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Research article

Some Opial type inequalities in (p, q)-calculus

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Abstract: In this paper, we establish 5 kinds of integral Opial-type inequalities in (p, q)-calculus by means of Hölder's inequality, Cauchy inequality, an elementary inequality and some analysis technique. First, we investigated the Opial inequalities in (p, q)-calculus involving one function and its (p, q) derivative. Furthermore, Opial inequalities in (p, q)-calculus involving two functions and two functions with their (p, q) derivatives are given. Our results are (p, q)-generalizations of some known inequalities, such as Opial-type integral inequalities and (p, q)-Wirtinger inequality.

Keywords: (p, q)-derivative; (p, q)-integral; (p, q)-calculus; Opial inequality; Opial-type integral inequality
Mathematics Subject Classification: 26D15, 81P68

1. Introduction

In 1960, Opial [1] presented an inequality involving integral of a function and its derivative as follows

Theorem 1.1. *Let $x \in C^1[0, h]$ be such that $x(t) > 0$ in $(0, h)$. Then, the following inequalities hold:*

i) If $x(0) = x(h) = 0$, then

$$\int_0^h |x(t)x'(t)|dt \leq \frac{h}{4}\int_0^h |x'(t)|^2 dt. \tag{1.1}$$

ii) If $x(0) = 0$, then

$$\int_0^h |x(t)x'(t)|dt \leq \frac{h}{2}\int_0^h |x'(t)|^2 dt. \tag{1.2}$$

In (1.1), the constant $\frac{h}{4}$ is the best possible.

Since then, the study of generalizations, extensions and discretizations for inequalities (1.1) and (1.2) of Opial type inequalities has grown into a substantial field, with many important applications in theory of differential equations, approximations and probability, among others. For more details, we cite the readers to [2–7] and the references therein.

Recently, Mirkovi et.al [8] established a new integral inequality of the q-Opial type as follows:

Theorem 1.2. *Let $f \in C^1[0,1]$ be q-decreasing function with $f(bq^0) = 0$. Then, for any $p \geq 0$,*

$$\int_a^b |D_q f(x)||f(x)|^p d_q x \leq (b-a)^p \int_a^b |D_q f(x)|^{p+1} d_q x. \tag{1.3}$$

In [9], Alp et al. gave the following q-Opial type inequality for quantum integral :

Theorem 1.3. *Let $x(t) \in C^1[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then, the following inequality holds :*

$$\int_0^h |(x(t) + x(qt))D_q x(t)| d_q t \leq \frac{h}{1+q} \int_0^h |D_q x(t)|^2 d_q t. \tag{1.4}$$

The (p,q)-calculus is known as two parameter quantum calculus, a generalization of q-calculus, whose applications play important roles in physics, chemistry, orthogonal polynomials and number theory [10,11]. In [12], Mursaleen et al. applied (p,q)-calculus in approximation theory and investigated first (p,q)-analogue of Bernstein operators. Recently, Sadjang in [13] studied the (p,q)-derivative, the (p,q)-integration, and obtained some of their properties and the fundamental theorem of (p,q)-calculus. About the recent results of (p,q)-calculus, please see [14–18].

Inspired by the above mentioned works [8,9,13], in this paper, we will establish some (p,q)-Opial type inequalities by using (p,q)-calculus and analysis technique. If $p = 1$ and $q \to 1^-$, then all the results we have obtained in this paper reduce to the classical cases.

2. Preliminaries

In what follows, p, q are two real numbers satisfying $0 < q < p \leq 1$, and (p,q)-bracket is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p-q} = p^{n-1} + qp^{n-2} + \cdots + q^{n-1}, \quad n \in \mathbb{N}.$$

Definition 2.1. ([13]) *The (p,q)-derivative of the function f is defined as*

$$D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0,$$

and $(D_{p,q} f)(0) = \lim_{x \to 0} D_{p,q} f(x) = f'(0)$, privided that f is differentiable at 0.

Definition 2.2. ([13]) *Let f be an arbitrary function and a be a positive real number, the (p,q)-integral of f from 0 to a is defined by*

$$\int_0^a f(x) d_{p,q} x = (p-q)a \sum_{i=0}^{\infty} \frac{q^i}{p^{i+1}} f\left(\frac{q^i}{p^{i+1}} a\right).$$

Also for two nonnegative numbers such that $a < b$, we have

$$\int_a^b f(x)d_{p,q}x = \int_0^b f(x)d_{p,q}x - \int_0^a f(x)d_{p,q}x.$$

Lemma 2.1. ([13]) *The (p,q)-derivative fulfills the following product rules*

$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x),$$

$$D_{p,q}(f(x)g(x)) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x).$$

Theorem 2.2. ([13]) *(Fundamental theorem of (p,q)-calculus). If $F(x)$ is a (p,q)-antiderivative of $f(x)$ and $F(x)$ is continuous at $x = 0$, we have*

$$\int_a^b f(x)d_{p,q}x = F(b) - F(a),$$

where $0 \leq a < b \leq \infty$.

3. Main results

Theorem 3.1. *Let $f \in C^1[0,h]$ be such that $f(t) > 0$ in $(0,h)$. If $f(0) = 0$ and $m \in \mathbb{N}$. Then, the following inequality holds:*

$$\int_0^h \left| \sum_{j=0}^{m} f^{m-j}(px) f^j(qx) \right| |D_{p,q}f(x)| d_{p,q}x \leq h^m \int_0^h |D_{p,q}f(x)|^{m+1} d_{p,q}x. \tag{3.1}$$

Proof. Choosing $g(x)$ as

$$g(x) = \int_0^x |D_{p,q}f(t)| d_{p,q}t, \tag{3.2}$$

such that

$$|D_{p,q}f(x)| = D_{p,q}g(x), \tag{3.3}$$

and for $x \in [0,h]$, we have

$$|f(x)| = \left| \int_0^x D_{p,q}f(t)d_{p,q}t \right| \leq \int_0^x |D_{p,q}f(t)| d_{p,q}t = g(x), \tag{3.4}$$

$$|f(px)| = \left| \int_0^{px} D_{p,q}f(t)d_{p,q}t \right| \leq \int_0^{px} |D_{p,q}f(t)| d_{p,q}t = g(px), \tag{3.5}$$

and

$$|f(qx)| = \left| \int_0^{qx} D_{p,q}f(t)d_{p,q}t \right| \leq \int_0^{qx} |D_{p,q}f(t)| d_{p,q}t = g(qx). \tag{3.6}$$

By Lemma 2.1, we can obtain that

$$D_{p,q}f^{m+1}(x) = \sum_{j=0}^{m} f^{m-j}(px)f^{j}(qx)D_{p,q}f(x). \tag{3.7}$$

Thus, by (3.2)–(3.7), it follows that

$$\begin{aligned}\int_0^h \left|\sum_{j=0}^{m} f^{m-j}(px)f^{j}(qx)D_{p,q}f(x)\right| d_{p,q}x &\le \int_0^h \sum_{j=0}^{m} |f(px)|^{m-j}|f(qx)|^{j}|D_{p,q}f(x)|d_{p,q}x \\ &\le \int_0^h \sum_{j=0}^{m} g^{m-j}(px)g^{j}(qx)D_{p,q}g(x)d_{p,q}x \\ &= \int_0^h D_{p,q}g^{m+1}(x)d_{p,q}x = g^{m+1}(h).\end{aligned} \tag{3.8}$$

By using the Hölder's inequality and (3.8) with (3.2) for (p,q)-integral with indices $m+1$ and $\frac{m+1}{m}$, we obtain

$$\begin{aligned}&\int_0^h \left|\sum_{j=0}^{m} f^{m-j}(px)f^{j}(qx)D_{p,q}f(x)\right| d_{p,q}x \\ &\le g^{m+1}(h) = \left[\int_0^h |D_{p,q}f(x)|d_{p,q}x\right]^{m+1} \\ &\le \left[\left(\int_0^h d_{p,q}x\right)^{\frac{m}{m+1}} \left(\int_0^h |D_{p,q}f(x)|^{m+1}d_{p,q}x\right)^{\frac{1}{m+1}}\right]^{m+1} \\ &= h^m \int_0^h |D_{p,q}f(x)|^{m+1}d_{p,q}x,\end{aligned}$$

which proves the theorem.

Remark 3.1. *If $p = 1$ and $m = 1$, then Theorem 3.1 reduces to Theorem 3.3 in [9]. If $p = 1$ and $m = \alpha$, then Theorem 3.1 reduces to Theorem 3.9 in [9]. Moreover, if $p = 1$, $m = 1$, and $q \to 1^-$, then (3.1) reduces to (1.2) in Theorem 1.1.*

Theorem 3.2. *Let $f \in C^1[0,h]$ be such that $f(t) > 0$ in $(0,h)$. If $f(0) = f(h) = 0$ and $m \in \mathbb{N}$. Then, the following inequality holds :*

$$\int_0^h \left|\sum_{j=0}^{m} f^{m-j}(px)f^{j}(qx)\right| |D_{p,q}f(x)|d_{p,q}x \le \left(\frac{h}{1+q}\right)^m \int_0^h |D_{p,q}f(x)|^{m+1}d_{p,q}x. \tag{3.9}$$

Proof. Let $g(x)$ be as in (3.2) and $w(x)$ be as follows

$$w(x) = \int_x^h |D_{p,q}f(t)|d_{p,q}t. \tag{3.10}$$

Then, we obtain by the condition $f(h) = 0$ that

$$|D_{p,q}f(x)| = -D_{p,q}w(x), \tag{3.11}$$

and for $x \in [0, h]$, we have

$$|f(x)| = \left| \int_x^h D_{p,q} f(t) d_{p,q} t \right| \leq \int_x^h |D_{p,q} f(t)| d_{p,q} t = w(x), \tag{3.12}$$

$$|f(px)| = \left| \int_{px}^h D_{p,q} f(t) d_{p,q} t \right| \leq \int_{px}^h |D_{p,q} f(t)| d_{p,q} t = w(px), \tag{3.13}$$

and

$$|f(qx)| = \left| \int_{qx}^h D_{p,q} f(t) d_{p,q} t \right| \leq \int_{qx}^h |D_{p,q} f(t)| d_{p,q} t = w(qx). \tag{3.14}$$

From (3.8), one has

$$\int_0^{\frac{h}{1+q}} \left| \sum_{j=0}^{m} f^{m-j}(px) f^j(qx) \right| |D_{p,q} f(x)| d_{p,q} x \leq g^{m+1} \left(\frac{h}{1+q} \right). \tag{3.15}$$

Similarly, by (3.10)–(3.14), we can get that

$$\begin{aligned} \int_{\frac{h}{1+q}}^h \left| \sum_{j=0}^{m} f^{m-j}(px) f^j(qx) \right| |D_{p,q} f(x)| d_{p,q} x &\leq - \int_{\frac{h}{1+q}}^h \sum_{j=0}^{m} w^{m-j}(px) w^j(qx) D_{p,q} w(x) d_{p,q} x \\ &= w^{m+1} \left(\frac{h}{1+q} \right). \end{aligned} \tag{3.16}$$

Adding (3.15) to (3.16), we obtain

$$\int_0^h \left| \sum_{j=0}^{m} f^{m-j}(px) f^j(qx) \right| D_{p,q} f(x) d_{p,q} x \leq g^{m+1} \left(\frac{h}{1+q} \right) + w^{m+1} \left(\frac{h}{1+q} \right). \tag{3.17}$$

Using the Hölder's inequality with indices $m+1$ and $\frac{m+1}{m}$, we can write that

$$\begin{aligned} g^{m+1} \left(\tfrac{h}{1+q} \right) &= \left[\int_0^{\frac{h}{1+q}} |D_{p,q} f(t)| d_{p,q} t \right]^{m+1} \\ &\leq \left[\left(\int_0^{\frac{h}{1+q}} d_{p,q} t \right)^{\frac{m}{m+1}} \left(\int_0^{\frac{h}{1+q}} |D_{p,q} f(t)|^{m+1} d_{p,q} t \right)^{\frac{1}{m+1}} \right]^{m+1} \\ &= \left(\tfrac{h}{1+q} \right)^m \int_0^{\frac{h}{p+q}} |D_{p,q} f(t)|^{m+1} d_{p,q} t. \end{aligned} \tag{3.18}$$

Similarly, we obtain

$$\begin{aligned} w^{m+1} \left(\frac{h}{1+q} \right) &= \left[\int_{\frac{h}{1+q}}^h |D_{p,q} f(t)| d_{p,q} t \right]^{m+1} \\ &\leq \left[\left(\int_{\frac{h}{1+q}}^h d_{p,q} t \right)^{\frac{m}{m+1}} \left(\int_{\frac{h}{1+q}}^h |D_{p,q} f(t)|^{m+1} d_{p,q} t \right)^{\frac{1}{m+1}} \right]^{m+1} \\ &= \left(\frac{hq}{1+q} \right)^m \int_{\frac{h}{p+q}}^h |D_{p,q} f(t)|^{m+1} d_{p,q} t \leq \left(\frac{h}{1+q} \right)^m \int_{\frac{h}{p+q}}^h |D_{p,q} f(t)|^{m+1} d_{p,q} t. \end{aligned} \tag{3.19}$$

Therefore, (3.17)–(3.19) imply that (3.9) holds.

Remark 3.2. *If $p = 1$ and $m = 1$, then Theorem 3.2 reduces to Theorem 3.1 in [9]. In Theorem 3.2 if we take $p = 1$, $m = 1$, and $q \to 1^-$, we recapture the inequality (1.1).*

Theorem 3.3. *Let $m > 0$. Assume that $\mu(t)$ is a nonnegative and continuous function on $[0, h]$, $m \in \mathbb{N}$, and $f \in C^1[0, 1]$ with $f(0) = 0$. Then, the following inequality holds:*

$$\int_0^h \mu(x)|f(x)|^m |D_{p,q}f(x)| d_{p,q}x \le h^{\frac{m^2}{m+1}} \left(\int_0^h \mu(x)^{\frac{m+1}{m}} d_{p,q}x\right)^{\frac{m}{m+1}} \int_0^h |D_{p,q}f(x)|^{m+1} d_{p,q}x. \tag{3.20}$$

Proof. Using the Hölder's inequality for (p, q)-integral with indices $\frac{m+1}{m}$ and $m + 1$, we obtain

$$\begin{aligned} \int_0^h \mu(x) \quad & |f(x)|^m |D_{p,q}f(x)| d_{p,q}x \\ & \le \left(\int_0^h \mu(x)^{\frac{m+1}{m}} |f(x)|^{m+1} d_{p,q}x\right)^{\frac{m}{m+1}} \cdot \left(\int_0^h |D_{p,q}f(x)|^{m+1} d_{p,q}x\right)^{\frac{1}{m+1}}. \end{aligned} \tag{3.21}$$

By using (3.4) and from the Hölder's inequality for (p, q)-integral with indices $\frac{m+1}{m}$ and $m + 1$, we get

$$\begin{aligned} \int_0^h \mu(x)^{\frac{m+1}{m}} \quad & |f(x)|^{m+1} d_{p,q}x \\ & \le \int_0^h \mu(x)^{\frac{m+1}{m}} \left[\int_0^h |D_{p,q}f(t)| d_{p,q}t\right]^{m+1} d_{p,q}x \\ & \le \int_0^h \mu(x)^{\frac{m+1}{m}} d_{p,q}x \cdot \left[\left(\int_0^h d_{p,q}t\right)^{\frac{m}{m+1}} \left(\int_0^h |D_{p,q}f(t)|^{m+1} d_{p,q}t\right)^{\frac{1}{m+1}}\right]^{m+1} \\ & = h^m \int_0^h \mu(x)^{\frac{m+1}{m}} d_{p,q}x \int_0^h |D_{p,q}f(x)|^{m+1} d_{p,q}x. \end{aligned} \tag{3.22}$$

Substituting (3.22) into (3.21), we have

$$\int_0^h \mu(t)|f(x)|^m |D_{p,q}f(x)| d_{p,q}x \le h^{\frac{m^2}{m+1}} \left(\int_0^h \mu(x)^{\frac{m+1}{m}} d_{p,q}x\right)^{\frac{m}{m+1}} \int_0^h |D_{p,q}f(x)|^{m+1} d_{p,q}x.$$

This completes the proof.

Theorem 3.4. *If f and g are absolutely continuous functions on $[0, h]$, and $f(0) = g(0) = 0$, then*

$$\int_0^h [f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x)] d_{p,q}x \le \frac{h}{2} \int_0^h [(D_{p,q}f(x))^2 + (D_{p,q}g(x))^2] d_{p,q}x. \tag{3.23}$$

Proof. As in (3.4), we have

$$|f(x)| \le \int_0^x |D_{p,q}f(t)| d_{p,q}t, \quad |g(x)| \le \int_0^x |D_{p,q}g(t)| d_{p,q}t. \tag{3.24}$$

By using Lemma 2.1, Theorem 2.2, (3.24) and Cauchy inequality, we get

$$\int_0^h [f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x)] d_{p,q}x$$

$$= \int_0^h D_{p,q}(f(x)g(x))d_{p,q}x = f(h)g(h) - f(0)g(0)$$

$$= f(h)g(h) \le |f(h)||g(h)| \le \int_0^h |D_{p,q}f(x)|d_{p,q}x \cdot \int_0^h |D_{p,q}g(x)|d_{p,q}x$$

$$\le \frac{1}{2}\left[\left(\int_0^h |D_{p,q}f(x)|d_{p,q}x\right)^2 + \left(\int_0^h |D_{p,q}g(x)|d_{p,q}x\right)^2\right]$$

$$\le \frac{1}{2}\left[\int_0^h d_{p,q}x \cdot \int_0^h |D_{p,q}f(x)|^2 d_{p,q}x + \int_0^h d_{p,q}x \cdot \int_0^h |D_{p,q}g(x)|^2 d_{p,q}x\right]$$

$$= \frac{h}{2}\int_0^h [(D_{p,q}f(x))^2 + (D_{p,q}g(x))^2]d_{p,q}x.$$

Theorem 3.5. *Let $m, r > 0$. Assume that f and g are absolutely continuous functions on $[0, h]$, $f(0) = g(0) = 0$, $f(h) = g(h) = 0$, and $f(t), g(t) > 0$ in $(0, h)$. Then*

$$\int_0^h |f(x)|^m |g(x)|^r d_{p,q}x \le \frac{\left(\frac{h}{2}\right)^{m+r}}{m+r}\left(m\int_0^h |D_{p,q}f(x)|^{m+r}d_{p,q}x + r\int_0^h |D_{p,q}g(x)|^{m+r}d_{p,q}x\right). \tag{3.25}$$

Proof. Let

$$y(x) = \int_0^x |D_{p,q}f(t)|d_{p,q}t, \quad z(x) = \int_x^h |D_{p,q}f(t)|d_{p,q}t, \quad t \in [0, h]. \tag{3.26}$$

Then, we have

$$\begin{aligned} &|D_{p,q}f(x)| = D_{p,q}y(x) = -D_{p,q}z(x), \\ &|f(x)| = \left|\int_0^x D_{p,q}f(t)d_{p,q}t\right| \le \int_0^x |D_{p,q}f(t)|d_{p,q}t = y(x), \end{aligned} \tag{3.27}$$

and

$$|f(x)| = \left|\int_x^h D_{p,q}f(t)d_{p,q}t\right| \le \int_x^h |D_{p,q}f(t)|d_{p,q}t = z(x). \tag{3.28}$$

By (3.26)–(3.28), we obtain

$$|f(x)| \le \frac{y(x) + z(x)}{2} = \frac{1}{2}\int_0^h |D_{p,q}f(t)|d_{p,q}t. \tag{3.29}$$

Similarly, we have

$$|g(x)| \le \frac{1}{2}\int_0^h |D_{p,q}g(t)|d_{p,q}t. \tag{3.30}$$

On the other hand, the following elementary inequality in [8] holds :

$$mA^{m+r}+rB^{m+r}-(m+r)A^mB^r\geq 0,\quad A,B\geq 0,\quad m,r>0. \tag{3.31}$$

From (3.29)–(3.31), we get

$$|f(x)|^m|g(x)|^r\leq \frac{\left(\frac{1}{2}\right)^{m+r}}{m+r}\left[m\left(\int_0^h|D_{p,q}f(t)|d_{p,q}t\right)^{m+r}+r\left(\int_0^h|D_{p,q}g(t)|d_{p,q}t\right)^{m+r}\right]. \tag{3.32}$$

By using Hölder's inequality on the right side of (3.32) with indices $\frac{m+r}{m+r-1}$, $m+r$, we obtain

$$|f(x)|^m|g(x)|^r\leq \frac{\left(\frac{1}{2}\right)^{m+r}}{m+r}\left[mh^{m+r-1}\int_0^h|D_{p,q}f(t)|^{m+r}d_{p,q}t+rh^{m+r-1}\int_0^h|D_{p,q}g(t)|^{m+r}d_{p,q}t\right]. \tag{3.33}$$

Integrating (3.33) on $[0,h]$, we have

$$\int_0^h|f(x)|^m|g(x)|^rd_{p,q}x\leq \frac{\left(\frac{h}{2}\right)^{m+r}}{m+r}\left(m\int_0^h|D_{p,q}f(t)|^{m+r}d_{p,q}t+r\int_0^h|D_{p,q}g(t)|^{m+r}d_{p,q}t\right),$$

and the proof is completed.

Remark 3.3. *If $m=r>0$ and $f(x)=g(x)$, then the inequality (3.25) reduces to the following (p,q)-Wirtinger inequality:*

$$\int_0^h|f(x)|^{2m}d_{p,q}x\leq\left(\frac{h}{2}\right)^{2m}\int_0^h|D_{p,q}f(x)|^{2m}d_{p,q}x.$$

4. Example

In the following, we will give an example to illustrate our main result.

Example 4.1. *Let $p=\frac{2}{3}$, $q=\frac{1}{2}$ and $m=5$. Set $\mu(t)=t$, it is clear that $\mu(t)$ is a nonnegative and continuous function on $[0,\frac{\pi}{2}]$. Set $h=\frac{\pi}{2}$, and $f(x)=\sin x$, then $f\in C^1[0,1]$ with $f(0)=0$. Thus, by Theorem 3.3, we have*

$$\int_0^{\frac{\pi}{2}}x|\sin x|^5|D_{\frac{2}{3},\frac{1}{2}}\sin x|d_{\frac{2}{3},\frac{1}{2}}x\leq\left(\frac{\pi}{2}\right)^{\frac{25}{6}}\left(\int_0^{\frac{\pi}{2}}x^{\frac{6}{5}}d_{\frac{2}{3},\frac{1}{2}}x\right)^{\frac{5}{6}}\int_0^{\frac{\pi}{2}}|D_{\frac{2}{3},\frac{1}{2}}\sin x|^6d_{\frac{2}{3},\frac{1}{2}}x.$$

5. Conclusion

It is known that (p,q)-calculus is a generalization of q-calculus. In this paper, we have established 5 new kinds of general Opial type integral inequalities in (p,q)-calculus. The methods we used to establish our results are quite simple and in virtue of some basic observations and applications of some fundamental inequalities and analysis technique. First, we investigated the Opial inequalities in (p,q)-calculus involving one function and its (p,q) derivative. Furthermore, Opial inequalities in (p,q)-calculus involving two functions and two functions with their (p,q) derivatives are given. We also discussed several particular cases. Our results are (p,q)-generalizations of Opial-type integral inequalities and (p,q)-Wirtinger inequality. An example is given to illustrate the effectiveness of our main result.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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