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Research article

Stacked book graphs are cycle-antimagic

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Abstract: A family of subgraphs of a finite, simple and connected graph G is called an *edge covering* of G if every edge of graph G belongs to at least one of the subgraphs. In this manuscript, we define the edge covering of a stacked book graph and its uniform subdivision by cycles of different lengths. If every subgraph of G is isomorphic to one graph H (say) and there is a bijection $\phi : V(G) \cup E(G) \rightarrow \{1, 2, \dots, |V(G)| + |E(G)|\}$ such that $wt_\phi(H)$ forms an arithmetic progression then such a graph is called (α, d)-H-antimagic.
In this paper, we prove super (α, d)-cycle-antimagic labelings of stacked book graphs and r subdivided stacked book graph.

Keywords: book graph; stacked book graph $SB_{(p,q)}$; r subdivided stacked book graph $SB_{(p,q)}(r)$; super (α, d)-C_4-antimagic labeling; super (α, d)-$C_{4(r+1)}$-antimagic labeling
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1. Introduction and Preliminaries

Let G be a finite, simple and connected graph. A family of subgraphs $H_1, H_2, \dots, H_t$ is called an *edge-covering* if every edge from $E(G)$ belongs to at least one of the subgraphs H_i, $i = 1, 2, \dots, t$. When $H_i, i = 1, 2, \dots, t$ is isomorphic to a given graph H, then graph G admits an H-*covering*. G is called an (α, d)-*H-antimagic* if there exists a total labeling $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, v + e\}$ with the H-weights,

$$wt_\phi(H) = \sum_{v \in V(H)} \phi(v) + \sum_{e \in E(H)} \phi(e),$$

forming an arithmetic progression $\alpha, \alpha + d, \alpha + 2d, \ldots, \alpha + (t - 1)d$, where $\alpha > 0$ and $d \geq 0$ are two integers and t is the number of all subgraphs of G isomorphic to H. Moreover, G is said to be *super* (α, d)-*H-antimagic* if $\phi(V(G)) = \{1, 2, 3, \ldots, |V(G)|\}$.

The H-supermagic graph was first introduced by Gutiérrez and Lladó in [3]. Some other results can be seen from [4, 7–11]. An (a, d)-H-antimagic labeling was introduced by Inayah *et al.* [5]. The further results on antimagic labeling are discussed in [2, 6, 16]. In [12], authors discussed the supermagic and super $(\alpha, 1)$-C_4-antimagic labeling of book graph and its disjoint union. The super (a, d)-C_3-antimagicness of a corona graph for differences $d \in \{0, 1, \ldots, 5\}$ is discussed in [1]. M. A. Umar [14] study the existence of the super cycle-antimagic labeling of ladder graphs for differences $d \in \{0, 1, \ldots, 15\}$. M.A.Umar *et al.* [13] gives the super (α, d)-C_4-antimagic labeling of book graphs for differences $d = 1, 2, \ldots, 13$.

In this research manuscript, we investigated the existence of super $(\alpha, 1)$-C_4-antimagic labeling of stacked book graphs $SB_{(p,q)}$ that can be thought of as generalization of a book graph and super $(\alpha, 1)$-$C_{4(r+1)}$-antimagic labeling of its r subdivided graph $SB_{(p,q)}(r)$.

2. Cycle-antimagic labeling of stacked book graphs

A *Cartesian product* of two graphs G_1 and G_2, denoted by $G_1 \square G_2$, is the graph with vertex set $V(G_1) \square V(G_2)$, where two vertices (u, u') and (v, v') are adjacent if and only if $u = v$ and $u'v' \in E(G_2)$ or $u' = v'$ and $uv \in E(G_1)$.

A *stacked book* graph denoted by $SB_{(p,q)}$ is defined as the cartesian product of a *star graph* S_p on $p + 1$ vertices with a *path* P_q on q vertices. i.e., $SB_{(p,q)} \cong S_{p+1} \square P_q$, where the symbol $\square$ used to denote the cartesian product of two graphs. The stacked book graph $SB_{(p,q)}$ contains $q(p + 1)$ vertices and $q(2p + 1) - (p + 1)$ edges.
The vertex set $V(SB_{(p,q)})$ have the elements $\{c^{(j)}, x_i^{(j)} : 1 \leq i \leq p, 1 \leq j \leq q\}$ and the edge set $E(SB_{(p,q)})$ have the elements

$$\cup_{j=1}^{q} \left(\cup_{i=1}^{p} \{c^{(j)} x_i^{(j)}\} \right) \cup \left(\cup_{i=1}^{p} \left(\cup_{j=1}^{q-1} \{c^{(j)} c^{(j+1)}, x_i^{(j)} x_i^{(j+1)}\} \right) \right)$$

A typical picture of stacked book graph $SB_{(p,q)}$ is given in Figure 1:

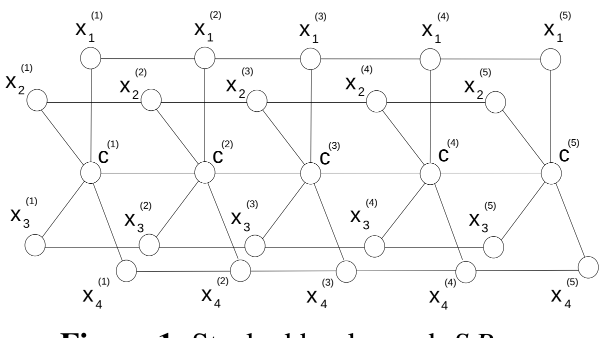

Figure 1. Stacked book graph $SB_{(4,5)}$.

Clearly stacked book graph $SB_{(p,q)}$ admits C_4-covering. It will be worth noting for $p = 1$, the stacked book graph $SB_{(1,q)}$ is a *ladder* graph $P_2 \square P_q$, for $p = 2$, the stacked book graph $SB_{(2,q)}$ is a *grid* graph $P_2 \square P_q$ and for $q = 2$, the stacked book graph $SB_{(p,2)}$ is a *book* graph $P_p \square P_2$. Ming-Ju Lee *et al.* describe the super $(\alpha, 1)$-cycle-antimagic labeling of grid graph $P_p \square P_q$ in [15]. M. A. Umar *et al.* [12] give the supermagic and super $(\alpha, 1)$-C_4-antimagic labeling of book graph and its disjoint union while [13] describes the super (α, d)-C_4-antimagic labeling of book graphs for differences $d = 1, 2, \ldots, 13$. Therefore we consider $p, q \geq 3$ in this paper.

Let $C_4^{i,j}$ be the $(i, j)^{\text{th}}$ cycle for $1 \leq i \leq p, 1 \leq j \leq q-1$ in $SB_{(p,q)}$. Each $(i, j)^{\text{th}}$-cycle $C_4^{i,j}$ in $SB_{(p,q)}$ has the vertex set $\{c^{(j)}, c^{(j+1)}, x_i^{(j)}, x_i^{(j+1)}\}$ and the edge set $\{c^{(j)}c^{(j+1)}, x_i^{(j)}x_i^{(j+1)}, c^{(j)}x_i^{(j)}, c^{(j+1)}x_i^{(j+1)}\}$.
The corresponding $C_4^{i,j}$-weight under a total labeling ϕ would be:

$$\begin{aligned} wt_\phi(C_4^{i,j}) &= \sum_{v \in V(C_4^{i,j})} \phi(v) + \sum_{e \in E(C_4^{i,j})} \phi(e). \\ &= \sum_{k=j}^{j+1} \left(\phi(x_i^{(k)}) + \phi(c^{(k)}) + \phi(c^{(k)}x_i^{(k)})\right) + \left(\phi(x_i^{(j)}x_i^{(j+1)}) + \phi(c^{(j)}c^{(j+1)})\right) \end{aligned}$$

For our convenience, throughout this paper by $i = \overline{1, p}$, we mean $i = 1, 2, \ldots, p$ and vice versa.

Theorem 1. *Let $p, q \geq 3$ be positive integers and S_p be a star on $p+1$ vertices. Then stacked book graph $SB_{(p,q)}$ admits a super $(\alpha, 1)$-C_4-antimagic labeling.*

Proof. The total labeling ϕ_0 have the form:

$$\phi_0(c^{(j)}) = \begin{cases} \lceil \frac{q}{2} \rceil + \frac{j}{2} & j \equiv 0 \pmod 2 \\ \frac{j+1}{2} & j \equiv 1 \pmod 2 \end{cases}$$

$$\begin{aligned} \phi_0(x_i^{(j)}) &= q + i + (j-1)p & i &= \overline{1,p}, j = \overline{1,q} \\ \phi_0(c^{(j)}x_i^{(j)}) &= p(2q+1-j) + q + 1 - i & i &= \overline{1,p}, j = \overline{1,q} \\ \phi_0(c^{(j)}c^{(j+1)}) &= 2q(p+1) - j & i &= \overline{1,p}, j = \overline{1,q-1} \\ \phi_0(x_i^{(j)}x_i^{(j+1)}) &= q(2p+1) + i(q-1) + j & i &= \overline{1,p}, j = \overline{1,q-1} \end{aligned}$$

Evidently,

$$\begin{aligned} \phi_0(x_i^{(j)}) + \phi(c^{(j)}x_i^{(j)}) &= 2q(p+1) + 1 \\ \phi_0(c^{(j)}) + \phi(c^{(j+1)}) + \phi(c^{(j)}c^{(j+1)}) &= \lceil \frac{q}{2} \rceil + 2q(p+1) + 1 \end{aligned} \tag{2.1}$$

and therefore,

$$wt_{\phi_0}(C_4^{i,j}) \backslash \, \phi(x_i^{(j)}x_i^{(j+1)}) = 6q(p+1) + 3 + \lceil \frac{q}{2} \rceil \tag{2.2}$$

Equations (2.1) and (2.2) gives:

$$wt_{\phi_0}(C_4^{i,j}) = 8q(p+1) + 2 + \lceil \frac{q}{2} \rceil + j + (i-1)(q-1) \tag{2.3}$$

For convenience, define $wt_{\phi_0}(\text{partial}) = 8q(p+1) + 2 + \lceil \frac{q}{2} \rceil$.
Therefore the $C_4^{i,j}$-weights are:

$$
\begin{aligned}
wt_{\phi_0}(C_4^{1,1}) &= \{wt_{\phi_0}(\text{partial})\} + 1 \\
wt_{\phi_0}(C_4^{1,2}) &= \{wt_{\phi_0}(\text{partial})\} + 2 \\
&. \\
&. \\
&. \\
wt_{\phi_0}(C_4^{1,q-1}) &= \{wt_{\phi_0}(\text{partial})\} + q - 1 \\
wt_{\phi_0}(C_4^{2,1}) &= \{wt_{\phi_0}(\text{partial})\} + 1 + (q-1) \\
wt_{\phi_0}(C_4^{2,2}) &= \{wt_{\phi_0}(\text{partial})\} + 2 + (q-1) \\
&. \\
&. \\
&. \\
wt_{\phi_0}(C_4^{2,q-1}) &= \{wt_{\phi_0}(\text{partial})\} + 2(q-1) \\
wt_{\phi_0}(C_4^{2,q-1}) &= \{wt_{\phi_0}(\text{partial})\} + 2(q-1) \\
&. \\
&. \\
&. \\
wt_{\phi_0}(C_4^{p,1}) &= \{wt_{\phi_0}(\text{partial})\} + 1 + (P-1)(q-1) \\
wt_{\phi_0}(C_4^{p,2}) &= \{wt_{\phi_0}(\text{partial})\} + 2 + (P-1)(q-1) \\
&. \\
&. \\
&. \\
wt_{\phi_0}(C_4^{p,q-1}) &= \{wt_{\phi_0}(\text{partial})\} + p(q-1) \qquad (2.4)
\end{aligned}
$$

which makes the total labeling ϕ_0 a super $(\alpha, 1)$-C_4-antimagic labeling and the proof is complete. □

3. Cycle-antimagic labeling of *r*-subdivided stacked book graph

Let G be a graph and $r \geq 1$ be a positive integer. By $G(r)$, we define r-subdivided graph of G constructed by inserting r new vertices into every edge of G.
In this way, $SB_{(p,q)}(r)$ is the r-subdivided graph of stacked book graph with the vertex set $\{c^{(j)}, x_i^{(j)}, u_r^{(i,j)} : 1 \leq i \leq p, 1 \leq j \leq q\} \cup \{\epsilon_r^{(j)}, \delta_r^{(i,j)} : 1 \leq i \leq p, 1 \leq j \leq q-1\}$ and the edge set

$$\{c^{(j)}u_1^{(i,j)}, x_i^{(j)}u_r^{(i,j)}, u_k^{(i,j)}u_{k+1}^{(i,j)} : 1 \leq i \leq p, 1 \leq j \leq q, 1 \leq k \leq r-1\} \cup$$

$$\{c^{(j)}\epsilon_1^{(j)}, c^{(j+1)}\epsilon_r^{(j)}, x_i^{(j)}\delta_1^{(i,j)}, x_i^{(j+1)}\delta_r^{(i,j)}, \epsilon_k^{(j)}\epsilon_{k+1}^{(j)}, \delta_k^{(i,j)}\delta_{k+1}^{(i,j)} : 1 \leq i \leq p, 1 \leq j \leq q-1, 1 \leq k \leq r-1\}$$

where $u_r^{(i,j)}$, $\epsilon_r^{(j)}$, $\delta_r^{(i,j)}$ are r new vertices inserted into the edges $c^{(j)}x_i^{(j)}$, $c^{(j)}c^{(j+1)}$ and $x_i^{(j)}x_i^{(j+1)}$ respectively. Clearly r-subdivided stacked book graph $SB_{(p,q)}(r)$ admits $C_{4(r+1)}$-covering.
Let $C^{i,j}_{4(r+1)}$ be the $(i,j)^{\text{th}}$-cycle for $1 \le i \le p, 1 \le j \le q-1$ in $SB_{(p,q)}(r)$. Each $(i,j)^{\text{th}}$-cycle $C^{i,j}_{4(r+1)}$ in $SB_{(p,q)}(r)$ has the vertex set

$$\{c^{(j)}, c^{(j+1)}, x_i^{(j)}, x_i^{(j+1)}\}, \cup_{k=1}^{r}\{u_k^{(i,j)}, u_k^{(i,j+1)}, \epsilon_k^{(j)}, \delta_k^{(i,j)}\}$$

and the edge set $\{u_k^{(i,j)}u_{k+1}^{(i,j)}, u_k^{(i,j+1)}u_{k+1}^{(i,j+1)}, \epsilon_k^{(j)}\epsilon_{k+1}^{(j)}, \delta_k^{(i,j)}\delta_{k+1}^{(i,j)} : 1 \le k \le r-1\} \cup$

$$\{c^{(j)}u_1^{(i,j)}, c^{(j+1)}u_1^{(i,j+1)}, x_i^{(j)}u_r^{(i,j)}, x_i^{(j+1)}u_r^{(i,j+1)}, c^{(j)}\epsilon_1^{(j)}, c^{(j+1)}\epsilon_r^{(j)}, x_i^{(j)}\delta_1^{(i,j)}, x_i^{(j+1)}\delta_r^{(i,j)}\}$$

$C^{i,j}_{4(r+1)}$-weight under a total labeling ϕ would be:

$$\begin{aligned}
wt_\phi(C^{i,j}_{4(r+1)}) &= \sum_{v\in V(C^{i,j}_{4(r+1)})} \phi(v) + \sum_{e\in E(C^{i,j}_{4(r+1)})} \phi(e). \\
&= \sum_{s=j}^{j+1}\left(\phi(c^{(s)}) + \phi(x_i^{(s)})\right) + \sum_{s=1}^{r}\left(\sum_{t=j}^{j+1}\left(\phi(u_s^{(i,t)})\right) + \phi(\epsilon_s^{(j)}) + \phi(\delta_s^{(i,j)})\right) + \\
&+ \sum_{s=j}^{j+1}\left(\phi(c^{(s)}u_1^{(i,s)}) + \phi(x_i^{(s)}u_r^{(i,s)})\right) + \phi(c^{(j)}\epsilon_1^{(j)}) + \phi(c^{(j+1)}\epsilon_r^{(j)}) + \phi(x_i^{(j)}\delta_1^{(i,j)}) + \\
&+ \phi(x_i^{(j+1)}\delta_r^{(i,j)}) + \sum_{k=1}^{r-1}\left(\phi(\epsilon_k^{(j)}\epsilon_{k+1}^{(j)}) + \phi(\delta_k^{(i,j)}\delta_{k+1}^{(i,j)})\right) + \sum_{s=j}^{j+1}\left(\sum_{k=1}^{r-1}\phi(u_k^{(i,s)}u_{k+1}^{(i,s)})\right) \\
&= \text{Partial}_1 + 2\text{Partial}_2 + \text{Partial}_3
\end{aligned} \tag{3.1}$$

where

$$\text{Partial}_1 = \phi(c^{(j)}\epsilon_1^{(j)}) + \phi(c^{(j+1)}\epsilon_r^{(j)}) + \sum_{s=j}^{j+1}\phi(c^{(s)}) + \sum_{k=1}^{r}\phi(\epsilon_k^{(j)}) + \sum_{k=1}^{r-1}\phi(\epsilon_k^{(j)}\epsilon_{k+1}^{(j)}) \tag{3.2}$$

$$\text{Partial}_2 = \phi(x_i^{(j)}) + \phi(x_i^{(j)}u_r^{(i,j)}) + \phi(c^{(j)}u_1^{(i,j)}) + \sum_{k=1}^{r}\phi(u_k^{(i,j)}) + \sum_{k=1}^{r-1}\phi(u_k^{(i,j)}u_{k+1}^{(i,j)}) \tag{3.3}$$

$$\text{Partial}_3 = \phi(x_i^{(j)}\delta_1^{(i,j)}) + \phi(x_i^{(j+1)}\delta_r^{(i,j)}) + \sum_{k=1}^{r}\phi(\delta_k^{(i,j)}) + \sum_{k=1}^{r-1}\phi(\delta_k^{(i,j)}\delta_{k+1}^{(i,j)}) + \tag{3.4}$$

Theorem 2. *Let $p, q \ge 3$ and $r \ge 1$ be positive integers and $SB_{(p,q)}(r)$ be r-subdivided stacked book graph then $SB_{(p,q)}(r)$ admits a super $(\beta, 1)$-$C_{4(r+1)}$-antimagic labeling.*

Proof. The total labeling ϕ have the form:

$$\phi(c^{(j)}) = \begin{cases} \lceil \frac{q}{2} \rceil + \frac{j}{2} & j \equiv 0 \pmod 2 \\ \frac{j+1}{2} & j \equiv 1 \pmod 2 \end{cases}$$

$$\phi(x_i^{(j)}) = j + iq \qquad i = \overline{1,p},\ j = \overline{1,q}$$

$$\begin{aligned}
\phi(\epsilon_k^{(j)}) &= pq + k(q-1) + 1 + j && j = \overline{1, q-1},\ k = \overline{1, r} \\
\phi(\delta_k^{(i,j)}) &= p + (q-1)(pk + r + i) + 1 + j && i = \overline{1, p},\ j = \overline{1, q-1},\ k = \overline{1, r} \\
\phi(u_k^{(i,j)}) &= r(p+1)(q-1) + q(pk + i) + j && i = \overline{1, p},\ j = \overline{1, q},\ k = \overline{1, r} \\
\phi(c^{(j)} u_1^{(i,j)}) &= r(p+1)(q-1) + q(p(r+2) + 2 - i) + 1 - j && i = \overline{1, p},\ j = \overline{1, q} \\
\phi(x_i^{(j)} u_r^{(i,j)}) &= r(p+1)(q-1) + q(p(r+3) + 2 - i) + 1 - j && i = \overline{1, p},\ j = \overline{1, q} \\
\phi(u_k^{(i,j)} u_{k+1}^{(i,j)}) &= r(p+1)(q-1) + q(p(2r+3-k) + 2 - i) + 1 - j && i = \overline{1, p},\ j = \overline{1, q},\ k = \overline{1, r-1} \\
\phi(c^{(j)} \epsilon_1^{(j)}) &= r(p+1)(q-1) + 2q(p(r+1) + 1) - j && j = \overline{1, q-1} \\
\phi(c^{(j+1)} \epsilon_r^{(j)}) &= r(p+1)(q-1) + q(2p(r+1) + 3) - (1 + j) && j = \overline{1, q-1} \\
\phi(\epsilon_k^{(j)} \epsilon_{k+1}^{(j)}) &= p(3qr + 2q - r) - (q-1)(k - 2r) + 3q - 1 - j && j = \overline{1, q-1},\ k = \overline{1, r-1} \\
\phi(x_i^{(j)} \delta_1^{(i,j)}) &= (q-1)[r(p+2) + i] + 2pq(r+1) + q + j && i = \overline{1, p},\ j = \overline{1, q-1} \\
\phi(x_i^{(j+1)} \delta_r^{(i,j)}) &= (q-1)[r(p+2) + 2p + 1 - i] + 2pq(r+1) + 2q - j && i = \overline{1, p},\ j = \overline{1, q-1} \\
\phi(\delta_k^{(i,j)} \delta_{k+1}^{(i,j)}) &= (q-1)[2(pr + p + r) + 1 - i - pk] + 2q[pr + p + 1] - j && i = \overline{1, p},\ j = \overline{1, q-1},\ k = \overline{1, r-1}
\end{aligned}$$

Using expressions (3.2), (3.3) and (3.4), we have:

$$\begin{aligned}
\text{Partial}_1 &= \lceil \frac{q}{2} \rceil + j + 1 + r[pq + \frac{(r+1)(q-1)}{2} + 1 + j] + \\
&+ 2r(p+1)(q-1) + 4pq(r+1) + 5q - 1 - 2j \\
&+ (r-1)[p(3qr + 2q - r) + 2r(q-1) + 3q - 1 - j - \frac{r(q-1)}{2}] \\
&= \lceil \frac{q}{2} \rceil + 1 + pqr(4 + 3r) + p(1+r)(2q - r) + r(2r+1)(q-1) + q(3r+2) \qquad (3.5)
\end{aligned}$$

$$\begin{aligned}
\text{Partial}_2 &= iq + j + 2r(p+1)(q-1) + pq(2r+5) + 2q(2-i) + 2 - 2j + \\
&+ r[r(p+1)(q-1) + qi + j + pq\frac{(r+1)}{2}] \\
&+ (r-1)[r(p+1)(q-1) + pq(2r+3) + q(2-i) + 1 - j - \frac{pqr}{2}] \\
&= r(p+1)(q-1)(1+2r) + (1+r)[2pq(1+r) + 1 + 2q] + 1 \qquad (3.6) \\
\text{Partial}_3 &= p + (q-1)(r+i) + 1 + j + pr\frac{(q-1)(r+1)}{2} - pr\frac{(q-1)(r-1)}{2} \\
&+ (q-1)[2r(p+2) + 2p + 1] + 4pq(r+1) + 3q - (r-1)j \\
&(r-1)(q-1)[2pr + 2p + 2r + 1 - i] + 2q(r-1)[pr + p + 1] \\
&= r(r+1)(q-1)(2p+3) + q(2pr^2 + 5pr + 2p + 2r + 1) + r + i(q-1) + j \\
&= \text{SPartial}_3 + i(q-1) + j \qquad (3.7)
\end{aligned}$$

where $\text{SPartial}_3 = r(r+1)(q-1)(2p+3) + q(2pr^2 + 5pr + 2p + 2r + 1) + r$

$$\begin{aligned}
wt_\phi(C_{4(r+1)}^{i,j}) &= \text{Partial}_1 + 2\text{Partial}_2 + \text{Partial}_3 \\
&= \text{Partial}_1 + 2\text{Partial}_2 + \text{SPartial}_3 + i(q-1) + j \qquad (3.8)
\end{aligned}$$

One can observe here that the $\text{Partial}_1 + 2\text{Partial}_2 + \text{SPartial}_3$ are independent of i and j. Equation (3.8) clearly shows that $wt_\phi(C^{i,j}_{4(r+1)})$ only depends on i and j. Equation (3.8) with equation (2.4) proves that $SB_{(p,q)}(r)$ admits a super $(\beta, 1)$-$C_{4(r+1)}$-antimagic labeling which completes the proof. □

4. Conclusion

In this manuscript, we prove results related to super $(\alpha, 1)$-C_4-antimagic labeling of stacked book graphs $SB_{(p,q)}$ and super $(\alpha, 1)$-$C_{4(r+1)}$-antimagic labeling of its r subdivided graph $SB_{(p,q)}(r)$. One can extend these results for other differences d and for disjoint union of stacked book graphs. One can also prove results about applications of graph labeling in data science and communication networks.

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Conflict of interests

The authors declare that there is no conflict of interests.

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