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## Research article

## Existence of infinitely many solutions for a nonlocal problem

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#### Abstract

In this paper, we deal with a class of fractional Hénon equation and by using the LyapunovSchmidt reduction method, under some suitable assumptions, we derive the existence of infinitely many solutions, whose energy can be made arbitrarily large. Compared to the previous works, we encounter some new challenges because of the nonlocal property for fractional Laplacian. But by doing some delicate estimates for the nonlocal term we overcome the difficulty and find infinitely many nonradial solutions.


Keywords: fractional Laplacian; critical exponent; reduction method
Mathematics Subject Classification: 35B33, 35J60

## 1. Introduction

In this paper, we consider the following nonlocal Hénon equation

$$
\begin{cases}\mathcal{A}_{s} u=|x|^{\alpha} u^{p}, \quad u>0 & x \in B_{1}(0),  \tag{1.1}\\ u=0, & \text { on } \partial B_{1}(0)\end{cases}
$$

with critical growth, where $\alpha>0$ is a positive constant, $p=\frac{n+2 s}{n-2 s}, n \geq 2+2 s, \frac{1}{2}<s<1, B_{1}(0)$ is the unit ball in $\mathbb{R}^{n}$ and $\mathcal{A}_{s}$ stands for the fractional Laplacian operator in $B_{1}(0)$ with zero Dirichlet boundary values on $\partial B_{1}(0)$.

Here, to define the fractional Laplacian operator $\mathcal{A}_{s}$ in $B_{1}(0)$, let $\left\{\lambda_{k}, \varphi_{k}\right\}$ be the eigenvalues and corresponding eigenfunctions of the Laplacian operator $-\Delta$ in $B_{1}(0)$ with zero Dirichlet boundary values on $\partial B_{1}(0)$, namely, $\left\{\lambda_{k}, \varphi_{k}\right\}$ satisfies

$$
\begin{cases}-\Delta \varphi_{k}=\lambda_{k} \varphi_{k}, & \text { in } B_{1}(0) \\ \varphi_{k}=0, & \text { on } \partial B_{1}(0)\end{cases}
$$

with $\left\|\varphi_{k}\right\|_{L^{2}\left(B_{1}(0)\right)}=1$. Then we can define the fractional Laplacian operator $\mathcal{A}_{s}: H_{0}^{s}\left(B_{1}(0)\right) \rightarrow$ $H_{0}^{-s}\left(B_{1}(0)\right)$ as

$$
\mathcal{A}_{s} u=\sum_{k=1}^{\infty} \lambda_{k}^{s} c_{k} \varphi_{k},
$$

where the fractional Sobolev space $H_{0}^{s}\left(B_{1}(0)\right)(0<s<1)$ is given by

$$
H_{0}^{s}\left(B_{1}(0)\right)=\left\{u=\sum_{k=1}^{\infty} c_{k} \varphi_{k} \in L^{2}\left(B_{1}(0)\right): \sum_{k=1}^{\infty} \lambda_{k}^{s} c_{k}^{2}<\infty\right\}
$$

and equipped with the following inner product

$$
\left\langle\sum_{k=1}^{\infty} c_{k} \varphi_{k}, \sum_{k=1}^{\infty} d_{k} \varphi_{k}\right\rangle_{H_{0}^{s}\left(B_{1}(0)\right)}=\sum_{k=1}^{\infty} \lambda_{k}^{s} c_{k} d_{k}
$$

Form the above definitions, it immediately follows that for any $u, v \in H_{0}^{s}\left(B_{1}(0)\right)$,

$$
\langle u, v\rangle_{H_{0}^{s}\left(B_{1}(0)\right)}=\int_{B_{1}(0)} \mathcal{A}_{s}^{\frac{1}{2}} u \mathcal{A}_{s}^{\frac{1}{2}} v=\int_{B_{1}(0)} \mathcal{A}_{s} u \cdot v .
$$

It is well known that the nonlinear fractional equations appear in diverse areas including physics, biological modeling and mathematical finances and have attracted the considerable attention in the recent period. Also in recent years, there have been many investigations for the related fractional problem $\mathcal{A}_{s} u=f(u)$, where $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is a certain function. But a complete review of the available results in this context goes beyond the aim of this paper. Here we just mention some very recent papers which study fractional equations involving the critical sobolev exponent (cf. [3, 7, 19, 20, 21]).

On the other hand, our main interest in the present paper is motivated by some works that have appeared in recent years related to the classical local Hénon equation of this kind,

$$
\begin{cases}-\Delta u=|x|^{\alpha} u^{p}, & u>0  \tag{1.2}\\ u=0, & x \in B_{1}(0), \\ \text { on } \partial B_{1}(0) .\end{cases}
$$

Among pioneer works we mention Ni [13], where the author established a compactness result of $H_{0, \text { rad }}^{1}\left(B_{1}(0)\right) \hookrightarrow L^{p+1}\left(B_{1}(0)\right)$ and thus got the existence of one positive radial solution for (1.2) if $p \in\left(1, \frac{n+2+2 \alpha}{n-2}\right)$. Later, in [18], Smets, Su and Willem established some symmetry breaking phenomenon and obtained the non-radial property of the ground state solution of (1.2) if $1<p<\frac{n+2}{n-2}$ and $\alpha$ is large enough. When $n \geq 3$ and $p=\frac{n+2}{n-2}-\sigma$, Cao and Peng [1] verified that the ground state solutions of (1.2) are non-radial and blow up as $\sigma \rightarrow 0$. Meanwhile, when $p=\frac{n+2}{n-2}$, Serra [17] showed that (1.2) has a non-radial solution if $n \geq 4$ and $\alpha$ is large enough. More recently, Wei and Yan [23] proved the existence of infinitely many non-radial solutions for (1.2) for any $\alpha>0$. For other results related to the Hénon problem (1.2), one can refer to $[2,11,14,15]$ and the references therein.

Up to our knowledge, not much is obtained for the existence of multiple solutions of equation (1.2) with fractional operator. Motivated by [23] and [12], we want to exploit the finite dimensional reduction method to investigate the existence of infinitely many non-radial solutions for (1.1). To achieve our aim, we will study the following more general problem

$$
\begin{cases}\mathcal{A}_{s} u=\Phi(|x|) u^{p}, \quad u>0 & x \in B_{1}(0),  \tag{1.3}\\ u=0, & \text { on } \partial B_{1}(0),\end{cases}
$$

where $\Phi(r)$ is a bounded function defined in $[0,1]$. It is easy to check that a necessary condition for the existence of a solution of (1.3) is that $\Phi(r)$ is positive somewhere from Pohozaev identity (see[16]). At this point we call attention to the recent work of [12], where we studied (1.3) in $\mathbb{R}^{n}$ and proved that if $n>2+2 s, 0<s<1$ and $\Phi(|x|)$ satisfies

$$
\begin{equation*}
\Phi(r)=\Phi_{0}-\frac{a}{r^{m}}+O\left(\frac{1}{r^{m+\theta}}\right) \text { as } r \in\left(r_{0}-\delta, r_{0}+\delta\right) \tag{1.4}
\end{equation*}
$$

with $\max \left\{2, n-2 s-2 \cdot \frac{(n-2 s)^{2}}{n+2 s}\right\}<m<n-2 s, \Phi_{0}>0, r_{0}>0, \theta>0, \delta>0$, then (1.3) has infinitely many non-radial solutions. It is worth mentioning that from assumption (1.4), $r_{0}$ is a local maximum point of $\Phi(r)$ and then a critical point of $\Phi(r)$. Also the function $r^{\alpha}$ achieves its maximum on $[0,1]$ at $r_{0}=1$ but $r_{0}=1$ is not a critical point of $r^{\alpha}$. However we will verify that if $\Phi(r)$ is increasing near $r_{0}=1$, through $r_{0}=1$ is not a critical point of $\Phi(r)$, the zero Dirichlet boundary condition makes it possible to construct infinitely many solutions of (1.3). Now we state our main result as follows:

Theorem 1.1. Suppose that $n \geq 2+2 s, \frac{1}{2}<s<1$. If $\Phi(1)>0$ and $\Phi^{\prime}(1)>0$, then problem (1.3) has infinitely many non-radial solutions. Particularly, the Hénon equation (1.1) has infinitely many non-radial solutions.

In the end of this part, let us outline the main idea in the proof of Theorem 1.1.
Given any $\varepsilon>0$ and $y^{0} \in \mathbb{R}^{n}$, let

$$
U_{\varepsilon, y^{0}}(x)=\sigma_{n, s}\left(\frac{\varepsilon}{\varepsilon^{2}+\left|x-y^{0}\right|^{2}}\right)^{\frac{n-2 s}{2}}
$$

 solves the following critical fractional equation

$$
\begin{equation*}
\mathcal{A}_{s} u=u^{p}, \lim _{|x| \rightarrow \infty} u(x)=0, u>0 \text { in } \mathbb{R}^{n} . \tag{1.5}
\end{equation*}
$$

Also, very recently, J. DÁvila, M. del Pino and Y. Sire [5] obtained the non-degeneracy of $U_{\varepsilon, y^{0}}(x)$. More precisely, if we define the corresponding functional of (1.5) as

$$
I_{0}(u)=\frac{1}{2} \int_{\mathbb{R}^{n}}\left|\mathcal{A}_{s}^{\frac{1}{2}} u\right|^{2}-\frac{1}{p} \int_{\mathbb{R}^{n}}|u|^{p},
$$

then $I_{0}$ possesses a finite-dimensional manifold $Z$ of least energy critical points, given by

$$
Z=\left\{U_{\varepsilon, y^{0}}: \varepsilon>0, y^{0} \in \mathbb{R}^{n}\right\}
$$

and

$$
\operatorname{ker}_{0}^{\prime \prime}(u)=\operatorname{span}_{\mathbb{R}}\left\{\frac{\partial U_{\varepsilon, y^{0}}}{\partial y_{1}^{0}}, \cdots, \frac{\partial U_{\varepsilon, y^{0}}}{\partial y_{n}^{0}}, \frac{\partial U_{\varepsilon, y^{0}}}{\partial \varepsilon}\right\}, \quad \forall U_{\varepsilon, y^{0}} \in Z .
$$

Now let us fix a positive integer $k \geq k_{0}$, where $k_{0}$ is large, which is to be determined later and set

$$
v=k^{\frac{n-2 s+1}{n-2 s}}
$$

to be the scaling parameter. Using the transformation $u(x) \mapsto v^{-\frac{n-2 s}{2}} u\left(\frac{x}{v}\right)$, (1.3) becomes

$$
\begin{cases}\mathcal{A}_{s} u=\Phi\left(\frac{|x|}{v}\right) u^{p}, & u>0  \tag{1.6}\\ u=0, & x \in B_{v}(0), \\ u & \text { on } \partial B_{v}(0)\end{cases}
$$

Since $U_{\varepsilon, y^{0}}$ is not zero on $\partial B_{\nu}(0)$, we define $P U_{\varepsilon, y^{0}}$ as the solution of the following problem

$$
\begin{equation*}
\mathcal{A}_{s} P U_{\varepsilon, y^{0}}=\mathcal{A}_{s} U_{\varepsilon, y^{0}} \text { in } B_{\nu}(0), \quad P U_{\varepsilon, y^{0}}=0 \text { on } \partial B_{\nu}(0), \tag{1.7}
\end{equation*}
$$

and we will use the solution $P U_{\varepsilon, y^{0}}$ to build up the approximate solutions for (1.6).
For $x=\left(x^{\prime}, x^{\prime \prime}\right) \in \mathbb{R}^{2} \times \mathbb{R}^{n-2}$, we define

$$
\begin{aligned}
& H_{k}=\left\{u: u \in H_{0}^{s}\left(B_{v}(0)\right), u \text { is even in } x_{j}, j=2, \cdots, n,\right. \\
&\left.\quad u\left(r \cos \theta, r \sin \theta, x^{\prime \prime}\right)=u\left(r \cos \left(\theta+\frac{2 i \pi}{k}\right), r \sin \left(\theta+\frac{2 i \pi}{k}\right), x^{\prime \prime}\right)\right\} .
\end{aligned}
$$

Also, we denote

$$
U_{\varepsilon, r}(x)=\sum_{i=1}^{k} P U_{\varepsilon, x^{i}}(x),
$$

where

$$
x^{i}=\left(r \cos \frac{2(i-1) \pi}{k}, r \sin \frac{2(i-1) \pi}{k}, 0\right), \quad i=1, \cdots, k
$$

with 0 is the zero vector in $\mathbb{R}^{n-2}$. And throughout this paper, we always assume that $r \in\left[v\left(1-\frac{r_{0}}{k}\right), v(1-\right.$ $\left.\left.\frac{r_{1}}{k}\right)\right], \varepsilon_{0} \leq \varepsilon \leq \varepsilon_{1}$ for some constants $r_{1}>r_{0}>0$ and $\varepsilon_{1}>\varepsilon_{0}>0$.

To prove Theorem 1.1, it suffices to verify the following result:
Theorem 1.2. Under the assumption of Theorem 1.1, there is an integer $k_{0}>0$, such that for any integer $k \geq k_{0}$, (1.6) has a solution $u_{k}$ of the form

$$
u_{k}=U_{\varepsilon, r_{k}}(x)+\omega_{k}
$$

where $\omega_{k} \in H_{k}$, and as $k \rightarrow+\infty,\left\|\omega_{k}\right\|_{L^{\infty}\left(B_{v}(0)\right)} \rightarrow 0, r_{k} \in\left[v\left(1-\frac{r_{0}}{k}\right), v\left(1-\frac{r_{1}}{k}\right)\right], \varepsilon_{0} \leq \varepsilon \leq \varepsilon_{1}$.
We want to point out that compared with [23], due to the fact that the fractional Laplacian operator is nonlocal and very few things on this topic are known about the fractional Laplacian, we have to face much difficulties in the reduction process and need some more delicate estimates in the proof of our results.

The rest of the paper is organized as follows. In Section 2, we will carry out a reduction procedure and we prove our main result in Section 3. Finally, in Appendix, some basic estimates and an energy expansion for the functional corresponding to problem (1.6) will be established.

## 2. Finite-dimension reduction

In this section, we perform a finite-dimensional reduction. Let

$$
\begin{equation*}
\|u\|_{*}=\sup _{x \in B_{r}(0)}\left(\sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n-2 s}{2}+\tau}}\right)^{-1}|u(x)| \tag{2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\|f\|_{* *}=\sup _{x \in B_{v}(0)}\left(\sum_{i=1}^{n} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n+2 s}{2}+\tau}}\right)^{-1}|f(x)|, \tag{2.2}
\end{equation*}
$$

where $\tau=\frac{n-2 s}{n-2 s+1}$. For this choice of $\tau$, we find that

$$
\sum_{i=2}^{k} \frac{1}{\left|x^{i}-x^{1}\right|^{\tau}} \leq \frac{C k^{\tau}}{\mathcal{v}^{\tau}} \sum_{i=2}^{k} \frac{1}{i^{\tau}} \leq C .
$$

Let

$$
Z_{i, 1}=\frac{\partial P U_{\varepsilon, x^{i}}}{\partial r}, \quad Z_{i, 2}=\frac{\partial P U_{\varepsilon, x^{i}}}{\partial \varepsilon} .
$$

Now we consider

$$
\left\{\begin{array}{l}
\mathcal{A}_{s} \varphi_{k}-p \Phi\left(\frac{|x|}{v}\right) U_{\varepsilon, r}^{p-1} \varphi_{k}=g_{k}+\sum_{l=1}^{2} c_{l} \sum_{i=1}^{k} U_{\varepsilon, x^{i}}^{p-1} Z_{i, l}, \quad \text { in } B_{v}(0),  \tag{2.3}\\
\varphi_{k} \in H_{k}, \\
\left\langle U_{\varepsilon, i^{i}}^{p-1} Z_{i, l}, \varphi_{k}\right\rangle=0, \quad i=1, \ldots, k, l=1,2
\end{array}\right.
$$

for some numbers $c_{l}$, where $\langle u, v\rangle=\int_{B_{v}(0)} u v$. Then we have
Lemma 2.1. Suppose that $\varphi_{k}$ solves (2.3) for $g=g_{k}$. If $\left\|g_{k}\right\|_{* *}$ goes to zero as $k$ goes to infinity, so does $\left\|\varphi_{k}\right\|_{*}$.

Proof. We will argue by an indirect method. Suppose by contradiction that there exist $k \rightarrow+\infty, g=$ $g_{k}, \varepsilon_{k} \in\left[\varepsilon_{0}, \varepsilon_{1}\right], r_{k} \in\left[v\left(1-\frac{r_{0}}{k}\right), v\left(1-\frac{r_{1}}{k}\right)\right]$ and $\varphi_{k}$ solving (2.3) for $g=g_{k}, \varepsilon=\varepsilon_{k}, r=r_{k}$ with $\left\|g_{k}\right\|_{* *} \rightarrow 0$ and $\left\|\varphi_{k}\right\|_{*} \geq c>0$. We may assume that $\left\|\varphi_{k}\right\|_{*}=1$. Also for simplicity, we drop the subscript $k$.

Note that we can rewrite (2.3) as

$$
\begin{align*}
\varphi(x)= & p \int_{B_{v}(0)} \frac{1}{|y-x|^{n-2 s}} \Phi\left(\frac{|y|}{v}\right) U_{\varepsilon, r}^{p-1}(y) \varphi(y) d y \\
& +\int_{B_{v}(0)} \frac{1}{|y-x|^{n-2 s}}\left(g(y)+\sum_{l=1}^{2} c_{l} \sum_{i=1}^{k} U_{x_{i}, \varepsilon}^{p-1}(y) Z_{i, l}(y)\right) d y . \tag{2.4}
\end{align*}
$$

Now we estimate each terms in (2.4). Analogous to Lemma A.3, we have

$$
\begin{align*}
& \left|\int_{B_{v}(0)} \frac{1}{|y-x|^{n-2 s}} \Phi\left(\frac{|y|}{v}\right) U_{\varepsilon, r}^{p-1}(y) \varphi(y) d y\right| \\
& \leq C\|\varphi\|_{*} \int_{B_{v}(0)} \frac{1}{|y-x|^{n-2 s}} U_{\varepsilon, r}^{p-1}(y) \sum_{i=1}^{k} \frac{1}{\left(1+\left|y-x^{i}\right|\right)^{\frac{n-2 s}{2}+\tau}} d y  \tag{2.5}\\
& \leq C\|\varphi\|_{*} \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n-2 s}{2}+\tau+\epsilon}} .
\end{align*}
$$

Meanwhile, by Lemma A.2, we get

$$
\begin{align*}
\left|\int_{B_{v}(0)} \frac{1}{|x-y|^{n-2 s}} g(y) d y\right| & \leq C\|g\|_{* *} \int_{B_{v}(0)} \frac{1}{|x-y|^{n-2 s}} \sum_{i=1}^{k} \frac{1}{\left(1+\left|y-x^{i}\right|\right)^{\frac{n+2 s}{2}+\tau}} d y  \tag{2.6}\\
& \leq C\|g\|_{* *} \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n-2 s}{2}+\tau}}
\end{align*}
$$

and

$$
\begin{align*}
& \left|\int_{B_{v}(0)} \frac{1}{|x-y|^{n-2 s}} \sum_{i=1}^{k} U_{\varepsilon, x^{i}}^{p-1}(y) Z_{i, l}(y) d y\right| \\
& \leq C \int_{B_{v}(0)} \frac{1}{|x-y|^{n-2 s}} \sum_{i=1}^{k} U_{\varepsilon, x^{i}}^{p}(y) d y  \tag{2.7}\\
& \leq C \int_{B_{v}(0)} \frac{1}{|x-y|^{n-2 s}} \sum_{i=1}^{k} \frac{1}{\left(1+\left|y-x^{i}\right|\right)^{n+2 s}} d y \\
& \leq C \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n-2 s}{2}+\tau}} .
\end{align*}
$$

Next, we estimate $c_{l}, l=1,2$. Multiplying (2.3) by $Z_{1, t}(t=1,2)$ and integrating, we see that $c_{l}$ satisfies

$$
\begin{equation*}
\sum_{l=1}^{2} c_{l} \sum_{i=1}^{k}\left\langle U_{\varepsilon, x^{i}}^{p-1} Z_{i, l}, Z_{1, t}\right\rangle=\left\langle\mathcal{A}_{s} \varphi-p \Phi\left(\frac{|y|}{v}\right) U_{\varepsilon, r}^{p-1} \varphi, Z_{1, t}\right\rangle-\left\langle g, Z_{1, t}\right\rangle . \tag{2.8}
\end{equation*}
$$

First, it follows from Lemma A. 1 that

$$
\begin{aligned}
\left|\left\langle g, Z_{1, t}\right\rangle\right| & \leq C\|g\|_{* * *} \int_{B_{v}(0)} \frac{1}{\left(1+\left|y-x^{1}\right|\right)^{n-2 s}} \sum_{i=1}^{k} \frac{1}{\left(1+\left|y-x^{i}\right|\right)^{n+2 s}+\tau} \\
& \leq C\|g\|_{* * *}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& \left\langle\mathcal{A}_{s} \varphi-p \Phi\left(\frac{|y|}{v}\right) U_{\varepsilon, r}^{p-1} \varphi, Z_{1, t}\right\rangle \\
& =\left\langle\mathcal{A}_{s} Z_{1, t}-p \Phi\left(\frac{|y|}{v}\right) U_{\varepsilon, r}^{p-1} Z_{1, t}, \varphi\right\rangle \\
& =\left\langle p U_{x^{1}, \varepsilon}^{p-1} Z_{1, t}-p \Phi\left(\frac{|y|}{v}\right) U_{\varepsilon, r}^{p-1} Z_{1, t}, \varphi\right\rangle \\
& \leq C\|\varphi\|_{*} \int_{B_{v}(0)}\left|\Phi\left(\frac{|y|}{v}\right)-1\right|\left(\sum_{i=1}^{k} \frac{1}{\left(1+\left|y-x^{i}\right|\right)^{n-2 s}}\right)^{p-1} \frac{1}{\left(1+\left|y-x^{1}\right|\right)^{n-2 s}} \\
& \quad \cdot \sum_{i=1}^{k} \frac{1}{\left(1+\left|y-x^{i}\right|\right)^{n-2 s+\tau}}=: J_{0} .
\end{aligned}
$$

Define

$$
\Omega_{i}=\left\{x=\left(x^{\prime}, x^{\prime \prime}\right) \in B_{v}(0):\left\langle\frac{x^{\prime}}{\left|x^{\prime}\right|}, \frac{\left(x^{i}\right)^{\prime}}{\left|\left(x^{i}\right)^{\prime}\right|}\right\rangle \geq \cos \frac{\pi}{k}\right\}, i=1,2, \cdots, k .
$$

Observe that for $y \in \Omega_{1},\left|y-x^{i}\right| \geq\left|y-x^{1}\right|$ and then

$$
\begin{aligned}
\sum_{i=2}^{k} \frac{1}{\left(1+\left|y-x^{i}\right|\right)^{n-2 s}} & \leq C \frac{1}{\left(1+\left|y-x^{1}\right|\right)^{\frac{n-2 s}{2}}} \sum_{i=2}^{k} \frac{1}{\left(1+\left|y-x^{i}\right|\right)^{\frac{n-2 s}{2}}} \\
& \leq C \frac{1}{\left(1+\left|y-x^{1}\right|\right)^{n-2 s-\tau}} \sum_{i=2}^{k} \frac{1}{\left|x^{1}-x^{i}\right|^{\tau}} \\
& \leq C\left(\frac{k}{v}\right)^{\tau} \frac{1}{\left(1+\left|y-x^{1}\right|\right)^{n-2 s-\tau}}
\end{aligned}
$$

which implies

$$
\left(\sum_{i=1}^{k} \frac{1}{\left(1+\left|y-x^{i}\right|\right)^{n-2 s}}\right)^{\frac{4 s}{n-2 s}} \leq C \frac{1}{\left(1+\left|y-x^{1}\right|\right)^{4 s-\frac{4 s \tau}{n-2 s}}}
$$

and

$$
\sum_{i=1}^{k} \frac{1}{\left(1+\left|y-x^{i}\right|\right)^{\frac{n-2 s}{2}+\tau}} \leq C \frac{1}{\left(1+\left|y-x^{1}\right|\right)^{\frac{n-2 s}{2}}}
$$

As a result, from (2.9), we have

$$
\begin{equation*}
J_{0} \leq C k\|\varphi\|_{*} \int_{B_{v}(0)}\left|\Phi\left(\frac{|y|}{v}\right)-1\right| \frac{1}{\left(1+\left|y-x^{1}\right|\right)^{\frac{3 n}{2}+s-\frac{4 s}{n-2 s}}} . \tag{2.10}
\end{equation*}
$$

Using the same argument used for proving (A.7), it follows from (2.9) and (2.10) that

$$
\begin{equation*}
\left\langle\mathcal{A}_{s} \varphi-p \Phi\left(\frac{|y|}{v}\right) U_{\varepsilon, r}^{p-1} \varphi, Z_{1, t}\right\rangle=o\left(\|\varphi\|_{*}\right) . \tag{2.11}
\end{equation*}
$$

Finally, we have

$$
\begin{aligned}
& \sum_{i=1}^{k}\left\langle U_{\varepsilon, x^{i}}^{p-1} Z_{i, l}, Z_{1, t}\right\rangle \\
\leq & C \sum_{i=1}^{k} \int_{B_{v}(0)}\left(\frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n-2 s}}\right)^{p} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{n-2 s}} d x \\
\leq & C .
\end{aligned}
$$

Hence using (2.8), we get

$$
\begin{equation*}
c_{l}=o\left(\|\varphi\|_{*}\right)+O\left(\|g\|_{* *}\right) . \tag{2.12}
\end{equation*}
$$

So, combining (2.4)-(2.7) and (2.12), one has

$$
\begin{equation*}
\|\varphi\|_{*} \leq\left(\|g\|_{* *}+\frac{\sum_{i=1}^{k} \frac{1}{\left(1+\left|y-x^{i}\right|\right)^{\frac{n-2 s+\tau+\epsilon}{2}}}}{\sum_{i=1}^{k} \frac{1}{\left(1+\mid y-x^{i}\right)^{\frac{n-2 s}{2}+\tau}}}\right) . \tag{2.13}
\end{equation*}
$$

Being $\|\varphi\|_{*}=1$, we obtain from (2.13) that there is $R>0$ such that

$$
\begin{equation*}
\|\varphi(x)\|_{L^{\infty}\left(B_{R}\left(x^{i}\right)\right)} \geq a>0, \tag{2.14}
\end{equation*}
$$

for some $i$. But, by using (2.3), $\tilde{\varphi}(x)=\varphi\left(x-x^{i}\right)$ converges, uniformly in any compact set, to a solution $\phi$ of the following equation

$$
\mathcal{A}_{s} \phi-p U_{\varepsilon, 0}^{p-1} \phi=0, \text { in } \mathbb{R}^{n}
$$

for some $\varepsilon \in\left[\varepsilon_{0}, \varepsilon_{1}\right]$. Due to the non-degeneracy of $U_{\varepsilon, 0}$, we can infer that $\phi=0$, which yields a contradiction with (2.14) and then this proof has been proved.

From Lemma 2.1, arguing as proving Proposition 4.1 in [6] or Proposition 2.2 in [12], we can show the following result.

Proposition 2.2. There exists $k_{0}>0$ and a constant $C>0$, independent of $k$, such that for all $k \geq k_{0}$, and $g \in L^{\infty}\left(\mathbb{R}^{n}\right)$, problem (2.3) has a unique solution $\varphi=L_{k}(g)$. Also

$$
\left\|L_{k}(g)\right\|_{*} \leq C\|g\|_{* *}
$$

and

$$
\left|c_{l}\right| \leq C\|g\|_{* *} .
$$

To prove our results, we consider

$$
\left\{\begin{array}{l}
\mathcal{A}_{s}\left(U_{\varepsilon, r}+\varphi\right)=\Phi\left(\frac{|x|}{v}\right)\left(U_{\varepsilon, r}+\varphi\right)^{p}+\sum_{l=1}^{2} c_{l} \sum_{i=1}^{k} U_{\varepsilon, x^{i}}^{p-1} Z_{i, l}, \quad \text { in } B_{v}(0)  \tag{2.15}\\
\varphi \in H_{k}, \\
\left\langle U_{\varepsilon, x^{i}}^{p-1} Z_{i, l}, \varphi\right\rangle=0, \quad i=1, \ldots, k, l=1,2
\end{array}\right.
$$

In order to use the contraction mapping theorem to prove that (2.15) is uniquely solvable in the set that $\|\varphi\|_{*}$ is small, we rewrite (2.15) as

$$
\left\{\begin{array}{l}
\mathcal{A}_{s} \varphi-p \Phi\left(\frac{|x|}{v}\right) U_{\varepsilon, r}^{p-1} \varphi=N(\varphi)+l_{k}+\sum_{l=1}^{2} c_{l} \sum_{i=1}^{k} U_{\varepsilon, x^{i}}^{p-1} Z_{i, l}, \quad \text { in } B_{v}(0),  \tag{2.16}\\
\varphi \in H_{k}, \\
\left\langle U_{\varepsilon, x^{i}}^{p-1} Z_{i, l}, \varphi\right\rangle=0, \quad i=1, \ldots, k, l=1,2
\end{array}\right.
$$

where

$$
N(\varphi)=\Phi\left(\frac{|x|}{v}\right)\left(\left(U_{\varepsilon, r}+\varphi\right)^{p}-U_{\varepsilon, r}^{p}-p U_{\varepsilon, r}^{p-1} \varphi\right)
$$

and

$$
l_{k}=\Phi\left(\frac{|x|}{v}\right) U_{\varepsilon, r}^{p}-\sum_{i=1}^{k} U_{\varepsilon, x}^{p} .
$$

Lemma 2.3. There holds

$$
\|N(\varphi)\|_{* *} \leq C\|\varphi\|_{*}^{\min \{p, 2\}}
$$

Proof. Firstly, we deal with the case $p \leq 2$. By Hölder inequality, we find

$$
\begin{aligned}
|N(\varphi)| & \leq C\|\varphi\|_{*}^{p}\left(\sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n-2 s}{2}+\tau}}\right)^{p} \\
& \leq C\|\varphi\|_{*}^{p} \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n+2 s}{2}+\tau}}\left(\sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\tau}}\right)^{\frac{4 s}{n-2 s}} \\
& \leq C\|\varphi\|_{*}^{p} \sum_{i=1}^{k} \frac{1}{\left(1+\left|y-x^{i}\right|\right)^{\frac{n+2 s}{2}+\tau}} .
\end{aligned}
$$

Using the same argument as above, if $p>2$, we also have

$$
\begin{aligned}
|N(\varphi)| \leq & C\|\varphi\|_{*}^{2}\left(\sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n-2 s}}\right)^{p-2}\left(\sum_{j=1}^{k} \frac{1}{\left(1+\left|x-x^{j}\right|\right)^{\frac{n-2 s}{2}+\tau}}\right)^{2} \\
& +\|\varphi\|_{*}^{p}\left(\sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n-2 s}{2}+\tau}}\right)^{p} \\
\leq & C\left(\|\varphi\|_{*}^{p}+\|\varphi\|_{*}^{2}\right)\left(\sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n-2 s}{2}+\tau}}\right)^{p} \\
\leq & C\|\varphi\|_{*}^{2} \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n+2 s}{2}+\tau}},
\end{aligned}
$$

which completes our proof.
Lemma 2.4. Assume that $r \in\left[v\left(1-\frac{r_{0}}{k}\right), v\left(1-\frac{r_{1}}{k}\right)\right]$. Then there is a small $\epsilon>0$, such that

$$
\left\|l_{k}\right\|_{* *} \leq C\left(\frac{1}{v}\right)^{\frac{1}{2}+\epsilon} .
$$

Proof. Recall that

$$
\Omega_{i}=\left\{x=\left(x^{\prime}, x^{\prime \prime}\right) \in B_{\nu}(0):\left\langle\frac{x^{\prime}}{\left|x^{\prime}\right|}, \frac{\left(x^{i}\right)^{\prime}}{\left|\left(x^{i}\right)^{\prime}\right|}\right\rangle \geq \cos \frac{\pi}{k}\right\}
$$

and

$$
\begin{aligned}
l_{k}= & \Phi\left(\frac{|x|}{v}\right)\left(U_{\varepsilon, r}^{p}-\sum_{i=1}^{k}\left(P U_{\varepsilon, x^{i}}\right)^{p}\right) \\
& +\Phi\left(\frac{|x|}{v}\right)\left(\sum_{i=1}^{k}\left(P U_{\varepsilon, x^{i}}\right)^{p}-\sum_{i=1}^{k} U_{\varepsilon, x^{i}}^{p}\right)+\sum_{i=1}^{k} U_{\varepsilon, x^{i}}^{p}\left(\Phi\left(\frac{|x|}{v}\right)-1\right) \\
=: & J_{1}+J_{2}+J_{3} .
\end{aligned}
$$

By symmetry, we can suppose that $x \in \Omega_{1}$ and for any $x \in \Omega_{1}$,

$$
\left|x-x^{i}\right| \geq\left|x-x^{1}\right|
$$

Thus

$$
\begin{equation*}
\left|J_{1}\right| \leq C \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{4 s}} \sum_{i=2}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n-2 s}}+C\left(\sum_{i=2}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n-2 s}}\right)^{p} . \tag{2.17}
\end{equation*}
$$

By Lemma A.1, taking any $1<\theta \leq \frac{n+2 s}{2}$, we obtain that for any $x \in \Omega_{1}$,

$$
\begin{aligned}
& \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{4 s}} \sum_{i=2}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n-2 s}} \leq \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{\frac{n+2 s}{2}}} \sum_{i=2}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n+2 s} 2} \\
& \leq C \sum_{i=2}^{k}\left[\frac{1}{\left(1+\left|x-x^{1}\right|\right)^{n+2 s-\theta}}+\frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n+2 s-\theta}}\right] \frac{1}{\left|x^{i}-x^{1}\right|^{\theta}} \\
& \leq C \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{n+2 s-\theta}} \sum_{i=2}^{k} \frac{1}{\left|x^{i}-x^{1}\right|^{\theta}} \\
& \leq \frac{C}{\left(1+\left|x-x^{i}\right|\right)^{n+2 s-\theta}}\left(\frac{k}{v}\right)^{\theta} .
\end{aligned}
$$

Choosing $\theta>\frac{n-2 s+1}{2}$ with $n+2 s-\theta \geq \frac{n+2 s}{2}+\tau$, we have

$$
\begin{equation*}
\frac{1}{\left(1+\left|x-x^{1}\right|\right)^{4 s}} \sum_{i=2}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n-2 s}} \leq \frac{C}{\left(1+\left|x-x^{1}\right|\right)^{n+2 s}+\tau}\left(\frac{1}{v}\right)^{\frac{1}{2}+\epsilon} \tag{2.18}
\end{equation*}
$$

On the other hand, for $x \in \Omega_{1}$, by Lemma A. 1 again, we get

$$
\begin{aligned}
& \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n-2 s}} \\
\leq & \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{\frac{n-2 s}{2}}} \frac{1}{\left(1+\left|x-x^{i}\right| \mid\right)^{\frac{n-2 s}{2}}} \\
\leq & \frac{C}{\left|x^{i}-x^{1}\right|^{\frac{n-2 s}{2}-\frac{n-2 s}{n+2} \tau}}\left(\frac{1}{\left(1+\left|x-x^{1}\right|\right)^{\frac{n-2 s}{2}+\frac{n-2 s}{n+2 s} \tau}}+\frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n-2 s}{2}+\frac{n-2 s}{n+2 s} \tau}}\right) \\
\leq & \frac{C}{\left|x^{i}-x^{1}\right|^{\frac{n-2 s}{2}-\frac{n-2 s}{n+2 s} \tau}} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{\frac{n-2 s}{2} \tau+\frac{n-2 s}{n+2 s} \tau}} .
\end{aligned}
$$

So

$$
\sum_{i=2}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n-2 s}} \leq C\left(\frac{k}{v}\right)^{\frac{n-2 s}{2}-\frac{n-2 s}{n+2 s} \tau} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{\frac{n-2 s}{2}+\frac{n-2 s}{n+2 s} \tau}}
$$

which gives

$$
\begin{equation*}
\left(\sum_{i=2}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n-2 s}}\right)^{p} \leq C\left(\frac{k}{v}\right)^{\frac{n+2 s}{2}-\tau} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{\frac{n+2 s}{2}+\tau}} . \tag{2.19}
\end{equation*}
$$

Combining (2.17)-(2.19), we have

$$
\left\|J_{1}\right\|_{* *} \leq C\left(\frac{1}{v}\right)^{\frac{1}{2}+\epsilon} .
$$

Now, we estimate $J_{2}$. Let $H(x, y)$ be the regular part of the Green function for $\mathcal{A}_{s}$ in $B_{1}(0)$ with the Dirichlet boundary condition (see the definition in Appendix) and let $\bar{x}_{*}^{i}$ be the reflection point of $\bar{x}^{i}$ with respect to $\partial B_{1}(0)$, where $\bar{x}^{i}=\frac{x^{i}}{v}$. Then

$$
\frac{H\left(\bar{x}, \bar{x}^{i}\right)}{v^{n-2 s}}=\frac{C}{v^{n-2 s}\left|\bar{x}-\bar{x}_{*}^{i}\right|^{n-2 s}} \leq \frac{C}{\left(1+\left|x-x^{i}\right|\right)^{n-2 s}} .
$$

Take $\kappa=1-\theta$ with $\theta>0$ small. By (A.1), we have

$$
\begin{aligned}
\left|J_{2}\right| & \leq \sum_{i=1}^{k} \frac{C}{\left(1+\left|x-x^{i}\right|\right)^{4 s}} \frac{H\left(\bar{x}, \bar{x}^{i}\right)}{v^{n-2 s}} \\
& \leq \sum_{i=1}^{k} \frac{C}{\left(1+\left|x-x^{i}\right|\right)^{4 s+\kappa(n-2 s)}}\left(\frac{H\left(\bar{x}, \bar{x}^{i}\right)}{v^{n-2 s}}\right)^{\kappa} \\
& \leq C\left(\frac{1}{v \ell}\right)^{\kappa(n-2 s)} \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{4 s+\kappa(n-2 s)}} \\
& \leq C\left(\frac{1}{v}\right)^{\frac{1}{2}+\epsilon} \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{4 s+\kappa(n-2 s)}} \\
& \leq C\left(\frac{1}{v}\right)^{\frac{1}{2}+\epsilon} \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n+2 s}{2}+\tau}},
\end{aligned}
$$

since $\theta \frac{n-2 s}{n-2 s+1}>\frac{1}{2}$ for $n \geq 2+2 s$.
Finally, we estimate $J_{3}$. For any $x \in \Omega_{1}$ and $i=2, \cdots, k$, applying Lemma A.1, we get

$$
\begin{aligned}
U_{\varepsilon, x^{i}}^{p}(x) & \leq C \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{\frac{n+2 s}{2}}} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n+2 s}{2}}} \\
& \leq C\left(\frac{1}{\left(1+\left|x-x^{1}\right|\right)^{\frac{n+2 s}{2}}+\tau}+\frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n+2 s}{2}}+\tau}\right) \frac{1}{\left|x^{i}-x^{1}\right|^{\frac{n+2 s}{2}-\tau}} \\
& \leq C \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{\frac{n+2 s}{2}+\tau}} \frac{1}{\left|x^{i}-x^{1}\right|^{\frac{n+2 s}{2}-\tau}},
\end{aligned}
$$

which gives that

$$
\begin{align*}
\left|\sum_{i=2}^{k}\left(\Phi\left(\frac{|x|}{v}\right)-1\right) U_{\varepsilon, x^{i}}^{p}\right| & \leq C \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n+2 s}+\tau} \sum_{i=1}^{k} \frac{1}{\left|x^{i}-x^{1}\right| \frac{n+2 s}{2}-\tau} \\
& \leq C \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{\frac{n+2 s}{2}+\tau}}\left(\frac{k}{v}\right)^{\frac{n+2 s}{2}-\tau}  \tag{2.20}\\
& \leq C\left(\frac{1}{v}\right)^{\frac{1}{2}+\epsilon} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{\frac{n+2 s}{2}+\tau}} .
\end{align*}
$$

On the other hand, if $x \in \Omega_{1}$ and $||x|-v| \geq \delta v$, where $\delta>0$ is a fixed constant, then

$$
\left||x|-\left|x^{1}\|\geq\| x\right|-v\right|-\| x^{1}|-v| \geq \frac{1}{2} \delta v
$$

and thus

$$
\begin{equation*}
\left|\left(\Phi\left(\frac{|x|}{v}\right)-1\right) U_{\varepsilon, x^{1}}^{p}\right| \leq \frac{C}{v^{\frac{n+2 s}{2}-\tau}} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{\frac{n+2 s}{2}+\tau}} . \tag{2.21}
\end{equation*}
$$

If $x \in \Omega_{1}$ and $||x|-v| \leq \delta v$, we find

$$
\begin{align*}
\left|\Phi\left(\frac{|x|}{v}\right)-1\right| & \leq C\left|\frac{|x|}{v}-1\right| \leq \frac{C}{v}\left(\left(|x|-\left|x^{1}\right|\right)+\left(\left|x^{1}\right|-v\right)\right) \\
& \leq \frac{C}{v}\left(|x|-\left|x^{1}\right|\right)+\frac{C}{k}  \tag{2.22}\\
& \leq \frac{C}{v}\left(|x|-\left|x^{1}\right|\right)+\frac{C}{v^{\frac{1}{2}+\epsilon}}
\end{align*}
$$

and $\left\|x\left|-\left|x^{1}\|\leq\| x\right|-v\right|+|v-| x^{1}\right\| \leq 2 \delta v$. Thus,

$$
\begin{aligned}
\frac{|x|-\left|x^{1}\right|}{v} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{n+2 s}} & \leq \frac{C}{v^{\frac{1}{2}+\epsilon}} \frac{\| x\left|-\left|x^{1}\right|\right|^{\frac{1}{2}-\epsilon}}{\left(1+\left|x-x^{1}\right|\right)^{n+2 s}} \\
& \leq C\left(\frac{1}{v}\right)^{\frac{1}{2}+\epsilon} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{\frac{n+2 s}{2}+\tau}},
\end{aligned}
$$

which, together with (2.20)-(2.22), yields that

$$
\left\|J_{3}\right\|_{* *} \leq C\left(\frac{1}{v}\right)^{\frac{1}{2}+\epsilon} .
$$

This finished this proof.

Proposition 2.5. There is an integer $k_{0}>0$, such that for each $k \geq k_{0}, \varepsilon_{0} \leq \varepsilon \leq \varepsilon_{1}, r \in\left[v\left(1-\frac{r_{0}}{k}\right), v(1-\right.$ $\left.\left.\frac{r_{1}}{k}\right)\right]$, (2.16) has a unique solution $\varphi=\varphi(r, \varepsilon)$ satisfying

$$
\|\varphi\|_{*} \leq C\left(\frac{1}{v}\right)^{\frac{1}{2}+\epsilon}, \quad\left|c_{l}\right| \leq C\left(\frac{1}{v}\right)^{\frac{1}{2}+\epsilon}
$$

where $\epsilon>0$ is a small constant.
Proof. Recall that $v=k^{\frac{n-2 s+1}{n-2 s}}$ and set

$$
\mathcal{N}=\left\{w: w \in C^{\alpha_{0}}\left(B_{v}(0)\right) \cap H_{k},\|w\|_{*} \leq \frac{C}{v^{\frac{1}{2}}}, \int_{B_{v}(0)} U_{\varepsilon, i^{i}}^{p-1} Z_{i, l} w=0\right\}
$$

where $0<\alpha_{0}<s$ and $i=1,2, \ldots, k, l=1,2$. From Proposition 2.2, solving (2.16) is equivalent to solving

$$
\varphi=\mathcal{B}(\varphi):=L_{k}(N(\varphi))+L_{k}\left(l_{k}\right),
$$

where $L_{k}$ is defined in Proposition 2.2.
Firstly, we find

$$
\begin{aligned}
\|\mathcal{B}(\varphi)\|_{*} & \leq C\|N(\varphi)\|_{* *}+C\left\|l_{k}\right\|_{* *} \leq C\|\varphi\|_{*}^{\min \{p, 2\}}+C\left(\frac{1}{v}\right)^{\frac{1}{2}+\epsilon} \\
& \leq C\left(\frac{1}{v}\right)^{\frac{1}{2}(\min \{p, 2\})}+C\left(\frac{1}{v} \frac{{ }^{\frac{1}{2}+\epsilon}}{}\right. \\
& \leq C\left(\frac{1}{v}\right)^{\frac{1}{2}+\epsilon} \leq \frac{C}{v^{\frac{1}{2}}}
\end{aligned}
$$

which tells that $\mathcal{B}$ maps $\mathcal{N}$ to $\mathcal{N}$.
On the other hand, since

$$
\left|N^{\prime}(t)\right| \leq C|t|^{\min \{p-1,1\}}
$$

we get

$$
\begin{aligned}
& \left|L_{k}\left(N\left(\varphi_{1}\right)\right)-L_{k}\left(N\left(\varphi_{2}\right)\right)\right| \\
\leq & C\left(\left|\varphi_{1}\right|^{\min \{p-1,1\}}+\left|\varphi_{2}\right|^{\min \{p-1,1\}}\right)\left|\varphi_{1}-\varphi_{2}\right| \\
\leq & C\left(\left\|\varphi_{1}\right\|_{*}^{\min \{p-1,1\}}+\left\|\varphi_{2}\right\|_{*}^{\min \{p-1,1\}}\right)\left\|\varphi_{1}-\varphi_{2}\right\|_{*}\left(\sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n-2 s}{2}+\tau}}\right)^{\min \{p, 2\}} .
\end{aligned}
$$

Taking into account that

$$
\left(\sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n-2 s}{2}+\tau}}\right)^{\min \{p, 2\}} \leq \sum_{i=1}^{k} \frac{C}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n+2 s}{2}+\tau}},
$$

we have

$$
\begin{aligned}
& \left\|\mathcal{B}\left(\varphi_{1}\right)-\mathcal{B}\left(\varphi_{2}\right)\right\|_{*} \leq C\left\|N\left(\varphi_{1}\right)-N\left(\varphi_{2}\right)\right\|_{* *} \\
\leq & C\left(\left\|\varphi_{1}\right\|_{*}^{\min \{p-1,1\}}+\left\|\varphi_{2}\right\|_{*}^{\min (p-1,1)}\right)\left|\varphi_{1}-\varphi_{2}\right|_{*} \\
\leq & \frac{1}{2}\left\|\varphi_{1}-\varphi_{2}\right\|_{*} .
\end{aligned}
$$

Thus $\mathcal{B}$ is a contraction map.
Therefore, Applying the contraction mapping theorem, we can find a unique $\varphi=\varphi(r, \varepsilon) \in \mathcal{N}$ such that

$$
\varphi=\mathcal{B}(\varphi)
$$

and

$$
\|\varphi\|_{*} \leq C\left(\frac{1}{v}\right)^{\frac{1}{2}+\epsilon} .
$$

Moreover, we get the estimate of $c_{l}$ from (2.12).

## 3. Proof of Theorem 1.2

Let $F(\ell, \varepsilon)=I\left(U_{\varepsilon, r}+\varphi\right)$, where $r=\left|x^{1}\right|, \ell=1-\frac{r}{v}, \varphi$ is the function obtained in Proposition 2.5, and

$$
I(u)=\frac{1}{2} \int_{B_{v}(0)}\left|\mathcal{A}_{s}^{\frac{1}{2}} u\right|^{2}-\frac{1}{p+1} \int_{B_{v}(0)} \Phi\left(\frac{|x|}{v}\right)|u|^{p+1} .
$$

Proposition 3.1. We have

$$
\begin{aligned}
F(\ell, \varepsilon) & =I\left(U_{\varepsilon, r}\right)+O\left(\frac{k}{v^{1+\epsilon}}\right) \\
& =k\left(A+\frac{A_{1} H\left(\overline{x^{1}}, \overline{x^{1}}\right)}{\varepsilon^{n-2 s} v^{n-2 s}}+A_{2} \Phi^{\prime}(1) \ell-\sum_{i=2}^{k} \frac{A_{1} G\left(\overline{x^{i}}, \overline{x^{1}}\right)}{\varepsilon^{n-2 s} v^{n-2 s}}+O\left(\frac{1}{v^{1+\epsilon}}\right)\right)
\end{aligned}
$$

where $A, A_{1}, A_{2}$ are some positive constants, $\epsilon>0$ is a small constant.

## Proof. Since

$$
\left\langle I^{\prime}\left(U_{\varepsilon, r}+\varphi\right), \varphi\right\rangle=0, \quad \forall \varphi \in \mathcal{N},
$$

there is $t \in(0,1)$ such that

$$
\begin{aligned}
F(\ell, \varepsilon)= & I\left(U_{\varepsilon, r}+\varphi\right) \\
= & I\left(U_{\varepsilon, r}\right)-\frac{1}{2} D^{2} I\left(U_{\varepsilon, r}+t \varphi\right)(\varphi, \varphi) \\
= & I\left(U_{\varepsilon, r}\right)-\frac{1}{2} \int_{B_{v}(0)}\left(\left|\mathcal{A}_{s}^{\frac{1}{2}} \varphi\right|^{2}-p \Phi\left(\frac{|x|}{v}\right)\left(U_{\varepsilon, r}+t \varphi\right)^{p-1} \varphi^{2}\right) \\
= & I\left(U_{\varepsilon, r}\right)-\frac{1}{2} \int_{B_{v}(0)}\left(N(\varphi)+l_{k}\right) \varphi \\
& +\frac{p}{2} \int_{B_{v}(0)} \Phi\left(\frac{|x|}{v}\right)\left(\left(U_{\varepsilon, r}+t \varphi\right)^{p-1}-U_{\varepsilon, r}^{p-1}\right) \varphi^{2} \\
= & I\left(U_{\varepsilon, r}\right)+O\left(\int _ { B _ { v } ( 0 ) } \left(|\varphi|^{p+1}+\left|N(\varphi)\left\|\varphi\left|+\left|l_{k} \| \varphi\right|\right)\right) .\right.\right.\right.
\end{aligned}
$$

Firstly,

$$
\begin{aligned}
& \int_{B_{v}(0)}\left(\left|N ( \varphi ) \left\|\varphi\left|+\left|l_{k} \| \varphi\right|\right)\right.\right.\right. \\
\leq & C\left(\|N(\varphi)\|_{* *}+\left\|l_{k}\right\|_{* *}\right)\|\varphi\|_{*} \int_{B_{v}(0)} \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n+2 s}{2}+\tau}} \sum_{j=1}^{k} \frac{1}{\left(1+\left|x-x^{j}\right|\right)^{\frac{n-2 s}{2}+\tau}} .
\end{aligned}
$$

From $\sum_{i=2}^{k} \frac{1}{\left|x^{i}-x^{1}\right|^{\tau}} \leq C$ and Lemma A.1, we obtain

$$
\begin{aligned}
& \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n+2 s}{2}+\tau}} \sum_{j=1}^{k} \frac{1}{\left(1+\left|x-x^{j}\right|\right)^{\frac{n-2 s}{2}+\tau}} \\
= & \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n+2 \tau}}+\sum_{j=1}^{k} \sum_{j \neq i} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n+2 s}{2}+\tau}} \frac{1}{\left(1+\left|x-x^{j}\right|\right)^{\frac{n-2 s}{2}+\tau}} \\
\leq & \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n+2 \tau}}+C \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n+\tau}} \sum_{j=2}^{k} \frac{1}{\left|x^{j}-x^{1}\right|^{\tau}} \\
\leq & C \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n+\tau}} .
\end{aligned}
$$

Therefore, we see

$$
\begin{aligned}
\int_{B_{v}(0)}\left(\left|N ( \varphi ) \left\|\varphi\left|+\left|l_{k} \| \varphi\right|\right)\right.\right.\right. & \leq C\left(\|N(\varphi)\|_{* *}+\left\|l_{k}\right\|_{* *}\right)\|\varphi\|_{*} \int_{B_{v}(0)} \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n+\tau}} \\
& \leq C k\left(\|N(\varphi)\|_{* *}+\left\|l_{k}\right\|_{* *}\right)\|\varphi\|_{*} \leq C k\left(\frac{1}{v^{1+\epsilon}}\right)
\end{aligned}
$$

On the other hand, by Hölder inequality, we have

$$
\begin{aligned}
\int_{B_{v}(0)}|\varphi|^{p+1} & \leq C \|\left.\varphi\right|_{*} ^{p+1} \int_{B_{v}(0)}\left(\sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n-2 s}+\tau}\right)^{p+1} \\
& \leq C \|\left.\varphi\right|_{*} ^{p+1} \int_{B_{v}(0)}\left(\sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n+\tau}}\right)\left(\sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\tau}}\right)^{p} \\
& \leq C \|\left.\varphi\right|_{*} ^{p+1} \int_{B_{v}(0)} \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n+\tau}} \\
& \leq C k\|\varphi\|_{*}^{p+1} \leq C k\left(\frac{1}{v^{1+\epsilon}}\right) .
\end{aligned}
$$

Combining the estimates above and applying Proposition A.5, we have proved

$$
F(\ell, \varepsilon)=k\left(A+\frac{A_{1} H\left(\bar{x}^{1}, \bar{x}^{1}\right)}{\varepsilon^{n-2 s} v^{n-2 s}}+A_{2} \Phi^{\prime}(1) \ell-\sum_{i=2}^{k} \frac{A_{1} G\left(\overline{x^{i}}, \bar{x}^{1}\right)}{\varepsilon^{n-2 s} v^{n-2 s}}+O\left(\frac{1}{v^{1+\epsilon}}\right)\right) .
$$

Proposition 3.2. We have

$$
\frac{\partial F(\ell, \varepsilon)}{\partial \varepsilon}=k(n-2 s) A_{1}\left(-\frac{H\left(\bar{x}^{1}, \bar{x}^{1}\right)}{\varepsilon^{n+1-2 s} \nu^{n-2 s}}+\sum_{i=2}^{k} \frac{G\left(\bar{x}^{i}, \bar{x}^{1}\right)}{\varepsilon^{n+1-2 s} v^{n-2 s}}+O\left(\frac{1}{v^{1+\epsilon}}\right)\right),
$$

and

$$
\frac{\partial F(\ell, \varepsilon)}{\partial \ell}=k\left(A_{1} \frac{\frac{\partial H\left(\bar{x}^{1}, \bar{x}^{1}\right)}{\partial \ell}}{\varepsilon^{n-2 s} \nu^{n-2 s}}+A_{2} \Phi^{\prime}(1)-A_{1} \sum_{i=2}^{k} \frac{\frac{\partial G\left(\bar{x}^{i}, \bar{x}^{1}\right)}{\partial \ell}}{\varepsilon^{n-2 s} v^{n-2 s}}+O\left(\frac{1}{\nu^{\epsilon}}\right)\right),
$$

where $A_{1}, A_{2}$ are the same constants as in Proposition 3.1, and $\epsilon>0$ is a small constant. Proof. Note that

$$
\begin{aligned}
\frac{\partial F(\ell, \varepsilon)}{\partial \varepsilon}= & \left\langle I^{\prime}\left(U_{\varepsilon, r}+\varphi\right), \frac{\partial U_{\varepsilon, r}}{\partial \varepsilon}+\frac{\partial \varphi}{\partial \varepsilon}\right\rangle \\
= & \left\langle I^{\prime}\left(U_{\varepsilon, r}+\varphi\right), \frac{\partial U_{\varepsilon, r}}{\partial \varepsilon}\right\rangle+\sum_{l=1}^{2} \sum_{i=1}^{k} c_{l}\left\langle U_{\varepsilon, x^{i}}^{p-1} Z_{i, l}, \frac{\partial \varphi}{\partial \varepsilon}\right\rangle \\
= & \frac{\partial I\left(U_{\varepsilon, r}\right)}{\partial \varepsilon}-\int_{B_{v}(0)} \Phi\left(\frac{|x|}{v}\right)\left[\left(U_{\varepsilon, r}+\varphi\right)^{p}-U_{\varepsilon, r}^{p}\right] \frac{\partial U_{\varepsilon, r}}{\partial \varepsilon} \\
& +\sum_{l=1}^{2} \sum_{i=1}^{k} c_{l}\left\langle U_{\varepsilon, x^{i}}^{p-1} Z_{i, l}, \frac{\partial \varphi}{\partial \varepsilon}\right\rangle .
\end{aligned}
$$

Considering

$$
\left\langle U_{\varepsilon, x^{i}}^{p-1} Z_{i, l}, \frac{\partial \varphi}{\partial \varepsilon}\right\rangle=-\left\langle\frac{\partial\left(U_{\varepsilon, x^{i}}^{p-1} Z_{i, l}\right)}{\partial \varepsilon}, \varphi\right\rangle,
$$

and from Proposition 2.5, we can get

$$
\begin{aligned}
& \left|\sum_{i=1}^{k} c_{l}\left\langle U_{\varepsilon, x^{i}}^{p-1} Z_{i, l}, \frac{\partial \varphi}{\partial \varepsilon}\right\rangle\right| \\
\leq & C \mid c_{l}\| \| \varphi \|_{*} \int_{B_{r}(0)} \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n-2 s}{2}+\tau}} \sum_{j=1}^{k} \frac{1}{\left(1+\left|x-x^{j}\right|\right)^{n+2 s}} \\
\leq & C \frac{k}{v^{1+\epsilon}} .
\end{aligned}
$$

On the other hand, being $\varphi \in \mathcal{N}$, we have

$$
\begin{aligned}
& \int_{B_{v}(0)} \Phi\left(\frac{|x|}{v}\right)\left[\left(U_{\varepsilon, r}+\varphi\right)^{p}-U_{\varepsilon, r}^{p}\right] \frac{\partial U_{\varepsilon, r}}{\partial \varepsilon} \\
= & \int_{B_{v}(0)} p \Phi\left(\frac{|x|}{v}\right) U_{\varepsilon, r}^{p-1} \frac{\partial U_{\varepsilon, r}}{\partial \varepsilon} \varphi+O\left(\int_{B_{v}(0)}|\varphi|^{2}\right) \\
= & \int_{B_{v}(0)} p \Phi\left(\frac{|x|}{v}\right)\left(U_{\varepsilon, r}^{p-1} \frac{\partial U_{\varepsilon, r}}{\partial \varepsilon} \varphi-\sum_{i=1}^{k} U_{\varepsilon, x^{\prime}} \frac{\partial U_{\varepsilon, x^{i}}}{\partial \varepsilon}\right) \varphi \\
& +p \sum_{i=1}^{k} \int_{B_{v}(0)}\left(\Phi\left(\frac{|x|}{v}\right)-1\right) U_{\varepsilon, x^{i}}^{p-1} \frac{\partial U_{\varepsilon, x^{i}}}{\partial \varepsilon} \varphi+O\left(\int_{B_{v}(0)}|\varphi|^{2}\right) \\
= & k \int_{\Omega_{1}} p \Phi\left(\frac{|x|}{v}\right)\left(U_{\varepsilon, r}^{p-1} \frac{\partial U_{\varepsilon, r}}{\partial \varepsilon} \varphi-\sum_{i=1}^{k} U_{\varepsilon, x^{1}} \frac{\partial U_{\varepsilon, x^{i}}}{\partial \varepsilon}\right) \varphi \\
& +k \int_{B_{v}(0)} p\left(\Phi\left(\frac{|x|}{v}\right)-1\right) U_{\varepsilon, x^{1}}^{p-1} \frac{\partial U_{\varepsilon, x^{1}}}{\partial \varepsilon} \varphi+O\left(\int_{B_{v}(0)}|\varphi|^{2}\right), \\
\mid & \left.\int_{\Omega_{1}} \Phi\left(\frac{|x|}{v}\right)\left(U_{\varepsilon, r}^{p-1} \frac{\partial U_{\varepsilon, r}}{\partial \varepsilon} \varphi-\sum_{i=1}^{k} U_{\varepsilon, x^{i}} \frac{\partial U_{\varepsilon, x^{i}}}{\partial \varepsilon}\right) \varphi \right\rvert\, \\
\leq & C \int_{\Omega_{1}}\left(U_{\varepsilon, r}^{p-1}\left(U_{\varepsilon, x^{1}}-P U_{\varepsilon, x^{1}}\right)+U_{\varepsilon, x^{1}}^{p-1} \sum_{i=2}^{k} U_{\varepsilon, x^{i}}+\sum_{i=2}^{k} U_{\varepsilon, x^{i}}^{p}\right)|\varphi| \\
\leq & \frac{C}{v^{1+\epsilon}}
\end{aligned}
$$

and

$$
\begin{aligned}
& \left|\int_{B_{v}(0)}\left(\Phi\left(\frac{|x|}{v}\right)-1\right) U_{\varepsilon, x^{1}}^{p-1} \frac{\partial U_{\varepsilon, x^{1}}}{\partial \varepsilon} \varphi\right| \\
\leq & \left|\int_{|x x|-v \mid \leq \sqrt{v}}+\int_{||x|-v| \geq \sqrt{v}}\left(\Phi\left(\frac{|x|}{v}\right)-1\right) U_{\varepsilon, x^{1}}^{p-1} \frac{\partial U_{\varepsilon, x^{1}}}{\partial \varepsilon} \varphi\right| \\
\leq & \frac{C}{v^{1+\epsilon}} .
\end{aligned}
$$

Thus, using the estimates above,

$$
\frac{\partial F(\ell, \varepsilon)}{\partial \varepsilon}=\frac{\partial I\left(U_{\varepsilon, r}\right)}{\partial \varepsilon}+O\left(\frac{k}{v^{1+\epsilon}}\right),
$$

from which and Proposition A.6, the conclusion follows.
Finally, noting that $\frac{\partial}{\partial \ell}=-v \frac{\partial}{\partial r}$ and arguing as before, we can get the estimate $\frac{\partial F(\ell, \varepsilon)}{\partial \ell}$.
To prove Theorem 1.2, now we estimate $H\left(\bar{x}^{1}, \bar{x}^{1}\right)$ and $G\left(\bar{x}^{i}, \bar{x}^{1}\right), i \geq 2$. Let $\bar{x}_{*}^{1}=\left(\frac{1}{\left|x^{1}\right|}, 0, \cdots, 0\right)$ be the reflection of $\bar{x}^{1}$ with respect to the unit sphere. Then

$$
H\left(\bar{x}^{1}, \bar{x}^{1}\right)=\frac{C}{\left|\bar{x}^{1}-\bar{x}_{*}^{1}\right| n-2 s}=\frac{C}{2^{n-2 s} \ell^{n-2 s}}
$$

for some constant $C>0$.
On the other hand,

$$
\left|\bar{x}^{i}-\bar{x}_{*}^{1}\right|=\sqrt{\left|\bar{x}^{i}-\bar{x}^{1}\right|^{2}+4 \ell^{2}-4 \ell\left|\bar{x}^{i}-\bar{x}^{1}\right| \cos \theta_{i}},
$$

where $\theta_{i}$ is the angle between $\bar{x}^{i}-\bar{x}^{1}$ and $(1,0, \cdots, 0)$ and then $\theta_{i}=\frac{\pi}{2}+\frac{(i-1) \pi}{k}$. Thus

$$
\begin{aligned}
G\left(\bar{x}^{i}, \bar{x}^{1}\right) & =\frac{C}{\left|\bar{x}^{i}-\bar{x}^{1}\right|^{n-2 s}}-\frac{C}{\left|\bar{x}^{i}-\bar{x}_{*}^{1}\right|^{n-2 s}} \\
& =\frac{C}{\left|\bar{x}^{i}-\bar{x}^{1}\right|^{n-2 s}}\left(1-\frac{1}{1+\frac{4 \frac{4{ }^{2}+4\left|\bar{x}^{i}-\bar{x}^{1}\right| \sin \frac{(i-1) \pi}{k}}{\left|\bar{x}^{i}-\bar{x}\right|^{2}}}{\frac{n-2 s}{2}}} .\right.
\end{aligned}
$$

As

$$
\left|x^{1}-x^{i}\right|=2\left|x^{1}\right| \sin \frac{(i-1) \pi}{k}, i=2, \cdots, k
$$

$\ell k \rightarrow c>0$ and

$$
0<c<\frac{\sin \frac{(i-1) \pi}{k}}{\frac{(i-1) \pi}{k}} \leq c^{\prime}, i=2, \cdots,\left[\frac{k}{2}\right],
$$

we find

$$
\frac{c_{1}}{i^{2}} \leq \frac{4 \ell^{2}+4 \ell\left|\bar{x}^{i}-\bar{x}^{1}\right| \sin \frac{(i-1) \pi}{k}}{\left|\bar{x}^{i}-\bar{x}^{1}\right|^{2}} \leq \frac{c_{2}}{i^{2}}
$$

for some constants $c_{2} \geq c_{1}>0$.
So, there exists a constant $A_{3}>0$ such that

$$
\begin{aligned}
\sum_{i=2}^{k} G\left(\bar{x}^{i}, \bar{x}^{1}\right) & =\sum_{i=2}^{k} \frac{C}{\left|\bar{x}^{i}-\bar{x}^{1}\right|^{n-2 s}}\left(1+O\left(\frac{1}{\left(1+\frac{c}{i^{2}}\right)^{n-2 s}}\right)\right) \\
& =\sum_{i=2}^{k} \frac{C}{\left|\bar{x}^{i}-\bar{x}^{1}\right|^{n-2 s}}\left(\tilde{c}+O\left(\frac{1}{i^{2}}\right)\right) \\
& =A_{3} k^{n-2 s}+O\left(\frac{1}{k}\right)
\end{aligned}
$$

Therefore, it follows from Propositions 3.1 and 3.2 that there are constants $B_{1}, B_{2}, B_{3}$ such that

$$
\begin{equation*}
F(\ell, \varepsilon)=k\left(A+\frac{B_{1}}{\varepsilon^{n-2 s} v^{n-2 s} \ell^{n-2 s}}+B_{2} \ell-\frac{B_{3} k^{n-2 s}}{\varepsilon^{n-2 s} v^{n-2 s}}+O\left(\frac{1}{v^{1+\epsilon}}\right)\right), \tag{3.1}
\end{equation*}
$$

$$
\begin{equation*}
\frac{\partial F(\ell, \varepsilon)}{\partial \varepsilon}=k\left(-\frac{B_{1}(n-2 s)}{\varepsilon^{n+1-2 s} v^{n-2 s} \ell^{n-2 s}}+\frac{B_{3}(n-2 s) k^{n-2 s}}{\varepsilon^{n+1-2 s} v^{n-2 s}}+O\left(\frac{1}{v^{1+\epsilon}}\right)\right) \tag{3.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial F(\ell, \varepsilon)}{\partial \ell}=k\left(-\frac{B_{1}(n-2 s)}{\varepsilon^{n-2 s} v^{n-2 s} \ell^{n+1-2 s}}+B_{2}+O\left(\frac{1}{\nu^{\epsilon}}\right)\right) . \tag{3.3}
\end{equation*}
$$

Proof of Theorem 1.2. Recall that $\ell=1-\frac{r}{v}$ and $v=k^{\frac{n-2 s+1}{n-2 s}}$. Denote $L=\ell k$. Then from (3.2) and (3.3), $\frac{\partial F(\ell, \varepsilon)}{\partial \varepsilon}=0$ and $\frac{\partial F(\ell, \varepsilon)}{\partial \ell}=0$ are equivalent to

$$
\begin{equation*}
\frac{-B_{1}(n-2 s)}{\varepsilon^{n+1-2 s} L^{n-2 s}}+\frac{B_{3}(n-2 s)}{\varepsilon^{n+1-2 s}}+O\left(\frac{1}{\nu^{\epsilon}}\right)=0 \tag{3.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{-B_{1}(n-2 s)}{\varepsilon^{n-2 s} L^{n+1-2 s}}+B_{2}+O\left(\frac{1}{v^{\epsilon}}\right)=0 \tag{3.5}
\end{equation*}
$$

respectively.
Denote

$$
h_{1}(L, \varepsilon)=\frac{-B_{1}(n-2 s)}{\varepsilon^{n+1-2 s} L^{n-2 s}}+\frac{B_{3}(n-2 s)}{\varepsilon^{n+1-2 s}}
$$

and

$$
h_{2}(L, \varepsilon)=\frac{-B_{1}(n-2 s)}{\varepsilon^{n-2 s} L^{n+1-2 s}}+B_{2} .
$$

Thus $h_{1}=0$ and $h_{2}=0$ have a unique solution

$$
L_{0}=\left(\frac{B_{1}}{B_{3}}\right)^{\frac{1}{n-2 s}}, \quad \varepsilon_{0}=\left(\frac{B_{1}(n-2 s)}{B_{2} L_{0}^{n+1-2 s}}\right)^{\frac{1}{n-2 s}} .
$$

Moreover, it is easy to verify that

$$
\frac{\partial h_{1}\left(L_{0}, \varepsilon_{0}\right)}{\partial \varepsilon}=0, \quad \frac{\partial h_{2}\left(L_{0}, \varepsilon_{0}\right)}{\partial L}>0
$$

and

$$
\frac{\partial h_{1}\left(L_{0}, \varepsilon_{0}\right)}{\partial L}=\frac{\partial h_{2}\left(L_{0}, \varepsilon_{0}\right)}{\partial \varepsilon}>0,
$$

which means that $h_{1}=0$ and $h_{2}=0$ at $\left(L_{0}, \varepsilon_{0}\right)$ is invertible. So, (3.4) and (3.5) have a solution near $\left(L_{0}, \varepsilon_{0}\right)$.

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## Conflict of interest

The author declares that there are no conflicts of interest.

## References

1. D. Cao, S. Peng, The asymptotic behavior of the ground state solutions for Hénon equation, J. Math. Anal. Appl., 278 (2003), 1-17.
2. D. Cao, S. Peng, S. Yan, Asymptotic behavior of ground state solutions for Hénon equation, IMA J. Appl. Math., 74 (2009), 468-480.
3. W. Chen, C. Li, B. Ou, Classification of solutions for an integral equation, Comm. Pure Appl. Math., 59 (2006), 330-343.
4. W. Choi, S. Kim, K.-A. Lee, Asymptotic behavior of solutions for nonlinear elliptic problems with the fractional Laplacian, J. Funct. Anal., 266 (2014), 6531-6598.
5. J. DÁvia, M. Del Pino, Y. Sire, Non degeneracy of the bubble in the critical case for nonlocal equations, Proc. Amer. Math. Soc., 141 (2013), 3865-3870.
6. M. del Pino, P. Felmer, M. Musso, Two-bubble solutions in the super-critical Bahri-Coron's problem, Calc. Var. Partial Dif., 16 (2003), 113-145.
7. S. Dipierro, G. Palatucci, E. Valdinoci, Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian, Matematiche, 68 (2013), 201-216.
8. R.-L. Frank, E.-H. Lieb, Inversion positivity and the sharp Hardy-Littlewood-Sobolev inequality, Calc. Var. Partial Dif., 39 (2010), 85-99.
9. R.-L. Frank, E.-H. Lieb, A new rearrangement-free proof of the sharp Hardy-Littlewood- Sobolev inequality, Spectral Theory, Function Spaces and Inequalities, 219 (2012), 55-67.
10. E.-H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. Math., 118 (1983), 349-374.
11. N. Hirano, Existence of positive solutions for the Hénon equation involving critical Sobolev exponent, J. Differ. Equations 247 (2009), 1311-1333.
12. W. Long, J. Yang, Positive or sign-changing solutions for a critical semilinear non-local equation, Z. Angew. Math. Phys., 67 (2016), 45.
13. W.-M. Ni, A nonlionear Dirichlet problem on the unit ball and its application, Indiana Univ. Math. J., 6 (1982), 801-807.
14. S. Peng, Multiple boundary concentrating solutions to Dirichlet problem of Hénon equation, Acta Math. Appl. Sin., 22 (2006), 137-162.
15. A. Pistoia, E. Serra, Multi-peak solutions for the Hénon equation with slightly subcritical growth, Math. Z., 256 (2007), 75-97.
16. X. Ros-Oton, J. Serra, The Pohozaev identity for the fractional Laplacian, Arch. Ration. Mech. Anal., 213 (2014), 587-628.
17. E. Serra, Non-radial positive solutions for the Hénon equation with critical growth, Calc. Var. Partial Dif., 23 (2005), 301-326.
18. D. Smets, J. Su, M. Willem, Non-radial ground states for the Hénon equation, Comm. Contemp. Math., 4 (2002), 467-480.
19. R. Servadei, E. Valdinoci, A Brezis-Nirenberg result for non-local critical equations in low dimension, Commun. Pure Appl. Anal., 12 (2013), 2445-2464.
20. J. Tan, The Brezis-Nirenberg type problem involving the square root of the Laplacian, Calc. Var. Partial Dif., 42 (2012), 21-41.
21. J. Tan, Positive solutions for nonlocal ellipitc problems, Discrete Contin. Dyn. Syst., 33 (2013), 837-859.
22. J. Wei, S. Yan, Infinitely many positive solutions for the prescribed scalar curvature problem on $\mathbb{S}^{N}$, J. Funct. Anal., 258 (2010), 3048-3081.
23. J. Wei, S. Yan, Infinitely many nonradial solutions for the Hénon equation with critical growth, Rev. Mat. Iberoam., 29 (2013), 997-1020.

## A. Energy expansion

In this section, we will give some basic estimates and the energy expansion for the approximate solutions. First, recall that

$$
x^{i}=\left(r \cos \frac{2(i-1) \pi}{k}, r \sin \frac{2(i-1) \pi}{k}, 0\right), i=1, \cdots, k,
$$

where $0 \in \mathbb{R}^{n-2}, r \in\left[v\left(1-\frac{r_{0}}{k}\right), v\left(1-\frac{r_{1}}{k}\right)\right]$, and

$$
I(u)=\frac{1}{2} \int_{B_{v}(0)}\left|\mathcal{A}_{s}^{\frac{1}{2}} u\right|^{2}-\frac{1}{p+1} \int_{B_{v}(0)} \Phi\left(\frac{|x|}{v}\right)|u|^{p+1} .
$$

Now, we introduce the following two lemmas which are important in this sequel and have been proved in [22] and [12] respectively.

Lemma A.1. For any constant $0<\sigma \leq \min \{\alpha, \beta\}$, there exists a constant $C>0$, such that

$$
\frac{1}{\left(1+\left|y-x^{i}\right|\right)^{\alpha}} \frac{1}{\left(1+\left|y-x^{j}\right|\right)^{\beta}} \leq \frac{C}{\left|x^{i}-x^{j}\right| \sigma}\left(\frac{1}{\left(1+\left|y-x^{i}\right|\right)^{\alpha+\beta-\sigma}}+\frac{1}{\left(1+\left|y-x^{j}\right|\right)^{\alpha+\beta-\sigma}}\right) .
$$

Lemma A.2. For any constant $0<\kappa<n-2 s$, there is a constant $C>0$ such that

$$
\int_{B_{v}(0)} \frac{1}{\left.|y|\right|^{n-2 s}} \frac{1}{(1+|x-y|)^{2 s+\kappa}} d y \leq \frac{C}{(1+|x|)^{\kappa}} .
$$

Note that

$$
U_{\varepsilon, r}(x)=\sum_{i=1}^{k} P U_{\varepsilon, x^{i}}(x)
$$

and

$$
U_{\varepsilon, x^{i}}(x)=C_{n, s} \frac{\varepsilon^{\frac{n-2 s}{2}}}{\left(1+\varepsilon^{2}\left|x-x^{i}\right|^{2}\right)^{\frac{n-2 s}{2}}}
$$

for some suitable $C_{n, s}$. We have

Lemma A.3. There is a small $\epsilon>0$ and some constant $C>0$ such that

$$
\int_{B_{v}(0)} \frac{1}{|x-y|^{n-2 s}} U_{\varepsilon, r}^{\frac{4 s}{n-2 s}}(y) \sum_{i=1}^{k} \frac{1}{\left(1+\left|y-x^{i}\right|\right)^{\frac{n-2 s}{2}+\tau}} d y \leq C \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{\frac{n-2 s}{2}+\tau+\epsilon}} .
$$

Proof. We can find this proof in [12]. Here we just need to use

$$
U_{\varepsilon, r}(x) \leq C \sum_{i=1}^{k} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n-2 s}}
$$

Let $G(x, y)$ be the Green function of $\mathcal{A}_{s}$ in $B_{1}(0)$ with the Dirichlet boundary condition (see [4]), namely, $G(x, y)$ solves

$$
\mathcal{A}_{s} G(\cdot, y)=\delta_{y} \text { in } B_{1}(0), \quad G(\cdot, y)=0 \text { on } B_{1}(0),
$$

and the regular part of $G$ is given by

$$
H(x, y)=\frac{\alpha_{n, s}}{|x-y|^{n-2 s}}-G(x, y),
$$

where $\alpha_{n, s}=\frac{1}{\left|S^{n-1}\right|} \frac{2^{1-2 s} \Gamma\left(\frac{n-2 s}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma(s)}$.
Let $\bar{x}^{i}=\frac{x^{i}}{v}$. Then
Proposition A.4. We have

$$
\begin{equation*}
U_{\varepsilon, x^{i}}(x)-P U_{\varepsilon, x^{i}}(x)=\frac{1}{\varepsilon^{\frac{n-2 s}{2}} v^{n-2 s}} H\left(\bar{x}, \bar{x}^{i}\right)+O\left(\frac{1}{(\nu \ell)^{n+2-2 s}}\right), \tag{A.1}
\end{equation*}
$$

where $\ell=1-\left|\bar{x}^{i}\right|=1-\frac{r}{v}$.
Proof. First, letting $\varphi_{x^{i}, \varepsilon}=U_{\varepsilon, x^{i}}-P U_{\varepsilon, x^{i}}$, from the equations satisfied by $U_{\varepsilon, x^{i}}$ and $P U_{\varepsilon, x^{i}}$, we can get that

$$
\left\{\begin{array}{l}
\mathcal{A}_{s} \varphi_{x^{i}, \varepsilon}=0 \text { in } B_{v}(0),  \tag{A.2}\\
\varphi_{x^{i}, \varepsilon}=C_{n, s} \frac{1}{\varepsilon^{(n-2 s) / 2}\left|x-x^{i n}\right|^{n-2 s}}+O\left(\frac{1}{\varepsilon^{n+2-2 s}\left|x-x^{i n}\right|^{n+2-2 s}}\right) \quad \text { on } \partial B_{v}(0) .
\end{array}\right.
$$

Denote by $\tilde{G}(x, y)$ the Green function of $\mathcal{A}_{s}$ in $B_{\nu}(0)$ with the Dirichlet boundary condition and by $\tilde{H}(x, y)$ the regular part of $\tilde{G}(x, y)$. So we can find

$$
\left\{\begin{array}{l}
\mathcal{A}_{s} \tilde{H}\left(x, x^{i}\right)=0 \text { in } B_{v}(0), \\
\tilde{H}\left(x, x^{i}\right)=\frac{\alpha_{n, s}}{\mid x-x^{i \mid n-2 s}} \text { on } \partial B_{v}(0),
\end{array}\right.
$$

which, together with (A.2), yields that

$$
\left\{\begin{array}{l}
\mathcal{A}_{s}\left(\varphi_{x^{i}, \varepsilon}-\frac{1}{\varepsilon^{(n-2 s) / 2}} \tilde{H}\left(x, x^{i}\right)\right)=0 \quad \text { in } B_{\gamma}(0), \\
\varphi_{x^{i}, \varepsilon}-\frac{1}{\varepsilon^{(n-2 s) / 2}} \tilde{H}\left(x, x^{i}\right)=O\left(\frac{1}{\varepsilon^{n+2-2 s\left|x-x^{i n}\right|^{n+2-2 s}}}\right) \text { on } \partial B_{\nu}(0) .
\end{array}\right.
$$

As a result,

$$
\begin{equation*}
\left\|\varphi_{x^{i}, \varepsilon}-\frac{1}{\varepsilon^{(n-2 s) / 2}} \tilde{H}\left(x, x^{i}\right)\right\|_{L^{\infty}} \leq \frac{C}{\varepsilon^{n+2-2 s} d\left(x^{i}, \partial B_{v}(0)\right)^{n+2-2 s}}=\frac{C}{\varepsilon^{n+2-2 s}\left|V-\left|x^{i}\right|^{n+2-2 s}\right.} \tag{A.3}
\end{equation*}
$$

for some constant $C>0$.
Using (A.3), we have proved

$$
U_{\varepsilon, x^{i}}(x)-P U_{\varepsilon, x^{i}}(x)=\frac{1}{\varepsilon^{\frac{n-2 s}{2}} \nu^{n-2 s}} H\left(\bar{x}, \bar{x}^{i}\right)+O\left(\frac{1}{(v \ell)^{n+2-2 s}}\right) .
$$

Proposition A.5. There holds

$$
I\left(U_{\varepsilon, r}\right)=k\left(A+\frac{A_{1} H\left(\bar{x}^{1}, \bar{x}^{1}\right)}{\varepsilon^{n-2 s} v^{n-2 s}}+A_{2} \Phi^{\prime}(1) \ell-\sum_{i=2}^{k} \frac{A_{1} G\left(\bar{x}^{i}, \bar{x}^{1}\right)}{\varepsilon^{n-2 s} v^{n-2 s}}+O\left(\frac{1}{v^{1+\epsilon}}\right)\right),
$$

where $A, A_{1}, A_{2}$ are some positive constants, and $\epsilon$ is a small constant.
Proof. Note that

$$
I\left(U_{\varepsilon, r}\right)=\frac{1}{2} \int_{B_{v}(0)}\left|\mathcal{A}_{s}^{\frac{1}{2}} U_{\varepsilon, r}\right|^{2}-\frac{1}{p+1} \int_{B_{v}(0)} \Phi\left(\frac{|x|}{v}\right)\left|U_{\varepsilon, r}\right|^{p+1} .
$$

Using the symmetry and (A.1), we have

$$
\begin{align*}
\int_{B_{v}(0)} \left\lvert\, \mathcal{A}_{s}^{\frac{1}{2}} U_{\varepsilon,\left.\right|^{2}}\right. & =\sum_{i=1}^{k} \sum_{j=1}^{k} \int_{B_{v}(0)} U_{\varepsilon, x^{j}}^{p} P U_{\varepsilon, x^{i}} \\
& =k \sum_{i=1}^{k} \int_{B_{v}(0)} U_{\varepsilon, x^{1}}^{p} P U_{\varepsilon, x^{i}}  \tag{A.4}\\
& =k\left(\int_{B_{v}(0)} U_{\varepsilon, x^{1}}^{p+1}-\int_{B_{v}(0)} U_{\varepsilon, x^{1}}^{p}\left(U_{\varepsilon, x^{1}}-P U_{\varepsilon, x^{1}}\right)+\sum_{i=2}^{k} \int_{B_{v}(0)} U_{\varepsilon, x^{1}}^{p} P U_{\varepsilon, x^{i}}\right) \\
& =k\left(\int_{\mathbb{R}^{n}} U_{1,0}^{p+1}-\frac{B_{0} H\left(\bar{x}^{1}, \bar{x}^{1}\right)}{\varepsilon^{n-2 s} v^{n-2 s}}+\sum_{i=2}^{k} \frac{B_{0} G\left(\bar{x}^{i}, \bar{x}^{1}\right)}{\varepsilon^{n-2 s} v^{n-2 s}}+O\left(\frac{k}{v}\right)^{n+2-2 s}\right),
\end{align*}
$$

where $B_{0}=\int_{\mathbb{R}^{n}} U_{1,0}^{p}$.
Recalling that

$$
\Omega_{i}=\left\{x=\left(x^{\prime}, x^{\prime \prime}\right) \in B_{v}(0):\left\langle\frac{x^{\prime}}{\left|x^{\prime}\right|}, \frac{\left(x^{i}\right)^{\prime}}{\left|\left(x^{i}\right)^{\prime}\right|}\right\rangle \geq \cos \frac{\pi}{k}\right\}, i=1,2, \cdots, k,
$$

we can obtain that

$$
\begin{align*}
\int_{B_{v}(0)} \Phi\left(\frac{|x|}{v}\right) U_{\varepsilon, r}^{p+1}= & k \int_{\Omega_{1}} \Phi\left(\frac{|x|}{v}\right)\left(P U_{\varepsilon, x^{1}}\right)^{p+1}+k p \int_{\Omega_{1}} \sum_{i=2}^{k}\left(P U_{\varepsilon, x^{1}}\right)^{p} P U_{\varepsilon, x^{i}} \\
& +k O\left(\int_{\Omega_{1}}\left|\Phi\left(\frac{|x|}{v}\right)-1\right| \sum_{i=2}^{k} U_{\varepsilon, x^{1}}^{p} U_{\varepsilon, x^{i}}+\int_{\Omega_{1}} U_{\varepsilon, x^{1}}^{(p+1) / 2}\left(\sum_{i=2}^{k} U_{\varepsilon, x^{i}}\right)^{(p+1) / 2}\right) . \tag{A.5}
\end{align*}
$$

Next, we estimate each terms in (A.5). For $x \in \Omega_{1},\left|x-x^{i}\right| \geq\left|x-x^{1}\right|$, we obtain for any $\kappa \in(1, n-2 s)$,

$$
\sum_{i=2}^{k} U_{\varepsilon, x^{i}} \leq C \sum_{i=2}^{k} \frac{1}{\left|x^{1}-x^{i}\right|^{k}} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{n-2 s-\kappa}}
$$

which implies that if $\kappa$ close to $n-2 s$,

$$
\int_{\Omega_{1}} U_{\varepsilon, x^{1}}^{p+1 / 2}\left(\sum_{i=2}^{k} U_{\varepsilon, x^{i}}\right)^{p+1 / 2}=O\left(\frac{k}{v}\right)^{\kappa \cdot \frac{n}{n-2 s}}=O\left(\frac{1}{v^{1+\epsilon}}\right) .
$$

On the other hand, from (A.1) again, we have

$$
\begin{aligned}
& \int_{\Omega_{1}} \sum_{i=2}^{k}\left(P U_{\varepsilon, x^{1}}\right)^{p} P U_{\varepsilon, x^{i}} \\
= & \left.\int_{\Omega_{1}} \sum_{i=2}^{k}\left(P \frac{\left.C_{n, s^{\frac{n-2 s}{2}}}^{\left(1+\varepsilon^{2}\left|x-x^{i}\right|^{2}\right)^{\frac{n-2 s}{2}}}\right)^{p}\left(\frac { 1 } { \varepsilon ^ { \frac { n - 2 s } { 2 } } \nu ^ { n - 2 s } } G \left(\bar{x}, \bar{x}^{i}\right.\right.}{}\right)+O\left(\frac{1}{(\nu \ell)^{n+2-2 s}}\right)\right) \\
= & \varepsilon^{\frac{n+2 s}{2}} \int_{\tilde{\Omega}_{1}} \sum_{i=2}^{k}\left(P \frac{C_{n, s}}{\left(1+|y|^{2}\right)^{\frac{n-2 s}{2}}}\right)^{p}\left(\frac{1}{\varepsilon^{\frac{n-2 s}{2}} v^{n-2 s}} G\left(\frac{y}{\varepsilon}+x^{i}, \bar{x}^{i}\right)+O\left(\frac{1}{(\nu \ell)^{n+2-2 s}}\right)\right) \varepsilon^{-n} d y \\
= & \int_{B_{v}(0)} U_{1,0}^{p} \sum_{i=2}^{k} \frac{G\left(\bar{x}, \bar{x}^{i}\right)}{\varepsilon^{n-2 s} \nu^{n-2 s}}+O\left(\frac{1}{v^{1+\epsilon}}\right),
\end{aligned}
$$

where $\tilde{\Omega}_{1}=\left\{\frac{y}{\varepsilon}+x^{1} \in \Omega_{1}\right\}$.
Now we estimate $\int_{\Omega_{1}}\left|\Phi\left(\frac{|x|}{v}\right)-1\right| U_{\varepsilon, x^{1}}^{p} \sum_{i=2}^{k} U_{\varepsilon, x^{i}}$. For $x \in \Omega_{1}$ and $\| x|-v| \geq \delta v$, where $\delta>0$ is a fixed constant, we have

$$
\left\|x\left|-\left|x^{1}\|\geq\| x\right|-v\right|-\right\| x^{1}|-v| \geq \frac{1}{2} \delta v
$$

and from Lemma A.1,

$$
\begin{align*}
& \int_{\Omega_{1}}\left|\Phi\left(\frac{|x|}{v}\right)-1\right| U_{\varepsilon, x^{1}}^{p} \sum_{i=2}^{k} U_{\varepsilon, x^{i}} \\
& \leq C \int_{\Omega_{1}} \sum_{i=2}^{k} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{n+2 s}} \frac{1}{\left(1+\left|x-x^{i}\right|\right)^{n-2 s}}  \tag{A.6}\\
& \leq C\left(\frac{1}{v}\right)^{1+s} \sum_{i=2}^{k} \frac{1}{\left|x^{i}-x^{1}\right|^{n-1-2 s}} \int_{\Omega_{1}} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{n+s}} \\
& \leq C\left(\frac{1}{v}\right)^{1+\epsilon} .
\end{align*}
$$

If $x \in \Omega_{1}$ and $||x|-v| \leq \delta v$, we have

$$
\left|\Phi\left(\frac{|x|}{v}\right)-1\right| \leq C\left|\frac{|x|}{v}-1\right| \leq \frac{C}{v}\left(\left(|x|-\left|x^{1}\right|\right)+\left(\left|x^{1}\right|-v\right)\right)
$$

$$
\leq \frac{C}{v}\left(|x|-\left|x^{1}\right|\right)+\frac{C}{k},
$$

and $\left\|x\left|-\left|x^{1}\|\leq\| x\right|-v\right|+|v-| x^{1}\right\| \leq 2 \delta v$. Choosing $\kappa \in(1, n-2 s)$, we obtain that

$$
\begin{aligned}
& \int_{\Omega_{1}}\left|\Phi\left(\frac{|x|}{v}\right)-1\right| U_{\varepsilon, x^{1}}^{p} \sum_{i=2}^{k} U_{\varepsilon, x^{i}} \\
& \leq C \int_{\Omega_{1}}\left(\frac{1}{v}\left(|x|-\left|x^{1}\right|\right)+\frac{1}{k}\right) \sum_{i=2}^{k} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{n+2 s}} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{n-2 s}} \\
& \leq \frac{C}{v} \int_{\Omega_{1}} \sum_{i=2}^{k} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{n+2 s-1}} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{n-2 s}} \\
& \quad+\frac{C}{k} \sum_{i=2}^{k} \frac{1}{\left|x^{i}-x^{1}\right|^{\kappa}} \int_{\Omega_{1}} \frac{1}{\left(1+\left|x-x^{1}\right|\right)^{2 n-\kappa}} \\
& \leq C\left(\frac{1}{v}\right)^{1+\epsilon}
\end{aligned}
$$

This, together with (A.6), tells us that

$$
\begin{equation*}
\int_{\Omega_{1}}\left|\Phi\left(\frac{|x|}{v}\right)-1\right| U_{\varepsilon, x^{1}}^{p} \sum_{i=2}^{k} U_{\varepsilon, x^{i}} \leq C\left(\frac{1}{v}\right)^{1+\epsilon} \tag{A.7}
\end{equation*}
$$

Finally,

$$
\begin{aligned}
& \int_{\Omega_{1}} \Phi\left(\frac{|x|}{v}\right)\left(P U_{\varepsilon, x^{1}}\right)^{p+1} \\
= & \int_{\Omega_{1}}\left(P U_{\varepsilon, x^{1}}\right)^{p+1}+\int_{\Omega_{1}}\left(\Phi\left(\frac{|x|}{v}\right)-1\right) U_{\varepsilon, x^{1}}^{p+1}+O\left(\int_{\Omega_{1}}\left|\Phi\left(\frac{|x|}{v}\right)-1\right| U_{\varepsilon, x^{1}}^{p} \frac{H\left(\bar{x}, \bar{x}^{1}\right)}{v^{n-2 s}}\right) \\
= & \int_{B_{v}(0)} U_{1,0}^{p+1}-\frac{(p+1) B_{0} H\left(\bar{x}^{1}, \bar{x}^{1}\right)}{\varepsilon^{n-2 s} v^{n-2 s}}+\int_{\Omega_{1}}\left(\Phi\left(\frac{|x|}{v}\right)-1\right) U_{\varepsilon, x^{1}}^{p+1}+O\left(\frac{1}{v^{1+\epsilon}}\right) .
\end{aligned}
$$

But

$$
\begin{aligned}
\int_{\Omega_{1}}\left(\Phi\left(\frac{|x|}{v}\right)-1\right) U_{\varepsilon, x^{1}}^{p+1} & =\int_{\Omega_{1}}\left(\Phi\left(\left|\bar{x}^{1}\right|\right)-1\right) U_{\varepsilon, x^{1}}^{p+1}+\int_{\Omega_{1}}\left(\Phi\left(\frac{|x|}{v}\right)-\Phi\left(\left|\bar{x}^{1}\right|\right)\right) U_{\varepsilon, x^{1}}^{p+1} \\
& =\int_{\Omega_{1}}\left(\Phi\left(\left|\bar{x}^{1}\right|\right)-1\right) U_{\varepsilon, x^{1}}^{p+1}+O\left(\frac{1}{v^{1+\epsilon}}\right) \\
& =-\Phi^{\prime}(1) \ell \int_{B_{v}(0)} U_{1,0}^{p+1}+O\left(\frac{1}{v^{1+\epsilon}}\right) .
\end{aligned}
$$

Thus, from the estimates above, we have proved

$$
\int_{B_{v}(0)} \Phi\left(\frac{|x|}{v}\right) U_{\varepsilon, r}^{p+1}=k\left(\int_{B_{v}(0)} U_{1,0}^{p+1}-\Phi^{\prime}(1) \ell \int_{B_{v}(0)} U_{1,0}^{p+1}-\frac{(p+1) B_{0} H\left(\bar{x}^{1}, \bar{x}^{1}\right)}{\varepsilon^{n-2 s} v^{n-2 s}}\right.
$$

$$
\left.+(p+1) \sum_{i=2}^{k} \frac{B_{0} G\left(\bar{x}^{i}, \bar{x}^{1}\right)}{\varepsilon^{n-2 s} v^{n-2 s}}+O\left(\frac{1}{v^{1+\epsilon}}\right)\right),
$$

which, together with (A.4), implies that our desired result holds.

Similar to Proposition A.5, we also have

## Proposition A.6.

$$
\frac{\partial I\left(U_{\varepsilon, r}\right)}{\partial \varepsilon}=k(n-2 s) A_{1}\left(-\frac{H\left(\bar{x}^{1}, \bar{x}^{1}\right)}{\varepsilon^{n+1-2 s} v^{n-2 s}}+\sum_{i=2}^{k} \frac{G\left(\bar{x}^{i}, \bar{x}^{1}\right)}{\varepsilon^{n+1-2 s} v^{n-2 s}}+O\left(\frac{1}{v^{1+\epsilon}}\right)\right),
$$

and

$$
\frac{\partial I\left(U_{\varepsilon, r}\right)}{\partial r}=k\left(A_{1} \frac{\frac{\partial H\left(\bar{x}^{1}, x^{1}\right)}{\partial r}}{\varepsilon^{n-2 s} v^{n-2 s}}-A_{2} \Phi^{\prime}(1) \frac{1}{v}-A_{1} \sum_{i=2}^{k} \frac{\frac{\partial G\left(\bar{i}^{i}, \bar{x}^{1}\right)}{\partial r}}{\varepsilon^{n-2 s} v^{n-2 s}}+O\left(\frac{1}{v^{1+\epsilon}}\right)\right),
$$

where $A_{1}, A_{2}$ are the same constants as in Proposition A.5.
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