Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 5903–5915.
DOI:10.3934/math.2020378
Received: 15 May 2020
Accepted: 13 July 2020
Published: 15 July 2020

*Research article*

# Stability of an $n$-variable mixed type functional equation in probabilistic modular spaces

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**Abstract:** In this research paper, we solve a new $n$-variable mixed type additive-quadratic functional equation and prove the Ulam stability of the new $n$-variable mixed type additive-quadratic functional equation in probabilistic modular spaces by using fixed point method.

**Keywords:** Ulam stability; functional equation; probabilistic modular space; fixed point method
**Mathematics Subject Classification:** 39B72, 68U10, 94A08, 47H10, 39B52

## 1. Introduction and preliminaries

A mapping $f : U \to V$ is called additive if $f$ satisfies the Cauchy functional equation

$$f(x + y) = f(x) + f(y) \tag{1.1}$$

for all $x, y \in U$. It is easy to see that the additive function $f(x) = ax$ is a solution of the functional equation (1.1) and every solution of the functional equation (1.1) is said to be an additive mapping. A mapping $f : U \to V$ is called quadratic if $f$ satisfies the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.2}$$

for all $x, y \in U$. A mapping $f : U \to V$ is quadratic if and only if there exist a symmetric biadditive mapping $B : U^2 \to V$ such that $f(x) = B(x, x)$ and this $B$ is unique, refer (see [1, 10]). It is easy to see that the quadratic function $f(x) = ax^2$ is a solution of the functional equation (1.2) and every solution of the functional equation (1.2) is said to be a quadratic mapping.

Mixed type functional equation is an advanced development in the field of functional equations. A single functional equation has more than one nature is known as mixed type functional equation. Further, in the development of mixed type functional equations, atmost only few functional equations have been obtained by many researchers (see [3, 6, 9, 11, 12, 16, 17, 22, 24]).

Let $G$ be a group and $H$ be a metric group with a metric $d(.,.)$. Given $\epsilon > 0$ does there exists a $\delta > 0$ such that if a function $f : G \to H$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then is there exist a homomorphism $a : G \to H$ with $d(f(x), a(x)) < \epsilon$ for all $x \in G$? This problem for the stability of functional equations was raised by Ulam [23] and answerd by Hyers [7]. Later, it was developed by Rassias [20], Rassias [18, 21] and Gv̆uruta [5].

The probabilistic modular space was introduced by Nourouzi [14] in 2007. Later, it was developed by K. Nourouzi [4, 15].

**Definition 1.1.** Let $V$ be a real vector space. If $\mu : V \to \Delta$ fulfills the following conditions

(i) $\mu(v)(0) = 0$,
(ii) $\mu(v)(t) = 1$ for all $t > 0$, if and only if $v = \gamma$ ($\gamma$ is the null vector in $V$),
(iii) $\mu(-v)(t) = \mu(v)(t)$,
(iv) $\mu(au + bv)(r + t) \geq \mu(v)(r) \wedge \mu(v)(t)$

for all $u, v \in V$, $a, b, r, t \in \mathbb{R}^+$, $a + b = 1$, then a pair $(V, \mu)$ is called a probabilistic modular space and $(V, \mu)$ is $b$-homogeneous if $\rho(av)(t) = \mu(v)(\frac{t}{|a|^b})$ for all $v \in V, t > 0$, $a \in \mathbb{R}\backslash\{0\}$. Here $\Delta$ is $g : \mathbb{R} \to \mathbb{R}^+$ the set of all nondecreasing functions with $\inf_{t\in\mathbb{R}} g(t) = 0$ and $\sup_{t\in\mathbb{R}} g(t) = 1$. Also, the function min is denoted by $\wedge$.

**Example 1.2.** Let $V$ be a real vector space and $\mu$ be a modular on $X$. Then a pair $(V, \mu)$ is a probabilistic modular space, where

$$\mu(v)(t) = \begin{cases} \frac{t}{t+\rho(x)}, & t > 0\ , v \in V \\ 0, & t \leq 0\ , v \in V. \end{cases}$$

In 2002, Rassias [19] studied the Ulam stability of a mixed-type functional equation

$$g\left(\sum_{i=1}^{3} x_i\right) + \sum_{i=1}^{3} g(x_i) = \sum_{1\leq i\leq j\leq 3} g(x_i + x_j).$$

Later, Nakmalachalasint [13] generalized the above functional equation and obtained an $n$-variable mixed-type functional equation of the form

$$g\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} g(x_i) = \sum_{1\leq i\leq j\leq n} g(x_i + x_j)$$

for $n > 2$ and investigated its Ulam stability.

In 2005, Jun and Kim [8] introduced a generalized $AQ$-functional equation of the form

$$g(x + ay) + ag(x - y) = g(x - ay) + ag(x + y)$$

for $a \neq 0, \pm 1$.

In 2013, Zolfaghari *et al.* [25] investigated the Ulam stability of a mixed type functional equation in probabilistic modular spaces. In the same year, Cho *et al.* [2] introduced a fixed point method to prove the Ulam stability of $AQC$-functional equations in $\beta$-homogeneous probabilistic modular spaces.

Motivated from the notion of probabilistic modular spaces and by the mixed type functional equations, we introduce a new mixed type functional equation satisfied by the solution $f(x) = x + x^2$ of the form

$$\begin{aligned}\sum_{i=1,\ j=i+1}^{n-1} &\Big(f(2x_i + x_j)\Big) + f(2x_n + x_1) \\ &- 2\left[\sum_{i=1,\ j=i+1}^{n-1} \Big(f(x_i + x_j)\Big) + f(x_n + x_1)\right] = \sum_{i=1}^{n} f(-x_i),\end{aligned} \tag{1.3}$$

for $n \in \mathbb{N}$ and investigate its Ulam stability in probabilistic modular spaces.

This paper is organized as follows: In Section 1, we provide a necessary introduction of this paper. In Sections 2 and 3, we obtain the general solution of the functional equation (1.3) in even case and in odd case, respectively. In Sections 4 and 5, we investigate the Ulam stability of (1.3) in probabilistic modular space by using fixed point theory for even and odd cases, respectively and the conclusion is given in Section 6.

## 2. General solution of a mixed type functional equation for even case

Let $U$ and $V$ be real vector spaces. In this section, we obtain the general solution of a mixed type functional equation (1.3) for even case of the form

$$\begin{aligned}\sum_{i=1,\ j=i+1}^{n-1} &\Big(f(2x_i + x_j)\Big) + f(2x_n + x_1) \\ &- 2\left[\sum_{i=1,\ j=i+1}^{n-1} \Big(f(x_i + x_j)\Big) + f(x_n + x_1)\right] = \sum_{i=1}^{n} f(x_i)\end{aligned} \tag{2.1}$$

for $n \in \mathbb{N}$.

**Theorem 2.1.** *Let $f : U \to V$ satisfy the functional equation (2.1). If $f$ is an even mapping, then $f$ is quadratic.*

*Proof.* Assume that $f : U \to V$ is even and satisfies the functional equation (2.1). Replacing $(x_1, x_2, \ldots, x_n)$ by $(0, 0, \ldots, 0)$ and by $(x_1, 0, \ldots, 0)$ in (2.1), we obtain $f(0) = 0$ and

$$f(2x_1) = 4f(x_1) \tag{2.2}$$

for all $x_1 \in U$, respectively. Again, replacing $(x_1, x_2, x_3, \ldots, x_n)$ by $(x_1, x_1, 0, \ldots, 0)$ in (2.1), we have

$$f(3x_1) = 9f(x_1) \tag{2.3}$$

for all $x_1 \in U$. Now, from (2.2) and (2.3), we get

$$f(nx_1) = n^2 f(x_1),$$

for all $x_1 \in U$. Replacing $(x_1, x_2, x_3, x_4, \ldots, x_n)$ by $(x_1, x_2, x_2, 0, \ldots, 0)$ in (2.1), we obtain

$$f(2x_1 + x_2) + f(x_1 + 2x_2) = 4f(x_1 + x_2) + f(x_1) + f(x_2) \tag{2.4}$$

for all $x_1, x_2 \in U$. Replacing $(x_1, x_2, x_3, x_4, \ldots, x_n)$ by $(x_1, x_2, 0, 0, \ldots, 0)$ in (2.1), we get

$$f(2x_1 + x_2) + f(x_2) = 2f(x_1 + x_2) + 2f(x_1) \tag{2.5}$$

for all $x_1, x_2 \in U$. Replacing $x_2$ by $-x_2$ in (2.5), using the evenness of $f$ and again adding the resultant to (2.5), we get

$$f(2x_1 + x_2) + f(2x_1 - x_2) + 2f(x_2) = 2f(x_1 + x_2) + 2f(x_1 - x_2) + 4f(x_1) \tag{2.6}$$

for all $x_1, x_2 \in U$. Replacing $(x_1, x_2)$ by $(x_1 + x_2, x_1 - x_2)$ in (2.6), we get

$$f(3x_1 + x_2) + f(x_1 + 3x_2) = 4f(x_1 + x_2) - 2f(x_1 - x_2) + 8f(x_1) + 8f(x_2) \tag{2.7}$$

for all $x_1, x_2 \in U$. Letting $(x_1, x_2)$ by $(x_1, x_1 + x_2)$ in (2.5), we get

$$f(3x_1 + x_2) + f(x_1 + x_2) = 2f(2x_1 + x_2) + 2f(x_1) \tag{2.8}$$

for all $x_1, x_2 \in U$. Replacing $x_1$ by $x_2$ and $x_2$ by $x_1$ in (2.8), we have

$$f(x_1 + 3x_2) + f(x_1 + x_2) = 2f(x_1 + 2x_2) + 2f(x_2) \tag{2.9}$$

for all $x_1, x_2 \in U$. Now, adding (2.8) and (2.9), we obtain

$$\begin{aligned} &f(3x_1 + x_2) + f(x_1 + 3x_2) + 2f(x_1 + x_2) \\ &\quad = 2f(2x_1 + x_2) + 2f(x_1 + 2x_2) + 2f(x_1) + 2f(x_2) \end{aligned} \tag{2.10}$$

for all $x_1, x_2 \in U$. Using (2.4), (2.7) and (2.10), we obtain (1.2). Hence the mapping $f$ is quadratic. □

## 3. General solution of a mixed type functional equation for odd case

Let $U$ and $V$ be real vector spaces. In this section, we obtain the general solution of a mixed type functional equation (1.3) for even case of the form

$$\begin{aligned} &\sum_{i=1,\quad j=i+1}^{n-1} \Big(f(2x_i + x_j)\Big) + f(2x_n + x_1) \\ &\quad - 2\left[\sum_{i=1,\quad j=i+1}^{n-1} \Big(f(x_i + x_j)\Big) + f(x_n + x_1)\right] = -\sum_{i=1}^{n} f(x_i) \end{aligned} \tag{3.1}$$

for $n \in \mathbb{N}$.

**Theorem 3.1.** *Let $f : U \to V$ satisfy the functional equation (3.1). If $f$ is an odd mapping, then $f$ is additive.*

*Proof.* Assume that $f$ is odd and satisfies the functional equation (3.1). Replacing $(x_1, x_2, \ldots, x_n)$ by $(0, 0, \ldots, 0)$ and $(x_1, 0, \ldots, 0)$ in (3.1), we obtain $f(0) = 0$ and

$$f(2x_1) = 2f(x_1) \tag{3.2}$$

for all $x_1 \in U$, respectively. Again, replacing $(x_1, x_2, x_3, \ldots, x_n)$ by $(x_1, x_1, 0, \ldots, 0)$ in (3.1), we have

$$f(3x_1) = 9f(x_1) \tag{3.3}$$

for all $x_1 \in U$. Now, from (3.2) and (3.3), we get

$$f(nx_1) = nf(x_1)$$

for all $x_1 \in U$. Replacing $(x_1, x_2, x_3, x_4, \ldots, x_n)$ by $(x_1, x_2, 0, 0, \cdots, 0)$ in (3.1), we get

$$f(2x_1 + x_2) - 2f(x_1 + x_2) = -f(x_2) \tag{3.4}$$

for all $x_1, x_2 \in U$. Replacing $x_2$ by $-x_2$ in (3.4), using the oddness of $g$ and again adding the resultant to (3.4), we get

$$f(2x_1 + x_2) + f(2x_1 - x_2) = 2f(x_1 + x_2) + 2f(x_1 - x_2) \tag{3.5}$$

for all $x_1, x_2 \in U$. Replacing $(x_1, x_2)$ by $(x_1 + x_2, x_1 - x_2)$ in (3.5), we get

$$f(3x_1 + x_2) + f(x_1 + 3x_2) = 4f(x_1) + 4f(x_2) \tag{3.6}$$

for all $x_1, x_2 \in U$. Replacing $x_1$ by $x_2$ and $x_2$ by $x_1$ in (3.4), we have

$$f(2x_1 + x_2) + f(x_1 + 2x_2) = 4f(x_1 + x_2) - f(x_1) - f(x_2) \tag{3.7}$$

for all $x_1, x_2 \in U$. Replacing $(x_1, x_2)$ by $(x_1, x_1 + x_2)$ in (3.4), we get

$$f(3x_1 + x_2) - 2f(2x_1 + x_2) = -f(x_1 + x_2) \tag{3.8}$$

for all $x_1, x_2 \in U$. Replacing $x_1$ by $x_2$ and $x_2$ by $x_1$ in (3.8) and adding the resultant equation to (3.8), we obtain

$$f(3x_1 + x_2) + f(x_1 + 3x_2) - 2f(2x_1 + x_2) - 2f(x_1 + 2x_2) = -2f(x_1 + x_2) \tag{3.9}$$

for all $x_1, x_2 \in U$. Using (3.6), (3.7) and (3.9), we obtain (1.1). Hence the mapping $f$ is additive. □

## 4. Stability of a mixed type functional equation for even case

In this section, we prove the Ulam stability of the $n$-variablel mixed type functional equation (1.3) for even case in probabilistic modular spaces (PM-spaces) by using fixed point technique.

For a mapping $f : M \to (V, \mu)$, consider

$$\begin{aligned} S_e(x, y) = &\sum_{i=1,\ j=i+1}^{n-1} \Big( f(2x_i + x_j) \Big) + f(2x_n + x_1) \\ &- 2\left[ \sum_{i=1,\ j=i+1}^{n-1} \Big( f(x_i + x_j) \Big) + f(x_n + x_1) \right] - \sum_{i=1}^{n} f(x_i) \end{aligned}$$

for $n \in \mathbb{N}$.

**Theorem 4.1.** *Let $M$ be a linear space and $(V,\mu)$ be a $\mu$-complete $b$-homogeneous PM-space. Suppose that a mapping $f : M \to (V,\mu)$ satisfies an inequality*

$$\mu(S_e(x_1, x_2, \ldots, x_n)) \geq \rho(x_1, x_2, \ldots, x_n)(t) \tag{4.1}$$

*for all $x_1, x_2, \ldots, x_n \in M$ and a given mapping $\rho : M \times M \to \Delta$ such that*

$$\rho(2^a x, 0, \ldots, 0)(2^{2ba}Nt) \geq \rho(x, 0, \ldots, 0)(t) \tag{4.2}$$

*for all $x \in M$ and*

$$\rho(2^{am}x_1, 2^{am}x_2, \ldots, 2^{am}x_n)(2^{2bam}t) = 1 \tag{4.3}$$

*for all $x_1, x_2, \ldots, x_n \in M$ and a constant $0 < N < \frac{1}{2^b}$. Then there exists a unique quadratic mapping $T : M \to (V,\mu)$ satisfying (2.1) and*

$$\mu(T(x) - f(x))\left(\frac{t}{2^{2b}N^{\frac{a-1}{2}}(1-2^bN)}\right) \geq \rho(x, 0, \ldots, 0)(t) \tag{4.4}$$

*for all $x \in M$.*

*Proof.* Replacing $(x_1, x_2, \ldots, x_n)$ by $(x, 0, \ldots, 0)$ in (4.1), we obtain

$$\mu(f(2x) - 2^2 f(x))(t) \geq \rho(x, 0, \ldots, 0)(t) \tag{4.5}$$

for all $x \in M$. This implies

$$\begin{aligned}\mu\left(\frac{f(2x)}{2^2} - f(x)\right)(t) &= \mu\left(f(2x) - 2^2 f(x)\right)(2^{2b}t) \\ &\geq \rho(x, 0, \ldots, 0)(2^{2b}t)\end{aligned} \tag{4.6}$$

for all $x \in M$. Replacing $x$ by $2^{-1}x$ in (4.6), we obtain

$$\begin{aligned}\mu\left(\frac{f(2^{-1}x)}{2^{-2}} - f(x)\right)(t) &= \mu\left(\frac{f(x)}{2^2} - f(2^{-1}x)\right)\left(\frac{t}{2^{2b}}\right) \\ &\geq \rho(2^{-1}x, 0, \ldots, 0)\left(2^{2b}N^{-1}\frac{Nt}{2^{2b}}\right) \\ &\geq \rho(x, 0, \ldots, 0)\left(2^{2b}N^{-1}t\right).\end{aligned} \tag{4.7}$$

From (4.6) and (4.7), we obtain

$$\mu\left(\frac{f(2^a x)}{2^{2a}} - f(x)\right)(t) \geq \Psi(x)(t) := \rho(x, 0, \ldots, 0)\left(2^{2b}N^{\frac{a-1}{2}}t\right) \tag{4.8}$$

for all $x \in M$.

Consider $P := \{f : M \to (U,\mu) | f(0) = 0\}$ and define $\eta$ on $P$ as follows:

$$\eta(f) = \inf\{l > 0 : \mu(f(x))(lt) \geq \Psi(x)(t), \forall x \in M\}.$$

It is simple to prove that $\eta$ is modular on $N$ and indulges the $\Delta_2$-condition with $2^b = \kappa$ and Fatou property. Also, $N$ is $\eta$-complete (see [25]). Consider the mapping $Q : P_\eta \to P_\eta$ defined by $QT(x) := \frac{T(2^a x)}{2^{2a}}$ for all $T \in P_\eta$.

Let $f, j \in P_\eta$ and $l > 0$ be an arbitrary constant with $\eta(f - j) \leq l$. From the definition of $\eta$, we get

$$\mu(f(x) - j(x))(lt) \geq \Psi(x)(t)$$

for all $x \in M$. This implies

$$\begin{aligned}
&\mu\left(Qf(x) - Qj(x)\right)(Nlt)\\
&\quad = \mu\left(2^{-2a} f(2^a x) - 2^{-2a} j(2^a x)\right)(Nlt)\\
&\quad = \mu\left(f(2^a x) - j(2^a x)\right)\left(2^{2ba} Nlt\right)\\
&\quad \geq \Psi(2^a x)(2^{2ba} Nt)\\
&\quad \geq \Psi(x)(t)
\end{aligned}$$

for all $x \in M$. Hence $\eta(Qf - Qj) \leq N\eta(f - j)$ for all $f, j \in P_\eta$, which means that $Q$ is an $\eta$-strict contraction. Replacing $x$ by $2^a x$ in (4.8), we have

$$\mu\left(\frac{f(2^{2a} x)}{2^{2a}} - f(2^a x)\right)(t) \geq \Psi(2^a x)(t) \tag{4.9}$$

for all $x \in M$ and therefore

$$\begin{aligned}
&\mu\left(2^{-2(2a)} f(2^{2a} x) - 2^{-2a} f(2^a x)\right)(Nt)\\
&\quad = \mu\left(2^{-2a} f(2^{2a} x) - f(2^a x)\right)(2^{2ba} Nt)\\
&\quad \geq \Psi(2^a x)(2^{2ba} Nt) \geq \Psi(x)(t)
\end{aligned} \tag{4.10}$$

for all $x \in E$. Now

$$\begin{aligned}
&\mu\left(\frac{f(2^{2a} x)}{2^{2(2a)}} - f(x)\right)\left(2^b(Nt + t)\right)\\
&\quad \geq \mu\left(\frac{f(2^{2a} x)}{2^{2(2a)}} - \frac{f(2^a x)}{2^{2a}}\right)(Nt) \wedge \mu\left(\frac{f(2^a x)}{2^{2a}} - f(x)\right)(t)\\
&\quad \geq \Psi(x)(t)
\end{aligned} \tag{4.11}$$

for all $x \in M$. In (4.11), replacing $x$ by $2^a x$ and $2^b(Nt + t)$ by $2^{2\beta a} 2^b(N^2 t + Nt)$, we obtain

$$\begin{aligned}
&\mu\left(\frac{f(2^{3a} x)}{2^{2(2a)}} - f(2^a x)\right)\left(2^{2ba} 2^b(N^2 t + Nt)\right)\\
&\quad \geq \Psi(2^a x)(2^{2bjNt}) \geq \Psi(x)(t)
\end{aligned} \tag{4.12}$$

for all $x \in M$. Therefore,

$$\mu\left(\frac{f(2^{3a} x)}{2^{3(2a)}} - \frac{f(2^a x)}{2^{2a}}\right)\left(2^b(N^2 t + Nt)\right) \geq \Psi(x)(t) \tag{4.13}$$

for all $x \in M$. This implies

$$\begin{aligned}
&\mu\left(\frac{f(2^{3a}x)}{2^{3(2a)}} - f(x)\right)\left(2^b(2^b(N^2t + Nt) + t)\right) \qquad (4.14)\\
&\geq \mu\left(\frac{f(2^{3a}x)}{2^{3(2a)}} - \frac{f(2^a x)}{2^{2a}}\right)\left(2^b(N^2t + Nt)\right) \wedge \mu\left(\frac{f(2^a x)}{2^{2a}} - f(x)\right)(t)\\
&\geq \Psi(x)(t)
\end{aligned}$$

for all $x \in M$. Generalizing the above inequality, we get

$$\mu\left(\frac{f(2^{am}x)}{2^{2(am)}} - f(x)\right)\left((2^bN)^{m-1}t + 2^b\sum_{i=1}^{m-1}(2^bN)^{i-1}t\right) \geq \Psi(x)(t) \qquad (4.15)$$

for all $x \in M$ and a positive integer $m$. Hence we have

$$\begin{aligned}
\eta(Q^m f - f) &\leq (2^bN)^{m-1} + 2^b\sum_{i=1}^{m-1}(2^bN)^{i-1} \qquad (4.16)\\
&\leq 2^b\sum_{i=1}^{m}(2^bN)^{i-1} \leq \frac{2^b}{1-2^bN}.
\end{aligned}$$

Now, one can easily prove that $\{Q^m(f)\}$ is $\eta-$converges to $T \in P_\eta$ (see [25]). Thus (4.16) becomes

$$\eta(T - f) \leq \frac{2^b}{1-2^bN}, \qquad (4.17)$$

which implies

$$\mu\left(T(x) - f(x)\right)\left(\frac{2^b}{1-2^bN}t\right) \geq \Psi(x)(t) = \rho(x, 0, \ldots, 0)\left(2^b2^{2b}N^{\frac{a-1}{2}}t\right) \qquad (4.18)$$

for all $x \in M$ and hence we have

$$\mu\left(T(x) - f(x)\right)\left(\frac{t}{2^{2b}N^{\frac{a-1}{2}}(1-2^bN)}\right) \geq \rho(x, 0, \ldots, 0)(t) \qquad (4.19)$$

for all $x \in M$ and hence the inequality (4.4) holds. One can easily prove the uniqueness of $T$ (see [25]). □

## 5. Stability of a mixed type functional equation for odd case

In this section, we prove the Ulam stability of the $n$-variable mixed type functional equation (1.3) for odd case in probabilistic modular spaces (PM-spaces) by using fixed point technique.

For a mapping $f : M \to (U, \mu)$, consider

$$S_o(x_1, x_2, \ldots, x_n) = \sum_{i=1,\ \ j=i+1}^{n-1}\left(f(2x_i + x_j)\right) + f(2x_n + x_1)$$

$$-2\left[\sum_{i=1,\quad j=i+1}^{n-1}\Big(f(x_i+x_j)\Big)+f(x_n+x_1)\right]+\sum_{i=1}^{n}f(x_i)$$

for $n \in \mathbb{N}$.

**Theorem 5.1.** *Let $M$ be a linear space and $(V,\mu)$ be a $\mu$-complete $b$-homogeneous PM-space. Suppose that a mapping $f : M \to (V,\mu)$ satisfies an inequality*

$$\mu(S_o(x_1,x_2,\ldots,x_n)) \geq \rho(x_1,x_2,\ldots,x_n)(t) \tag{5.1}$$

*for all $x_1,x_2,\ldots,x_n \in M$ and a given mapping $\rho : M\times M \to \Delta$ such that*

$$\rho(2^a x,0,\ldots,0)(2^{ba}Nt) \geq \rho(x,0,\ldots,0)(t) \tag{5.2}$$

*for all $x \in M$ and*

$$\rho(2^{am}x_1,2^{am}x_2,\ldots,2^{am}x_n)(2^{bam}t) = 1 \tag{5.3}$$

*for all $x_1,x_2,\ldots,x_n \in M$ and a constant $0 < N < \frac{1}{2^b}$. Then there exists a unique additive mapping $A : M \to (V,\mu)$ satisfying (3.1) and*

$$\mu(A(x)-f(x))\left(\frac{t}{2^b N^{\frac{a-1}{2}}(1-2^bN)}\right) \geq \rho(x,0,\ldots,0)(t) \tag{5.4}$$

*for all $x \in M$.*

*Proof.* Replacing $(x_1,x_2,\ldots,x_n)$ by $(x,0,\ldots,0)$ in (5.1), we obtain

$$\mu(f(2x)-2f(x))(t) \geq \rho(x,0,\ldots,0)(t) \tag{5.5}$$

for all $x \in M$. This implies

$$\begin{aligned}\mu\left(\frac{f(2x)}{2}-f(x)\right)(t) &= \mu\left(f(2x)-2f(x)\right)(2^bt)\\ &\geq \rho(x,0,\ldots,0)(2^bt)\end{aligned} \tag{5.6}$$

for all $x \in M$. Replacing $x$ by $2^{-1}x$ in (5.6), we obtain

$$\begin{aligned}\mu\left(\frac{f(2^{-1}x)}{2^{-1}}-f(x)\right)(t) &= \mu\left(\frac{f(x)}{2}-f(2^{-1}x)\right)\left(\frac{t}{2^b}\right)\\ &\geq \rho(2^{-1}x,0,\ldots,0)\left(2^bN^{-1}\frac{Nt}{2^b}\right)\\ &\geq \rho(x,0,\ldots,0)\left(2^bN^{-1}t\right).\end{aligned} \tag{5.7}$$

From (5.6) and (5.7), we obtain

$$\mu\left(\frac{f(2^ax)}{2^a}-f(x)\right)(t) \geq \Psi(x)(t) := \rho(x,0,\ldots,0)\left(2^bN^{\frac{a-1}{2}}t\right) \tag{5.8}$$

for all $x \in M$.

Consider $P := \{f : M \to (U, \mu) | f(0) = 0\}$ and define $\eta$ on $P$ as follows:

$$\eta(f) = \inf\{l > 0 : \mu(f(x))(lt) \geq \Psi(x)(t), \forall x \in M\}.$$

It is simple to prove that $\eta$ is modular on $N$ and indulges the $\Delta_2$-condition with $2^b = \kappa$ and Fatou property. Also, $N$ is $\eta$-complete (see [25]). Consider a mapping $Q : P_\eta \to P_\eta$ defined by $QA(x) := \frac{A(2^a x)}{2^a}$ for all $A \in P_\eta$.

Let $f, j \in P_\eta$ and $l > 0$ be an arbitrary constant with $\eta(f - j) \leq l$. From the definition of $\eta$, we get

$$\mu(f(x) - j(x))(lt) \geq \Psi(x)(t)$$

for all $x \in M$. This implies

$$\begin{aligned}
&\mu\left(Qf(x) - Qj(x)\right)(Nlt) \\
&\quad = \mu\left(2^{-a} f(2^a x) - 2^{-a} j(2^a x)\right)(Nlt) \\
&\quad = \mu\left(f(2^a x) - j(2^a x)\right)\left(2^{ba} Nlt\right) \\
&\quad \geq \Psi(2^a x)(2^{ba} Nt) \\
&\quad \geq \Psi(x)(t)
\end{aligned}$$

for all $x \in M$. Hence $\eta(Qf - Qj) \leq N\eta(f - j)$ for all $f, j \in P_\eta$, which means that $Q$ is an $\eta$-strict contraction. Replacing $x$ by $2^a x$ in (5.8), we get

$$\mu\left(\frac{f(2^{2a} x)}{2^a} - f(2^a x)\right)(t) \geq \Psi(2^a x)(t) \tag{5.9}$$

for all $x \in M$ and thus

$$\begin{aligned}
&\mu\left(2^{-2a} f(2^{2a} x) - 2^{-a} f(2^a x)\right)(Nt) \\
&\quad = \mu\left(2^{-a} f(2^{2a} x) - f(2^a x)\right)(2^{ba} Nt) \\
&\quad \geq \Psi(2^a x)(2^{ba} Nt) \geq \Psi(x)(t),
\end{aligned} \tag{5.10}$$

for all $x \in E$. Now

$$\begin{aligned}
&\mu\left(\frac{f(2^{2a} x)}{2^{2a}} - f(x)\right)\left(2^b(Nt + t)\right) \\
&\quad \geq \mu\left(\frac{f(2^{2a} x)}{2^{2a}} - \frac{f(2^a x)}{2^a}\right)(Nt) \wedge \mu\left(\frac{f(2^a x)}{2^a} - f(x)\right)(t) \\
&\quad \geq \Psi(x)(t)
\end{aligned} \tag{5.11}$$

for all $x \in M$. In (5.11), replacing $x$ by $2^a x$ and $2^b(Nt + t)$ by $2^{ba} 2^b(N^2 t + Nt)$, we obtain

$$\begin{aligned}
&\mu\left(\frac{f(2^{3a} x)}{2^{2a}} - f(2^a x)\right)\left(2^{ba} 2^b(N^2 t + Nt)\right) \\
&\quad \geq \Psi(2^a x)(2^{bjNt}) \geq \Psi(x)(t)
\end{aligned} \tag{5.12}$$

for all $x \in M$. Therefore,

$$\mu\left(\frac{f(2^{3a}x)}{2^{3a}} - \frac{f(2^a x)}{2^a}\right)\left(2^b(N^2t + Nt)\right) \geq \Psi(x)(t) \tag{5.13}$$

for all $x \in M$. This implies

$$\begin{aligned}
&\mu\left(\frac{f(2^{3a}x)}{2^{3a}} - f(x)\right)\left(2^b(2^b(N^2t + Nt) + t)\right) \\
&\quad\geq \mu\left(\frac{f(2^{3a}x)}{2^{3a}} - \frac{f(2^a x)}{2^a}\right)\left(2^b(N^2t + Nt)\right) \wedge \mu\left(\frac{f(2^a x)}{2^a} - f(x)\right)(t) \\
&\quad\geq \Psi(x)(t)
\end{aligned} \tag{5.14}$$

for all $x \in M$. Generalizing the above inequality, we have

$$\mu\left(\frac{f(2^{am}x)}{2^{am}} - f(x)\right)\left((2^bN)^{m-1}t + 2^b\sum_{i=1}^{m-1}(2^bN)^{i-1}t\right) \geq \Psi(x)(t) \tag{5.15}$$

for all $x \in M$ and a positive integer $m$. Hence we have

$$\begin{aligned}
\eta(Q^m f - f) &\leq (2^bN)^{m-1} + 2^b\sum_{i=1}^{m-1}(2^bN)^{i-1} \\
&\leq 2^b\sum_{i=1}^{m}(2^bN)^{i-1} \leq \frac{2^b}{1-2^bN}.
\end{aligned} \tag{5.16}$$

Now, one can easily prove that $\{Q^m(f)\}$ is $\eta$-convergent to $A \in P_\eta$ (see [25]). Thus (4.16) becomes

$$\eta(A - f) \leq \frac{2^b}{1-2^bN}, \tag{5.17}$$

which leads

$$\mu\left(A(x) - f(x)\right)\left(\frac{2^b}{1-2^bN}t\right) \geq \Psi(x)(t) = \rho(x, 0, \ldots, 0)\left(2^b2^bN^{\frac{a-1}{2}}t\right) \tag{5.18}$$

for all $x \in M$ and hence we have

$$\mu\left(A(x) - f(x)\right)\left(\frac{t}{2^bN^{\frac{a-1}{2}}(1-2^bN)}\right) \geq \rho(x, 0, \ldots, 0)(t) \tag{5.19}$$

for all $x \in M$ and hence the inequality (5.4) holds. One can easily prove the uniqueness of $A$ (see [25]). □

## 6. Conclusion

In this paper, we introduced a new $n$-variable mixed type functional equation satisfied by the solution $f(x) = ax + bx^2$. Mainly, we obtained its general solution and investigated its Ulam stability in $PM$-spaces by using fixed point theory and we hope that this research work is a further improvement in the field of functional equations.

## Acknowledgments

This work was supported by Incheon National University Research Grant 2020-2021.


## Conflict of interest

The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

The authors declare that they have no competing interests.

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