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Research article

Piecewise reproducing kernel-based symmetric collocation approach for linear stationary singularly perturbed problems

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Abstract: The aim of this paper is to develop an accurate symmetric collocation scheme for a class of linear stationary singular perturbation problems with two boundary layers. To adapt to the character of solutions, piecewise reproducing kernels is constructed. In the boundary layers intervals, inverse multiquadrics kernel function is employed. In the regular interval, exponential kernel function is used. On the basis of the piecewise reproducing kernels, a new symmetric collocation technique is presented for the considered linear stationary singular perturbation problems. Results of numerical tests illustrate that our method is easy to implement and is uniformly effective for any small ε.

Keywords: reproducing kernel method; singular perturbation problems; piecewise kernels
Mathematics Subject Classification: 65L60, 65R20

1. Introduction

In this study, we focus on the singular perturbation problems (SPPs) with turning points

$$\begin{cases} \varepsilon v''(x) + a_0(x)v'(x) + a_1(x)v(x) = g(x), \\ v(0) = \tau_0, \;\; v(1) = \tau_1, \end{cases} \tag{1.1}$$

where $0 \le x \le 1$, $0 < \varepsilon \ll 1$, $a_0(\gamma) = 0$ for a point in interval $(0, 1)$.

The points such that $a_0(x) = 0$ are called turning points. Usually, turning points problems lead to boundary layers or interior layers. And therefore, for SPPs with turning points, the numerical solutions is a more challenging job. Under certain assumptions in [1], problem (1) has a unique solution with two boundary layers at left and right end points of the interval $[0, 1]$. The goal of the work is to present an accurate symmetric collocation scheme for this class of SPPs.

SPPs arise in the physical theory of semiconductors, chemical reactions, biological sciences, etc. Solutions of SPPs, unlike regular problems, have boundary or interior layers. The derivatives of the solution in these subintervals grow without bound as $\varepsilon \to 0$. For SPPs, the use of classical numerical techniques for regular problems can not lead to accurate numerical solutions. In the last decade, there have been many effective numerical methods for SPPs [1–12]. However, most of the existing work has considered SPPs without tuning points, while much less is known on SPPs with tuning points. In Ref. [1, 4–8], some existing numerical techniques have been proposed for SPPs with turning points.

Reproducing kernel method (RKM) is a powerful technique for operator equations. Over the last decades, the method has achieved important developments in fractional differential equations, singular integral equations, fuzzy differential equations, and so on [8, 9, 13–29]. Unfortunately, the direct employ of the method can not produce ideal approximate solutions to SPPs with turning points. In [8], Geng and Qian based on the RKM and the idea of domain decomposition, present an effective method for SPPs with turning points and having two boundary layers. However, the implement of the method is complicated. In this paper, we shall construct a piecewise reproducing kernel function (RKF) based on the inverse multiquadrics kernel functions and the exponential kernel functions. Then we present a symmetric collocation method for SPPs with turning points.

2. Piecewise reproducing kernels

In this section, following the idea in [20], we will introduce a piecewise reproducing kernel function which is used to deal with the character of solutions to SPPs (1) .

Definition 2.1. *A Hilbert space H consists of complex value functions defined on E, is named a reproducing kernel Hilbert space (RKHS) if for each $x \in E$, the evaluation functional $e_x(f) = f(x)$ is continuous.*

The application of Riesz Representation Theorem for Hilbert space yields the following theorem.

Theorem 2.1. *For the above space H and each $x \in E$, there exists a unique $K_x \in H$ such that $(f, K_x) = f(x)$ for all $f \in H$.*

Definition 2.2. *Let H be a Hilbert function space defined in E. The function $K : E \times E \to R$ defined by $K(x, y) = K_x(y)$ is known as a RKF of space H.*

Definition 2.3. *For asymmetric function $K : E\times E \to R$, it is known as a positive definite kernel (PDK) if for any $n \in N$, $x_1, x_2, \ldots, x_n \in E, c_1, c_2, \ldots, c_n \in R$, $\sum\limits_{i,j=1}^{n} c_i c_j K(x_i, x_j) \geq 0$.*

Theorem 2.2. *The RKF of a RKHS is positive definite. Also, every PDK can define a unique RKHS, of which it is the RKF.*

Theorem 2.3. *Both inverse multiquadrics kernel function $IMQ(x, y) = \frac{1}{\sqrt{(x-y)^2+\gamma^2}}$ and exponential kernel function $G(x, y) = e^{\alpha xy} (\alpha > 0)$ are strictly positive definite.*

By Theorem 2.2, $IMQ_1(x, y) = IMQ(x, y), x, y \in [0, \delta_1]$ is the RKF of a RKHS $H_1[0, \delta_1]$, $G(x, y), x, y \in [\delta_1, \delta_2]$ is the RKF of a RKHS $H_2[\delta_1, \delta_2]$ and $IMQ_2(x, y) = IMQ(x, y), x, y \in [\delta_2, 1]$ is the RKF of a RKHS $H_3[\delta_2, 1]$.

Define piecewise kernel function

$$K(x,y)=\begin{cases} IMQ_1(x,y), & x,y\in[0,\delta_1),\\ G(x,y), & x,y\in[\delta_1,\delta_2],\\ IMQ_2(x,y), & x,y\in(\delta_2,1],\\ 0, & others.\end{cases} \tag{2.1}$$

and Hilbert functions space

$$H[0,1]=\{w(x)|w(x)=\begin{cases} w_1(x), & x\in[0,\delta_1),\\ w_2(x), & x,y\in[\delta_1,\delta_2],\\ w_3(x), & x,y\in(\delta_2,1].\end{cases} \quad w_1\in H_1[0,\delta_1], w_2\in H_2[\delta_1,\delta_2], w_3\in H_3[\delta_2,1]\}. \tag{2.2}$$

The inner product in this space is

$$(u,v)_H=(u_1,v_1)_{H_1}+(u_2,v_2)_{H_2}+(u_3,v_3)_{H_3}.$$

Theorem 2.4. *$H[0,1]$ is a RKHS with RKF $K(x,y)$.*

Proof. We will prove $K(x,y)$ given in (2) is a RKF. For each $v(y)\in H$, if $x\in[0,\delta_1)$,

$$(v(y),K(x,y))_H=(v_1(y),IMQ_1(x,y))_{H_1}+(v_2(y),0)_{H_2}+(v_3(y),0)_{H_3}=v_1(x).$$

Similarly, if $x\in[\delta_1,\delta_2]$,

$$(v(y),K(x,y))_H=v_2(x).$$

And if $x\in(\delta_2,1]$,

$$(v(y),K(x,y))_H=v_3(x).$$

This follows that $(v(y),K(x,y))_H=v(x)$, for $x\in[0,1]$. Therefore, $H[0,1]$ is a RKHS with RKF $K(x,y)$. □

3. Piecewise RKF-based symmetric collocation method

Setting $\delta_1=\overline{K}\varepsilon$ and $\delta_2=1-\overline{K}\varepsilon$, the interval $[0,1]$ is then divided into three subintervals $[0,\delta_1]$, $[\delta_1,\delta_2]$ and $[\delta_2,1]$, where $[0,\delta_1]$ and $[\delta_2,1]$ are the boundary layers regions, and $[\delta_1,\delta_2]$ is the regular region. Eq.(1) is equivalent to

$$\begin{cases} \varepsilon v''(x)+a_0(x)v'(x)+a_1(x)v(x)=g(x),\\ v(0)=\tau_0,\ v(\delta_1^-)-v(\delta_1^+)=0,\ v'(\delta_1^-)-v'(\delta_1^+)=0,\\ v(\delta_2^-)-v(\delta_2^+)=0,\ v'(\delta_2^-)-v'(\delta_2^+)=0,\ v(1)=\tau_1.\end{cases} \tag{3.1}$$

Define $Lv=\varepsilon v''+a_0(x)v'+a_1(x)v$, $B_1v=v(0)$, $B_2v=v(\delta_1^-)-v(\delta_1^+)$,$B_3v=v'(\delta_1^-)-v'(\delta_1^+)$, $B_4v=v(\delta_2^-)-v(\delta_2^+)$, $B_5v=v'(\delta_2^-)-v'(\delta_2^+)$, $B_6v=v(1)$.

Choosing N distinct scattered points in $[0,1]$, such as $\{x_1,x_2,\dots,x_N\}$. We construct basis functions by the piecewise RKF $K(x,y)$. Let $\psi_i(x)=L_yK(x,y)|_{y=x_i}$, $i=1,2,\dots,N$, $\psi_{N+j}(x)=B_{j,y}K(x,y)$, $j=$

$1,2\ldots,6$, here $B_{j,y}$ denotes operator B_j acts on the function of y. The collocation solution $v_N(x)$ for Eq. (1) may be written as

$$v_N(x) = \sum_{i=1}^{N+6} \beta_i \psi_i(x), \tag{3.2}$$

where $\{\beta_i\}_{i=1}^{N+6}$ are constants to be determined.

We require $v_N(x)$ to satisfy the governing differential equation at all centers x_j, $j = 1,2,\ldots,N$, i.e.

$$Lv_N(x_k) = \sum_{i=1}^{N+6} \beta_i L\psi_i(x_k) = g(x_k),\ \ k = 1,2,\ldots,N. \tag{3.3}$$

In addition, let $v_N(x)$ satisfy boundary conditions

$$\begin{cases} B_1 v_N(x) = \sum\limits_{i=1}^{N+6} \beta_i B_1\psi_i(x) = \tau_0, \\ B_2 v_N(x) = \sum\limits_{i=1}^{N+6} \beta_i B_2\psi_i(x) = 0, \\ B_3 v_N(x) = \sum\limits_{i=1}^{N+6} \beta_i B_3\psi_i(x) = 0, \\ B_4 v_N(x) = \sum\limits_{i=1}^{N+6} \beta_i B_4\psi_i(x) = 0, \\ B_5 v_N(x) = \sum\limits_{i=1}^{N+6} \beta_i B_5\psi_i(x) = 0, \\ B_6 v_N(x) = \sum\limits_{i=1}^{N+6} \beta_i B_6\psi_i(x) = \tau_1. \end{cases} \tag{3.4}$$

System (6) and (7) of linear equations can be reduced to the matrix form:

$$A\beta = \mathbf{g}, \tag{3.5}$$

where $\beta = (\beta_1,\beta_2,\ldots,\beta_{N+6})^\top$, $\mathbf{g} = (g(x_1), g(x_2),\ldots, g(x_N), \tau_0, 0, 0, 0, 0, \tau_1)^\top$. Clearly, A is a symmetric matrix.

Define functionals $\lambda_i = \delta_{x_i} \circ L, i = 1,2,\ldots,N$, $\lambda_{N+j} = B_j$, $j = 1,2,\ldots,6$, i.e. $\lambda_i(v) = Lv(x_i), i = 1,2,\ldots,N$, $\lambda_{N+j}(v) = B_j(v)$, $j = 1,2,\ldots,6$.

Theorem 3.1. *If the functionals $\lambda_i, i = 1,2,\ldots,N+6$ are linearly independent, then the matrix A is invertible. That is, system (8) has a unique solution.*

The present method is outlined as follows:

Step 1: Construct basis functions $\psi_i(x)$ using the kernel function $K(x,y)$, operator L and $B_j, j = 1,2,\ldots,6$.

Step 2: Set collocation approximation $v_N(x) = \sum\limits_{i=1}^{N+6} \beta_i\psi_i(x)$.

Step 3: Let v_N satisfies the equation and boundary conditions in (4).

Step 4: Obtain the coefficients in v_N and then get approximate solution v_N.

Theorem 3.2. *If $a_0(x), a_1(x)$ and $g(x) \in C^2[0,1]$, then*

$$\| v(x) - v_N(x) \|_\infty = \max_{x\in[0,1]} | v(x) - v_N(x) | \leq c\, h^2,$$

where $c > 0$ is a real number and $h = \max_{1\leq i\leq N-1} | x_i - x_{i+1} |$.

Proof. Because $v_N(x) \in H[0,1]$ and satisfies formula (8), we have

$$v_N(x) \in C^1[0,1].$$

Although $v_N(x)$ only belongs to C^1 on the entire interval $[0,1]$, from the assumptions of the Theorem and the regularity of functions in $H_1[0,\delta_1]$, $H_2[\delta_1,\delta_2]$ and $H_3[\delta_2,1]$, we have

$$Lv_N(x) - g(x) \in C^2$$

on three subintervals $[0,\delta_1]$, $[\delta_1,\delta_2]$ and $[\delta_2,1]$. respectively.
By applying [28], $| v(x) - v_N(x) | \leq c\, h^2$ on three subintervals. Hence,

$$\| v(x) - v_N(x) \|_\infty = \max_{x\in[0,1]} | v(x) - v_N(x) | \leq c\, h^2.$$

□

4. Numerical experiments

Define the maximum absolute errors $E_N = \| v - v_N \|_\infty \triangleq \max_{x\in[0,1]} | v(x) - v_N(x) |$.

Example 1

We applied our method to the SPPs given by [1,9]

$$\varepsilon v''(x) - 2(2x-1)v'(x) - 4v(x) = 0, \quad x \in (0,1), \tag{4.1}$$

with the boundary conditions $v(0) = v(1) = 1$. Its true solution is $v(x) = e^{\frac{-2x(x-1)}{\varepsilon}}$. Choosing $\delta_1 = \overline{K}\varepsilon,\ \delta_2 = 1-\overline{K}\varepsilon,\ \overline{K} = 10, \gamma = 2\varepsilon,\ \alpha = 1,\ x_i = \begin{cases} \frac{\delta_1 * (i-1)}{N_1}, i = 1,2,\ldots,N_1, \\ \delta_1 + \frac{(\delta_2-\delta_1)*(i-N_1)}{N_2-N_1+1}, i = N_1+1,\ldots,N_2, \\ \delta_2 + \frac{(1-\delta_2)*(i-N_2)}{N-N_2}, i = N_2+1,\ldots,N, \end{cases}$ $(N_1 = 59, N_2 = 69, N = 128)$ in the present method, Table 1 shows the maximum absolute errors compared with the methods in [1,9]. If the zero mean normally distributed white noise with standard derivation of 0.01 is added to the right function $g(x) = 0$, the obtained errors are depicted in Figure 1. It is found from Table and Figure 1 that our novel approach has higher accuracy and is robust.

Table 1. Example 1: Comparison of errors E_N.

ε	$N = 128$([1])	$N = 128$([9])	$N = 128$(Our method)
2^{-8}	3.68×10^{-3}	3.33×10^{-4}	3.88×10^{-5}
2^{-10}	8.26×10^{-3}	4.70×10^{-3}	3.86×10^{-5}
2^{-12}	8.01×10^{-3}	8.20×10^{-2}	3.85×10^{-5}
2^{-16}	–	–	3.85×10^{-5}
2^{-20}	–	–	3.85×10^{-5}
2^{-24}	–	–	3.85×10^{-5}

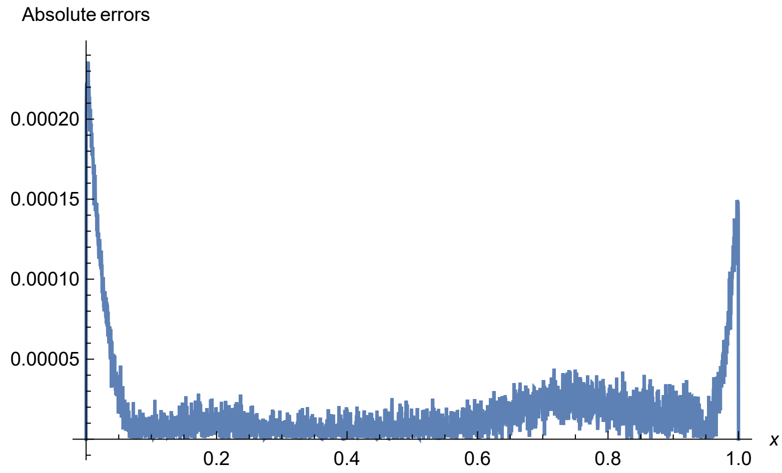

Figure 1. Graphics of the absolute errors on the entire interval for $\varepsilon = 2^{-12}$ with noise data.

Example 2 Next, we consider the SPPs given by [1, 9]

$$\begin{cases} \varepsilon v''(x) - 2(2x-1)v'(x) - 4v(x) = g(x), \quad x \in (0,1), \\ v(0) = v(1) = 1, \end{cases} \tag{4.2}$$

where $g(x) = 4(4x-1)$. Its exact solution $v(x) = -2x + \left(\frac{\operatorname{erf}\left(\frac{2x-1}{\sqrt{2}\sqrt{\varepsilon}}\right)}{\operatorname{erf}\left(\frac{1}{\sqrt{2}\sqrt{\varepsilon}}\right)} + 2 \right) e^{-\frac{2(1-x)x}{\varepsilon}}$. Taking $\delta_1 = \overline{K}\varepsilon$, $\delta_2 = 1 - \overline{K}\varepsilon$, $\overline{K} = 10$, $\gamma = 2\varepsilon$, $\alpha = 1$, $x_i = \begin{cases} \frac{\delta_1 * (i-1)}{N_1}, i = 1, 2, \ldots, N_1, \\ \delta_1 + \frac{(\delta_2 - \delta_1)*(i-N_1)}{N_2 - N_1 + 1}, i = N_1 + 1, \ldots, N_2, \\ \delta_2 + \frac{(1-\delta_2)*(i-N_2)}{N-N_2}, i = N_2 + 1, \ldots, N, \end{cases}$ $(N_1 = 59, N_2 = 69, N = 128)$ in the present method, Table 2 shows the maximum absolute errors compared with the methods in [1, 9]. Figure 2 shows the absolute errors on the left boundary layer for $\varepsilon = 10^{-6}$. Figure 3 shows the absolute errors on the right boundary layer for $\varepsilon = 10^{-6}$. Figure 4 shows the absolute errors on the regular interval for $\varepsilon = 10^{-6}$.

Table 2. Example 2: Comparison of errors E_N.

ε	$N = 256$([1])	$N = 256$([9])	$N = 128$(Our method)
2^{-8}	1.33×10^{-3}	2.26×10^{-4}	1.29×10^{-5}
2^{-12}	1.22×10^{-3}	5.02×10^{-4}	1.28×10^{-5}
2^{-16}	–	–	1.28×10^{-5}
2^{-20}	–	–	1.28×10^{-5}
2^{-24}	–	–	1.28×10^{-5}

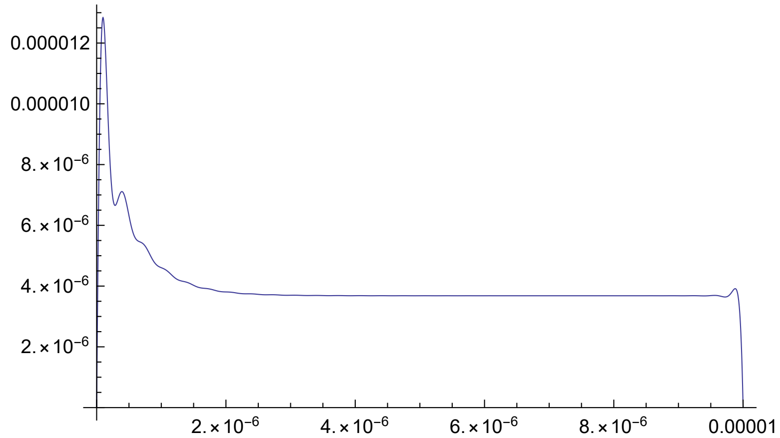

Figure 2. Graphics of the absolute errors on the left boundary layer for $\varepsilon = 10^{-6}$.

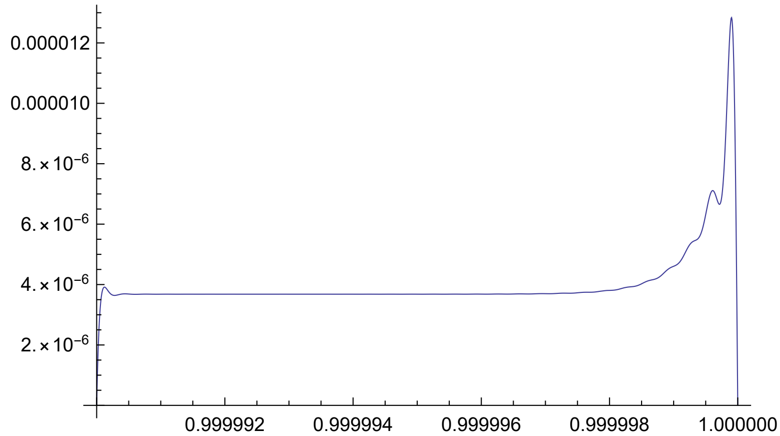

Figure 3. Graphics of the absolute errors on the right boundary layer for $\varepsilon = 10^{-6}$.

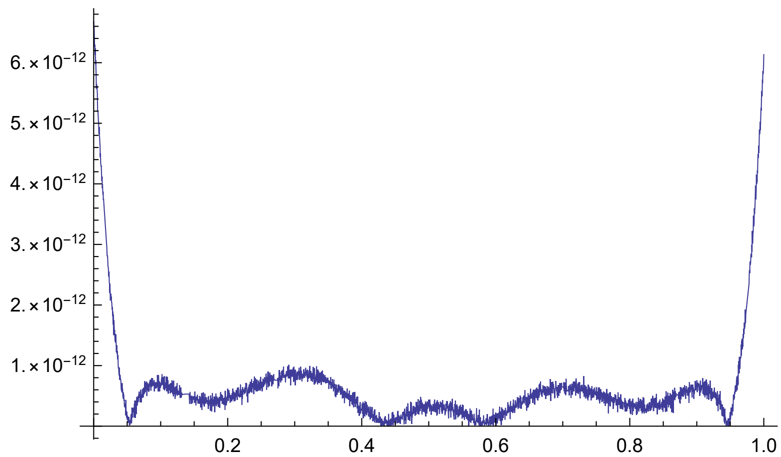

Figure 4. Graphics of the absolute errors on the regular interval for $\varepsilon = 10^{-6}$.

5. Conclusions

By using the inverse multiquadrics RKF and the exponential RKF, a new piecewise RKF is builded. Take advantage of the piecewise RKFs, a new symmetric collocation scheme is presented for linear stationary SPPs. The significant advantage of the present novel technique is that it can adapt to the character of boundary layers of the solution to SPPs. Our method can give accurate approximation to the solutions of the considered SPPs on both the boundary intervals and the regular interval. The results of numerical tests illustrate that the method is uniformly effective for any small ε.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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