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Research article

Existence of three periodic solutions for a quasilinear periodic boundary value problem

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Abstract: In this paper, we prove the existence of at least three periodic solutions for the quasilinear periodic boundary value problem

$$\begin{cases} -p(x')x'' + \alpha(t)x = \lambda f(t,x) \text{ a.e. } t \in [0,1], \\ x(1) - x(0) = x'(1) - x'(0) = 0 \end{cases}$$

under appropriate hypotheses via a three critical points theorem of B. Ricceri. In addition, we give an example to illustrate the validity of our result.

Keywords: periodic solutions; three solutions; critical point; periodic boundary problem
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1. Introduction and main results

In this paper, we consider the quasilinear periodic boundary value problem

$$\begin{cases} -p(x')x'' + \alpha(t)x = \lambda f(t,x) \text{ a.e. } t \in [0,1], \\ x(1) - x(0) = x'(1) - x'(0) = 0 \end{cases} \tag{1.1}$$

where $f(t,x) : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is an L^1-Carathéodory function which is 1-periodic in t and λ is a positive parameter.

We need the following assumptions:

(Q_1) $p : \mathbf{R} \to (0,+\infty)$ is a continuous function such that there exist two positive numbers $M \geq m$ and

$$m \leq p(x) \leq M, \ \forall \ x \in \mathbf{R}. \tag{1.2}$$

(Q_2) $\alpha(t) \in C(\mathbf{R})$ is a 1-periodic positive weight function, that is, there exist $\alpha_1 \geq \alpha_0 > 0$ such that

$$\alpha_0 \leq \alpha(t) \leq \alpha_1, \ \forall\, t \in [0, 1]. \tag{1.3}$$

Since the three critical points theorem was obtained by Ricceri [11], it has been one of the most frequently applied abstract multiplicity theorems. After that, Averna-Bonanno [1] and Ricceri [12, 13] have given some general three critical points theorems due to Ricceri [11]. These three critical points theorems in [1, 11–13] are widely used to solve differential equations (see, for example [1–10, 13]). In particular, in [5], using the three critical points theorem of [11], Bonanno and Livrea have studied the existence and multiplicity of solutions for the periodic boundary value problem

$$\begin{cases} -x'' + A(t)x = \lambda b(t)\nabla G(x), \ t \in [0, T], \\ x(T) - x(0) = x'(T) - x'(0) = 0, \end{cases} \tag{1.4}$$

where $A(t) = (a_{i,j}(t))_{n\times n}$ is positive definite matrix for all $t \in [0, T], a_{i,j}(t) \in C([0, T], \mathbf{R}), G \in C^1(\mathbf{R}^n, \mathbf{R})$ and $b(t) \in L^1([0, T]) \setminus \{0\}$ that is a.e. nonnegative. Noticing that when $p(x') \equiv 1, T = 1$ and $f(t, x) = b(t)g(x)$, the n-dimensional problem (1.4) from [5] reduces to the one-dimensional problem (1.1) in case $n = 1$. Recently, in [7], using two general three critical points theorems of [1] and [12], Li et al. have studied the existence of three periodic solutions for p-Hamiltonian systems

$$\begin{cases} -(|x'|^{p-2}x')' + A(t)|x|^{p-2}x = \lambda\nabla F(t, x) + \mu\nabla G(t, x), \ t \in [0, T], \\ x(T) - x(0) = x'(T) - x'(0) = 0, \end{cases} \tag{1.5}$$

where $\lambda, \mu \in [0, +\infty), p > 1, A(t) = (a_{i,j}(t))_{n\times n}$ is positive definite matrix for all $t \in [0, T], a_{i,j}(t) \in C([0, T], \mathbf{R}), F : [0, T] \times \mathbf{R}^n \to \mathbf{R}$ is a function such that $F(\cdot, x)$ is continuous in $[0, T]$ for all $x \in \mathbf{R}^n$ and $F(t, \cdot)$ is a C^1-function in $\mathbf{R}^n$ for a.e. $t \in [0, T]$, and $G \in [0, T] \times \mathbf{R}^n \to \mathbf{R}$ is measurable in $[0, T]$ and C^1 in $\mathbf{R}^n$. Noticing that when $p = 2, T = 1, n = 1$ and $\mu = 0$, problem (1.5) becomes problem (1.1) as $p(x') \equiv 1$.

It is well known, the second order Hamiltonian systems satisfying periodic boundary conditions is motivated by celestial mechanics(see [15]). Finding periodic solutions for the system is a classic problem. The authors of [5] and [7] have proved the existence of three periodic solutions for this system. We also want to point out that, in [5] and [7], the nonlinear terms in the differential equations of the problems studied there do not depend on the derivatives of the unknown functions, i.e., $p(x') \equiv 1$, in (1.1). So we are interested in problem (1.1).

On the other hand, in [3], using the three critical points theorem of [11], Afrouzi and Heidarkhani established a three solutions result for the following quasilinear two point boundary value problem

$$\begin{cases} -x'' = \lambda h(x')f(t, x), \quad t \in [a, b], \\ x(a) = x(b) = 0, \end{cases} \tag{1.6}$$

where $f : [a, b] \times \mathbf{R} \to \mathbf{R}$ is an L^1-Carathéodory function, $h : \mathbf{R} \to (0, +\infty)$ is a continuous function and $\lambda > 0, a, b \in \mathbf{R}$, and extend the main result of [8] to problem (1.6). Inspired by the ideas of [3, 8], we discuss the existence of three periodic solutions for problem (1.1).

The aim of this paper is to establish some new criteria for problem (1.1) to have at least three periodic solutions by applying the three critical points theorem due to B. Ricceri. In addition, we give an example to illustrate the validity of our result.

Next we state our results.

Theorem 1.1. *Let* $g(y) = \int_0^y \left(\int_0^\tau p(\xi)d\xi\right) d\tau$ $(\forall y \in \mathbf{R})$ *and* $F(t,x) = \int_0^x f(t,\xi)d\xi$. *Assume that* $f : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ *is an* L^1-*Carathéodory function with* 1-*periodic in t,* $p : \mathbf{R} \to (0,+\infty)$ *satisfies* (Q_1) *and* $\alpha(t) \in C(\mathbf{R})$ *satisfies* (Q_2). *Assume that there exist three positive constants* c, d *and* s *with* $s < 2$, $g(2d) + g(-2d) + \frac{\alpha_0 d^2}{2} > c^2 \min\{m, \alpha_0\}$, *and a function* $\gamma(t) \in L^1([0,1])$ *such that*
(i) $f(t,x) \geq 0$ *for each* $(t,x) \in [0,1]\times[0,\frac{3d}{2}]$;
(ii) $F(t,x) \leq \gamma(t)(1+|x|^s)$ *for all* $x \in \mathbf{R}$ *and a.e.* $t \in [0,1]$;
(iii)

$$\sup_{(t,x)\in[0,1]\times[-c,c]} F(t,x) < \frac{c^2\min\{m,\alpha_0\}}{g(2d)+g(-2d)+4\alpha_1 d^2}\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,d)dt. \tag{1.7}$$

Then, there exist an open set $\Lambda \in [0,+\infty)$ *and a positive number* r_0 *such that for every* $\lambda \in \Lambda$, *problem* (1.1) *has at least three periodic solutions whose norms in Z are less than* r_0, *where*

$$Z = \{x : [0,1] \to \mathbf{R} | x \text{ is absolutely continuous}, x(1) = x(0), x' \in L^2([0,1])\}.$$

When $f(t,x) = f_1(t)f_2(x)$, we have the following result by using Theorem 1.1.

Theorem 1.2. *Assume that* $f_1 : \mathbf{R} \to \mathbf{R}^+$ *is a* 1-*periodic continuous function,* $f_2 : \mathbf{R} \to \mathbf{R}$ *is a continuous function,* $p : \mathbf{R} \to (0,+\infty)$ *satisfies* (Q_1) *and* $\alpha(t) \in C(\mathbf{R})$ *satisfies* (Q_2). *Assume that there exist three positive constants* c, d *and* s *with* $s < 2$, $g(2d)+g(-2d)+\frac{\alpha_0 d^2}{2} > c^2\min\{m,\alpha_0\}$, *and a positive constant* γ *such that*
(i) $f_2(x) \geq 0$ *for each* $x \in [0,\frac{3d}{2}]$;
(ii) $\int_0^x f_2(\xi)d\xi \leq \gamma(1+|x|^s)$ *for all* $x \in \mathbf{R}$ *and a.e.* $t \in [0,1]$;
(iii)

$$\begin{aligned}&\max_{t\in[0,1]} f_1(t)\cdot\max_{x\in[-c,c]}\int_0^x f_2(\xi)d\xi \\ < \;& \frac{c^2\min\{m,\alpha_0\}}{g(2d)+g(-2d)+4\alpha_1 d^2}\int_{\frac{1}{4}}^{\frac{3}{4}} f_1(t)dt\cdot\int_0^d f_2(\xi)d\xi.\end{aligned} \tag{1.8}$$

Then, there exist an open set $\Lambda \in [0,+\infty)$ *and a positive number* r_0 *such that for every* $\lambda \in \Lambda$, *problem*

$$\begin{cases} -p(x')x'' + \alpha(t)x = \lambda f_1(t)f_2(x) \text{ a.e. } t \in [0,1], \\ x(1)-x(0) = x'(1)-x'(0) = 0 \end{cases} \tag{1.9}$$

has at least three periodic solutions whose norms in Z are less than r_0.

Remark 1. If we take $p(x') \equiv 1$ in Theorem 1.2, then we can get the corresponding results of problem (1.4) as $T = 1, n = 1$. Inspecting the conditions of Theorem 3.1 in [5], it is not difficult to find that its hypothesis is different from that of our Theorem 1.2, so this is a new result.

Furthermore, when $\alpha(t)$ and $f(t,x)$ don't depend on t, we have the following autonomous version of Theorem 1.1.

Theorem 1.3. *Assume that* $f : \mathbf{R} \to \mathbf{R}$ *is a continuous function and* $p : \mathbf{R} \to (0,+\infty)$ *satisfies* (Q_1). *Assume that there exist three positive constants* c, d *and* s *with* $s < 2$, $g(2d)+g(-2d)+\frac{\alpha_0 d^2}{2} > c^2\min\{m,\alpha_0\}$, *and a positive constant* γ *such that*
(i) $f(x) \geq 0$ *for each* $x \in [0,\frac{3d}{2}]$;

(ii) $\int_0^x f(\xi)d\xi \le \gamma(1+|x|^s)$ *for all* $x \in \mathbf{R}$;
(iii)

$$\sup_{x\in[-c,c]} \int_0^x f(\xi)d\xi < \frac{c^2 \min\{m,\alpha_0\}}{2g(2d)+2g(-2d)+8\alpha_1 d^2} \int_0^d f(\xi)d\xi. \tag{1.10}$$

Then, there exist an open set $\Lambda \in [0,+\infty)$ *and a positive number* r_0 *such that for every* $\lambda \in \Lambda$, *problem*

$$\begin{cases} -p(x')x'' + \alpha x = \lambda f(x), \\ x(1)-x(0) = x'(1)-x'(0) = 0 \end{cases} \tag{1.11}$$

has at least three periodic solutions whose norms in Z *are less than* r_0, *where* $\alpha > 0$.

We postpone the proofs to the next section and turn to give an example to illustrate the validity of Theorem 1.1.

Example 1. Let $p(y) = 2-\cos y$, $\alpha(t) \equiv 1$ and

$$f(t,x) = \begin{cases} |\sin(\pi t)|e^x, & \text{if } (t,x) \in [0,1]\times(-\infty,16], \\ |\sin(\pi t)|(\sqrt{x}+e^{16}-4), & \text{if } (t,x) \in [0,1]\times(16,+\infty). \end{cases}$$

Then, we have $m = 1, M = 3, \alpha_0 = \alpha_1 = 1$,

$$g(y) = y^2 + \cos y - 1,$$

and

$$\begin{aligned} F(t,x) &= \int_0^x f(t,\xi)d\xi \\ &= \begin{cases} |\sin(\pi t)|(e^x-1), & \text{if } (t,x) \in [0,1]\times(-\infty,16], \\ |\sin(\pi t)|(\frac{2}{3}\sqrt{x^3}+xe^{16}-4x+A), & \text{if } (t,x) \in [0,1]\times(16,+\infty), \end{cases} \end{aligned}$$

where $A = -15e^{16} + \frac{61}{3}$. If $d = 16$ and $c = \sqrt{\frac{4}{\min\{m,\alpha_0\}}} = 2$, then

$$g(2d)+g(-2d)+\frac{\alpha_0 d^2}{2} = 2174 + 2\cos 32 > c^2\min\{m,\alpha_0\} = 4,$$

and

$$\begin{aligned} \sup_{(t,x)\in[0,1]\times[-2,2]} F(t,x) &\le (e^2-1) \\ &< \frac{\sqrt{2}(e^{16}-1)}{3070+2\cos 32} \\ &\le \frac{4}{3070+2\cos 32}\int_{\frac{1}{4}}^{\frac{3}{4}} |\sin(\pi t)|(e^{16}-1)dt \\ &= \frac{c^2\min\{m,\alpha_0\}}{g(2d)+g(-2d)+4\alpha_1 d^2}\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,d)dt. \end{aligned}$$

This shows that (1.7) of Theorem 1.1 holds. Further, if $\gamma(t) \equiv e^{16}$ and $s = \frac{2}{3}$, then all the assumptions of Theorem 1.1 are satisfied. Hence, there exist an open interval $\Lambda \in [0, +\infty)$ and a positive number r_0 such that for every $\lambda \in \Lambda$, problem (1.1) has at least three periodic solutions whose norms in Z are less than r_0.

Remark 2 If we take $p(x') \equiv 1$ in Theorem 1.1, then we can get the corresponding results of problem (1.5) as $T = 1, p = 2, n = 1$ and $\mu = 0$. Let $F(t, x)$ and $\alpha(t)$ be the functions in Example 1 respectively, and then after a simple calculation, it is not difficult to verify that

$$\begin{cases} -x'' + \alpha(t)x = \lambda \nabla F(t, x), \ t \in [0, 1], \\ x(1) - x(0) = x'(1) - x'(0) = 0, \end{cases}$$

satisfy the conditions of Theorem 1 of [7]. Moreover, Example 1 does not satisfy the assumptions of Theorem 1 of [7], so Theorem 1.1 represents a development of Theorem 1 of [7] in some sense.

2. Variational setting and proof of Theorems

For the reader's convenience, we first recall here the three critical points theorem of [12] and Proposition 3.1 of [13].

Lemma 2.1. ([12], Theorem 1) *Let Z be a separable and reflexive real Banach space, $\Phi : Z \to \mathbf{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on Z^* and Φ is bounded on each bounded subset of Z; $\Psi : Z \to \mathbf{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

$$\lim_{\|x\| \to +\infty} (\Phi(x) - \lambda \Psi(x)) = +\infty \tag{2.1}$$

for all $\lambda \in [0, +\infty)$, and that there exists $\beta \in \mathbf{R}$ such that

$$\sup_{\lambda \geq 0} \inf_{x \in Z} (\Phi(x) + \lambda(\beta - \Psi(x))) < \inf_{x \in Z} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\beta - \Psi(x))). \tag{2.2}$$

Then, there exist an open set $\Lambda \in [0, +\infty)$ and a positive number r_0 such that for every $\lambda \in \Lambda$, the equation

$$\Phi'(x) - \lambda \Psi'(x) = 0$$

has at least three solutions whose norms in Z are less than r_0.

Proposition 2.2. ([13], Proposition 3.1) *Let Z be a nonempty set, and Φ, Ψ two real functions on Z. Assume that there are $r > 0$ and $x_0, x_1 \in Z$ such that*

$$\Phi(x_1) = \Psi(x_1) = 0, \ \Phi(x_0) > r, \ \text{and} \ \sup_{x \in \Phi^{-1}((-\infty, r])} \Psi(x) < r \frac{\Psi(x_0)}{\Phi(x_0)}.$$

Then, for each $\beta \in \mathbf{R}$ satisfying

$$\sup_{x \in \Phi^{-1}((-\infty, r])} \Psi(x) < \beta < r \frac{\Psi(x_0)}{\Phi(x_0)},$$

one has

$$\sup_{\eta \geq 0} \inf_{x \in Z} (\Phi(x) + \lambda(\beta - \Psi(x))) < \inf_{x \in Z} \sup_{\lambda \geq 0} (\Phi(x) + \lambda(\beta - \Psi(x))).$$

Remark 3 We recall that a L^1-Carathéodory function $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ is defined by

(C_1) $t \to f(t,x)$ is measurable for every $x \in \mathbf{R}$;

(C_2) $x \to f(t,x)$ is continuous for almost every $t \in [0,T]$;

(C_3) for every $\rho > 0$ there exists a function $l_\rho \in L^1([0,T])$ such that

$$\sup_{|x|\le\rho} |f(t,x)| \le l_\rho(t) \text{ for almost every } t \in [0,T].$$

Next, we establish the variational setting for problem (1.1).

Throughout the sequel, the Sobolev space Z is defined by

$$Z = \{x : [0,1] \to \mathbf{R} | x \text{ is absolutely continuous}, x(1) = x(0), x' \in L^2([0,1])\}$$

with the norm

$$\|x\| = \left(\int_0^1 (|x'|^2 + |x|^2)dt\right)^{\frac{1}{2}}.$$

Clearly, Z is a Hilbert space and $Z^* = Z$, where Z^* is the dual space of Z.

Setting

$$g(y) = \int_0^y \left(\int_0^\tau p(\xi)d\xi\right)d\tau, \text{ for every } y \in \mathbf{R}, \tag{2.3}$$

we have

$$g'(y) = \int_0^y p(\xi)d\xi, \text{ and } g''(y) = p(y), \text{ for every } y \in \mathbf{R}. \tag{2.4}$$

Proposition 2.3. *Assume that $p(\cdot)$ satisfies* (Q_1) *, then g' is strongly monotone.*

Proof. By (Q_1) and (2.4), we have

$$g'(y) - g'(z) = \begin{cases} -\int_y^z p(\xi)d\xi \le m(y-z) < 0, & \text{if } y < z; \\ \int_z^y p(\xi)d\xi \ge m(y-z) > 0, & \text{if } y > z. \end{cases}$$

It follows that

$$(g'(y) - g'(z))(y-z) \ge m(y-z)^2$$

for all $y, z \in \mathbf{R}$. This shows that g' is strongly monotone. □

Put

$$\Phi(x) = \int_0^1 g(x'(t))dt + \frac{1}{2}\int_0^1 \alpha(t)|x(t)|^2 dt, \text{ for every } x \in Z.$$

Proposition 2.4. *Assume that $p(\cdot)$ and $\alpha(t)$ satisfy* (1.2) *and* (1.3) *respectively, then*

(1) *Φ is well-defined in Z;*

(2) *Φ is Gâteaux differentiable in Z;*

(3) *Φ' is a Lipschitzian operator;*

(4) *Φ is convex in Z.*

Proof. From (1.2) and (1.3), we have

$$\int_0^1 g(x'(t))dt = \int_0^1 \left(\int_0^{x'(t)} \left(\int_0^{\tau} p(\xi)d\xi \right) d\tau \right) dt \le M \int_0^1 \left(\int_0^{x'(t)} \tau d\tau \right) dt = \frac{M}{2} \int_0^1 |x'(t)|^2 dt$$

and

$$\Phi(x) \le \frac{M}{2} \int_0^1 |x'(t)|^2 dt + \frac{\alpha_1}{2} \int_0^1 |x(t)|^2 dt \le \frac{1}{2} \max\{M, \alpha_1\} \|x\|^2,$$

for every $x \in Z$, which implies that Φ is well-defined in Z.

Taken that $x, y \in Z$ and $\{a_n\} \in \mathbf{R}\backslash\{0\}$ with $\lim_{n\to+\infty} a_n = 0$. By the mean value theorem of differential calculus, for a.e. $t \in [0,1]$, we can see that there exist $u_n(t)$ and $v_n(t)$ such that

$$\lim_{n\to+\infty} u_n(t) = x'(t), \quad \lim_{n\to+\infty} v_n(t) = x(t)$$

and

$$\frac{g(x'(t) + a_n y'(t)) - g(x'(t))}{a_n} = g'(u_n(t))y'(t),$$
$$\frac{\frac{1}{2}\alpha(t)(x(t) + a_n y(t))^2 - \frac{1}{2}\alpha(t)(x(t))^2}{a_n} = \alpha(t)v_n(t)y(t).$$

Then, by (1.2) and (1.3), if n is large enough, we have

$$\begin{aligned} |g'(u_n(t))y'(t)| &\le M(|x'(t)| + |y'(t)|)|y'(t)|, \\ |\alpha(t)v_n(t)y(t)| &\le \alpha_1(|x(t)| + |y(t)|)|y(t)| \end{aligned} \tag{2.5}$$

for almost every $t \in [0,1]$. Again using the Lebesgue's theorem, from the continuity of g' and the arbitrariness of $\{a_n\}$, we know that Φ is Gâteaux differentiable in Z with

$$\Phi'(x)(y) = \int_0^1 g'(x'(t))y'(t)dt + \int_0^1 \alpha(t)x(t)y(t)dt. \tag{2.6}$$

If we fix $x, y, z \in Z$ with $\|z\| \le 1$. Let $u(t)$ be such that

$$|g'(x'(t)) - g'(y'(t))| \le |g''(u(t))||x'(t) - y'(t)|$$

for every $t \in [0,1]$. Thus, by the Hölder inequality, (1.2), (1.3) and (2.4), we have

$$\begin{aligned} &\left| \int_0^1 (g'(x'(t)) - g'(y'(t)))z'(t)dt + \int_0^1 \alpha(t)(x(t) - y(t))z(t)dt \right| \\ \le\ & \int_0^1 |g''(u(t))||x'(t) - y'(t)| \cdot |z'(t)|dt + \alpha_1 \int_0^1 |x(t) - y(t)| \cdot |z(t)|dt \\ \le\ & M\|x' - y'\|_{L^2} \cdot \|z'\|_{L^2} + \alpha_1 \|x - y\|_{L^2} \cdot \|z\|_{L^2} \\ \le\ & \max\{M, \alpha_1\}\|x - y\| \cdot \|z\|, \end{aligned}$$

which shows that

$$\|\Phi'(x) - \Phi'(y)\|_{Z^*} \le \max\{M, \alpha_1\}\|x - y\|,$$

for every $x, y \in Z$ and Φ' is Lipschitzian.

Finally, from (1.2) and (2.4), we know that g is convex. Noticing that $\alpha(t)x^2$ is convex in x, we have Φ is convex in Z. □

For every $x \in Z$, put

$$F(t,x) = \int_0^x f(t,\xi)d\xi, \text{ and } \Psi(x) = \int_0^1 \left(\int_0^{x(t)} f(t,\xi)d\xi\right)dt.$$

Since $f : [0,1]\times\mathbf{R} \to \mathbf{R}$ is an L^1-Carathéodory function, we know that Ψ is a well-defined and Gâteaux differentiable functional with

$$\Psi'(x)(y) = \int_0^1 f(t,x(t))y(t)dt$$

for every $x, y \in Z$. Since the embeddings $Z \hookrightarrow L^q$ $(q \geq 1)$ and $Z \hookrightarrow L^\infty$ are compact (See R. A. Adams [16]), we have $\Psi' : Z \to Z^*$ is a continuous and compact operator.

Next, we consider the functional $I : Z \to \mathbf{R}$ defined by

$$I(x) = \Phi(x) - \lambda\Psi(x) \tag{2.7}$$

for every $x \in Z$, where $\lambda > 0$. Clearly, I is Gâteaux differentiable. If $x \in Z$ is a critical point for I, we have

$$\int_0^1 g'(x'(t))y'(t)dt + \int_0^1 \alpha(t)x(t)y(t)dt = \lambda \int_0^1 f(t,x(t))y(t)dt$$

for each $y \in Z$. This implies that $g' \circ x'$ has a weak derivative which equals $\alpha(t)x(t) - \lambda f(t,x(t))$ and is thus continuous, so $g' \circ x'$ is $C^1([0,1])$. Since g' is an invertible C^1-function, it follows that x' is also in $C^1([0,1])$, hence x is in $C^2([0,1])$.

Set

$$e(t) = -g'(x'(t)) + \int_0^t \alpha(\tau)x(\tau)d\tau - \lambda \int_0^t f(\tau,x(\tau))d\tau - C$$

such that $\int_0^1 e(t)dt = 0$. Let $y(t) = \int_0^t e(\tau)d\tau$. Then $y(t) \in Z$ and $\int_0^1 |e(t)|^2 dt = 0$, that is, $e(t) = 0$ for a.e. $t \in [0,1]$. This shows that

$$\begin{aligned} -(g' \circ x')'(t) + \alpha(t)x(t) &= -g''(x')x''(t) + \alpha(t)x(t) \\ &= -p(x'(t))x''(t) + \alpha(t)x(t) = \lambda f(t,x(t)) \end{aligned}$$

for all $t \in [0,1]$. Hence we conclude that x is a solution of problem (1.1) belongs to $C^2([0,1])$.

Proof of Theorem 1.1 Consider the functional $I(x) = \Phi(x) - \lambda\Psi(x)$ for every $x \in Z$ and $\lambda > 0$. From Proposition 2.4, we know that Φ is well-defined, Gâteaux differentiable and convex functional in Z, and Φ' is a Lipschitzian operator, which implies that Φ is a sequentially weakly lower semicontinuous via Theorem 1.2 of [15]. Further, we claim that Φ admits a continuous inverse on Z^*. In fact, by (1.2), (1.3), Proposition 2.3 and (2.6), we have

$$\begin{aligned} &\langle \Phi'(x) - \Phi'(y), x - y\rangle \\ = &\int_0^1 (g'(x'(t)) - g'(y'(t)), x'(t) - y'(t))dt + \int_0^1 \alpha(t)|x(t) - y(t)|^2 dt \end{aligned}$$

$$\begin{aligned} &\geq \int_0^1 m|x'(t)-y'(t)|^2dt + \int_0^1 \alpha_0|x(t)-y(t)|^2dt \\ &\geq \min\{m,\alpha_0\}\|x-y\|^2, \end{aligned}$$

for all $x, y \in Z$, which shows that Φ' is uniformly monotone in Z. Put $y = 0$, then we have

$$\begin{aligned} &\min\{m,\alpha_0\}\|x\|^2 \leq |\langle\Phi'(x), x\rangle| \leq \|\Phi'(x)\|_{Z^*} \cdot \|x\| \\ \Rightarrow\ &\min\{m,\alpha_0\}\|x\| \leq \|\Phi'(x)\|_{Z^*}, \end{aligned}$$

which shows that Φ' is coercive in Z. Since Φ' is a Lipschitzian operator, Φ' is hemicontinuous in Z. By Theorem 26. A of [14] we can see that Φ admits a continuous inverse on Z^*. From the estimation formula in the proof of Proposition 2.4 $\Phi(x) \leq \frac{1}{2}\max\{M,\alpha_1\}\|x\|^2$, we see that Φ is bounded on each bounded subset of Z.

On the other hand, as we saw in above, $\Psi : Z \to \mathbf{R}$ a continuously G$\hat{a}$teaux differentiable functional whose G$\hat{a}$teaux derivative is compact. Based on the previous discussion of I, we know that the critical points of $I = \Phi - \lambda\Psi$ in Z are the solutions of problem (1.1). Therefore, we only need to verify that both assumptions of Lemma 2.1 are valid.

From (1.2) and (1.3), it follows that

$$\Phi(x) \geq \frac{1}{2}\min\{m,\alpha_0\}\|x\|^2,$$

for all $x \in Z$. By assumption (ii), we see that

$$\lim_{\|x\|\to+\infty} I(x) = \lim_{\|x\|\to+\infty} \Phi(x) - \lambda\Psi(x) = +\infty,$$

and so (2.1) of Lemma 2.1 holds.

Next, We want to prove the validity of (2.2) in Lemma 2.1 by using Proposition 2.2. For $x \in Z$, taking into account

$$|x(t)| \leq |\int_{t_1}^t x'(\tau)d\tau| + |x(t_1)| \leq \int_0^1 |x'(\tau)|d\tau + |x(t_1)|$$

and

$$\begin{aligned} |x(t)| &\leq \int_0^1 |x'(\tau)|d\tau + \int_0^1 |x(t_1)|dt_1 \\ &\leq \left(\int_0^1 |x'(\tau)|^2d\tau\right)^{\frac{1}{2}} + \left(\int_0^1 |x(\tau)|^2d\tau\right)^{\frac{1}{2}}, \end{aligned}$$

we have

$$\max_{t\in[0,1]} |x(t)| \leq \sqrt{2}\|x\|.$$

Thus, for each $r > 0$, we can obtain

$$\Phi^{-1}((-\infty, r]) \subseteq \left\{x \in Z | \max_{t\in[0,1]} |x(t)| \leq \sqrt{\frac{4r}{\min\{m,\alpha_0\}}}\right\},$$

which shows that

$$\sup_{x\in\Phi^{-1}((-\infty,r])}\Psi(x)\le\sup_{(t,x)\in[0,1]\times[-c,c]}F(t,x),\tag{2.8}$$

where $c=\sqrt{\frac{4r}{\min\{m,\alpha_0\}}}$.

Put $x_1=0$ and

$$x_0(t)=\begin{cases} d, & \text{if } t\in[0,\frac{1}{4}],\\ 2dt+\frac{d}{2}, & \text{if } t\in[\frac{1}{4},\frac{1}{2}],\\ -2dt+\frac{5d}{2}, & \text{if } t\in[\frac{1}{2},\frac{3}{4}],\\ d, & \text{if } t\in[\frac{3}{4},1].\end{cases}$$

Then we have $\Phi(x_1)=\Psi(x_1)=0$, $x_0\in Z$,

$$\begin{aligned}\Phi(x_0) &= \frac{1}{4}g(2d)+\frac{1}{4}g(-2d)+\frac{1}{2}\int_0^1\alpha(t)|x_0(t)|^2dt\\ &\le \frac{1}{4}g(2d)+\frac{1}{4}g(-2d)+\frac{\alpha_1}{2}\int_0^1|x_0(t)|^2dt\\ &= \frac{1}{4}g(2d)+\frac{1}{4}g(-2d)+\frac{31\alpha_1d^2}{2\times24}\\ &< \frac{1}{4}g(2d)+\frac{1}{4}g(-2d)+\alpha_1d^2,\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}\Phi(x_0) &\ge \frac{1}{4}g(2d)+\frac{1}{4}g(-2d)+\frac{\alpha_0}{2}\int_0^1|x_0(t)|^2dt\\ &= \frac{1}{4}g(2d)+\frac{1}{4}g(-2d)+\frac{31\alpha_0d^2}{2\times24}\\ &> \frac{1}{4}g(2d)+\frac{1}{4}g(-2d)+\frac{\alpha_0d^2}{8}.\end{aligned}\tag{2.10}$$

From $\min_{t\in[\frac{1}{4},\frac{3}{4}]}\{x_0(t)\}=d$, $\max_{t\in[\frac{1}{4},\frac{3}{4}]}\{x_0(t)\}=\frac{3d}{2}$ and the assumptions (i), we have

$$\begin{aligned}\Psi(x_0) &= \int_0^{\frac{1}{4}}\int_0^d f(t,\xi)d\xi dt+\int_{\frac{1}{4}}^{\frac{3}{4}}\int_0^{x_0(t)}f(t,\xi)d\xi dt+\int_{\frac{3}{4}}^1\int_0^d f(t,\xi)d\xi dt\\ &\ge \int_{\frac{1}{4}}^{\frac{3}{4}}\int_0^d f(t,\xi)d\xi dt=\int_{\frac{1}{4}}^{\frac{3}{4}}F(t,d)dt.\end{aligned}\tag{2.11}$$

Moreover, choose $r=\frac{c^2\min\{m,\alpha_0\}}{4}$, and recall from the assumptions of Theorem 1.1 that

$$g(2d)+g(-2d)+\frac{\alpha_0d^2}{2}>c^2\min\{m,\alpha_0\}.$$

From (1.7), (2.8), (2.9),(2.10) and (2.11) we obtain that

$$\Phi(x_0)>\frac{g(2d)}{4}+\frac{g(-2d)}{4}+\frac{\alpha_0d^2}{8}>r>0$$

and

$$\sup_{x\in\Phi^{-1}((-\infty,r])}\Psi(x) < \frac{c^2\min\{m,\alpha_0\}}{g(2d)+g(-2d)+4\alpha_1 d^2}\int_{\frac{1}{4}}^{\frac{3}{4}}F(t,d)dt \le r\frac{\Psi(x_0)}{\Phi(x_0)}.$$

By Proposition 2.2, we know that (2.2) of Lemma 2.1 holds. So, the proof is complete. □

Proof of Theorem 1.2 Let $f(t,x) = f_1(t)f_2(x)$. Noting that

$$\sup_{(t,x)\in[0,1]\times[-c,c]}F(t,x) = \max_{t\in[0,1]}f_1(t)\cdot\max_{x\in[-c,c]}\int_0^x f_2(\xi)d\xi,$$

it is easy to verify that the assumptions of Theorem 1.1 hold. So, the proof is complete. □

Proof of Theorem 1.3 Let $f(t,x) = f(x)$. Noting that

$$\sup_{(t,x)\in[0,1]\times[-c,c]}F(t,x) = \max_{x\in[-c,c]}\int_0^x f(\xi)d\xi,$$

it is easy to see that the assumptions of Theorem 1.1 hold. So, the proof is complete. □

3. Conclusions

Periodic solutions of Hamiltonian systems are important in applications. For second order Hamiltonian systems or p-Hamiltonian systems subject to periodic boundary conditions, there are many works reported on the existence of three periodic solutions. But the results on the multiplicity of periodic solutions of quasilinear periodic boundary value problem are very rare. In this paper, we study a quasilinear second order differential equation involving periodic boundary condition. Using a three critical points theorem obtained by B. Ricceri, we establish some new existence theorem of at least three periodic solutions for the quasilinear periodic boundary value problem (1.1) under appropriate hypotheses. In addition, we give an example to illustrate the validity of our results.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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