^k e^{\sqrt{M}t} + \widetilde{C}_2^k e^{-\sqrt{M}t}\Big] I_{(\xi_k, \xi_{k+1}]}(t) + \hat{h}_2(t), \tag{3.28}$$

$$\begin{aligned} \mathrm{E}\|\Lambda h_1 - \Lambda h_2\|^2 \leq {} & 3\mathrm{E}\left\| e^{\sqrt{M}t} \sum_{k=0}^{\infty} (C_1^k - \widetilde{C}_1^k) I_{(\xi_k, \xi_{k+1}]}(t) \right\|^2 \\ & + 3\mathrm{E}\left\| e^{-\sqrt{M}t} \sum_{k=0}^{\infty} (C_2^k - \widetilde{C}_2^k) I_{(\xi_k, \xi_{k+1}]}(t) \right\|^2 \\ & + 3\mathrm{E} \parallel \hat{h}_1(t) - \hat{h}_2(t) \parallel^2 . \end{aligned} \tag{3.29}$$

Among them,

$$
\begin{aligned}
&\mathrm{E}\left\|\sum_{k=0}^{\infty} e^{\sqrt{M}t}(C_1^k-\widetilde{C}_1^k)I_{(\xi_k,\xi_{k+1}]}(t)\right\|^2\\
&\quad=\mathrm{E}\Bigg\|\sum_{k=0}^{\infty} e^{\sqrt{M}t}\Big[\delta_k^-(C_1-\widetilde{C}_1)-e^{-2\sqrt{M}\xi_1}\delta_k^-(C_2-\widetilde{C}_2)\\
&\quad+\sum_{n=1}^{k}\Delta_{n,k}^-b_n(\tau_n)\Big[(h_1(\xi_n)-h_2(\xi_n))-(\hat{h}_1(\xi_n)-\hat{h}_2(\xi_n))\Big]\Big]I_{(\xi_k,\xi_{k+1}]}(t)\Bigg\|^2\\
&\quad\le\frac{3}{4}\mathrm{E}\|e^{\sqrt{M}t}(C_1-\widetilde{C}_1)\|^2+\frac{3}{4}\mathrm{E}\|e^{\sqrt{M}t}e^{-2\sqrt{M}\xi_1}(C_2-\widetilde{C}_2)\|^2\\
&\quad+3\mathrm{E}\left\|e^{\sqrt{M}t}\sum_{k=0}^{\infty}\sum_{n=1}^{k}\Delta_{n,k}^-b_n(\tau_n)\Big[(h_1(\xi_n)-h_2(\xi_n))-(\hat{h}_1(\xi_n)-\hat{h}_2(\xi_n))\Big]I_{(\xi_k,\xi_{k+1}]}(t)\right\|^2,
\end{aligned}
\tag{3.30}
$$

and we have

$$A^+(t)=\widetilde{A}^+(t), \tag{3.31}$$

$$A^-(t)=\widetilde{A}^-(t). \tag{3.32}$$

$$
\begin{aligned}
&\mathrm{E}\|B^-(t)-\widetilde{B}^-(t)\|^2\\
&\quad=\mathrm{E}\left\|\sum_{k=0}^{\infty}\sum_{n=1}^{k}\Delta_{n,k}^-b_n(\tau_n)\Big[(h_1(\xi_n)-h_2(\xi_n))-(\hat{h}_1(\xi_n)-\hat{h}_2(\xi_n))\Big]I_{(\xi_k,\xi_{k+1}]}(t)\right\|^2\\
&\quad\le\sup_{t\in J}\Big[\mathrm{E}\|h_1(t)-h_2(t)\|^2+\mathrm{E}\|\hat{h}_1(t)-\hat{h}_2(t)\|^2\Big]\mathrm{E}\left\|\sum_{k=0}^{\infty}\sum_{n=1}^{k}\Delta_{n,k}^-b_n(\tau_n)I_{(\xi_k,\xi_{k+1}]}(t)\right\|^2\\
&\quad\le B_1^2\sup_{t\in J}\Big[\mathrm{E}\|h_1(t)-h_2(t)\|^2+\mathrm{E}\|\hat{h}_1(t)-\hat{h}_2(t)\|^2\Big],
\end{aligned}
\tag{3.33}
$$

$$
\begin{aligned}
&\mathrm{E}\|B^+(t)-\widetilde{B}^+(t)\|^2\\
&\quad=\mathrm{E}\left\|\sum_{k=0}^{\infty}\sum_{n=1}^{k}\Delta_{n,k}^+b_n(\tau_n)\Big[(h_1(\xi_n)-h_2(\xi_n))-(\hat{h}_1(\xi_n)-\hat{h}_2(\xi_n))\Big]I_{(\xi_k,\xi_{k+1}]}(t)\right\|^2\\
&\quad\le\sup_{t\in J}\Big[\mathrm{E}\|h_1(t)-h_2(t)\|^2+\mathrm{E}\|\hat{h}_1(t)-\hat{h}_2(t)\|^2\Big]\mathrm{E}\left\|\sum_{k=0}^{\infty}\sum_{n=1}^{k}\Delta_{n,k}^+b_n(\tau_n)I_{(\xi_k,\xi_{k+1}]}(t)\right\|^2\\
&\quad\le B_2^2\sup_{t\in J}\Big[\mathrm{E}\|h_1(t)-h_2(t)\|^2+\mathrm{E}\|\hat{h}_1(t)-\hat{h}_2(t)\|^2\Big],
\end{aligned}
\tag{3.34}
$$

and

$$\begin{aligned}
\mathrm{E}\|\hat{h}_1(t)-\hat{h}_2(t)\ \|^2 \le\ & 2\mathrm{E}\left(\left\|\frac{e^{-\sqrt{M}t}}{2\sqrt{M}}\right\|^2\left\|\int_0^t e^{\sqrt{M}s}[\sigma_1(s)-\sigma_2(s)]\mathrm{d}s\right\|^2\right)\\
&+2\mathrm{E}\left(\left\|\frac{e^{\sqrt{M}t}}{2\sqrt{M}}\right\|^2\left\|\int_0^t e^{-\sqrt{M}s}[\sigma_1(s)-\sigma_2(s)]\mathrm{d}s\right\|^2\right)\\
\le\ & 2\mathrm{E}\left(\left\|\frac{e^{-\sqrt{M}t}}{2\sqrt{M}}\right\|^2\left\|\int_0^t e^{\sqrt{M}s}[f(s,h_1(s),h_1'(s))\right.\right.\\
&\left.\left.-f(s,h_2(s),h_2'(s))+M(h_1(s)-h_2(s))]\mathrm{d}s\right\|^2\right)\\
&+2\mathrm{E}\left(\left\|\frac{e^{\sqrt{M}t}}{2\sqrt{M}}\right\|^2\left\|\int_0^t e^{-\sqrt{M}s}[f(s,h_1(s),h_1'(s))\right.\right.\\
&\left.\left.-f(s,h_2(s),h_2'(s))+M(h_1(s)-h_2(s))]\mathrm{d}s\right\|^2\right),
\end{aligned} \tag{3.35}$$

combing with (H_5), we have

$$\begin{aligned}
&\mathrm{E}\left\|\int_0^t e^{-\sqrt{M}s}[f(s,h_1(s),h_1'(s))-f(s,h_2(s),h_2'(s))+M(h_1(s)-h_2(s))]\mathrm{d}s\right\|^2\\
&\le 2\int_0^t K\Psi(s,\mathrm{E}\|h_1(s)-h_2(s)\|^2,\mathrm{E}\|h_1'(s)-h_2'(s)\|^2)\mathrm{d}s\\
&+2\int_0^t M\mathrm{E}\|h_1(s)-h_2(s)\|^2\mathrm{d}s.
\end{aligned} \tag{3.36}$$

From (3.31) and (3.32), we can easily know that $|Q|$ is dependent with $h(t)$. So, based on the above discussion, we can know when $\|\ h_1(t)-h_2(t)\ \|\to 0$, $\|\hat{h}_1(t)-\hat{h}_2(t)\|\to 0$, $\|C_1-\widetilde{C}_1\|\to 0$ and $\|C_2-\widetilde{C}_2\|\to 0$. And then ,we can get when $\|\ h_1(t)-h_2(t)\ \|\to 0$, $\mathrm{E}\ \|\ \Lambda h_1(t)-\Lambda h_2(t)\ \|^2\to 0$.

Then

$$\begin{aligned}
\Lambda' h(t)=&\sum_{k=0}^{\infty}(\sqrt{M}C_1^k e^{\sqrt{M}t}-\sqrt{M}C_2^k e^{-\sqrt{M}t})I_{(\xi_k,\xi_{k+1}]}(t)\\
&+\frac{1}{2}e^{-\sqrt{M}t}\int_0^t e^{\sqrt{M}s}\sigma(s)\mathrm{d}s+\frac{1}{2}e^{\sqrt{M}t}\int_0^t e^{-\sqrt{M}s}\sigma(s)\mathrm{d}s,
\end{aligned} \tag{3.37}$$

hence,

$$\begin{aligned}
\mathrm{E}\ \|\ \Lambda' h_1(t)-\Lambda' h_2(t)\ \|^2 \le\ & 4\sqrt{M}\mathrm{E}\left\|\sum_{k=0}^{\infty}(C_1^k-\widetilde{C}_1^k)e^{\sqrt{M}t}I_{(\xi_k,\xi_{k+1}]}(t)\right\|^2\\
&+4\sqrt{M}\mathrm{E}\left\|\sum_{k=0}^{\infty}(C_2^k-\widetilde{C}_2^k)e^{-\sqrt{M}t}I_{(\xi_k,\xi_{k+1}]}(t)\right\|^2\\
&+2\mathrm{E}\left\|e^{-\sqrt{M}t}\int_0^t e^{\sqrt{M}s}[\sigma_1(s)-\sigma_2(s)]\mathrm{d}s\right\|^2\\
&+2\mathrm{E}\left\|e^{\sqrt{M}t}\int_0^t e^{-\sqrt{M}s}[\sigma_1(s)-\sigma_2(s)]\mathrm{d}s\right\|^2,
\end{aligned} \tag{3.38}$$

which implies that

$$\lim_{\delta\to 0} \parallel \Lambda h_1(t) - \Lambda h_2(t) \parallel_{PC^1} = 0. \tag{3.39}$$

Thus, we have proved that Λ is a continuous operator.

Step(4). We prove that the functions in the set $\left\{u \in PC^1(J,\mathbb{R}) \mid u \in \Lambda(D)\right\}$ are uniformly bounded.

Because $u \in \Lambda(D)$, for any $u \in \left\{u \in PC^1(J,\mathbb{R}) \mid u \in \Lambda(D)\right\}$, there exists $h(t) \in D$ such that $u = \Lambda h(t)$,

$$\begin{aligned} \mathrm{E} \parallel \Lambda h(t) \parallel^2 \le\ & 3\mathrm{E}\left\| \sum_{k=0}^{\infty} C_1^k e^{\sqrt{M}t} I_{(\xi_k,\xi_{k+1}]}(t)\right\|^2 \\ & + 3\mathrm{E}\left\| \sum_{k=0}^{\infty} C_2^k e^{-\sqrt{M}t} I_{(\xi_k,\xi_{k+1}]}(t)\right\|^2 \\ & + 3\mathrm{E} \parallel \hat{h}(t) \parallel^2, \end{aligned} \tag{3.40}$$

suppose $r_1 = \frac{1}{M}$, $r_2 = \frac{e^{2\sqrt{M}}}{M}$ and we have

$$\begin{aligned} \mathrm{E} \parallel \hat{h}(t) \parallel^2 &\le r_1 \int_0^t M r_2 \mathrm{E}\|f(s,h(s),h'(s)) + Mh(s)\|^2 \mathrm{d}s \\ &\quad + r_2 \int_0^t M r_1 \mathrm{E}\|f(s,h(s),h'(s)) + Mh(s)\|^2 \mathrm{d}s \\ &\le 4Mr_1r_2 \int_0^t [\mathrm{E}\|f(s,h(s),h'(s))\|^2 + M\mathrm{E}\|h(s)\|^2]\mathrm{d}s \\ &\le 4Mr_1r_2 \int_0^t [\Theta(\mathrm{E}\|h(s)\|^2) + M\mathrm{E}\|h(s)\|^2]\mathrm{d}s, \end{aligned} \tag{3.41}$$

so, if $h(t)$ is the solution of the equation (1.4), we have

$$\mathrm{E}\|h(t)\|^2 \le 3(m_1 + m_2) + 12Mr_1r_2 \int_0^t \Theta(\mathrm{E}\|h(s)\|^2) + M\mathrm{E}\|h(s)\|^2 \mathrm{d}s. \tag{3.42}$$

Next, define $\phi(t) = \mathrm{E}\|h(t)\|^2$ and we have the inequality

$$\phi(t) \le 3(m_1 + m_2) + 12Mr_1r_2 \int_0^t \Theta(\phi(s)) + M\phi(s)\mathrm{d}s. \tag{3.43}$$

Define the right of the inequality (3.43) as the function $\varphi(t)$, we can get

$$\phi(t) \le \varphi(t), \qquad t \in J, \tag{3.44}$$

so,

$$\varphi'(t) \le 12Mr_1r_2[\Theta(\varphi(t)) + M\varphi(t)], \tag{3.45}$$

$$\begin{aligned} \ln\varphi(t) - \ln\varphi(0) &\le 12Mr_1r_2 \int_0^t \frac{\Theta(\varphi(s))}{\varphi(s)} + M\mathrm{d}s \\ &\le 12Mr_1r_2\left[\int_0^1 \frac{\Theta(\phi(s))}{\phi(s)}\mathrm{d}s + M\right], \end{aligned} \tag{3.46}$$

then we can easily get

$$
\begin{aligned}
\ln\phi(t) &\le 12Mr_1r_2\left[\int_0^1 \frac{\Theta(\phi(s))}{\phi(s)}\mathrm{d}s + M\right] + \ln(3m_1+3m_2) \\
&\le P\int_0^1 \frac{\Theta(\phi(t))}{\phi(t)}\mathrm{d}t + P^*.
\end{aligned}
\tag{3.47}
$$

For $\Delta^-_{n,k}$, $\Delta^+_{n,k}$, δ^-_k and δ^-_k are bounded, combining with

$$
\begin{aligned}
\mathrm{E}\parallel B^-(t)\parallel^2 &\le \mathrm{E}\left\|\sum_{k=0}^{\infty}\sum_{n=1}^{k}\Delta^-_{n,k}b_n(\tau_n)\Big[h(\xi_n)-\hat{h}(\xi_n)\Big]I_{(\xi_k,\xi_{k+1}]}(t)\right\|^2 \\
&\le \sup_{t\in J}\Big[\mathrm{E}\|h(t)\|^2+\mathrm{E}\|\hat{h}(t)\|^2\Big]\mathrm{E}\left\|\sum_{k=0}^{\infty}\sum_{n=1}^{k}\Delta^-_{n,k}b_n(\tau_n)I_{(\xi_k,\xi_{k+1}]}(t)\right\|^2,
\end{aligned}
\tag{3.48}
$$

and consider that $\Theta(s)$ satisfies the condition (H_4) and $D=[v_0(t),\omega_0(t)]$, where $v_0(t)$, $\omega_0(t)$ are all square integrable, so, $E\|\Lambda h(t)\|^2$ is bounded.

Then,

$$
\hat{h}'(t) = \frac{e^{-\sqrt{M}t}}{2}\int_0^t e^{\sqrt{M}s}\sigma(s)\mathrm{d}s + \frac{e^{\sqrt{M}t}}{2}\int_0^t e^{-\sqrt{M}s}\sigma(s)\mathrm{d}s,
\tag{3.49}
$$

and

$$
\begin{aligned}
\mathrm{E}\parallel \Lambda' h(t)\parallel^2 \le{}& 4\mathrm{E}\left\|\sum_{k=0}^{\infty}\sqrt{M}C_1^k e^{\sqrt{M}t}I_{(\xi_k,\xi_{k+1}]}(t)\right\|^2 + 4\mathrm{E}\left\|\sum_{k=0}^{\infty}\sqrt{M}C_2^k e^{-\sqrt{M}t}I_{(\xi_k,\xi_{k+1}]}(t)\right\|^2 \\
&+ 2\mathrm{E}\left\|e^{-\sqrt{M}t}\int_0^t e^{\sqrt{M}s}\sigma(s)\mathrm{d}s\right\|^2 + 2\mathrm{E}\left\|e^{\sqrt{M}t}\int_0^t e^{-\sqrt{M}s}\sigma(s)\mathrm{d}s\right\|^2.
\end{aligned}
\tag{3.50}
$$

Using the same way, we can prove that $E\parallel\Lambda' h(t)\parallel^2$ is bounded. Thus, we have proved that the functions in the set $\Big\{u(t)\in C^2(J,\mathbb{R})\mid u(t)\in\Lambda(D)\Big\}$ are uniformly bounded.

Step(5). We prove that the set $\{u(t)\mid u(t)=\Lambda h(t)\}$ is equicontinuous.

For every $u(t)\in\{u(t)\mid u(t)=\Lambda h(t)\}$, and every $t_1,t_2\in J$, $\|t_1-t_2\|<\delta$,

$$
u(t) = \sum_{k=0}^{\infty}\Big[C_1^k e^{\sqrt{M}t} + C_2^k e^{-\sqrt{M}t} + \hat{h}(t)\Big]I_{(\xi_k,\xi_{k+1}]}(t),
\tag{3.51}
$$

and,

$$\begin{aligned}
&\mathrm{E} \parallel u(t_1) - u(t_2) \parallel^2 \\
&\leq 3\mathrm{E}\left\| \sum_{k=0}^{\infty} C_1^k \Big[e^{\sqrt{M}t_1} I_{(\xi_k,\xi_{k+1}]}(t_1) - e^{\sqrt{M}t_2} I_{(\xi_k,\xi_{k+1}]}(t_2) \Big] \right\|^2 \\
&+ 3\mathrm{E}\left\| \sum_{k=0}^{\infty} C_2^k \Big[e^{-\sqrt{M}t_1} I_{(\xi_k,\xi_{k+1}]}(t_1) - e^{-\sqrt{M}t_2} I_{(\xi_k,\xi_{k+1}]}(t_2) \Big] \right\|^2 \\
&+ 3\mathrm{E} \parallel \hat{h}(t_1) - \hat{h}(t_2) \parallel^2 \\
&\leq 3\mathrm{E}\left\| \sum_{k=0}^{\infty} C_1^k \Big[e^{\sqrt{M}t_1} I_{(\xi_k,\xi_{k+1}]}(t_1) - e^{\sqrt{M}t_2} I_{(\xi_k,\xi_{k+1}]}(t_2) \Big] \right\|^2 \\
&+ 3\mathrm{E}\left\| \sum_{k=0}^{\infty} C_2^k \Big[e^{-\sqrt{M}t_1} I_{(\xi_k,\xi_{k+1}]}(t_1) - e^{-\sqrt{M}t_2} I_{(\xi_k,\xi_{k+1}]}(t_2) \Big] \right\|^2 \\
&+ 6\mathrm{E}\Big(\left\| \frac{e^{-\sqrt{M}t}}{2\sqrt{M}} \right\|^2 \left\| \int_{t_1}^{t_2} e^{\sqrt{M}s} \sigma(s) \mathrm{d}s \right\|^2 \Big) \\
&+ 6\mathrm{E}\Big(\left\| \frac{e^{\sqrt{M}t}}{2\sqrt{M}} \right\|^2 \left\| \int_{t_1}^{t_2} e^{-\sqrt{M}s} \sigma(s) \mathrm{d}s \right\|^2 \Big).
\end{aligned} \tag{3.52}$$

We suppose $t_1 \in (\xi_{k_1}, \xi_{k_1+1})$, $t_2 \in (\xi_{k_2}, \xi_{k_2+1})$, so when $\|t_1 - t_2\| < \delta$, $\|\xi_{k_1+1} - \xi_{k_2}\| < \delta$

$$\begin{aligned}
&\mathrm{E}\left\| \sum_{k=0}^{\infty} C_1^k \Big[e^{\sqrt{M}t_1} I_{(\xi_k,\xi_{k+1}]}(t_1) - e^{\sqrt{M}t_2} I_{(\xi_k,\xi_{k+1}]}(t_2) \Big] \right\|^2 \\
&\qquad = \mathrm{E}\|C_1^{k_1} e^{\sqrt{M}t_1} - C_1^{k_2} e^{\sqrt{M}t_2}\|^2.
\end{aligned} \tag{3.53}$$

So, it is easy to see that when $\delta \to 0$, $E \parallel u(t_1) - u(t_2) \parallel^2 \to 0$. Then, we consider

$$\begin{aligned}
u'(t) &= \sum_{k=0}^{\infty} (\sqrt{M} C_1^k e^{\sqrt{M}t} - \sqrt{M} C_2^k e^{-\sqrt{M}t}) I_{(\xi_k,\xi_{k+1}]}(t_1) \\
&+ \frac{1}{2} e^{-\sqrt{M}t} \int_0^t e^{\sqrt{M}s} \sigma(s) \mathrm{d}s + \frac{1}{2} e^{\sqrt{M}t} \int_0^t e^{-\sqrt{M}s} \sigma(s) \mathrm{d}s.
\end{aligned} \tag{3.54}$$

Therefore,

$$\begin{aligned}
&\mathrm{E} \parallel u'(t_1) - u'(t_2) \parallel^2 \\
&\leq 4\mathrm{E}\left\| \sum_{k=0}^{\infty} \sqrt{M} C_1^k \Big[e^{\sqrt{M}t_1} I_{(\xi_k,\xi_{k+1}]}(t_1) - e^{\sqrt{M}t_2} I_{(\xi_k,\xi_{k+1}]}(t_2) \Big] \right\|^2 \\
&+ 4\mathrm{E}\left\| \sum_{k=0}^{\infty} \sqrt{M} C_2^k \Big[e^{-\sqrt{M}t_1} I_{(\xi_k,\xi_{k+1}]}(t_1) - e^{-\sqrt{M}t_2} I_{(\xi_k,\xi_{k+1}]}(t_2) \Big] \right\|^2 \\
&+ 2\mathrm{E}\Big(\parallel e^{-\sqrt{M}t} \parallel^2 \left\| \int_{t_1}^{t_2} e^{\sqrt{M}s} \sigma(s) ds \right\|^2 \Big) \\
&+ 2\mathrm{E}\Big(\parallel e^{\sqrt{M}t} \parallel^2 \left\| \int_{t_1}^{t_2} e^{-\sqrt{M}s} \sigma(s) ds \right\|^2 \Big).
\end{aligned} \tag{3.55}$$

Using the same method, we can prove that when $\delta \to 0$, E $\| u'(t_1) - u'(t_2) \|^2 \to 0$. We have already proved that when $\|t_1 - t_2\| \to 0$, E $\| u(t_1) - u(t_2) \|^2 \to 0$. So

$$\| u(t_1) - u(t_2) \|_{PC^1} \to 0. \tag{3.56}$$

That is to say, the set $\{u(t) \mid u(t) = \Lambda h(t)\}$ is equicontinuous.

Using Lemma (2.5), we know that the set $\{u(t) \mid u(t) = \Lambda h(t)\}$ is a column compact set. It follows from Lemma (2.6) that the equation (1.4) has a solution in $D = [v_0(t), \omega_0(t)]$, where $t \in [0, 1]$. Thus, theorem (3.1) is established. □

4. Example

The main result could have many applications, now, we give an example to illustrate this theorem. We consider the following second order random impulsive differential equation with boundary value problems.

$$\begin{cases} -x''(t) = (-x(t)\sin(t) + t)^3, & t \in J', \\ x(\xi_k^+) = \dfrac{k}{3^k}\tau_k x(\xi_k^-), & k = 1, 2, \cdots, \\ x(0) - 2x'(0) = 1, \\ 2x(1) + x'(1) = 1. \end{cases} \tag{4.1}$$

Let $\tau_k \sim U(0, \frac{1}{2^k})$, then the probability density function of τ_k is

$$p(x) = \begin{cases} 2^k & x \in (0, \dfrac{1}{2^k}), \\ 0 & x \notin (0, \dfrac{1}{2^k}). \end{cases} \tag{4.2}$$

Set $\xi_0 = 0$, $\xi_{k+1} = \xi_k + \tau_{k+1}$. Obviously, $\{\xi_k\}$ is a process with independent increments and the impulsive moments ξ_k form a strictly increasing sequence. And for every $k \in \mathbb{N}$,

$$\xi_k < \xi_{k+1} \leq \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k+1}} < 1. \tag{4.3}$$

So in this example, $b_k(\tau_k) = \frac{k}{3^k}\tau_k$, τ_k is a random variable defined from Ω to $E_k = (0, d_k) = (0, \frac{1}{2^k})$. Suppose τ_i and τ_j are independent of each other when $i \neq j$, $x(\xi_k^+) = \lim_{t \to \xi_k + 0} x(t)$ and $x(\xi_k^-) = \lim_{t \to \xi_k - 0} x(t)$.

Taking $v_0(t) = 0$, $\omega_0(t) = 6\cos t$, we can easily prove that $v_0(t)$ is the lower solution and $\omega_0(t)$ is the upper solution of the equation (4.1). And for every $v_0(t) \leq x_2(t) < x_1(t) \leq \omega_0(t)$, we have

$$\begin{aligned} &[-x_1(t)\sin(t) + t]^3 - [-x_2(t)\sin(t) + t]^3 \\ &= -\sin t[x_1(t) - x_2(t)][(-x_1(t)\sin(t) + t)^2 \\ &+ (-x_1(t)\sin(t) + t)(-x_2(t)\sin(t) + t) + (-x_2(t)\sin(t) + t)^2] \\ &\geq -3[x_1(t) - x_2(t)][\omega_0(t) + 1]^2 \\ &\geq -147[x_1(t) - x_2(t)]. \end{aligned} \tag{4.4}$$

So, we can easily know that $M = 147$.

$$\sum_{n=1}^{k} \Delta^-_{n,k} b_n(\tau_n) \leq \sum_{n=1}^{k} 2^{n-1} \frac{n}{3^n} \tau_n = 3[1 - (\frac{2}{3})^k] - k(\frac{2}{3})^{k+1}. \tag{4.5}$$

So, we have proved that $\sum_{k=0}^{\infty} \Big\{ \sum_{n=1}^{k} \Delta^-_{n,k} b_n(\tau_n) \Big\} I_{(\xi_k, \xi_{k+1}]}(t) \leq 3$. That is to say $B_1 = 3$. In the same way we can prove that $B_2 = 3e^{3\sqrt{147}}$.

For every $v_0(t) \leq x_2(t) < x_1(t) \leq \omega_0(t)$, we have

$$\mathrm{E}\|[-x(t)\sin t + t]^3\|^2 \leq \mathrm{E}\|\omega_0(t)^3 + 3\omega_0(t)^2 + 3\omega_0(t) + 1\|^2 < \infty, \tag{4.6}$$

and

$$\begin{aligned} &\mathrm{E}\|[-x_1(t)\sin t + t]^3 - [-x_2(t)\sin t + t]^3\|^2 \\ &\leq \mathrm{E}\|[(x_1(t) - x_2(t))][(x_1(t)+1)^2 + (x_1(t)+1)(x_2(t)+1) + (x_2(t)+1)^2]\|^2. \end{aligned} \tag{4.7}$$

So, the equation (4.1) meets all the conditions of the theorem (3.1). We can get the solution of the equation of the equation (4.1) between $v_0(t) = 0$ and $\omega_0(t) = 6\cos t$ by constructing iterative sequences starting from v_0 an ω_0 respectively.

5. Conclusions

In this article, we study the existence of upper and lower solutions of second order random impulse equation (1.4). First, we study the solution form of the corresponding linear impulsive system (2.3) induced by system (1.4). Based on the form of the solution, we define the solution operator. Secondly, we prove that the fixed point of this operator is the solution of equation (1.4). Finally, we construct two monotone iterative sequences by the solution to (2.3). We then prove that they converge. Thus, it is concluded that there exists upper and lower solution to system (1.4). Impulsive differential equations have been studied in literature [7–10]. Random impulsive differential equations have also been discussed in the literature [12–14, 19, 27, 39]. In this paper, we extend the form of solutions to initial value problems of random impulsive differential equations to more general boundary value problems. The upper and lower methods are applied to Random impulsive differential equations and the related conclusions are generalized.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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