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Research article

Characterizations of ordered h-regular semirings by ordered h-ideals

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Abstract: The objective of this paper is to study the ordered h-regular semirings by the properties of their ordered h-ideals. It is proved that each h-regular ordered semiring is an ordered h-regular semiring but the converse does not follow. Important theorems relating to basic properties of the operator clousre and h-regular semirings are given. It is also proved that each regular ordered semiring is an ordered h-regular semiring but the converse does not hold. The classifications of the left and the right ordered h-regular semirings and the left and the right ordered h-weakly regular semirings are also presented.

Keywords: semiring; ordered semiring; ordered h-regular; ordered h-ideal
Mathematics Subject Classification: 16Y99, 16Y60

1. Introduction

Von Neumann gave the idea of regularity in rings in 1935 [1] and showed that if the semigroup, $(S,\cdot)$ is regular, then the ring $(S,+,\cdot)$ is also regular [1]. In 1951, Bourne showed if $\forall x \in S$ there exist $a, b \in S$ such that $x + xax = xbx$, then semiring $(S,+,\cdot)$ is also regular [2]. In structure theory of semirings, ideals play a vital role [3]. In [4], Xueling Ma and Jianming Zhan used the concept of h-ideals. They used the basic and main concept of h-ideals to prove many properties and results. Similarly, Jianming Zhan et al., in [5] also used h-ideals in their researches. This class of h-ideals has been used in many researches by different researchers. Ideals of semirings used in the structure theory play an important role in many aspects. Some properties of ideals are discussed in [6–8]. Gan and Jiang [9] studied the ordered semirings containing 0. Han and others in [10] discussed also the ordered semirings. Iizuka [11] introduced a new type of ideals namely h-ideals. In [12–14] they used h-ideals for many purposes related to their researches.

Main and basic concepts related to ordered semirings are given by Gan and Jiang [9]. The authors also derived some ideas related to minimal ideal, maximal ideal, ordered ideal of an ordered semiring and simple ordered semirings. Han, Kim and Neggers [10] also worked on semirings by partial ordered set. Munir and Shafiq [19] characterized the regular semirings through m-ideals. Satyt Patchakhieo and Bundit Pibalijommee [15] gave the basic definition of ordered semirings and left and right ordered ideal of the ordered semirings. They used two definitions in their properties and applications to prove their result.

Keeping in view the different characterizations of the regular semirings by the properties of the h-ideals, we were motivated to characterize the ordered h-regualr semirings by the properties of their ordered h-ideals. For this purpose, this paper represents ordered h-regular semirings along with their ordered h-ideals. In Section 2, we give some basic definitions which will be used in our further course of work. In Section 3, we characterize the ordered h-ideals semirings by their ordered h-ideals. In Section 4, we characterize the ordered h-regular semirings, and in Section 5, the characterization of the ordered h-weakly regular semirings is given. The conclusion of the paper is presented in the final Section 6.

2. Preliminaries

Definition 1. *A non-empty set S together with two binary operations $+$ and $\cdot$ satisfying the following properties:*
(C_1) $(S,+)$ *is a semigroup,*
(C_2) $(S,\cdot)$ *is a semigroup,*
(C_3) *Distributive laws hold in S, that is*

$$t_1 \cdot (t_2 + t_3) = t_1 \cdot t_2 + t_1 \cdot t_3,$$

and

$$(t_1 + t_2) \cdot t_3 = t_1 \cdot t_3 + t_2 \cdot t_3 \text{ for all } t_1, t_2, t_3 \in S,$$

is called a semiring, which is denoted by $(S,+,\cdot)$.

Definition 2. $(S,+,\cdot)$ *is additively commutative iff for all* $x_1, y_1 \in S, x_1 + y_1 = y_1 + x_1$. *S is multiplicatively commutative iff for all* $x_1, y_1 \in S, x_1 \cdot y_1 = y_1 \cdot x_1$. $(S,+,\cdot)$ *is called a commutative semiring iff it is both additively commutative and multiplicatively commutative. Suppose* $(S,+,\cdot)$ *is a semiring, if* $\forall a \in S; a + 0 = a = 0 + a$ *and* $a \cdot 0 = 0 = 0 \cdot a$, *then* $0 \in S$ *is called absorbing zero in S.*

Definition 3. *[15] Let $E \neq \emptyset$ and $(S,+,\cdot)$ is a semiring, $E \subseteq S$, is a left ideal or right ideal if these properties are satisfied:*
(I_1) $t_1 + t_2 \in E$ *for all* $t_1, t_2 \in E$.
(I_2) $SE \subseteq E$ *or* $ES \subseteq E$.
If E is left ideal and right ideal of S, then E is an ideal of S.

Definition 4. *[15] Suppose $(S, \leq)$ is a partially ordered set satisfying the following properties:*
(T_1) *$(S, +, \cdot)$ is semiring,*
(T_2) *if $x_1 \leq x_2$, then $x_1 + e \leq x_2 + e$,*
(T_3) *if $x_1 \leq x_2$, then $x_1 e \leq x_2 e$ and $e x_1 \leq e x_2$,*
for all $x_1, x_2, e \in S$, then, $(S, +, \cdot, \leq)$ is an ordered semiring.

Definition 5. *[15] Suppose $(S, +, \cdot, \leq)$ is an ordered semiring. Let $E \neq \emptyset$, $F \neq \emptyset$ be subsets of S, then we denote $(E] = \{g \in S | g \leq r \text{ for some } r \in E\}$ and $EF = \{gh | g \in E, h \in F\}$.*
We can write $(S, +, \cdot, \leq)$ as S.

Definition 6. *[15] Suppose S is an ordered semiring, $E \neq \emptyset$ and $E \subseteq S$ satisfies the following properties:*
(T_1) *E is left ideal or right ideal of S;*
(T_2) *if $g \leq w$ for some $w \in E$, then $g \in E$.*
Then E is a left ordered ideal or right ordered ideal.
If E is both left ordered ideal and right ordered ideal of S, then E is ordered ideal of S.

Definition 7. *Suppose S is an ordered semiring, if $x_1 \in S$, there exist $t \in S$ such that $x_1 \leq x_1 t x_1$, then S is called a regular ordered semiring.*

3. Ordered h-ideals semirings

In this section, we characterize the ordered h-ideals semirings by their ordered h-ideals.

Definition 8. *Suppose E is a nonempty subset of an ordered semiring S, then E is a left ordered h-ideal of S if the following properties are satisfied:*
(1) *E is a left ordered ideal of S,*
(2) *if $e + x_1 + t = x_2 + t$ for some $x_1, x_2 \in E, t \in E$, then $e \in E$.*

Similarly, we define the right ordered h-ideal.

If E is both a left ordered h-ideal and a right ordered h-ideal of S, then E is said to be an ordered h-ideal of S.

Definition 9. *Suppose $E \neq \emptyset$, $E \subseteq S$ and S is an ordered semiring, then the h-closure of E, denoted by $\overline{E}$, is defined by*

$$\overline{E} = \{g \in S, \text{ there exist } x_1, x_2 \in E, g + x_1 + h \leq x_2 + h, h \in E\}.$$

Definition 10. *Suppose S is an ordered semiring. If for every $x_1 \in S$, there exist $e, h, c \in S$ such that $x_1 + x_1 e x_1 + c \leq x_1 h x_1 + c,$. Then S is called h-regular ordered semiring.*

Definition 11. *Suppose S is an ordered semiring then $x_1 \in S$ is said to an ordered h-regular if $x_1 \in \overline{(x_1 S x_1]}$. If each element of S is ordered h-regular, then S is said to be an ordered h-regular semiring.*

It is easy to see that each h-regular ordered semiring is an ordered h-regular semiring but converse does not hold. We see this by the following example.

Example 1. *Suppose $S = \{t_1, t_2, t_3\}$. Define binary operations $\cdot$ and $+$ on S as:*

$+$	t_1	t_2	t_3
t_1	t_1	t_1	t_1
t_2	t_1	t_2	t_3
t_3	t_1	t_3	t_3

and

$\cdot$	t_1	t_2	t_3
t_1	t_2	t_2	t_2
t_2	t_2	t_2	t_2
t_3	t_2	t_2	t_2

We define order relation $\leq$ on S as follows :

$$\leq = \{(t_1,t_1), (t_2,t_2), (t_3,t_3), (t_1,t_2), (t_1,t_3), (t_2,t_3)\}.$$

*Then $(S, +, \cdot, \leq)$ is an ordered semiring. Furthermore, forall $a \in S$ **(1)** $a + t_1 + c \leq t_2 + c$, $c \in S$ **(2)** $t_1, t_2 \in (aSa]$ i.e. $t_1 \leq asa$, $t_2 \leq asa$, for some $asa \in aSa$. Hence S is an ordered h-regular semiring. On the other hand $t_3 + t_3at_3 + t_2 \leq t_3ct_3 + t_2$ has no solution, so S is not an h-regular ordered semiring.*

Lemma 1. *Suppose S is an ordered semiring and $E \subseteq S$ and $F \subseteq S$, where E and F are nonempty, then*
(1) *$(\overline{E}] \subseteq \overline{(E]}$.*
(2) *If $E \subseteq F$, then $\overline{E} \subseteq \overline{F}$.*
(3) *$\overline{(E]}F \subseteq \overline{(EF]}$ and $E\overline{(F]} \subseteq \overline{(EF]}$.*

Proof. **(1)** Let $g \in (\overline{E}]$. Then there exists $h \in \overline{E}$ such that $g \leq h$. Since $h \in \overline{E}$, then there exist $r_1, r_2 \in E$ such that $h + r_1 + k \leq r_2 + k$, $k \in E$. It follows that $g + r_1 + k \leq h + r_1 + k \leq r_2 + k$. Since $E \subseteq (E]$, $r_1, r_2 \in (E]$, $k \in (E]$, $g \in \overline{(E]}$, i.e. $(\overline{E}] \subseteq \overline{(E]}$.

(2) Consider $E \subseteq F$. Let $g \in \overline{E}$. Then, there exist $r_1, r_2 \in E$ such that $g + r_1 + k \leq r_2 + k$, $k \in E$. By the assumption, we get $r_1, r_2, k \in F$. This implies $g \in \overline{F}$, so $\overline{E} \subseteq \overline{F}$.

(3) Let $g \in \overline{(E]}$ and $w \in F$. So, there exist $p, q \in (E]$ such that $g + p + s \leq q + s$, $s \in (E]$. So, $gw + pw + sw \leq qw + sw$. Since $p, q, s \in (E]$, $p \leq r_1$ and $q \leq r_1^{/}$ and $s \leq r_1^{//}$,for some $r_1, r_1^{/}, r_1^{//} \in E$, so $pw \leq r_1w \in EF$ and $qw \leq r_1^{/}w \in EF$ and $sw \leq r_1^{//}w \in EF$. This implies that $gw \in \overline{(EF]}$. So $\overline{(E]}F \subseteq \overline{(EF]}$. Similarly we get $E\overline{(F]} \subseteq \overline{(EF]}$. □

Lemma 2. *[15] Suppose $E \subseteq S$, where E is nonempty and S is an ordered semiring. If E is closed under addition, then so are $(E]$, $\overline{(E]}$.*

Now we will use further throughout the section N (set of all positive integers). Let S be ordered semiring, $E \neq \emptyset$ and $E \subseteq S$, suppose $\sum_{finite} E$ be set of all finite sum of elements of E, and for $x \in S$, let $Nx = \{nx | n \in N\}$.

Lemma 3. *Suppose E and F are nonempty subsets of an ordered semiring S, with $E + E \subseteq E$ and $F + F \subseteq F$. Then*
(1) *$E \subseteq (E] \subseteq \overline{E} \subseteq \overline{(E]}$,*
(2) *$\overline{(E]} = \overline{\overline{(E]}}$, if E is left ordered h-ideal (or right ordered h-ideal) of S,*
(3) *$E + F \subseteq \overline{E} + \overline{F} \subseteq \overline{E + F}$,*
(4) *$\overline{(E]} + \overline{(F]} \subseteq \overline{\overline{(E]} + \overline{(F]}} \subseteq \overline{(E + F]}$,*
(5) *$\overline{E}\,\overline{F} \subseteq \overline{(E]}\,\overline{(F]}$,*
(6) *If E and F are two left ordered h-ideal and right ordered h-ideal of S, respectively then $\overline{(E]}\,\overline{(F]} \subseteq \overline{\left(\sum_{finite} EF\right]}$.*

Proof. **(1)** We see that $E \subseteq (E]$.

Let $g \in (E]$, so by definition of "( ]", there exists $r \in E$ such that

$$\begin{aligned} g &\leq r \\ g + r + r &\leq r + r + r. \end{aligned}$$

This implies that $g \in \overline{E} \Longrightarrow (E] \subseteq \overline{E}$.

Since $E \subseteq (E]$,

$$\Longrightarrow \overline{E} \subseteq \overline{(E]}.$$

(2) Let E is left ordered h-ideal (or right ordered h-ideal) of S.

By(i), $(E] \subseteq \overline{(E]}$; $\Longrightarrow \overline{(E]} \subseteq \overline{\overline{(E]}}$.

Let $g \in \overline{\overline{(E]}}$, then by definition of h -closure, there exist $h, k \in \overline{(E]}$ such that

$$g + h + s \leq k + s, s \in \overline{(E]}.$$

Since $h, k, s \in \overline{(E]}$, then by definition of h -closure, there exist $r_1, r_2, r_3, r_4, r_5, r_6 \in (E]$, such that

$$h + r_1 + s_1 \leq r_2 + s_1, s_1 \in (E]$$

$$k + r_3 + s_2 \leq r_4 + s_2, s_2 \in (E]$$

$$s + r_5 + s_3 \leq r_6 + s_3, s_3 \in (E]$$

$\Longrightarrow$

$$\begin{aligned} g + h + s + r_1 + s_1 + r_3 + s_2 + r_5 + s_3 &\leq k + s + r_1 + s_1 + r_3 + s_2 + r_5 + s_3 \\ &\leq r_4 + s_2 + s + r_1 + s_1 + r_5 + s_3 \\ &\leq r_4 + s_2 + r_6 + s_3 + r_1 + s_1 \\ &= r_1 + r_4 + r_6 + s_1 + s_2 + s_3 \end{aligned}$$

$\Longrightarrow$

$$g + (h + s + r_1 + r_3 + r_5) + (s_1 + s_2 + s_3) \leq (r_1 + r_4 + r_6) + (s_1 + s_2 + s_3).$$

Since $s_1, s_2, s_3 \in (E]$, then by definition of "( ]", there exist $s_1^{/}, s_2^{/}, s_3^{/} \in E$ such that $s_1 \leq s_1^{/}$ and $s_2 \leq s_2^{/}$ and $s_3 \leq s_3^{/}$.

$$\Longrightarrow s_1 + s_2 + s_3 \leq s_1^{/} + s_2^{/} + s_3^{/}$$

As E is a left ordered h-ideal (or right ordered h-ideal) of S, so E is a left ordered ideal (or right ordered ideal) of S.

Then by definition of left ordered ideal or right ordered ideal, we get

$$s_1 + s_2 + s_3 \in E.$$

Now, since $r_1, r_4, r_6 \in (E]$, so by using definition of "( ]", there exist $r_1^{/}, r_4^{/}, r_6^{/} \in E$, such that $r_1 \leq r_1^{/}$ and $r_4 \leq r_4^{/}$ and $r_6 \leq r_6^{/}$

$$\Longrightarrow r_1 + r_4 + r_6 \leq r_1^{/} + r_4^{/} + r_6^{/}.$$

Then by definition of left ordered ideal or right ordered ideal, we get

$$r_1 + r_4 + r_6 \in E.$$

Then by definition of left ordered ideal or right ordered ideal, we get $r_1, r_2, r_5, r_6, s_1, s_3 \in E$

$$\begin{aligned} h + s + (r_1 + r_5 + s_1 + s_3) &\leq (r_2 + r_6 + s_1 + s_3) \\ g + (h + s + r_1 + r_3 + r_5) + (s_1 + s_2 + s_3) &\leq (r_1 + r_4 + r_6) + (s_1 + s_2 + s_3) \end{aligned}$$

where $s_1 + s_2 + s_3 \in E$, then by definition of h-closure, we get $g \in \overline{E}$, then by **(1),** we get $g \in \overline{E} \subseteq \overline{(E]} \implies g \in \overline{(E]}, \implies \overline{\overline{(E]}} \subseteq \overline{(E]}$

$$\implies \overline{(E]} = \overline{\overline{(E]}}$$

(3) From **(1),** we have $E \subseteq \overline{E}$ and $F \subseteq \overline{F}$

$$\implies E + F \subseteq \overline{E} + \overline{F}.$$

Now we show $\overline{E} + \overline{F} \subseteq \overline{E + F}$. Suppose $g \in \overline{E} + \overline{F}$, so there exists $h \in \overline{E}$ and $k \in \overline{F}$ such that $g = h + k$.

Since $h \in \overline{E}, k \in \overline{F}$, so by using definition of h-closure, there exist $r, r^{/} \in E$ and $w, w^{/} \in F$ such that

$$h + r + s_1 \leq r^{/} + s_1, s_1 \in E,$$

and

$$k + w + s_2 \leq w^{/} + s_2, s_2 \in F.$$

$\implies$

$$\begin{aligned} g + r + s_1 + w + s_2 &= h + k + r + s_1 + w + s_2 \\ g + (r + w) + (s_1 + s_2) &\leq r^{/} + s_1 + w^{/} + s_2 \\ &= r^{/} + w^{/} + s_1 + s_2 \end{aligned}$$

$$g + (r + w) + (s_1 + s_2) \leq \left(r^{/} + w^{/}\right) + (s_1 + s_2)$$

As $(s_1 + s_2) \in E + F$, then by definition of h-closure, we get $g \in \overline{E + F}$,

$$\implies \overline{E} + \overline{F} \subseteq \overline{E + F}$$

(4) Let $g \in \overline{(E]} + \overline{(F]}$, then there exists $p \in \overline{(E]}, q \in \overline{(F]}$, such that $g = p + q$.

Now,

$$g + (p + q) + h = (p + q) + (p + q) + h \implies g + (p + q) + h = (p + p) + (q + q) + h.$$

Since $p + p \in \overline{(E]}$ and $q + q \in \overline{(F]}$, then by definition of h-closure, we get $g \in \overline{\overline{(E]} + \overline{(F]}}$,

$$\implies \overline{(E]} + \overline{(F]} \subseteq \overline{\overline{(E]} + \overline{(F]}}.$$

Suppose $g \in \overline{(E]} + \overline{(F]}$, so there exists $p \in \overline{(E]}$, $q \in \overline{(F]}$, such that $g = p + q$.

Since $p \in \overline{(E]}$ and $q \in \overline{(F]}$, so by using definition of h-closure, there exist $r, r' \in (E]$ and $w, w' \in (F]$ such that

$$p + r + s_1 \leq r' + s_1, s_1 \in (E],$$

and

$$q + w + s_2 \leq w' + s_2, s_2 \in (F].$$

$$g + r + w + s_1 + s_2 = p + q + r + w + s_1 + s_2$$

$$g + r + w + s_1 + s_2 \leq r' + w' + s_1 + s_2.$$

Since $(r + w), \left(r' + w'\right) \in (E + F]$,

$$g + (r + w) + (s_1 + s_2) \leq \left(r' + w'\right) + (s_1 + s_2).$$

This implies $g \in \overline{(E + F]}$, $\Longrightarrow$

$$\begin{aligned} \overline{(E]} + \overline{(F]} &\subseteq \overline{(E + F]} \\ \overline{\overline{(E]} + \overline{(F]}} &\subseteq \overline{\overline{(E + F]}} \\ \overline{\overline{(E]} + \overline{(F]}} &\subseteq \overline{(E + F]}. \end{aligned}$$

(5) By Lemma 1**,** we get

$$\overline{E}\,\overline{F} \subseteq \overline{(E]}\,\overline{(F]}.$$

(6) Let E, F are two left ordered h-ideal or right ordered h-ideal of S. We will prove that $\overline{(E]}\,\overline{(F]} \subseteq \overline{\left(\sum_{fij} EF\right]}$.

For this, let $g \in \overline{(E]}\,\overline{(F]}$ then $g = hk$, as $h \in \overline{(E]}, k \in \overline{(F]}$, then by definition of h-closure, there exist $p, p' \in (E]$ and $q, q' \in (F]$ such that

$$h + p + s_1 \leq p' + s_1, s_1 \in (E],$$

and

$$k + q + s_2 \leq q' + s_2, s_2 \in (F].$$

As

$$hk + pk + s_1k \leq p'k + s_1k.$$

Also

$$pk + pq + ps_2 \leq pq' + ps_2 \text{ and } p'k + p'q + p's_2 \leq p'q' + p's_2.$$

As

$$\begin{aligned} g &= hk \\ g + pk + pq + p^{/}q + \left(s_1k + p^{/}s_2 + ps_2\right) &= hk + pk + pq + p^{/}q + s_1k + p^{/}s_2 + ps_2 \\ &\leq p^{/}k + s_1k + pq + p^{/}q + p^{/}s_2 + ps_2 \\ &\leq p^{/}q^{/} + p^{/}s_2 + s_1k + pq + ps_2 \\ &= pq + p^{/}q^{/} + \left(s_1k + p^{/}s_2 + ps_2\right). \end{aligned}$$

Since E and F are left ordered ideal and right ordered ideal of S respectively, therefore,

$$pk + pq + p^{/}q \in \sum_{finite} EF, pq + p^{/}q^{/} \in \sum_{finite} EF,$$

and

$$s_1k + p^{/}s_2 + ps_2 \in \sum_{finite} EF.$$

This implies that

$$g + \left(pk + pq + p^{/}q\right) + \left(s_1k + p^{/}s_2 + ps_2\right) \leq \left(pq + p^{/}q^{/}\right) + \left(s_1k + p^{/}s_2 + ps_2\right).$$

So

$$g \in \overline{\sum_{finite} EF} \subseteq \overline{\left(\sum_{finite} EF\right]} \Longrightarrow g \in \overline{\left(\sum_{finite} EF\right]}.$$

$$\Longrightarrow \overline{(E]}\,\overline{(F]} \subseteq \overline{\left(\sum_{finite} EF\right]}.$$

□

Example 2. ***(i)*** *Every regular ordered semiring is an ordered h-regular semiring.*
(ii) Consider the semiring $(N, +, \cdot, \leq)$***, where*** N ***is the set of natural numbers. We define the relation*** $\leq$ ***on*** N ***by*** $g \leq h \Leftrightarrow g \geq h$ ***for all*** $g, h \in N$***. Then,*** $(N, \leq)$ ***is a partially ordered set, furthermore*** $(N, +, \cdot, \leq)$ ***is an ordered semiring. Since*** $g + ghg + s \leq ghg + s$ ***for all*** $g, h \in N, s \in N$***,*** $(N, +, \cdot, \leq)$ ***is an ordered*** h***-regular semiring. Moreover, since*** $2 \in N, 2 \nleq 2h2 = 4h$ ***for all*** $h \in N$***,*** $(N, +, \cdot, \leq)$ ***is not a regular ordered semiring. In addition, we get*** $(2N]$ ***is an ordered ideal of*** $(N, +, \cdot, \leq)$ ***which is not an*** h***-ideal, for the reason that*** $2 + 4 + 1 = 3 + 4$ *as* $1 \notin (2N]$.

Theorem 1. *Suppose* S *is an ordered semiring and* E *be left ideal or right ideal or ideal, then conditions given below are equivalent:*
(1) *E is left ordered h-ideal or right ordered h-ideal or ordered h-ideal of* S*;*
(2) *Let* $g \in S, g + r_1 + h \leq r_2 + h$ *for some* $r_1, r_2 \in E$*,* $h \in E$ *then* $g \in E$*;*
(3) $\overline{E} = E$.

Proof. **(1)**$\Longrightarrow$**(2)** Suppose E is a left ordered h-ideal. Suppose $g \in S$ such that $g + r_1 + h \leq r_2 + h$ for some $r_1, r_2 \in E$, $h \in E$ then by using definition of left ordered h-ideal, we get $g \in E$.

(2)⟹(3) Suppose **(2)** is true. Consider $g \in \overline{E}$, then there exist $r_1, r_2 \in E$ such that $g + r_1 + h \leq r_2 + h$, $h \in E$. By condition **(2),** we get $g \in E$. So, $\overline{E} \subseteq E$. Since $E \subseteq \overline{E}$, therefore $\overline{E} = E$.

(3)⟹(1) Assume that $\overline{E} = E$. Let $g \in S$ be such that $g + r_1 + h \leq r_2 + h$ for some $r_1, r_2 \in E$, $h \in E$. Then $g \in \overline{E}$. Since $\overline{E} = E$, so $g \in \overline{E} = E$. Thus $g \in E$. Since $g + r_1 + h \leq r_2 + h$ for some $r_1, r_2 \in E$, $h \in E$, then $g \in E$, so by the definition of left ordered h-ideal or right ordered h-ideal or ordered h-ideal, we get E is left ordered h-ideal or right ordered h-ideal or ordered h-ideal of S. □

Theorem 2. *Suppose S is an ordered semiring, then:*
(1) Intersection of any family of left ordered h-ideals of S is a left ordered h-ideal.
(2) Intersection of any family of right ordered h-ideals of S is a right ordered h-ideal.
(3) Intersection of any family of ordered h-ideals of S is an ordered h-ideal.

Proof. **(1)** Suppose E_n is a left ordered h-ideal of S for all $n \in J$, as $\cap_{n\in J} E_n \neq \emptyset$. Since E_n is a left ordered h-ideal, we get E_n is a left ordered ideal for all $n \in J$. Then $\cap_{n\in J} E_n$ is left ordered ideal. Consider there exist $g \in S$ and $r_1, r_2 \in \cap_{n\in J} E_n$, $h \in \cap_{n\in J} E_n$ is such that $g + r_1 + h = r_2 + h$. Since $\cap_{n\in J} E_n \subseteq E_n$ for all $n \in J$, we get, $r_1, r_2, h \in E_n$. Since E_n is a left ordered h-ideal and $r_1, r_2 \in E_n$, $g + r_1 + h = r_2 + h$, $h \in E_n$ for all $n \in J$, so by using definition of left ordered h-ideal, we get $g \in E_n$ for all $n \in J$. So $g \in \cap_{n\in J} E_n$. Therefore, $r_1, r_2 \in \cap_{n\in J} E_n$, $g + r_1 + h = r_2 + h$, $h \in \cap_{n\in J} E_n$. Then $g \in \cap_{n\in J} E_n$. By definition of left ordered h-ideal, we get $\cap_{n\in J} E_n$ is a left ordered h-idealof S.
(2) Suppose that E_n is a right ordered h-ideal of S for all $n \in J$, as $\cap_{n\in J} E_n \neq \emptyset$. Since E_n is right ordered h-ideal, we get, E_n is right ordered ideal for all $n \in J$. Then $\cap_{n\in J} E_n$ is right ordered ideal. Consider that there exist $g \in S$ and $r_1, r_2 \in \cap_{n\in J} E_n$, $h \in \cap_{n\in J} E_n$ such that $g + r_1 + h = r_2 + h$. Since $\cap_{n\in J} E_n \subseteq E_n$ for all $n \in J$, we have $r_1, r_2, h \in E_n$. Since E_n is a right ordered h-ideal and $r_1, r_2 \in E_n$, $g + r_1 + h = r_2 + h$, $h \in E_n$ for all $n \in J$. So by using the definition of right ordered h-ideal, we get $g \in E_n$ for all $n \in J$. So $g \in \cap_{n\in J} E_n$. Therefore, $r_1, r_2 \in \cap_{n\in J} E_n$, $g + r_1 + h = r_2 + h$, $h \in \cap_{n\in J} E_n$. Then $g \in \cap_{n\in J} E_n$. By definition of right ordered h-ideal, we have $\cap_{n\in J} E_n$ is a right ordered h-idealof S.
(3) From **(1)** and **(2),** we get $\cap_{n\in J} E_n$ is a left and right ordered h-ideal of S. Therefore, $\cap_{n\in J} E_n$ is an ordered h-ideal of S. Hence proved. □

Remark 1. (1) *The sum of two left ordered h-ideals or right ordered h-ideals is a left ordered h-ideal or right ordered h-ideal.*
(2) *The sum of two left ordered ideals or right ordered ideals is a left ordered ideal or right ordered ideal.*
(3) *The sum of two left ideals or right ideals is a left ideal or right ideal.*

Theorem 3. *Suppose S is an ordered semiring and $E \neq \emptyset, F \neq \emptyset, E \subseteq S, F \subseteq S$.*
(1) Consider E, F be two left ordered h-ideals, then $\overline{(E + F]}$ is smallest left ordered h-ideal containing $E \cup F$.
(2) Consider E, F be two right ordered h-ideals, then $\overline{(E + F]}$ is smallest right ordered h-ideal containing $E \cup F$.
(3) Consider E, F be two ordered h-ideals, then $\overline{(E + F]}$ is smallest ordered h-ideal containing $E \cup F$.

Proof. **(1)** Suppose E, F are two left ordered h-ideal of S. Suppose $g, h \in \overline{(E + F]}$, $s \in S$.

By the definition of h-closure, there exist $r, r^/, w, w^/ \in (E + F]$ such that

$$g + r + f_1 \leq r^/ + f_1, \ f_1 \in (E + F],$$

and

$$h + w + f_2 \leq w^{/} + f_2,\ f_2 \in (E+F]\,.$$

Hence

$$g + h + r + w + f_1 + f_2 \leq r^{/} + w^{/} + f_1 + f_2$$

and

$$sg + sr + sf_1 \leq sr^{/} + sf_1$$

As $(sf_1) \in (E+F]$, so by using the definition of h-closure, we get $(g+h) \in \overline{(E+F]}$ and $(sg) \in \overline{(E+F]}$.

This implies

$$\overline{(E+F]} \subseteq \overline{\overline{(E+F]}}$$

Let $g \in \overline{\overline{(E+F]}}$, then by definition of h-closure, there exist $i, v \in \overline{(E+F]}$ such that

$$g + i + f \leq v + f, \quad f \in \overline{(E+F]}.$$

Since $i, v, f \in \overline{(E+F]}$, then by definition of h-closure, there exist $r, r^{/}, w, w^{/}, d, d^{/} \in (E+F]$, such that

$$i + r + f_1 \leq r^{/} + f_1,\ f_1 \in (E+F]$$

$$v + w + f_2 \leq w^{/} + f_2,\ f_2 \in (E+F]$$

and

$$f + d + f_3 \leq d^{/} + f_3,\ f_3 \in (E]\,.$$

Now,

$$\begin{aligned} g + i + f + r + w + d + f_1 + f_2 + f_3 &\leq v + f + r + w + d + f_1 + f_2 + f_3 \\ &\leq w^{/} + f_2 + f + r + d + f_1 + f_3 \\ &= f + d + f_3 + w^{/} + f_2 + r + f_1 \\ &\leq d^{/} + f_3 + w^{/} + f_2 + r + f_1 \\ &= r + w^{/} + d^{/} + f_1 + f_2 + f_3. \end{aligned}$$

Since $\left(r + w^{/} + d^{/}\right), (i + f + r + w + d) \in E + F$ and $(f_1 + f_2 + f_3) \in E + F$, then by definition of h-closure, we get $g \in \overline{E+F} \subseteq \overline{(E+F]} \Longrightarrow \overline{\overline{(E+F]}} \subseteq \overline{(E+F]}$.

So, we get $\overline{\overline{(E+F]}} = \overline{(E+F]}$.

This shows that $\overline{(E+F]}$ is a left ordered h-ideal.

Suppose $g \in E \cup F$, then $g \in E$ or $g \in F$

As $g \in E$, then $g + (g + w) = (g + g) + w \in E + F$, for all $w \in F$. Thus $g \in \overline{(E+F]}$

As $g \in F$, then $(r + g) + g = r + (g + g) \in E + F$, for all $r \in E$. Thus $g \in \overline{(E+F]}$

Hence,

$$E \cup F \subseteq \overline{(E+F]}$$

Suppose L is a left ordered h-ideal containing $E \cup F$.

Then $E + F \subseteq L$ and hence $(E+F] \subseteq (L] = L$ implies that $\overline{(E+F]} \subseteq \overline{L} = L$

Therefore, $\overline{(E+F]}$ is the smallest left ordered h-ideal containing $E \cup F$.

(2) This is similar to **(1)**.

(3) From **(1)** and **(2)**, we prove that $\overline{(E+F]}$ is smallest left and right ordered h-ideal containing $E \cup F$. Therefore, $\overline{(E+F]}$ is smallest ordered h-ideal containing $E \cup F$. □

Theorem 4. *Suppose S is an ordered semiring and $E \neq \emptyset$, $E \subseteq S$. Then these properties hold.*
(1) Consider E a left ideal, then $\overline{(E]}$ is the smallest left ordered h-ideal containing E.
(2) Consider E a right ideal, then $\overline{(E]}$ is the smallest right ordered h-ideal containing E.
(3) Consdier E an ideal, then $\overline{(E]}$ is the smallest ordered h-ideal containing E.

Proof. Suppose E is a left ideal. We know that $\overline{(E]}$ is closed with respect to the operation of addition.

Suppose $g \in \overline{(E]}$, and $k \in E$, then by using definition of h -closure, there exist $r, w \in (E]$ such that

$$g + r + h \leq w + h, h \in (E].$$

Hence

$$kg + kr + kh \leq kw + kh.$$

So by using definition of "( ]", we have $kh \in (E]$. Since $(kr), (kw) \in (E]$, $kg + (kr) + (kh) \leq (kw) + (kh)$, $(kh) \in (E]$.

Then by definition of h-clousre, we get $kg \in \overline{(E]}$. Therefore,$\overline{(E]}$ is a left ordered h-ideal.

We know that $\overline{(E]}$ is a left ordered h-ideal containing E.

Suppose Q is a left ordered h-ideal containing E. So $(E] \subseteq (Q] = Q$. Then, $\overline{(E]} \subseteq \overline{Q} = Q$. Therefore, $\overline{(E]}$ is the smallest left ordered h-ideal containing E.

(2) This is similar to (1).

(3) From **(1)** and **(2)**, we prove that $\overline{(E]}$ is the smallest left and right ordered h-ideal containing E. Therefore, $\overline{(E]}$ is the smallest ordered h-ideal containing E. □

Corollary 1. *Suppose S is an ordered semiring, let $\emptyset \neq E \subseteq S$. We denote the smallest left ordered h-ideal containing E by $L_h(E)$, the smallest right ordered h-ideal containing E by $R_h(E)$, and the smallest ordered h-ideal of S containing E by $M_h(E)$. Then, the following results follows:*
(1) $L_h(E) = \overline{(\sum_{finite} E + \sum_{finite} SE]}$,
(2) $R_h(E) = \overline{(\sum_{finite} E + \sum_{finite} ES]}$,
(3) $M_h(E) = \overline{(\sum_{finite} E + \sum_{finite} SE + \sum_{finite} ES + \sum_{finite} SES]}$.

Proof. We want to prove $\sum_{finite} E + \sum_{finite} SE$ is a left ideal.

For this,

(I_1) Let $a, b \in \sum_{finite} E + \sum_{finite} SE$. Then

$$a + b \in \sum_{finite} E + \sum_{finite} SE$$

(I_2) Let $a \in \sum_{finite} E + \sum_{finite} SE, r \in S$

$\Longrightarrow$

$$\begin{aligned} ra &\in r\left(\sum_{finite} E + \sum_{finite} SE\right) \\ &= \sum_{finite} rE + \sum_{finite} rSE \\ &\subseteq \sum_{finite} E + \sum_{finite} SE \end{aligned}$$

$$\Longrightarrow ra \in \sum_{finite} E + \sum_{finite} SE$$

Therefore, $\sum_{finite} E + \sum_{finite} SE$ is a left ideal. By Theorem 4, we get

$$L_h(E) = \overline{(\sum_{finite} E + \sum_{finite} SE]}$$

We see that the proofs of **(2)** and **(3)**are similar to that of **(1)**. □

Corollary 2. *Suppose S is an ordered semiring, let $r \in S$. Then*
(1) $L_h(r) = \overline{(Nr + Sr]}$;
(2) $R_h(r) = \overline{(Nr + rS]}$;
(3) $M_h(r) = (Nr + Sr + rS + \sum_{finite} SrS]$, *where N = Set of natural numbers.*

4. Ordered h-regular semirings

We now give the characterization of ordered h-regular semirings by their ordered h-ideals.

Theorem 5. *Consider S is an ordered semiring. Then S is an ordered h-regular iff $E \cap F = \overline{(EF]}$, for all right ordered h-ideals E, left ordered h-ideals F of S.*

Proof. Suppose S is an ordered h-regular semiring and E is right ordered h-ideal, F is left ordered h-ideal of S. Then, we have $EF \subseteq E$ and $EF \subseteq F$. Thus, $(EF] \subseteq (E] = E$ and $(EF] \subseteq (F] = F$.

This implies $\overline{(EF]} \subseteq \overline{E} = E$ and $\overline{(EF]} \subseteq \overline{F} = F$. Thus $\overline{(EF]} \subseteq E \cap F$

Let $p \in E \cap F$. As S is an ordered h-regular, there exist $h, k \in (pSp]$, such that

$$p + h + o \leq k + o, o \in (pSp].$$

Since $h, k, o \in (pSp]$, then by definition of "( ]", there exist $s, j, j_1 \in S$ such that

$$h \leq psp,\ k \leq pjp,\ o \leq pj_1p.$$

Since E is a right ordered h-ideal, F is a left ordered h-ideal, we have $psp, pjp, pj_1p \in EF$. Since $h \leq psp \in EF, k \leq pjp \in EF, o \leq pj_1p \in EF$, so by using definition of "( ] ", we have $h, k, o \in (EF]$ so $p \in \overline{(EF]} \implies E \cap F \subseteq \overline{(EF]}$

$$\Longrightarrow E \cap F = \overline{(EF]}$$

Conversely, Consider $E \cap F = \overline{(EF]}$ for all right ordered h-ideals E of S, left ordered h-ideals F of S. Suppose $d \in S$, then by above Corollary 2 we get

$$L_h(d) = \overline{(Nd + Sd]} \text{ and } R_h(d) = \overline{(Nd + dS]}.$$

By assumption, $(R_h(d) \cap L_h(d) = \overline{(R_h(d)L_h(d)]})$. Now we show that $R_h(d)L_h(d) \subseteq \overline{(Sd]} \cap \overline{(dS]}$
Let $p \in R_h(d)$ and $h \in L_h(d)$.
Since $p \in R_h(d)$, then by the definition of h-closure,

$$\text{There exist} i, i^{/} \in (Nd + dS], \text{ such that } p + i + o_1 \leq i^{/} + o_1, o_1 \in (Nd + dS].$$

Since $h \in L_h(d)$, then by the definition of h-closure,

$$\text{There exist } v, v^{/} \in (Nd + Sd], \text{ such that } h + v + o_2 \leq v^{/} + o_2, o_2 \in (Nd + Sd],$$

$$ph + ih + o_1h \leq i^{/}h + o_1h.$$

Since $h, k, o \in (pSp]$, then by definition of "( ]", there exist $s, j, j_1 \in S$ such that

$$i \leq ed + ds, \quad i^{/} \leq fd + dj, \quad o_1 \leq gd + dj_1$$

It follows that

$$ih \leq edh + dsh = d\,(eh + sh) \in dS$$

$$i^{/}h \leq fdh + djh = d\,(fh + jh) \in dS$$

$$o_1h \leq gdh + dj_1h = d\,(gh + j_1h) \in dS.$$

Then by definition of "( ]", we get $ih, i^{/}h, o_1h \in (dS]$. Since $ih, i^{/}h \in (dS]$,

$$ph + ih + o_1h \leq i^{/}h + o_1h, \ o_1h \in (dS]\,.$$

Then by definition of h-closure, we get $ph \in \overline{(dS]}$. So,

$$R_h(d)L_h(d) \subseteq \overline{(dS]}$$

Similarly, we can show that

$$R_h(d)L_h(d) \subseteq \overline{(Sd]}$$

Therefore,

$$R_h(d)L_h(d) \subseteq \overline{(Sd]} \cap \overline{(dS]}.$$

Since $\overline{(dS]}$ is a right ordered h-ideal, $\overline{(Sd]}$ is a left ordered h-ideal and by assumption, we have,

$$\overline{(dS]} \cap \overline{(Sd]} \subseteq \overline{\left(\overline{(dS]} \cap \overline{(Sd]}\right]}.$$

Now we will show that $\overline{(dS]}\overline{(Sd]} \subseteq \overline{(dSd]}$. For this let $m \in \overline{(dS]}$ and $k \in \overline{(Sd]}$, then by definition of h-closure, there exist $l, l_1 \in (dS]$ and $q, q_1 \in (Sd]$, such that

$$m + l + o_3 \leq l_1 + o_3, o_3 \in (dS]$$

and

$$k + q + o_4 \leq q_1 + o_4, o_4 \in (Sd]$$

From above equations we get ,

$$\begin{aligned} mk + lk + o_3k &\leq l_1k + o_3k \\ lk + lq + lo_4 &\leq lq_1 + lo_4 \\ l_1k + l_1q + l_1o_4 &\leq l_1q_1 + l_1o_4 \\ o_3k + o_3q + o_3o_4 &\leq o_3q_1 + o_3o_4. \end{aligned}$$

Since S is a multiplicatively commutative ordered semiring,

$$\begin{aligned} mk + (lk + lq + l_1q) + (o_3k + lo_4 + l_1o_4) &\leq l_1k + o_3k + lq + l_1q + lo_4 + l_1o_4 \\ &\leq l_1q_1 + l_1o_4 + o_3k + lq + lo_4 \\ &= (l_1q_1 + lq) + (o_3k + lo_4 + l_1o_4). \end{aligned}$$

Since $l, l_1, o_3 \in (dS], q, q_1, o_4 \in (Sd]$, then by definition of “( ]”, there exist $s_1, s_2, s_3, l^{/}, l^{//}, l^{///} \in S$, such that

$$l \leq ds_1,\ l_1 \leq ds_2,\ o_3 \leq ds_3 \text{ and } q \leq l^{/}d,\ q_1 \leq l^{//}d,\ o_4 \leq l^{///}d.$$

Hence, we obtained,

$$\begin{aligned} lk + lq + l_1q &\leq l_1q + lo_4 + lq_1 + lo_4 \\ &\leq ds_2l^{/}d + ds_1l^{///}d + ds_1l^{//}d + ds_1l^{///}d \in dSd \end{aligned}$$

$\Longrightarrow$

$$lk + lq + l_1q \in (dSd]$$

and

$$\begin{aligned} o_3k + lo_4 + l_1o_4 &\leq o_3q + o_3o_4 + lo_4 + l_1o_4 + o_3q_1 + o_3o_4 \\ &\leq ds_3l^{/}d + ds_3l^{///}d + ds_1l^{///}d + ds_2l^{///}d + ds_3l^{//}d + ds_3l^{///}d \in dSd. \end{aligned}$$

Then by definition of “( ]”,

$$o_3k + lo_4 + l_1o_4 \in (dSd] \text{ and } l_1q_1 + lq \leq ds_2l^{//}d + ds_1l^{/}d \in dSd$$

$\Longrightarrow$

$$l_1q_1 + lq \in (dSd].$$

So, $mk \in \overline{(dSd]}$. Hence $\overline{(dS](Sd]} \subseteq \overline{(dSd]}$.

Hence

$$\overline{(dS]} \cap \overline{(Sd]} \subseteq \overline{\left(\overline{(dS]} \cap \overline{(Sd]}\right]} \subseteq \overline{\overline{(dSd]}} = \overline{(dSd]}.$$

Thus,

$$R_h(d) \cap L_h(d) = \overline{(R_h(d)L_h(d)]} \subseteq \overline{\left(\overline{(dS]} \cap \overline{(Sd]}\right]} \subseteq \overline{\overline{(dSd]}} = \overline{(dSd]}$$

It turns out, $d \in \overline{(dSd]}$. Hence, S is an ordered h-regular. □

Corollary 3. *Suppose S is a commutative ordered semiring. Then S is an ordered h-regular iff for each ordered h-ideal E of S, $E = \overline{(E^2]}$.*

Proof. Suppose S is an ordered h-regular. Consider E is an ordered h-ideal of S. Obviously we have $E = E \cap E = \overline{(E^2]}$.

Conversely, Suppose that for each ordered h-ideal E of S, $E = \overline{(E^2]}$. Consider $r \in S$. As S is a commutatively multiplication ordered semiring, we get

$$M_h(r) = L_h(r) = R_h(r).$$

So,

$$\begin{aligned} r &\in M_h(r) = \overline{(M_h(r)M_h(r)]} \\ &= \overline{(R_h(r)L_h(r)]} \\ &= \overline{\left(\overline{(Nr + rS]}\,\overline{(Nr + Sr]}\right]} \\ &\subseteq \overline{\left(\sum_{finite} \overline{(Nr + rS]}\,\overline{(Nr + Sr]}\right]} \\ &\subseteq \overline{\left(\overline{(rS]}\right]} = \overline{(rS]}. \end{aligned}$$

Since S is commutative i.e., $\overline{(rS]} = \overline{(Sr]}$, and $\overline{(rS]}$ is an ordered h-ideal, therefore

$$\begin{aligned} r &\in \overline{(rS]} = \overline{\left((\overline{(rS]}^2\right]} = \overline{\left(\overline{(rS]}\,\overline{(rS]}\right]} \\ &= \overline{\left(\overline{(rS]}\,\overline{(Sr]}\right]} \\ &\subseteq \overline{\left(\overline{(\sum_{finite} rSSr]}\right]} \subseteq \overline{\left(\overline{(rSr]}\right]} = \overline{(rSr]}. \end{aligned}$$

Thus, S is an ordered h-regular. □

Definition 12. *Suppose S is an ordered semiring, let $r \in S$. Suppose $r \in \overline{(Sr^2]}$, then r is said to be left ordered h-regular. Suppose $r \in \overline{(r^2S]}$, then r is called right ordered h-regular. Suppose each element of S is left or right ordered h-regular. Then, ordered semiring S is said to be a left or right ordered h-regular.*

Theorem 6. *Suppose S is left ordered h-regular semiring. Then*
(1) *for each left ordered h-ideal E of S, $\overline{(E^2]} = E$;*
(2) *$Q \cap E = \overline{(QE]}$, for each left ordered h-ideal E and each ordered h-ideal Q of S.*

Proof. **(1)** Suppose E is left ordered h-ideal of S. Then, we get $\overline{(E^2]} \subseteq \overline{(E]} = E$.

Suppose $r \in E$. As S is a left ordered h-regular, so $r \in \overline{(Sr^2]}$.

Since $\overline{(Sr^2]} \subseteq \overline{(SE^2]} \subseteq \overline{(E^2]}$, $\implies r \in \overline{(E^2]}$. Hence, $E \subseteq \overline{(E^2]} \implies \overline{(E^2]} = E$.

(2) Let E is left ordered h-ideal and Q is ordered h-ideal of S. Then, we get, $\overline{(QE]} \subseteq \overline{(Q]} = Q$ and $\overline{(QE]} \subseteq \overline{(E]} = E$. Hence, $\overline{(QE]} \subseteq Q \cap E$. Let $a \in Q \cap E$. As S is left ordered h-regular,

$$a \in \overline{(Sa^2]} \subseteq \overline{(SQE]} \subseteq \overline{(QE]}.$$

$\Longrightarrow a \in \overline{(QE]}$. Hence, $Q \cap E \subseteq \overline{(QE]}$. Thus,

$$\Longrightarrow Q \cap E = \overline{(QE]}.$$

□

Theorem 7. *Suppose S is a right ordered h-regular semiring. Then*
(1) for each right ordered h-ideal E of S, $\overline{(E^2]} = E$;
(2) $E \cap Q = \overline{(EQ]}$, for each right ordered h-ideal E and each ordered h-ideal Q of S.

Theorem 8. *Suppose S is an ordered semiring,then the conditions given below are equivalent:*
(1) for each left ordered h-ideal E, F of S, $E \cap F = \overline{(EF]}$.
(2) for each left ordered h-ideal E and each ordered h-ideal Q of S, $E \cap Q = \overline{(EQ]}$.
(3) S is left ordered h-regular and $R_h(E) \subseteq L_h(E)$ for all $\emptyset \neq E \subseteq S$

Proof. **(1) ⇒(2)** Let E is left ordered h-ideal of S and F is ordered h-ideal of S. Then $E \cap F = \overline{(EF]}$, F being left ordered h-ideal of S, we get $E \cap F = \overline{(EF]}$.

(2) ⇒(3) Let $\emptyset \neq E \subseteq S$. By assumption, we get $L_h(E) = L_h(E) \cap S = \overline{(L_h(E)S]}$. We have

$$\begin{aligned}
R_h(E) &= \overline{(\sum_{finite} E + \sum_{finite} ES]} \\
&\subseteq \overline{(\sum_{finite} L_h(E) + \sum_{finite} L_h(E)S]} \\
&= \overline{\left(\sum_{finite} \overline{(L_h(E)S]} + \sum_{finite} \overline{(L_h(E)S]}S\right]} \\
&\subseteq \overline{\left(\sum_{finite} \overline{(L_h(E)S]} + \sum_{finite} \overline{(L_h(E)SS]}\right]} \\
&= \overline{\left(\sum_{finite} \overline{(L_h(E)S]}\right]} \\
&= \overline{\sum_{finite} L_h(E)} \\
&= L_h(E).
\end{aligned}$$

Moreover, we show that $L_h(E) = M_h(L_h(E))$. Since $R_h(E) \subseteq L_h(E)$, we get, $L_h(E) \subseteq R_h(L_h(E)) \subseteq L_h(L_h(E)) = L_h(E)$.

Hence, $L_h(E) = R_h(L_h(E))$. It follows that $L_h(E) = M_h(L_h(E))$

Let $p \in S$. From assumption, we get

$$
\begin{aligned}
p &\in L_h(p) \cap M_h(p) = \overline{(L_h(p)M_h(p)]} \\
&= \overline{(L_h(p)M_h(L_h(p))]} \\
&= \overline{(L_h(p)L_h(p)]} \\
&\subseteq \overline{(Np^2 + \sum_{finite} Sp^2 + \sum_{finite} pSp + \sum_{finite} SpSp]} \\
&\subseteq \overline{(Np^2 + \sum_{finite} Sp^2 + \sum_{finite} R_h(p)p + \sum_{finite} SR_h(p)p]} \\
&\subseteq \overline{(Np^2 + \sum_{finite} Sp^2 + \sum_{finite} L_h(p)p + \sum_{finite} SL_h(p)p]} \\
&\subseteq \overline{(Np^2 + \sum_{finite} Sp^2 + \sum_{finite} L_h(p)p + \sum_{finite} L_h(p)p]} \\
&= \overline{(Np^2 + \sum_{finite} Sp^2 + \sum_{finite} L_h(p)p]} \\
&= \overline{(Np^2 + Sp^2 + L_h(p)p]} \\
&= \overline{\left(Np^2 + Sp^2 + \overline{(Np + Sp]}p\right]} \\
&\subseteq \overline{\left(Np^2 + Sp^2 + \overline{(Np^2 + Sp^2]}\right]} \\
&= \overline{\left(\overline{(Np^2 + Sp^2]}\right]} \\
&= \overline{(Np^2 + Sp^2]}
\end{aligned}
$$

$\Longrightarrow$

$$p \in \overline{(Np^2 + Sp^2]}$$

Since $p \in \overline{(Np^2 + Sp^2]}$, then by definition of h-closure, there exist $v, v' \in (Np^2 + Sp^2]$, such that

$$p + v + t_1 \leq v' + t_1, \text{ where } t_1 \in (Np^2 + Sp^2].$$

Since $v, v', t_1 \in (Np^2 + Sp^2]$, then by definition of "( ]",there exist $e, f, g \in N$ and $s, r, r_1 \in S$ such that

$$v \leq ep^2 + sp^2,\ v' \leq fp^2 + rp^2,\ t_1 \leq gp^2 + r_1p^2$$

In a similar way, we obtain

$$p^2 \in \overline{(Np^4 + Sp^4]}.$$

Then by definition of h-closure, there exist $u, u' \in (Np^4 + Sp^4]$, such that

$$p^2 + u + t_2 \leq u' + t_2 \qquad , t_2 \in (Np^4 + Sp^4].$$

Since $u, u', t_2 \in (Np^4 + Sp^4]$, then by definition of "( ]", there exist $e', f', g' \in N$ and $s', r', r_1' \in S$ such that

$$u \leq e'p^4 + s'p^4,\ u' \leq f'p^4 + r'p^4,\ t_2 \leq g'p^4 + r_1'p^4.$$

From $p^2 + u + t_2 \leq u^{/} + t_2$ we get

$$ep^2 + eu + et_2 \leq eu^{/} + et_2.$$

Now we have

$$\begin{aligned}
v + eu + et_2 &\leq ep^2 + sp^2 + eu + et_2 \\
&\leq eu^{/} + et_2 + sp^2 \\
&\leq ef^{/}p^4 + er^{/}p^4 + et_2 + sp^2 \\
v + eu + et_2 + fu + ft_2 &\leq ef^{/}p^4 + er^{/}p^4 + et_2 + sp^2 + fu + ft_2 \\
&\leq ef^{/}p^4 + er^{/}p^4 + sp^2 + f(e^{/}p^4 + s^{/}p^4) + et_2 + ft_2 \\
&\leq ef^{/}p^4 + er^{/}p^4 + sp^2 + fe^{/}p^4 + fs^{/}p^4 + e(gp^2 + r_1p^2) + f(gp^2 + r_1p^2) \\
&\leq ef^{/}p^4 + er^{/}p^4 + sp^2 + fe^{/}p^4 + fs^{/}p^4 + egp^2 + er_1p^2 + fgp^2 + fr_1p^2 \in Sp^2.
\end{aligned}$$

Then by definition of "( ]", $v + eu + et_2 + fu + ft_2 \in \left(Sp^2\right]$.
Now

$$\begin{aligned}
v^{/} + fu + ft_2 &\leq fp^2 + rp^2 + fu + ft_2 \\
v^{/} + eu + et_2 + fu + ft_2 &\leq fp^2 + rp^2 + eu + et_2 + sp^2 + fu + ft_2 \\
v^{/} + eu + et_2 + fu + ft_2 &\leq fp^2 + rp^2 + e(e^{/}p^4 + s^{/}p^4) + e(g^{/}p^4 + r_1^{/}p^4) + sp^2 + f(e^{/}p^4 + s^{/}p^4) \\
&\quad + f(g^{/}p^4 + r_1^{/}p^4) \\
v^{/} + eu + et_2 + fu + ft_2 &\leq fp^2 + rp^2 + ee^{/}p^4 + es^{/}p^4 + eg^{/}p^4 + er_1^{/}p^4 + sp^2 + fe^{/}p^4 + fs^{/}p^4 \\
+ fg^{/}p^4 + fr_1^{/}p^4 &\in Sp^2.
\end{aligned}$$

Then by definition of "( ]",

$$\Longrightarrow v^{/} + eu + et_2 + fu + ft_2 \in \left(Sp^2\right].$$

Now,

$$\begin{aligned}
et_2 + ft_2 &\leq e(g^{/}p^4 + r_1^{/}p^4) + f(g^{/}p^4 + r_1^{/}p^4) \\
&= eg^{/}p^4 + er_1^{/}p^4 + fg^{/}p^4 + fr_1^{/}p^4 \in Sp^2.
\end{aligned}$$

Then by definition of "( ]",

$$et_2 + ft_2 \in \left(Sp^2\right].$$

Now

$$p + (v + eu + et_2 + fu + ft_2) + (et_2 + ft_2) \leq \left(v^{/} + eu + et_2 + fu + ft_2\right) + (et_2 + ft_2)$$

$$\Longrightarrow p \in \overline{(Sp^2]}.$$

Hence, S is a left ordered h-regular.
(3) ⇒ (1)

Let E, F are left ordered h-ideals of S, then we get $\overline{(EF]} \subseteq \overline{(F]} = F$ We see that $E \subseteq R_h(E) \subseteq L_h(E) = E$. Hence, E is an ordered h-ideal. Thus $\overline{(EF]} \subseteq \overline{(E]} = E$. So, $\overline{(EF]} \subseteq E \cap F$

Suppose $p \in E \cap F$. By assumption, we get $p \in \overline{(Sp^2]}$. Since $\overline{(Sp^2]} \subseteq \overline{(SEF]} \subseteq \overline{(EF]}$, $p \in (EF]$. It turns out that $E \cap F \subseteq \overline{(EF]}$.

Therefore,

$$E \cap F = \overline{(EF]}$$

□

Theorem 9. *Suppose S is an ordered semiring. Then the conditions given below are equivalent:*
(1) for each right ordered h-ideal E, F of S, $E \cap F = \overline{(EF]}$.
(2) for each right ordered h-ideal E of S, each ordered h-ideal Q of S, $Q \cap E = \overline{(QE]}$.
(3) S is right ordered h-regular, $L_h(E) \subseteq R_h(E)$ for all $\emptyset \neq E \subseteq S$.

5. Ordered h-weakly regular semirings

Definition 13. *Suppose S is an ordered semiring, let $r \in S$. Suppose $r \in \overline{(\sum_{finite}(Sr)^2]}$, then r is said to be a left ordered h-weakly regular. Suppose $r \in \overline{(\sum_{finite}(rS)^2]}$, then r is said to be a right ordered h-weakly regular. Suppose each element in S is left or right ordered h-weakly regular, then ordered semiring S is said to be left or right ordered h-weakly regular.*

Theorem 10. *Suppose S is an ordered semiring, then the conditions given below are equivalent:*
(1) S is a left ordered h-weakly regular.
(2) for each left ordered h-ideal E of S, $\overline{(\sum_{finite} E^2]} = E$.
(3) for each left ordered h-ideal E of S and each ordered h-ideal Q of S, $Q \cap E = \overline{(\sum_{finite} QE]}$.

Proof. **(1)⇒(2)** Suppose E is a left ordered h-ideal of S. Then, we get, $\overline{(\sum_{finite} E^2]} \subseteq \overline{(E]} = E$

Let $a \in E$. By assumption, we have

$$a \in \overline{(\sum_{finite} SaSa]} \subseteq \overline{(\sum_{finite} SESE]} \subseteq \overline{(\sum_{finite} E^2]}.$$

Hence,

$$E \subseteq \overline{(\sum_{finite} E^2]}.$$

Thus,

$$\overline{(\sum_{finite} E^2]} = E$$

(2)⇒(1) Let $r \in S$. From assumption, Lemma 3, and Corollary 2. We get,

$$
\begin{aligned}
r &\in L_h(r) = \overline{(\sum_{finite} L_h(r)^2]} \\
&= \overline{\left(\sum_{finite} \overline{(Nr+Sr](Nr+Sr]}\right]} \\
&\subseteq \overline{\left(\sum_{finite} \overline{(\sum_{finite}(Nr+Sr)(Nr+Sr)]}\right]} \\
&\subseteq \overline{\left(\sum_{finite} \overline{(Sr]}\right]} \\
&= \overline{\left(\overline{(Sr]}\right]} \\
&= \overline{(Sr]}.
\end{aligned}
$$

Since $\overline{(Sr]}$ is left ordered h-ideal, we get $r \in \overline{(Sr]} = \overline{\left(\sum_{finite} \overline{(Sr]}^2\right]}$.

By Lemma 2 and Theorem 4, we have

$$
\begin{aligned}
\overline{\left(\sum_{finite} \overline{(Sr]}^2\right]} &\subseteq \overline{\left(\sum_{finite} \overline{(\sum_{finite} SrSr]}\right]} \\
&= \overline{\left(\overline{(\sum_{finite} SrSr]}\right]} \\
&= \overline{(\sum_{finite} (Sr)^2]}.
\end{aligned}
$$

Hence,

$$
r \in \overline{(\sum_{finite} (Sr)^2]}.
$$

Therefore, S is a left ordered h-weakly regular.

(2)⇒ (3) Suppose E is a left ordered h-ideal of S and Q is an ordered h-ideal of S. Then,

$$
\overline{(\sum_{finite} QE]} \subseteq \overline{(\sum_{finite} Q]} = Q,
$$

and

$$
\overline{(\sum_{finite} QE]} \subseteq \overline{(\sum_{finite} E]} = E.
$$

Hence, $\overline{(\sum_{finite} QE]} \subseteq Q \cap E$.

Let $a \in Q \cap E$. By assumption, we get,

$$
\begin{aligned}
a &\in L_h(a) = \overline{(\sum_{finite} L_h(a)^2]} \\
&\subseteq \overline{\left(\sum_{finite} M_h(a)L_h(a)\right]} \\
&\subseteq \overline{(\sum_{finite} QE]}.
\end{aligned}
$$

Hence,

$$Q \cap E \subseteq \overline{(\sum_{finite} QE]}.$$

Thus

$$Q \cap E = \overline{(\sum_{finite} QE]}$$

(3) ⇒(2) Suppose E is a left ordered h-ideal of S, then, we get $\overline{(\sum_{finite} E^2]} \subseteq \overline{(E]} = E$.

By Lemma 1, 2, Theorem 4 and Corollary 1, we get

$$
\begin{aligned}
E &= M_h(E) \cap E \\
&= \overline{(\sum_{finite} M_h(E)E]} \\
&= \overline{\left(\sum_{finite} \overline{(\sum_{finite} E + \sum_{finite} SE + \sum_{finite} ES + \sum_{finite} SES]}E\right]} \\
&\subseteq \overline{\left(\sum_{finite} \overline{(\sum_{finite} EE + \sum_{finite} SEE + \sum_{finite} ESE + \sum_{finite} SESE]}\right]} \\
&\subseteq \overline{\left(\sum_{finite} \overline{(\sum_{finite} E^2]}\right]} \\
&= \overline{\left(\overline{(\sum_{finite} E^2]}\right]} = \overline{(\sum_{finite} E^2]}.
\end{aligned}
$$

Thus,

$$\overline{(\sum_{finite} E^2]} = E.$$

□

Theorem 11. *Suppose S is an ordered semiring, then the conditions given below are equivalent:*
(1) *S is a right ordered h-weakly regular.*
(2) *for each right ordered h-ideal E of S, $\overline{(\sum_{finite} E^2]} = E$.*
(3) *for each right ordered h-ideal E of S and each ordered h-ideal Q of S, $E \cap Q = \overline{(\sum_{finite} EQ]}$.*

Proof. Straightforward. □

6. Conclusions

Concepts of the ordered *h*- ideals in semirings, alongside their essential properties, were presented. The classes of the semirings like ordered *h*-regular and ordered *h*-weakly regular semirings were characterized by the properties of the ordered *h*-ideals.

The ideas of the ordered *h*-ideals can be extended to the non associative structures like the ones in ([16–18, 20–22]). Moreover, ordered *h*-ideals can be extended for fuzzification in semiring theory.

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Conflict of interest

The authors declare no conflict of interest.

References

1. J. Von Neumann, *On regular rings*, Proc. Natl. Acad. Sci. U.S.A., **22** (1935), 707.
2. S. Bourne, *The Jacobson radical of a semiring*, Proc. Natl. Acad. Sci. U.S.A., **37** (1951), 63.
3. J. S. Golan, *Semirings and affine equations over them: Theory and applications*, Springer Science Business Media, **556** (2013).
4. X. Ma, J. Zhan, *Soft intersection h-ideals of hemirings and its applications*, Ital. J. Pur. Appl. Math., **32** (2014), 301–308.
5. J. Zhan, N. Çağman, A. Sezgin Sezer, *Applications of soft union sets to hemirings via SU-h-ideals*, J. Intell. Fuzzy Syst., **26** (2014), 1363–1370.
6. M. M. Arslanov, N. Kehayopulu, *A note on minimal and maximal ideals of ordered semigroups*, Lobachevskii J. Math., **11** (2002), 3–6.
7. Y. Cao, X. Xu, *On minimal and maximal left ideals in ordered semigroups*, Semigr. Forum, **60** (2000), 202–207.
8. V. N. Dixit, S. Dewan, *A note on quasi and bi-ideals in ternary semigroups*, Int. J. Math. Math. Sci., **18** (1995), 501–508.
9. G. A. N. Ai Ping, Y. L. Jiang, *On ordered ideals in ordered semirings*, J. Math. Res. Exposition, **31** (2011), 989–996.
10. J. S. Han, H. S. Kim, J. Neggers, *Semiring orders in a semiring*, Appl. Math. Inf. Sci., **6** (2012), 99–102.
11. K. Iizuka, *On the Jacobson radical of a semiring*, Tohoku Math. J., Second Ser., **11** (1959), 409–421.
12. Y. B. Jun, M. A. Öztürk, S. Z. Song, *On fuzzy h-ideals in hemirings*, Inf. Sci., **162** (2004), 211–226.

13. J. Zhan, *On properties of fuzzy left h-ideals in hemirings with t-norms*, Int. J. Math. Math. Sci., **19** (2005), 3127–3144.

14. J. Zhan, *Fuzzy h-ideals of hemirings*, Inf. Sci., **177** (2007), 876–886.

15. S. Patchakhieo, B. Pibaljommee, *Characterizations of ordered k-regular semirings by ordered k-ideals*, Asian-Eur. J. Math., **10** (2017), 1750020.

16. T. Shah, N. Kausar, I. Rehman, *Intuitionistic fuzzy normal subrings over a non-associative ring*, An. St. Univ. Ovidius Constanta, **20** (2012), 369–386.

17. N. Kausar, B. Islam, M. Javaid, et al. *Characterizations of non-associative rings by the properties of their fuzzy ideals*, J. Taibah Univ. Sci., **13** (2019), 820–833.

18. N. Kausar, M. Alesemi, S. Salahuddin, et al. *Characterizations of non-associative ordered semigroups by their intuitionistic fuzzy bi-ideals*, Discontinuity, Nonlinearity, Complex., **9** (2020), 257–275.

19. M. Munir, A. Shafiq, *A generalization of bi ideals in semirings*, Bull. Int. Math. Virt. Inst., **8** (2018), 123–133.

20. N. Kausar, M. Waqar, *Characterizations of non-associative rings by their intuitionistic fuzzy bi-ideals*, Eur. J. Pure Appl. Math., **12** (2019), 226–250.

21. N. Kausar, *Direct product of finite intuitionistic anti fuzzy normal subrings over non-associative rings*, Eur. J. Pure Appl. Math., **12** (2019), 622–648.

22. N. Kausar, B. Islam, S. Amjad, et al. *Intuitionistics fuzzy ideals with thresholds (α,β] in LA-rings*, Eur. J. Pure Appl. Math., **9** (2019), 906–943.