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Research article

Existence of infinitely many solutions for a nonlocal problem

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Abstract: In this paper, we deal with a class of fractional Hénon equation and by using the Lyapunov-Schmidt reduction method, under some suitable assumptions, we derive the existence of infinitely many solutions, whose energy can be made arbitrarily large. Compared to the previous works, we encounter some new challenges because of the nonlocal property for fractional Laplacian. But by doing some delicate estimates for the nonlocal term we overcome the difficulty and find infinitely many nonradial solutions.

Keywords: fractional Laplacian; critical exponent; reduction method **Mathematics Subject Classification:** 35B33, 35J60

1. Introduction

In this paper, we consider the following nonlocal Hénon equation

$$\begin{cases} \mathcal{A}_s u = |x|^{\alpha} u^p, \quad u > 0 \quad x \in B_1(0), \\ u = 0, \quad \text{on } \partial B_1(0) \end{cases}$$
(1.1)

with critical growth, where $\alpha > 0$ is a positive constant, $p = \frac{n+2s}{n-2s}$, $n \ge 2+2s$, $\frac{1}{2} < s < 1$, $B_1(0)$ is the unit ball in \mathbb{R}^n and \mathcal{A}_s stands for the fractional Laplacian operator in $B_1(0)$ with zero Dirichlet boundary values on $\partial B_1(0)$.

Here, to define the fractional Laplacian operator \mathcal{A}_s in $B_1(0)$, let $\{\lambda_k, \varphi_k\}$ be the eigenvalues and corresponding eigenfunctions of the Laplacian operator $-\Delta$ in $B_1(0)$ with zero Dirichlet boundary values on $\partial B_1(0)$, namely, $\{\lambda_k, \varphi_k\}$ satisfies

$$\begin{cases} -\Delta \varphi_k = \lambda_k \varphi_k, & \text{in } B_1(0), \\ \varphi_k = 0, & \text{on } \partial B_1(0) \end{cases}$$

with $\|\varphi_k\|_{L^2(B_1(0))} = 1$. Then we can define the fractional Laplacian operator \mathcal{A}_s : $H_0^s(B_1(0)) \rightarrow H_0^{-s}(B_1(0))$ as

$$\mathcal{A}_s u = \sum_{k=1}^{\infty} \lambda_k^s c_k \varphi_k,$$

where the fractional Sobolev space $H_0^s(B_1(0))(0 < s < 1)$ is given by

$$H_0^s(B_1(0)) = \left\{ u = \sum_{k=1}^{\infty} c_k \varphi_k \in L^2(B_1(0)) : \sum_{k=1}^{\infty} \lambda_k^s c_k^2 < \infty \right\}$$

and equipped with the following inner product

$$\left\langle \sum_{k=1}^{\infty} c_k \varphi_k, \sum_{k=1}^{\infty} d_k \varphi_k \right\rangle_{H_0^s(B_1(0))} = \sum_{k=1}^{\infty} \lambda_k^s c_k d_k.$$

Form the above definitions, it immediately follows that for any $u, v \in H_0^s(B_1(0))$,

$$\langle u,v\rangle_{H^s_0(B_1(0))}=\int_{B_1(0)}\mathcal{A}_s^{\frac{1}{2}}u\mathcal{A}_s^{\frac{1}{2}}v=\int_{B_1(0)}\mathcal{A}_s u\cdot v.$$

It is well known that the nonlinear fractional equations appear in diverse areas including physics, biological modeling and mathematical finances and have attracted the considerable attention in the recent period. Also in recent years, there have been many investigations for the related fractional problem $\mathcal{A}_s u = f(u)$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a certain function. But a complete review of the available results in this context goes beyond the aim of this paper. Here we just mention some very recent papers which study fractional equations involving the critical sobolev exponent (cf. [3, 7, 19, 20, 21]).

On the other hand, our main interest in the present paper is motivated by some works that have appeared in recent years related to the classical local Hénon equation of this kind,

$$\begin{cases} -\Delta u = |x|^{\alpha} u^{p}, \quad u > 0 \quad x \in B_{1}(0), \\ u = 0, \quad \text{on } \partial B_{1}(0). \end{cases}$$
(1.2)

Among pioneer works we mention Ni [13], where the author established a compactness result of $H_{0,rad}^1(B_1(0)) \hookrightarrow L^{p+1}(B_1(0))$ and thus got the existence of one positive radial solution for (1.2) if $p \in (1, \frac{n+2+2\alpha}{n-2})$. Later, in [18], Smets, Su and Willem established some symmetry breaking phenomenon and obtained the non-radial property of the ground state solution of (1.2) if $1 and <math>\alpha$ is large enough. When $n \ge 3$ and $p = \frac{n+2}{n-2} - \sigma$, Cao and Peng [1] verified that the ground state solutions of (1.2) are non-radial and blow up as $\sigma \to 0$. Meanwhile, when $p = \frac{n+2}{n-2}$, Serra [17] showed that (1.2) has a non-radial solution if $n \ge 4$ and α is large enough. More recently, Wei and Yan [23] proved the existence of infinitely many non-radial solutions for (1.2) for any $\alpha > 0$. For other results related to the Hénon problem (1.2), one can refer to [2, 11, 14, 15] and the references therein.

Up to our knowledge, not much is obtained for the existence of multiple solutions of equation (1.2) with fractional operator. Motivated by [23] and [12], we want to exploit the finite dimensional reduction method to investigate the existence of infinitely many non-radial solutions for (1.1). To achieve our aim, we will study the following more general problem

$$\begin{cases} \mathcal{A}_s u = \Phi(|x|)u^p, \quad u > 0 \quad x \in B_1(0), \\ u = 0, \quad \text{on } \partial B_1(0), \end{cases}$$
(1.3)

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where $\Phi(r)$ is a bounded function defined in [0, 1]. It is easy to check that a necessary condition for the existence of a solution of (1.3) is that $\Phi(r)$ is positive somewhere from Pohozaev identity (see[16]). At this point we call attention to the recent work of [12], where we studied (1.3) in \mathbb{R}^n and proved that if n > 2 + 2s, 0 < s < 1 and $\Phi(|x|)$ satisfies

$$\Phi(r) = \Phi_0 - \frac{a}{r^m} + O(\frac{1}{r^{m+\theta}}) \text{ as } r \in (r_0 - \delta, r_0 + \delta)$$
(1.4)

with $\max\{2, n - 2s - 2 \cdot \frac{(n-2s)^2}{n+2s}\} < m < n - 2s, \Phi_0 > 0, r_0 > 0, \theta > 0, \delta > 0$, then (1.3) has infinitely many non-radial solutions. It is worth mentioning that from assumption (1.4), r_0 is a local maximum point of $\Phi(r)$ and then a critical point of $\Phi(r)$. Also the function r^{α} achieves its maximum on [0, 1] at $r_0 = 1$ but $r_0 = 1$ is not a critical point of r^{α} . However we will verify that if $\Phi(r)$ is increasing near $r_0 = 1$, through $r_0 = 1$ is not a critical point of $\Phi(r)$, the zero Dirichlet boundary condition makes it possible to construct infinitely many solutions of (1.3). Now we state our main result as follows:

Theorem 1.1. Suppose that $n \ge 2 + 2s$, $\frac{1}{2} < s < 1$. If $\Phi(1) > 0$ and $\Phi'(1) > 0$, then problem (1.3) has infinitely many non-radial solutions. Particularly, the Hénon equation (1.1) has infinitely many non-radial solutions.

In the end of this part, let us outline the main idea in the proof of Theorem 1.1. Given any $\varepsilon > 0$ and $y^0 \in \mathbb{R}^n$, let

$$U_{\varepsilon,y^0}(x) = \sigma_{n,s} \left(\frac{\varepsilon}{\varepsilon^2 + |x - y^0|^2}\right)^{\frac{n-2s}{2}}$$

for $x \in \mathbb{R}^n$ and $\sigma_{n,s} = 2^{\frac{n-2s}{2}} \left(\frac{\Gamma(\frac{n+2s}{2})}{\Gamma(\frac{n-2s}{2})} \right)^{\frac{n-2s}{4s}}$. In 1983, Lieb [10] (also see [6, 7, 8, 9]) proved that $U_{\varepsilon,y^0}(x)$ solves the following critical fractional equation

$$\mathcal{A}_s u = u^p, \quad \lim_{|x| \to \infty} u(x) = 0, \quad u > 0 \quad in \quad \mathbb{R}^n.$$
(1.5)

Also, very recently, J. DÁvila, M. del Pino and Y. Sire [5] obtained the non-degeneracy of $U_{\varepsilon,y^0}(x)$. More precisely, if we define the corresponding functional of (1.5) as

$$I_0(u) = \frac{1}{2} \int_{\mathbb{R}^n} |\mathcal{A}_s^{\frac{1}{2}} u|^2 - \frac{1}{p} \int_{\mathbb{R}^n} |u|^p$$

then I_0 possesses a finite-dimensional manifold Z of least energy critical points, given by

$$Z = \{U_{\varepsilon, y^0} : \varepsilon > 0, y^0 \in \mathbb{R}^n\}$$

and

$$kerI_0''(u) = span_{\mathbb{R}}\left\{\frac{\partial U_{\varepsilon,y^0}}{\partial y_1^0}, \cdots, \frac{\partial U_{\varepsilon,y^0}}{\partial y_n^0}, \frac{\partial U_{\varepsilon,y^0}}{\partial \varepsilon}\right\}, \quad \forall \ U_{\varepsilon,y^0} \in Z.$$

Now let us fix a positive integer $k \ge k_0$, where k_0 is large, which is to be determined later and set

$$\nu = k^{\frac{n-2s+1}{n-2s}},$$

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to be the scaling parameter. Using the transformation $u(x) \mapsto v^{-\frac{n-2s}{2}}u(\frac{x}{v})$, (1.3) becomes

$$\begin{cases} \mathcal{A}_s u = \Phi(\frac{|x|}{v})u^p, \quad u > 0 \quad x \in B_v(0), \\ u = 0, \quad \text{on } \partial B_v(0). \end{cases}$$
(1.6)

Since U_{ε,v^0} is not zero on $\partial B_{\nu}(0)$, we define PU_{ε,v^0} as the solution of the following problem

$$\mathcal{A}_{s}PU_{\varepsilon,y^{0}} = \mathcal{A}_{s}U_{\varepsilon,y^{0}} \text{ in } B_{\nu}(0), \quad PU_{\varepsilon,y^{0}} = 0 \text{ on } \partial B_{\nu}(0), \tag{1.7}$$

and we will use the solution PU_{ε,y^0} to build up the approximate solutions for (1.6).

For $x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$, we define

$$H_k = \left\{ u : u \in H_0^s(B_\nu(0)), u \text{ is even in } x_j, j = 2, \cdots, n, \\ u(r\cos\theta, r\sin\theta, x'') = u(r\cos(\theta + \frac{2i\pi}{k}), r\sin(\theta + \frac{2i\pi}{k}), x'') \right\}.$$

Also, we denote

$$U_{\varepsilon,r}(x) = \sum_{i=1}^{k} P U_{\varepsilon,x^{i}}(x),$$

where

$$x^{i} = (r \cos \frac{2(i-1)\pi}{k}, r \sin \frac{2(i-1)\pi}{k}, 0), \ i = 1, \cdots, k$$

with 0 is the zero vector in \mathbb{R}^{n-2} . And throughout this paper, we always assume that $r \in [\nu(1-\frac{r_0}{k}), \nu(1-\frac{r_1}{k})]$, $\varepsilon_0 \le \varepsilon \le \varepsilon_1$ for some constants $r_1 > r_0 > 0$ and $\varepsilon_1 > \varepsilon_0 > 0$.

To prove Theorem 1.1, it suffices to verify the following result:

Theorem 1.2. Under the assumption of Theorem 1.1, there is an integer $k_0 > 0$, such that for any integer $k \ge k_0$, (1.6) has a solution u_k of the form

$$u_k = U_{\varepsilon, r_k}(x) + \omega_k$$

where $\omega_k \in H_k$, and as $k \to +\infty$, $\|\omega_k\|_{L^{\infty}(B_{\nu}(0))} \to 0$, $r_k \in \left[\nu(1-\frac{r_0}{k}), \nu(1-\frac{r_1}{k})\right]$, $\varepsilon_0 \le \varepsilon \le \varepsilon_1$.

We want to point out that compared with [23], due to the fact that the fractional Laplacian operator is nonlocal and very few things on this topic are known about the fractional Laplacian, we have to face much difficulties in the reduction process and need some more delicate estimates in the proof of our results.

The rest of the paper is organized as follows. In Section 2, we will carry out a reduction procedure and we prove our main result in Section 3. Finally, in Appendix, some basic estimates and an energy expansion for the functional corresponding to problem (1.6) will be established.

2. Finite-dimension reduction

In this section, we perform a finite-dimensional reduction. Let

$$||u||_{*} = \sup_{x \in B_{\nu}(0)} \Big(\sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{\frac{n-2s}{2}+\tau}} \Big)^{-1} |u(x)|$$
(2.1)

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and

$$||f||_{**} = \sup_{x \in B_{\nu}(0)} \Big(\sum_{i=1}^{n} \frac{1}{(1+|x-x^{i}|)^{\frac{n+2s}{2}+\tau}} \Big)^{-1} |f(x)|,$$
(2.2)

where $\tau = \frac{n-2s}{n-2s+1}$. For this choice of τ , we find that

$$\sum_{i=2}^{k} \frac{1}{|x^{i} - x^{1}|^{\tau}} \le \frac{Ck^{\tau}}{\nu^{\tau}} \sum_{i=2}^{k} \frac{1}{i^{\tau}} \le C$$

Let

$$Z_{i,1} = \frac{\partial P U_{\varepsilon,x^i}}{\partial r}, \quad Z_{i,2} = \frac{\partial P U_{\varepsilon,x^i}}{\partial \varepsilon}$$

Now we consider

$$\begin{cases} \mathcal{A}_{s}\varphi_{k} - p\Phi(\frac{|x|}{\nu})U_{\varepsilon,r}^{p-1}\varphi_{k} = g_{k} + \sum_{l=1}^{2}c_{l}\sum_{i=1}^{k}U_{\varepsilon,x^{i}}^{p-1}Z_{i,l}, & \text{in } B_{\nu}(0), \\ \varphi_{k} \in H_{k}, \\ \langle U_{\varepsilon,x^{i}}^{p-1}Z_{i,l},\varphi_{k}\rangle = 0, \quad i = 1, ..., k, l = 1, 2 \end{cases}$$

$$(2.3)$$

for some numbers c_l , where $\langle u, v \rangle = \int_{B_v(0)} uv$. Then we have

Lemma 2.1. Suppose that φ_k solves (2.3) for $g = g_k$. If $||g_k||_{**}$ goes to zero as k goes to infinity, so does $||\varphi_k||_{*}$.

Proof. We will argue by an indirect method. Suppose by contradiction that there exist $k \to +\infty$, $g = g_k, \varepsilon_k \in [\varepsilon_0, \varepsilon_1], r_k \in \left[\nu(1 - \frac{r_0}{k}), \nu(1 - \frac{r_1}{k})\right]$ and φ_k solving (2.3) for $g = g_k, \varepsilon = \varepsilon_k, r = r_k$ with $||g_k||_{**} \to 0$ and $||\varphi_k||_* \ge c > 0$. We may assume that $||\varphi_k||_* = 1$. Also for simplicity, we drop the subscript k.

Note that we can rewrite (2.3) as

$$\begin{aligned} \varphi(x) &= p \int_{B_{\nu}(0)} \frac{1}{|y-x|^{n-2s}} \Phi\Big(\frac{|y|}{\nu}\Big) U_{\varepsilon,r}^{p-1}(y) \varphi(y) dy \\ &+ \int_{B_{\nu}(0)} \frac{1}{|y-x|^{n-2s}} \Big(g(y) + \sum_{l=1}^{2} c_{l} \sum_{i=1}^{k} U_{x_{i},\varepsilon}^{p-1}(y) Z_{i,l}(y) \Big) dy. \end{aligned}$$

$$(2.4)$$

Now we estimate each terms in (2.4). Analogous to Lemma A.3, we have

$$\begin{split} & \left| \int_{B_{\nu}(0)} \frac{1}{|y-x|^{n-2s}} \Phi\Big(\frac{|y|}{\nu}\Big) U_{\varepsilon,r}^{p-1}(y) \varphi(y) dy \right| \\ & \leq C ||\varphi||_{*} \int_{B_{\nu}(0)} \frac{1}{|y-x|^{n-2s}} U_{\varepsilon,r}^{p-1}(y) \sum_{i=1}^{k} \frac{1}{(1+|y-x^{i}|)^{\frac{n-2s}{2}+\tau}} dy \\ & \leq C ||\varphi||_{*} \sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{\frac{n-2s}{2}+\tau+\epsilon}}. \end{split}$$

$$(2.5)$$

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Meanwhile, by Lemma A.2, we get

$$\begin{aligned} \left| \int_{B_{\nu}(0)} \frac{1}{|x-y|^{n-2s}} g(y) dy \right| &\leq C ||g||_{**} \int_{B_{\nu}(0)} \frac{1}{|x-y|^{n-2s}} \sum_{i=1}^{k} \frac{1}{(1+|y-x^{i}|)^{\frac{n+2s}{2}+\tau}} dy \\ &\leq C ||g||_{**} \sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{\frac{n-2s}{2}+\tau}} \end{aligned}$$
(2.6)

and

$$\left| \int_{B_{\nu}(0)} \frac{1}{|x-y|^{n-2s}} \sum_{i=1}^{k} U_{\varepsilon,x^{i}}^{p-1}(y) Z_{i,l}(y) dy \right|$$

$$\leq C \int_{B_{\nu}(0)} \frac{1}{|x-y|^{n-2s}} \sum_{i=1}^{k} U_{\varepsilon,x^{i}}^{p}(y) dy$$

$$\leq C \int_{B_{\nu}(0)} \frac{1}{|x-y|^{n-2s}} \sum_{i=1}^{k} \frac{1}{(1+|y-x^{i}|)^{n+2s}} dy$$

$$\leq C \sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{\frac{n-2s}{2}+\tau}}.$$
(2.7)

Next, we estimate c_l , l = 1, 2. Multiplying (2.3) by $Z_{1,l}(t = 1, 2)$ and integrating, we see that c_l satisfies

$$\sum_{l=1}^{2} c_{l} \sum_{i=1}^{k} \left\langle U_{\varepsilon,x^{i}}^{p-1} Z_{i,l}, Z_{1,t} \right\rangle = \left\langle \mathcal{A}_{s} \varphi - p \Phi(\frac{|y|}{\nu}) U_{\varepsilon,r}^{p-1} \varphi, Z_{1,t} \right\rangle - \left\langle g, Z_{1,t} \right\rangle.$$
(2.8)

First, it follows from Lemma A.1 that

$$\begin{aligned} |\langle g, Z_{1,t} \rangle| &\leq C ||g||_{**} \int_{B_{\nu}(0)} \frac{1}{(1+|y-x^{1}|)^{n-2s}} \sum_{i=1}^{k} \frac{1}{(1+|y-x^{i}|)^{\frac{n+2s}{2}+\tau}} \\ &\leq C ||g||_{**}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle \mathcal{A}_{s}\varphi - p\Phi(\frac{|y|}{\nu})U_{\varepsilon,r}^{p-1}\varphi, Z_{1,t} \rangle \\ &= \langle \mathcal{A}_{s}Z_{1,t} - p\Phi(\frac{|y|}{\nu})U_{\varepsilon,r}^{p-1}Z_{1,t}, \varphi \rangle \\ &= \langle pU_{x^{1},\varepsilon}^{p-1}Z_{1,t} - p\Phi(\frac{|y|}{\nu})U_{\varepsilon,r}^{p-1}Z_{1,t}, \varphi \rangle \\ &\leq C||\varphi||_{*} \int_{B_{\nu}(0)} \left| \Phi(\frac{|y|}{\nu}) - 1 \right| \Big(\sum_{i=1}^{k} \frac{1}{(1+|y-x^{i}|)^{n-2s}} \Big)^{p-1} \frac{1}{(1+|y-x^{1}|)^{n-2s}} \\ &\cdot \sum_{i=1}^{k} \frac{1}{(1+|y-x^{i}|)^{\frac{n-2s}{2}+\tau}} =: J_{0}. \end{aligned}$$

$$(2.9)$$

Define

$$\Omega_i = \left\{ x = (x', x'') \in B_{\nu}(0) : \langle \frac{x'}{|x'|}, \frac{(x^i)'}{|(x^i)'|} \rangle \ge \cos \frac{\pi}{k} \right\}, i = 1, 2, \cdots, k.$$

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Observe that for $y \in \Omega_1$, $|y - x^i| \ge |y - x^1|$ and then

$$\begin{split} \sum_{i=2}^{k} \frac{1}{(1+|y-x^{i}|)^{n-2s}} &\leq C \frac{1}{(1+|y-x^{1}|)^{\frac{n-2s}{2}}} \sum_{i=2}^{k} \frac{1}{(1+|y-x^{i}|)^{\frac{n-2s}{2}}} \\ &\leq C \frac{1}{(1+|y-x^{1}|)^{n-2s-\tau}} \sum_{i=2}^{k} \frac{1}{|x^{1}-x^{i}|^{\tau}} \\ &\leq C \Big(\frac{k}{\nu}\Big)^{\tau} \frac{1}{(1+|y-x^{1}|)^{n-2s-\tau}}, \end{split}$$

which implies

$$\Big(\sum_{i=1}^{k} \frac{1}{(1+|y-x^{i}|)^{n-2s}}\Big)^{\frac{4s}{n-2s}} \le C \frac{1}{(1+|y-x^{1}|)^{4s-\frac{4s\tau}{n-2s}}}$$

and

$$\sum_{i=1}^k \frac{1}{(1+|y-x^i|)^{\frac{n-2s}{2}+\tau}} \leq C \frac{1}{(1+|y-x^1|)^{\frac{n-2s}{2}}}$$

As a result, from (2.9), we have

$$J_0 \le Ck \|\varphi\|_* \int_{B_{\nu}(0)} \left| \Phi(\frac{|y|}{\nu}) - 1 \right| \frac{1}{(1+|y-x^1|)^{\frac{3n}{2}+s-\frac{4s}{n-2s}}}.$$
(2.10)

Using the same argument used for proving (A.7), it follows from (2.9) and (2.10) that

$$\langle \mathcal{A}_{s}\varphi - p\Phi(\frac{|y|}{\nu})U_{\varepsilon,r}^{p-1}\varphi, Z_{1,t}\rangle = o(||\varphi||_{*}).$$
(2.11)

Finally, we have

$$\sum_{i=1}^{k} \left\langle U_{\varepsilon,x^{i}}^{p-1} Z_{i,l}, Z_{1,t} \right\rangle$$

$$\leq C \sum_{i=1}^{k} \int_{B_{\nu}(0)} \left(\frac{1}{(1+|x-x^{i}|)^{n-2s}} \right)^{p} \frac{1}{(1+|x-x^{1}|)^{n-2s}} dx$$

$$\leq C.$$

Hence using (2.8), we get

$$c_l = o(\|\varphi\|_*) + O(\|g\|_{**}).$$
(2.12)

So, combining (2.4)-(2.7) and (2.12), one has

$$\|\varphi\|_{*} \leq \left(\|g\|_{**} + \frac{\sum_{i=1}^{k} \frac{1}{(1+|y-x^{i}|)^{\frac{n-2s}{2}+\tau+\epsilon}}}{\sum_{i=1}^{k} \frac{1}{(1+|y-x^{i}|)^{\frac{n-2s}{2}+\tau}}}\right).$$
(2.13)

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Being $\|\varphi\|_* = 1$, we obtain from (2.13) that there is R > 0 such that

$$\|\varphi(x)\|_{L^{\infty}(B_{R}(x^{i}))} \ge a > 0,$$
 (2.14)

for some *i*. But, by using (2.3), $\tilde{\varphi}(x) = \varphi(x - x^i)$ converges, uniformly in any compact set, to a solution ϕ of the following equation

$$\mathcal{A}_s \phi - p U_{\varepsilon 0}^{p-1} \phi = 0$$
, in \mathbb{R}^n

for some $\varepsilon \in [\varepsilon_0, \varepsilon_1]$. Due to the non-degeneracy of $U_{\varepsilon,0}$, we can infer that $\phi = 0$, which yields a contradiction with (2.14) and then this proof has been proved.

From Lemma 2.1, arguing as proving Proposition 4.1 in [6] or Proposition 2.2 in [12], we can show the following result.

Proposition 2.2. There exists $k_0 > 0$ and a constant C > 0, independent of k, such that for all $k \ge k_0$, and $g \in L^{\infty}(\mathbb{R}^n)$, problem (2.3) has a unique solution $\varphi = L_k(g)$. Also

$$||L_k(g)||_* \le C ||g||_{**}$$

and

$$|c_l| \le C ||g||_{**}$$

To prove our results, we consider

$$\begin{cases} \mathcal{A}_{s}(U_{\varepsilon,r}+\varphi) = \Phi(\frac{|x|}{\nu})(U_{\varepsilon,r}+\varphi)^{p} + \sum_{l=1}^{2} c_{l} \sum_{i=1}^{k} U_{\varepsilon,x^{i}}^{p-1} Z_{i,l}, & \text{ in } B_{\nu}(0), \\ \varphi \in H_{k}, \\ \langle U_{\varepsilon,x^{i}}^{p-1} Z_{i,l}, \varphi \rangle = 0, \quad i = 1, ..., k, l = 1, 2. \end{cases}$$

$$(2.15)$$

In order to use the contraction mapping theorem to prove that (2.15) is uniquely solvable in the set that $\|\varphi\|_*$ is small, we rewrite (2.15) as

$$\left\{ \begin{array}{l} \mathcal{A}_{s}\varphi - p\Phi\left(\frac{|x|}{\nu}\right)U_{\varepsilon,r}^{p-1}\varphi = N(\varphi) + l_{k} + \sum_{l=1}^{2}c_{l}\sum_{i=1}^{k}U_{\varepsilon,x^{i}}^{p-1}Z_{i,l}, \quad in \ B_{\nu}(0), \\ \varphi \in H_{k}, \\ \left\langle U_{\varepsilon,x^{i}}^{p-1}Z_{i,l},\varphi \right\rangle = 0, \quad i = 1, ..., k, l = 1, 2, \end{array} \right.$$

$$(2.16)$$

where

$$N(\varphi) = \Phi(\frac{|x|}{\nu}) \Big((U_{\varepsilon,r} + \varphi)^p - U_{\varepsilon,r}^p - p U_{\varepsilon,r}^{p-1} \varphi \Big)$$

and

$$l_k = \Phi(\frac{|x|}{\nu})U_{\varepsilon,r}^p - \sum_{i=1}^k U_{\varepsilon,x^i}^p$$

Lemma 2.3. There holds

$$||N(\varphi)||_{**} \le C ||\varphi||_{*}^{\min\{p,2\}}.$$

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Proof. Firstly, we deal with the case $p \le 2$. By Hölder inequality, we find

$$\begin{split} |N(\varphi)| &\leq C ||\varphi||_*^p \Big(\sum_{i=1}^k \frac{1}{(1+|x-x^i|)^{\frac{n-2s}{2}+\tau}} \Big)^p \\ &\leq C ||\varphi||_*^p \sum_{i=1}^k \frac{1}{(1+|x-x^i|)^{\frac{n+2s}{2}+\tau}} \Big(\sum_{i=1}^k \frac{1}{(1+|x-x^i|)^{\tau}} \Big)^{\frac{4s}{n-2s}} \\ &\leq C ||\varphi||_*^p \sum_{i=1}^k \frac{1}{(1+|y-x^i|)^{\frac{n+2s}{2}+\tau}}. \end{split}$$

Using the same argument as above, if p > 2, we also have

$$\begin{split} |N(\varphi)| &\leq C ||\varphi||_*^2 \Big(\sum_{i=1}^k \frac{1}{(1+|x-x^i|)^{n-2s}} \Big)^{p-2} \Big(\sum_{j=1}^k \frac{1}{(1+|x-x^j|)^{\frac{n-2s}{2}+\tau}} \Big)^2 \\ &+ ||\varphi||_*^p \Big(\sum_{i=1}^k \frac{1}{(1+|x-x^i|)^{\frac{n-2s}{2}+\tau}} \Big)^p \\ &\leq C (||\varphi||_*^p + ||\varphi||_*^2) \Big(\sum_{i=1}^k \frac{1}{(1+|x-x^i|)^{\frac{n-2s}{2}+\tau}} \Big)^p \\ &\leq C ||\varphi||_*^2 \sum_{i=1}^k \frac{1}{(1+|x-x^i|)^{\frac{n+2s}{2}+\tau}}, \end{split}$$

which completes our proof.

Lemma 2.4. Assume that $r \in \left[v(1-\frac{r_0}{k}), v(1-\frac{r_1}{k})\right]$. Then there is a small $\epsilon > 0$, such that

$$||l_k||_{**} \le C(\frac{1}{\nu})^{\frac{1}{2}+\epsilon}.$$

Proof. Recall that

$$\Omega_i = \left\{ x = (x', x'') \in B_{\nu}(0) : \left\langle \frac{x'}{|x'|}, \frac{(x^i)'}{|(x^i)'|} \right\rangle \ge \cos \frac{\pi}{k} \right\}$$

and

$$l_{k} = \Phi(\frac{|x|}{\nu}) \Big(U_{\varepsilon,r}^{p} - \sum_{i=1}^{k} (PU_{\varepsilon,x^{i}})^{p} \Big) \\ + \Phi(\frac{|x|}{\nu}) \Big(\sum_{i=1}^{k} (PU_{\varepsilon,x^{i}})^{p} - \sum_{i=1}^{k} U_{\varepsilon,x^{i}}^{p} \Big) + \sum_{i=1}^{k} U_{\varepsilon,x^{i}}^{p} \Big(\Phi(\frac{|x|}{\nu}) - 1 \Big) \\ =: J_{1} + J_{2} + J_{3}.$$

By symmetry, we can suppose that $x \in \Omega_1$ and for any $x \in \Omega_1$,

$$|x - x^i| \ge |x - x^1|.$$

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Thus

$$|J_1| \le C \frac{1}{(1+|x-x^1|)^{4s}} \sum_{i=2}^k \frac{1}{(1+|x-x^i|)^{n-2s}} + C \Big(\sum_{i=2}^k \frac{1}{(1+|x-x^i|)^{n-2s}}\Big)^p.$$
(2.17)

By Lemma A.1, taking any $1 < \theta \le \frac{n+2s}{2}$, we obtain that for any $x \in \Omega_1$,

$$\begin{split} &\frac{1}{(1+|x-x^{1}|)^{4s}}\sum_{i=2}^{k}\frac{1}{(1+|x-x^{i}|)^{n-2s}} \leq \frac{1}{(1+|x-x^{1}|)^{\frac{n+2s}{2}}}\sum_{i=2}^{k}\frac{1}{(1+|x-x^{i}|)^{\frac{n+2s}{2}}} \\ &\leq C\sum_{i=2}^{k}\left[\frac{1}{(1+|x-x^{1}|)^{n+2s-\theta}}+\frac{1}{(1+|x-x^{i}|)^{n+2s-\theta}}\right]\frac{1}{|x^{i}-x^{1}|^{\theta}} \\ &\leq C\frac{1}{(1+|x-x^{i}|)^{n+2s-\theta}}\sum_{i=2}^{k}\frac{1}{|x^{i}-x^{1}|^{\theta}} \\ &\leq \frac{C}{(1+|x-x^{i}|)^{n+2s-\theta}}\left(\frac{k}{\nu}\right)^{\theta}. \end{split}$$

Choosing $\theta > \frac{n-2s+1}{2}$ with $n + 2s - \theta \ge \frac{n+2s}{2} + \tau$, we have

$$\frac{1}{(1+|x-x^{1}|)^{4s}}\sum_{i=2}^{k}\frac{1}{(1+|x-x^{i}|)^{n-2s}} \le \frac{C}{(1+|x-x^{1}|)^{\frac{n+2s}{2}+\tau}}\left(\frac{1}{\nu}\right)^{\frac{1}{2}+\epsilon}.$$
(2.18)

On the other hand, for $x \in \Omega_1$, by Lemma A.1 again, we get

$$\leq \frac{1}{(1+|x-x^{i}|)^{n-2s}} \\ \leq \frac{1}{(1+|x-x^{1}|)^{\frac{n-2s}{2}}} \frac{1}{(1+|x-x^{i}|)^{\frac{n-2s}{2}}} \\ \leq \frac{C}{|x^{i}-x^{1}|^{\frac{n-2s}{2}-\frac{n-2s}{n+2s}\tau}} \Big(\frac{1}{(1+|x-x^{1}|)^{\frac{n-2s}{2}+\frac{n-2s}{n+2s}\tau}} + \frac{1}{(1+|x-x^{i}|)^{\frac{n-2s}{2}+\frac{n-2s}{n+2s}\tau}} \Big) \\ \leq \frac{C}{|x^{i}-x^{1}|^{\frac{n-2s}{2}-\frac{n-2s}{n+2s}\tau}} \frac{1}{(1+|x-x^{1}|)^{\frac{n-2s}{2}+\frac{n-2s}{n+2s}\tau}}.$$

So

$$\sum_{i=2}^{k} \frac{1}{(1+|x-x^{i}|)^{n-2s}} \leq C \left(\frac{k}{\nu}\right)^{\frac{n-2s}{2} - \frac{n-2s}{n+2s}\tau} \frac{1}{(1+|x-x^{1}|)^{\frac{n-2s}{2} + \frac{n-2s}{n+2s}\tau}},$$

which gives

$$\Big(\sum_{i=2}^{k} \frac{1}{(1+|x-x^{i}|)^{n-2s}}\Big)^{p} \le C\Big(\frac{k}{\nu}\Big)^{\frac{n+2s}{2}-\tau} \frac{1}{(1+|x-x^{1}|)^{\frac{n+2s}{2}+\tau}}.$$
(2.19)

Combining (2.17)-(2.19), we have

$$||J_1||_{**} \le C \Big(\frac{1}{\nu}\Big)^{\frac{1}{2}+\epsilon}.$$

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Now, we estimate J_2 . Let H(x, y) be the regular part of the Green function for \mathcal{A}_s in $B_1(0)$ with the Dirichlet boundary condition (see the definition in Appendix) and let \bar{x}_*^i be the reflection point of \bar{x}^i with respect to $\partial B_1(0)$, where $\bar{x}^i = \frac{x^i}{v}$. Then

$$\frac{H(\bar{x},\bar{x}^i)}{\nu^{n-2s}} = \frac{C}{\nu^{n-2s}|\bar{x}-\bar{x}^i_*|^{n-2s}} \le \frac{C}{(1+|x-x^i|)^{n-2s}}.$$

Take $\kappa = 1 - \theta$ with $\theta > 0$ small. By (A.1), we have

$$\begin{aligned} |J_2| &\leq \sum_{i=1}^{k} \frac{C}{(1+|x-x^i|)^{4s}} \frac{H(\bar{x},\bar{x}^i)}{\nu^{n-2s}} \\ &\leq \sum_{i=1}^{k} \frac{C}{(1+|x-x^i|)^{4s+\kappa(n-2s)}} \Big(\frac{H(\bar{x},\bar{x}^i)}{\nu^{n-2s}}\Big)^{\kappa} \\ &\leq C\Big(\frac{1}{\nu\ell}\Big)^{\kappa(n-2s)} \sum_{i=1}^{k} \frac{1}{(1+|x-x^i|)^{4s+\kappa(n-2s)}} \\ &\leq C\Big(\frac{1}{\nu}\Big)^{\frac{1}{2}+\epsilon} \sum_{i=1}^{k} \frac{1}{(1+|x-x^i|)^{4s+\kappa(n-2s)}} \\ &\leq C\Big(\frac{1}{\nu}\Big)^{\frac{1}{2}+\epsilon} \sum_{i=1}^{k} \frac{1}{(1+|x-x^i|)^{\frac{n+2s}{2}+\tau}}, \end{aligned}$$

since $\theta_{n-2s+1} > \frac{1}{2}$ for $n \ge 2+2s$. Finally, we estimate J_3 . For any $x \in \Omega_1$ and $i = 2, \dots, k$, applying Lemma A.1, we get

$$\begin{aligned} U_{\varepsilon,x^{i}}^{p}(x) &\leq C \frac{1}{(1+|x-x^{1}|)^{\frac{n+2s}{2}}} \frac{1}{(1+|x-x^{i}|)^{\frac{n+2s}{2}}} \\ &\leq C \Big(\frac{1}{(1+|x-x^{1}|)^{\frac{n+2s}{2}+\tau}} + \frac{1}{(1+|x-x^{i}|)^{\frac{n+2s}{2}+\tau}} \Big) \frac{1}{|x^{i}-x^{1}|^{\frac{n+2s}{2}-\tau}} \\ &\leq C \frac{1}{(1+|x-x^{1}|)^{\frac{n+2s}{2}+\tau}} \frac{1}{|x^{i}-x^{1}|^{\frac{n+2s}{2}-\tau}}, \end{aligned}$$

which gives that

$$\begin{aligned} \left| \sum_{i=2}^{k} \left(\Phi(\frac{|x|}{\nu}) - 1 \right) U_{\varepsilon,x^{i}}^{p} \right| &\leq C \frac{1}{(1+|x-x^{i}|)^{\frac{n+2s}{2}+\tau}} \sum_{i=1}^{k} \frac{1}{|x^{i} - x^{1}|^{\frac{n+2s}{2}-\tau}} \\ &\leq C \frac{1}{(1+|x-x^{1}|)^{\frac{n+2s}{2}+\tau}} \left(\frac{k}{\nu} \right)^{\frac{n+2s}{2}-\tau} \\ &\leq C (\frac{1}{\nu})^{\frac{1}{2}+\epsilon} \frac{1}{(1+|x-x^{1}|)^{\frac{n+2s}{2}+\tau}}. \end{aligned}$$
(2.20)

On the other hand, if $x \in \Omega_1$ and $||x| - v| \ge \delta v$, where $\delta > 0$ is a fixed constant, then

$$||x| - |x^{1}|| \ge ||x| - \nu| - ||x^{1}| - \nu| \ge \frac{1}{2}\delta\nu$$

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and thus

$$\left| \left(\Phi(\frac{|x|}{\nu}) - 1 \right) U_{\varepsilon, x^{1}}^{p} \right| \le \frac{C}{\nu^{\frac{n+2s}{2} - \tau}} \frac{1}{\left(1 + |x - x^{1}| \right)^{\frac{n+2s}{2} + \tau}}.$$
(2.21)

If $x \in \Omega_1$ and $||x| - \nu| \le \delta \nu$, we find

$$\begin{aligned} \left| \Phi(\frac{|x|}{\nu}) - 1 \right| &\leq C \left| \frac{|x|}{\nu} - 1 \right| \leq \frac{C}{\nu} ((|x| - |x^1|) + (|x^1| - \nu)) \\ &\leq \frac{C}{\nu} (|x| - |x^1|) + \frac{C}{k} \\ &\leq \frac{C}{\nu} (|x| - |x^1|) + \frac{C}{\nu^{\frac{1}{2} + \epsilon}} \end{aligned}$$
(2.22)

and $||x| - |x^1|| \le ||x| - \nu| + |\nu - |x^1|| \le 2\delta\nu$. Thus,

$$\frac{|x| - |x^1|}{\nu} \frac{1}{(1 + |x - x^1|)^{n+2s}} \le \frac{C}{\nu^{\frac{1}{2} + \epsilon}} \frac{||x| - |x^1||^{\frac{1}{2} - \epsilon}}{(1 + |x - x^1|)^{n+2s}} \le C(\frac{1}{\nu})^{\frac{1}{2} + \epsilon} \frac{1}{(1 + |x - x^1|)^{\frac{n+2s}{2} + \tau}}$$

which, together with (2.20)-(2.22), yields that

$$||J_3||_{**} \le C(\frac{1}{\nu})^{\frac{1}{2}+\epsilon}.$$

This finished this proof.

Proposition 2.5. There is an integer $k_0 > 0$, such that for each $k \ge k_0$, $\varepsilon_0 \le \varepsilon \le \varepsilon_1$, $r \in \left[\nu(1 - \frac{r_0}{k}), \nu(1 - \frac{r_1}{k})\right]$, (2.16) has a unique solution $\varphi = \varphi(r, \varepsilon)$ satisfying

$$\|\varphi\|_* \leq C(\frac{1}{\nu})^{\frac{1}{2}+\epsilon}, \quad |c_l| \leq C(\frac{1}{\nu})^{\frac{1}{2}+\epsilon},$$

where $\epsilon > 0$ is a small constant.

Proof. Recall that $v = k^{\frac{n-2s+1}{n-2s}}$ and set

$$\mathcal{N} = \left\{ w : w \in C^{\alpha_0}(B_{\nu}(0)) \cap H_k, \|w\|_* \le \frac{C}{\nu^{\frac{1}{2}}}, \int_{B_{\nu}(0)} U^{p-1}_{\varepsilon, x^i} Z_{i,l} w = 0 \right\},\$$

where $0 < \alpha_0 < s$ and i = 1, 2, ..., k, l = 1, 2. From Proposition 2.2, solving (2.16) is equivalent to solving

$$\varphi = \mathcal{B}(\varphi) := L_k(N(\varphi)) + L_k(l_k),$$

where L_k is defined in Proposition 2.2.

Firstly, we find

$$\begin{split} \|\mathcal{B}(\varphi)\|_{*} &\leq C \|N(\varphi)\|_{**} + C \|l_{k}\|_{**} \leq C \|\varphi\|_{*}^{\min\{p,2\}} + C (\frac{1}{\nu})^{\frac{1}{2}+\epsilon} \\ &\leq C (\frac{1}{\nu})^{\frac{1}{2}(\min\{p,2\})} + C (\frac{1}{\nu})^{\frac{1}{2}+\epsilon} \\ &\leq C (\frac{1}{\nu})^{\frac{1}{2}+\epsilon} \leq \frac{C}{\nu^{\frac{1}{2}}}, \end{split}$$

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which tells that \mathcal{B} maps \mathcal{N} to \mathcal{N} .

On the other hand, since

$$|N'(t)| \le C|t|^{\min\{p-1,1\}},$$

we get

$$\begin{aligned} &|L_{k}(N(\varphi_{1})) - L_{k}(N(\varphi_{2}))| \\ &\leq C(|\varphi_{1}|^{\min\{p-1,1\}} + |\varphi_{2}|^{\min\{p-1,1\}})|\varphi_{1} - \varphi_{2}| \\ &\leq C(||\varphi_{1}||^{\min\{p-1,1\}}_{*} + ||\varphi_{2}||^{\min\{p-1,1\}}_{*})||\varphi_{1} - \varphi_{2}||_{*} \Big(\sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{\frac{n-2s}{2}+\tau}}\Big)^{\min\{p,2\}}. \end{aligned}$$

Taking into account that

$$\Big(\sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{\frac{n-2s}{2}+\tau}}\Big)^{\min\{p,2\}} \le \sum_{i=1}^{k} \frac{C}{(1+|x-x^{i}|)^{\frac{n+2s}{2}+\tau}},$$

we have

$$\begin{split} \|\mathcal{B}(\varphi_{1}) - \mathcal{B}(\varphi_{2})\|_{*} &\leq C \|N(\varphi_{1}) - N(\varphi_{2})\|_{**} \\ &\leq C(\|\varphi_{1}\|_{*}^{\min\{p-1,1\}} + \|\varphi_{2}\|_{*}^{\min\{p-1,1\}})|\varphi_{1} - \varphi_{2}|_{*} \\ &\leq \frac{1}{2}\|\varphi_{1} - \varphi_{2}\|_{*}. \end{split}$$

Thus \mathcal{B} is a contraction map.

Therefore, Applying the contraction mapping theorem, we can find a unique $\varphi = \varphi(r, \varepsilon) \in \mathcal{N}$ such that

$$\varphi = \mathcal{B}(\varphi)$$

and

$$\|\varphi\|_* \le C(\frac{1}{\nu})^{\frac{1}{2}+\epsilon}.$$

Moreover, we get the estimate of c_l from (2.12).

3. Proof of Theorem 1.2

Let $F(\ell, \varepsilon) = I(U_{\varepsilon,r} + \varphi)$, where $r = |x^1|, \ell = 1 - \frac{r}{\nu}, \varphi$ is the function obtained in Proposition 2.5, and

$$I(u) = \frac{1}{2} \int_{B_{\nu}(0)} |\mathcal{A}_{s}^{\frac{1}{2}}u|^{2} - \frac{1}{p+1} \int_{B_{\nu}(0)} \Phi(\frac{|x|}{\nu}) |u|^{p+1}.$$

Proposition 3.1. We have

$$\begin{aligned} F(\ell,\varepsilon) &= I(U_{\varepsilon,r}) + O\left(\frac{k}{\nu^{1+\epsilon}}\right) \\ &= k\left(A + \frac{A_1H(\bar{x^1},\bar{x^1})}{\varepsilon^{n-2s}\nu^{n-2s}} + A_2\Phi'(1)\ell - \sum_{i=2}^k \frac{A_1G(\bar{x^i},\bar{x^1})}{\varepsilon^{n-2s}\nu^{n-2s}} + O(\frac{1}{\nu^{1+\epsilon}})\right) \end{aligned}$$

where A, A_1, A_2 are some positive constants, $\epsilon > 0$ is a small constant.

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Proof. Since

$$\langle I'(U_{\varepsilon,r}+\varphi),\varphi\rangle=0, \ \forall \varphi\in\mathcal{N},$$

there is $t \in (0, 1)$ such that

$$\begin{split} F(\ell,\varepsilon) &= I(U_{\varepsilon,r} + \varphi) \\ &= I(U_{\varepsilon,r}) - \frac{1}{2}D^2 I(U_{\varepsilon,r} + t\varphi)(\varphi,\varphi) \\ &= I(U_{\varepsilon,r}) - \frac{1}{2}\int_{B_{\nu}(0)} \left(\left|\mathcal{A}|_s^{\frac{1}{2}}\varphi\right|^2 - p\Phi(\frac{|x|}{\nu})(U_{\varepsilon,r} + t\varphi)^{p-1}\varphi^2\right) \\ &= I(U_{\varepsilon,r}) - \frac{1}{2}\int_{B_{\nu}(0)} (N(\varphi) + l_k)\varphi \\ &\quad + \frac{p}{2}\int_{B_{\nu}(0)} \Phi(\frac{|x|}{\nu})((U_{\varepsilon,r} + t\varphi)^{p-1} - U_{\varepsilon,r}^{p-1})\varphi^2 \\ &= I(U_{\varepsilon,r}) + O\Big(\int_{B_{\nu}(0)} \left(|\varphi|^{p+1} + |N(\varphi)||\varphi| + |l_k||\varphi|\Big)\Big). \end{split}$$

Firstly,

$$\begin{split} & \int_{B_{\nu}(0)} (|N(\varphi)||\varphi| + |l_{k}||\varphi|) \\ \leq & C(||N(\varphi)||_{**} + ||l_{k}||_{**})||\varphi||_{*} \int_{B_{\nu}(0)} \sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{\frac{n+2s}{2}+\tau}} \sum_{j=1}^{k} \frac{1}{(1+|x-x^{j}|)^{\frac{n-2s}{2}+\tau}}. \end{split}$$

From $\sum_{i=2}^{k} \frac{1}{|x^i - x^1|^{\tau}} \le C$ and Lemma A.1, we obtain

$$\begin{split} &\sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{\frac{n+2s}{2}+\tau}} \sum_{j=1}^{k} \frac{1}{(1+|x-x^{j}|)^{\frac{n-2s}{2}+\tau}} \\ &= \sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{n+2\tau}} + \sum_{j=1}^{k} \sum_{j\neq i} \frac{1}{(1+|x-x^{i}|)^{\frac{n+2s}{2}+\tau}} \frac{1}{(1+|x-x^{j}|)^{\frac{n-2s}{2}+\tau}} \\ &\leq \sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{n+2\tau}} + C \sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{n+\tau}} \sum_{j=2}^{k} \frac{1}{|x^{j}-x^{1}|^{\tau}} \\ &\leq C \sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{n+\tau}}. \end{split}$$

Therefore, we see

$$\begin{split} \int_{B_{\nu}(0)} (|N(\varphi)||\varphi| + |l_{k}||\varphi|) &\leq C(||N(\varphi)||_{**} + ||l_{k}||_{**})||\varphi||_{*} \int_{B_{\nu}(0)} \sum_{i=1}^{k} \frac{1}{(1 + |x - x^{i}|)^{n+\tau}} \\ &\leq Ck(||N(\varphi)||_{**} + ||l_{k}||_{**})||\varphi||_{*} \leq Ck\Big(\frac{1}{\nu^{1+\epsilon}}\Big). \end{split}$$

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On the other hand, by Hölder inequality, we have

$$\begin{split} \int_{B_{\nu}(0)} |\varphi|^{p+1} &\leq C ||\varphi||_{*}^{p+1} \int_{B_{\nu}(0)} \Big(\sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{\frac{n-2s}{2}+\tau}} \Big)^{p+1} \\ &\leq C ||\varphi||_{*}^{p+1} \int_{B_{\nu}(0)} \Big(\sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{n+\tau}} \Big) \Big(\sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{\tau}} \Big)^{p} \\ &\leq C ||\varphi||_{*}^{p+1} \int_{B_{\nu}(0)} \sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{n+\tau}} \\ &\leq C k ||\varphi||_{*}^{p+1} \leq C k \Big(\frac{1}{\nu^{1+\epsilon}} \Big). \end{split}$$

Combining the estimates above and applying Proposition A.5, we have proved

$$F(\ell,\varepsilon) = k \Big(A + \frac{A_1 H(\bar{x}^1, \bar{x}^1)}{\varepsilon^{n-2s} \nu^{n-2s}} + A_2 \Phi'(1)\ell - \sum_{i=2}^k \frac{A_1 G(\bar{x}^i, \bar{x}^1)}{\varepsilon^{n-2s} \nu^{n-2s}} + O(\frac{1}{\nu^{1+\epsilon}}) \Big).$$

Proposition 3.2. We have

$$\frac{\partial F(\ell,\varepsilon)}{\partial \varepsilon} = k(n-2s)A_1\Big(-\frac{H(\bar{x}^1,\bar{x}^1)}{\varepsilon^{n+1-2s}\nu^{n-2s}} + \sum_{i=2}^k \frac{G(\bar{x}^i,\bar{x}^1)}{\varepsilon^{n+1-2s}\nu^{n-2s}} + O(\frac{1}{\nu^{1+\epsilon}})\Big),$$

and

$$\frac{\partial F(\ell,\varepsilon)}{\partial \ell} = k \Big(A_1 \frac{\frac{\partial H(\bar{x}^1,\bar{x}^1)}{\partial \ell}}{\varepsilon^{n-2s} \nu^{n-2s}} + A_2 \Phi'(1) - A_1 \sum_{i=2}^k \frac{\frac{\partial G(\bar{x}^i,\bar{x}^1)}{\partial \ell}}{\varepsilon^{n-2s} \nu^{n-2s}} + O(\frac{1}{\nu^{\epsilon}}) \Big),$$

where A_1, A_2 are the same constants as in Proposition 3.1, and $\epsilon > 0$ is a small constant. *Proof.* Note that

$$\begin{split} \frac{\partial F(\ell,\varepsilon)}{\partial\varepsilon} &= \left\langle I'(U_{\varepsilon,r}+\varphi), \frac{\partial U_{\varepsilon,r}}{\partial\varepsilon} + \frac{\partial\varphi}{\partial\varepsilon} \right\rangle \\ &= \left\langle I'(U_{\varepsilon,r}+\varphi), \frac{\partial U_{\varepsilon,r}}{\partial\varepsilon} \right\rangle + \sum_{l=1}^{2} \sum_{i=1}^{k} c_{l} \left\langle U_{\varepsilon,x^{i}}^{p-1} Z_{i,l}, \frac{\partial\varphi}{\partial\varepsilon} \right\rangle \\ &= \frac{\partial I(U_{\varepsilon,r})}{\partial\varepsilon} - \int_{B_{\nu}(0)} \Phi(\frac{|x|}{\nu}) [(U_{\varepsilon,r}+\varphi)^{p} - U_{\varepsilon,r}^{p}] \frac{\partial U_{\varepsilon,r}}{\partial\varepsilon} \\ &+ \sum_{l=1}^{2} \sum_{i=1}^{k} c_{l} \left\langle U_{\varepsilon,x^{i}}^{p-1} Z_{i,l}, \frac{\partial\varphi}{\partial\varepsilon} \right\rangle. \end{split}$$

Considering

$$\left\langle U_{\varepsilon,x^{i}}^{p-1}Z_{i,l},\frac{\partial\varphi}{\partial\varepsilon}
ight
angle = -\left\langle rac{\partial(U_{\varepsilon,x^{i}}^{p-1}Z_{i,l})}{\partial\varepsilon},\varphi
ight
angle ,$$

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and from Proposition 2.5, we can get

$$\begin{split} & \left|\sum_{i=1}^{k} c_{l} \left\langle U_{\varepsilon, x^{i}}^{p-1} Z_{i, l}, \frac{\partial \varphi}{\partial \varepsilon} \right\rangle \right| \\ \leq & C |c_{l}| ||\varphi||_{*} \int_{B_{\nu}(0)} \sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{\frac{n-2s}{2}+\tau}} \sum_{j=1}^{k} \frac{1}{(1+|x-x^{j}|)^{n+2s}} \\ \leq & C \frac{k}{\nu^{1+\epsilon}}. \end{split}$$

On the other hand, being $\varphi \in \mathcal{N}$, we have

$$\begin{split} & \int_{B_{\nu}(0)} \Phi(\frac{|x|}{\nu}) [(U_{\varepsilon,r} + \varphi)^{p} - U_{\varepsilon,r}^{p}] \frac{\partial U_{\varepsilon,r}}{\partial \varepsilon} \\ &= \int_{B_{\nu}(0)} p \Phi(\frac{|x|}{\nu}) U_{\varepsilon,r}^{p-1} \frac{\partial U_{\varepsilon,r}}{\partial \varepsilon} \varphi + O\left(\int_{B_{\nu}(0)} |\varphi|^{2}\right) \\ &= \int_{B_{\nu}(0)} p \Phi(\frac{|x|}{\nu}) \left(U_{\varepsilon,r}^{p-1} \frac{\partial U_{\varepsilon,r}}{\partial \varepsilon} \varphi - \sum_{i=1}^{k} U_{\varepsilon,x^{i}} \frac{\partial U_{\varepsilon,x^{i}}}{\partial \varepsilon}\right) \varphi \\ &+ p \sum_{i=1}^{k} \int_{B_{\nu}(0)} \left(\Phi(\frac{|x|}{\nu}) - 1\right) U_{\varepsilon,x^{i}}^{p-1} \frac{\partial U_{\varepsilon,x^{i}}}{\partial \varepsilon} \varphi + O\left(\int_{B_{\nu}(0)} |\varphi|^{2}\right) \\ &= k \int_{\Omega_{1}} p \Phi(\frac{|x|}{\nu}) \left(U_{\varepsilon,r}^{p-1} \frac{\partial U_{\varepsilon,r}}{\partial \varepsilon} \varphi - \sum_{i=1}^{k} U_{\varepsilon,x^{i}} \frac{\partial U_{\varepsilon,x^{i}}}{\partial \varepsilon}\right) \varphi \\ &+ k \int_{B_{\nu}(0)} p \left(\Phi(\frac{|x|}{\nu}) - 1\right) U_{\varepsilon,x^{1}}^{p-1} \frac{\partial U_{\varepsilon,x^{1}}}{\partial \varepsilon} \varphi + O\left(\int_{B_{\nu}(0)} |\varphi|^{2}\right), \end{split}$$

$$\begin{split} & \Big| \int_{\Omega_1} \Phi(\frac{|x|}{\nu}) \Big(U_{\varepsilon,r}^{p-1} \frac{\partial U_{\varepsilon,r}}{\partial \varepsilon} \varphi - \sum_{i=1}^k U_{\varepsilon,x^i} \frac{\partial U_{\varepsilon,x^i}}{\partial \varepsilon} \Big) \varphi \Big| \\ \leq & C \int_{\Omega_1} \Big(U_{\varepsilon,r}^{p-1} (U_{\varepsilon,x^1} - PU_{\varepsilon,x^1}) + U_{\varepsilon,x^1}^{p-1} \sum_{i=2}^k U_{\varepsilon,x^i} + \sum_{i=2}^k U_{\varepsilon,x^i}^p \Big) |\varphi| \\ \leq & \frac{C}{\nu^{1+\epsilon}} \end{split}$$

and

$$\begin{split} & \left| \int_{B_{\nu}(0)} \left(\Phi(\frac{|x|}{\nu}) - 1 \right) U_{\varepsilon,x^{1}}^{p-1} \frac{\partial U_{\varepsilon,x^{1}}}{\partial \varepsilon} \varphi \right| \\ \leq & \left| \int_{|x|-\nu| \le \sqrt{\nu}} + \int_{|x|-\nu| \ge \sqrt{\nu}} \left(\Phi(\frac{|x|}{\nu}) - 1 \right) U_{\varepsilon,x^{1}}^{p-1} \frac{\partial U_{\varepsilon,x^{1}}}{\partial \varepsilon} \varphi \right| \\ \leq & \frac{C}{\nu^{1+\epsilon}}. \end{split}$$

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Thus, using the estimates above,

$$\frac{\partial F(\ell,\varepsilon)}{\partial \varepsilon} = \frac{\partial I(U_{\varepsilon,r})}{\partial \varepsilon} + O(\frac{k}{\nu^{1+\epsilon}}),$$

from which and Proposition A.6, the conclusion follows.

Finally, noting that $\frac{\partial}{\partial \ell} = -\nu \frac{\partial}{\partial r}$ and arguing as before, we can get the estimate $\frac{\partial F(\ell,\varepsilon)}{\partial \ell}$.

To prove Theorem 1.2, now we estimate $H(\bar{x}^1, \bar{x}^1)$ and $G(\bar{x}^i, \bar{x}^1), i \ge 2$. Let $\bar{x}^1_* = (\frac{1}{|\bar{x}^1|}, 0, \dots, 0)$ be the reflection of \bar{x}^1 with respect to the unit sphere. Then

$$H(\bar{x}^1, \bar{x}^1) = \frac{C}{|\bar{x}^1 - \bar{x}^1_*|^{n-2s}} = \frac{C}{2^{n-2s}\ell^{n-2s}}$$

for some constant C > 0.

On the other hand,

$$|\bar{x}^{i} - \bar{x}^{1}_{*}| = \sqrt{|\bar{x}^{i} - \bar{x}^{1}|^{2} + 4\ell^{2} - 4\ell|\bar{x}^{i} - \bar{x}^{1}|\cos\theta_{i}},$$

where θ_i is the angle between $\bar{x}^i - \bar{x}^1$ and $(1, 0, \dots, 0)$ and then $\theta_i = \frac{\pi}{2} + \frac{(i-1)\pi}{k}$. Thus

$$\begin{aligned} G(\bar{x}^{i}, \bar{x}^{1}) &= \frac{C}{|\bar{x}^{i} - \bar{x}^{1}|^{n-2s}} - \frac{C}{|\bar{x}^{i} - \bar{x}^{1}|^{n-2s}} \\ &= \frac{C}{|\bar{x}^{i} - \bar{x}^{1}|^{n-2s}} \Big(1 - \frac{1}{1 + \frac{4\ell^{2} + 4\ell|\bar{x}^{i} - \bar{x}^{1}|sin\frac{(i-1)\pi}{k}}{|\bar{x}^{i} - \bar{x}^{1}|^{2}}}\Big)^{\frac{n-2s}{2}}. \end{aligned}$$

As

$$|x^{1} - x^{i}| = 2|x^{1}|sin\frac{(i-1)\pi}{k}, i = 2, \cdots, k,$$

 $\ell k \to c > 0$ and

$$0 < c < \frac{\sin \frac{(i-1)\pi}{k}}{\frac{(i-1)\pi}{k}} \le c', \ i = 2, \cdots, [\frac{k}{2}],$$

we find

$$\frac{c_1}{i^2} \le \frac{4\ell^2 + 4\ell|\bar{x}^i - \bar{x}^1|\sin\frac{(i-1)\pi}{k}}{|\bar{x}^i - \bar{x}^1|^2} \le \frac{c_2}{i^2}$$

for some constants $c_2 \ge c_1 > 0$.

So, there exists a constant $A_3 > 0$ such that

$$\sum_{i=2}^{k} G(\bar{x}^{i}, \bar{x}^{1}) = \sum_{i=2}^{k} \frac{C}{|\bar{x}^{i} - \bar{x}^{1}|^{n-2s}} \left(1 + O\left(\frac{1}{(1 + \frac{c}{l^{2}})^{n-2s}}\right)\right)$$
$$= \sum_{i=2}^{k} \frac{C}{|\bar{x}^{i} - \bar{x}^{1}|^{n-2s}} (\tilde{c} + O(\frac{1}{l^{2}}))$$
$$= A_{3}k^{n-2s} + O(\frac{1}{k}).$$

Therefore, it follows from Propositions 3.1 and 3.2 that there are constants B_1, B_2, B_3 such that

$$F(\ell,\varepsilon) = k \Big(A + \frac{B_1}{\varepsilon^{n-2s} v^{n-2s} \ell^{n-2s}} + B_2 \ell - \frac{B_3 k^{n-2s}}{\varepsilon^{n-2s} v^{n-2s}} + O(\frac{1}{v^{1+\epsilon}}) \Big),$$
(3.1)

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$$\frac{\partial F(\ell,\varepsilon)}{\partial \varepsilon} = k \Big(-\frac{B_1(n-2s)}{\varepsilon^{n+1-2s} \nu^{n-2s} \ell^{n-2s}} + \frac{B_3(n-2s)k^{n-2s}}{\varepsilon^{n+1-2s} \nu^{n-2s}} + O(\frac{1}{\nu^{1+\epsilon}}) \Big)$$
(3.2)

and

$$\frac{\partial F(\ell,\varepsilon)}{\partial \ell} = k \Big(-\frac{B_1(n-2s)}{\varepsilon^{n-2s} \ell^{n+1-2s}} + B_2 + O(\frac{1}{\nu^{\epsilon}}) \Big).$$
(3.3)

Proof of Theorem 1.2. Recall that $\ell = 1 - \frac{r}{\nu}$ and $\nu = k^{\frac{n-2s+1}{n-2s}}$. Denote $L = \ell k$. Then from (3.2) and (3.3), $\frac{\partial F(\ell,\varepsilon)}{\partial \varepsilon} = 0$ and $\frac{\partial F(\ell,\varepsilon)}{\partial \ell} = 0$ are equivalent to

$$\frac{-B_1(n-2s)}{\varepsilon^{n+1-2s}L^{n-2s}} + \frac{B_3(n-2s)}{\varepsilon^{n+1-2s}} + O(\frac{1}{v^{\epsilon}}) = 0$$
(3.4)

and

$$\frac{-B_1(n-2s)}{\varepsilon^{n-2s}L^{n+1-2s}} + B_2 + O(\frac{1}{\nu^{\epsilon}}) = 0$$
(3.5)

respectively.

Denote

$$h_1(L,\varepsilon) = \frac{-B_1(n-2s)}{\varepsilon^{n+1-2s}L^{n-2s}} + \frac{B_3(n-2s)}{\varepsilon^{n+1-2s}}$$

and

$$h_2(L,\varepsilon) = \frac{-B_1(n-2s)}{\varepsilon^{n-2s}L^{n+1-2s}} + B_2.$$

Thus $h_1 = 0$ and $h_2 = 0$ have a unique solution

$$L_0 = \left(\frac{B_1}{B_3}\right)^{\frac{1}{n-2s}}, \quad \varepsilon_0 = \left(\frac{B_1(n-2s)}{B_2L_0^{n+1-2s}}\right)^{\frac{1}{n-2s}}.$$

Moreover, it is easy to verify that

$$\frac{\partial h_1(L_0,\varepsilon_0)}{\partial \varepsilon} = 0, \quad \frac{\partial h_2(L_0,\varepsilon_0)}{\partial L} > 0$$

and

$$\frac{\partial h_1(L_0,\varepsilon_0)}{\partial L} = \frac{\partial h_2(L_0,\varepsilon_0)}{\partial \varepsilon} > 0,$$

which means that $h_1 = 0$ and $h_2 = 0$ at (L_0, ε_0) is invertible. So, (3.4) and (3.5) have a solution near (L_0, ε_0) .

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Conflict of interest

The author declares that there are no conflicts of interest.

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A. Energy expansion

In this section, we will give some basic estimates and the energy expansion for the approximate solutions. First, recall that

$$x^{i} = \left(r\cos\frac{2(i-1)\pi}{k}, r\sin\frac{2(i-1)\pi}{k}, 0\right), \ i = 1, \cdots, k,$$

where $0 \in \mathbb{R}^{n-2}$, $r \in \left[\nu(1 - \frac{r_0}{k}), \nu(1 - \frac{r_1}{k})\right]$, and

$$I(u) = \frac{1}{2} \int_{B_{\nu}(0)} |\mathcal{A}_{s}^{\frac{1}{2}} u|^{2} - \frac{1}{p+1} \int_{B_{\nu}(0)} \Phi(\frac{|x|}{\nu}) |u|^{p+1}.$$

Now, we introduce the following two lemmas which are important in this sequel and have been proved in [22] and [12] respectively.

Lemma A.1. For any constant $0 < \sigma \le \min\{\alpha, \beta\}$, there exists a constant C > 0, such that

$$\frac{1}{(1+|y-x^i|)^{\alpha}}\frac{1}{(1+|y-x^j|)^{\beta}} \leq \frac{C}{|x^i-x^j|^{\sigma}}\Big(\frac{1}{(1+|y-x^i|)^{\alpha+\beta-\sigma}} + \frac{1}{(1+|y-x^j|)^{\alpha+\beta-\sigma}}\Big).$$

Lemma A.2. For any constant $0 < \kappa < n - 2s$, there is a constant C > 0 such that

$$\int_{B_{\nu}(0)} \frac{1}{|y|^{n-2s}} \frac{1}{(1+|x-y|)^{2s+\kappa}} dy \le \frac{C}{(1+|x|)^{\kappa}}$$

Note that

$$U_{\varepsilon,r}(x) = \sum_{i=1}^{k} PU_{\varepsilon,x^{i}}(x)$$

and

$$U_{\varepsilon,x^{i}}(x) = C_{n,s} \frac{\varepsilon^{\frac{n-2s}{2}}}{(1+\varepsilon^{2}|x-x^{i}|^{2})^{\frac{n-2s}{2}}}$$

for some suitable $C_{n,s}$. We have

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Lemma A.3. There is a small $\epsilon > 0$ and some constant C > 0 such that

$$\int_{B_{\nu}(0)} \frac{1}{|x-y|^{n-2s}} U_{\varepsilon,r}^{\frac{4s}{n-2s}}(y) \sum_{i=1}^{k} \frac{1}{(1+|y-x^{i}|)^{\frac{n-2s}{2}+\tau}} dy \le C \sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{\frac{n-2s}{2}+\tau+\epsilon}}$$

Proof. We can find this proof in [12]. Here we just need to use

$$U_{\varepsilon,r}(x) \le C \sum_{i=1}^{k} \frac{1}{(1+|x-x^{i}|)^{n-2s}}.$$

Let G(x, y) be the Green function of \mathcal{A}_s in $B_1(0)$ with the Dirichlet boundary condition (see [4]), namely, G(x, y) solves

$$\mathcal{A}_s G(\cdot, y) = \delta_y$$
 in $B_1(0)$, $G(\cdot, y) = 0$ on $B_1(0)$,

and the regular part of G is given by

$$H(x, y) = \frac{\alpha_{n,s}}{|x - y|^{n - 2s}} - G(x, y),$$

where $\alpha_{n,s} = \frac{1}{|S^{n-1}|} \frac{2^{1-2s}\Gamma(\frac{n-2s}{2})}{\Gamma(\frac{n}{2})\Gamma(s)}$. Let $\bar{x}^i = \frac{x^i}{y}$. Then

Proposition A.4. We have

$$U_{\varepsilon,x^{i}}(x) - PU_{\varepsilon,x^{i}}(x) = \frac{1}{\varepsilon^{\frac{n-2s}{2}}} V^{n-2s} H(\bar{x}, \bar{x}^{i}) + O(\frac{1}{(\nu\ell)^{n+2-2s}}),$$
(A.1)

where $\ell = 1 - |\bar{x}^i| = 1 - \frac{r}{v}$.

Proof. First, letting $\varphi_{x^i,\varepsilon} = U_{\varepsilon,x^i} - PU_{\varepsilon,x^i}$, from the equations satisfied by U_{ε,x^i} and PU_{ε,x^i} , we can get that

$$\begin{cases} \mathcal{A}_{s}\varphi_{x^{i},\varepsilon} = 0 \quad in \ B_{\nu}(0), \\ \varphi_{x^{i},\varepsilon} = C_{n,s} \frac{1}{\varepsilon^{(n-2s)/2}|x-x^{i}|^{n-2s}} + O(\frac{1}{\varepsilon^{n+2-2s}|x-x^{i}|^{n+2-2s}}) \quad on \ \partial B_{\nu}(0). \end{cases}$$
(A.2)

Denote by $\tilde{G}(x, y)$ the Green function of \mathcal{A}_s in $B_{\nu}(0)$ with the Dirichlet boundary condition and by $\tilde{H}(x, y)$ the regular part of $\tilde{G}(x, y)$. So we can find

$$\begin{cases} \mathcal{A}_s \tilde{H}(x, x^i) = 0 \quad in \ B_{\nu}(0), \\ \tilde{H}(x, x^i) = \frac{\alpha_{n,s}}{|x - x^i|^{n - 2s}} \quad on \ \partial B_{\nu}(0). \end{cases}$$

which, together with (A.2), yields that

$$\begin{cases} \mathcal{A}_s(\varphi_{x^i,\varepsilon} - \frac{1}{\varepsilon^{(n-2s)/2}}\tilde{H}(x,x^i)) = 0 \quad in \ B_v(0), \\ \varphi_{x^i,\varepsilon} - \frac{1}{\varepsilon^{(n-2s)/2}}\tilde{H}(x,x^i) = O(\frac{1}{\varepsilon^{n+2-2s}|x-x^i|^{n+2-2s}}) \quad on \ \partial B_v(0). \end{cases}$$

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As a result,

$$\|\varphi_{x^{i},\varepsilon} - \frac{1}{\varepsilon^{(n-2s)/2}}\tilde{H}(x,x^{i})\|_{L^{\infty}} \le \frac{C}{\varepsilon^{n+2-2s}d(x^{i},\partial B_{\nu}(0))^{n+2-2s}} = \frac{C}{\varepsilon^{n+2-2s}|\nu - |x^{i}||^{n+2-2s}}$$
(A.3)

for some constant C > 0.

Using (A.3), we have proved

$$U_{\varepsilon,x^{i}}(x) - PU_{\varepsilon,x^{i}}(x) = \frac{1}{\varepsilon^{\frac{n-2s}{2}} \nu^{n-2s}} H(\bar{x}, \bar{x}^{i}) + O(\frac{1}{(\nu \ell)^{n+2-2s}}).$$

Proposition A.5. There holds

$$I(U_{\varepsilon,r}) = k \Big(A + \frac{A_1 H(\bar{x}^1, \bar{x}^1)}{\varepsilon^{n-2s} \nu^{n-2s}} + A_2 \Phi'(1)\ell - \sum_{i=2}^k \frac{A_1 G(\bar{x}^i, \bar{x}^1)}{\varepsilon^{n-2s} \nu^{n-2s}} + O(\frac{1}{\nu^{1+\epsilon}}) \Big),$$

where A, A_1, A_2 are some positive constants, and ϵ is a small constant.

Proof. Note that

$$I(U_{\varepsilon,r}) = \frac{1}{2} \int_{B_{\nu}(0)} |\mathcal{A}_{s}^{\frac{1}{2}} U_{\varepsilon,r}|^{2} - \frac{1}{p+1} \int_{B_{\nu}(0)} \Phi(\frac{|x|}{\nu}) |U_{\varepsilon,r}|^{p+1}$$

Using the symmetry and (A.1), we have

$$\begin{split} \int_{B_{\nu}(0)} |\mathcal{A}_{s}^{\frac{1}{2}} U_{\varepsilon,r}|^{2} &= \sum_{i=1}^{k} \sum_{j=1}^{k} \int_{B_{\nu}(0)} U_{\varepsilon,x^{j}}^{p} P U_{\varepsilon,x^{i}} \\ &= k \sum_{i=1}^{k} \int_{B_{\nu}(0)} U_{\varepsilon,x^{1}}^{p} P U_{\varepsilon,x^{i}} \\ &= k \Big(\int_{B_{\nu}(0)} U_{\varepsilon,x^{1}}^{p+1} - \int_{B_{\nu}(0)} U_{\varepsilon,x^{1}}^{p} (U_{\varepsilon,x^{1}} - P U_{\varepsilon,x^{1}}) + \sum_{i=2}^{k} \int_{B_{\nu}(0)} U_{\varepsilon,x^{1}}^{p} P U_{\varepsilon,x^{i}} \Big) \\ &= k \Big(\int_{\mathbb{R}^{n}} U_{1,0}^{p+1} - \frac{B_{0}H(\bar{x}^{1}, \bar{x}^{1})}{\varepsilon^{n-2s}\nu^{n-2s}} + \sum_{i=2}^{k} \frac{B_{0}G(\bar{x}^{i}, \bar{x}^{1})}{\varepsilon^{n-2s}\nu^{n-2s}} + O(\frac{k}{\nu})^{n+2-2s} \Big), \end{split}$$
(A.4)

where $B_0 = \int_{\mathbb{R}^n} U_{1,0}^p$. Recalling that

$$\Omega_i = \left\{ x = (x', x'') \in B_{\nu}(0) : \langle \frac{x'}{|x'|}, \frac{(x^i)'}{|(x^i)'|} \rangle \ge \cos \frac{\pi}{k} \right\}, i = 1, 2, \cdots, k,$$

we can obtain that

$$\int_{B_{\nu}(0)} \Phi(\frac{|x|}{\nu}) U_{\varepsilon,r}^{p+1} = k \int_{\Omega_{1}} \Phi(\frac{|x|}{\nu}) (PU_{\varepsilon,x^{1}})^{p+1} + kp \int_{\Omega_{1}} \sum_{i=2}^{k} (PU_{\varepsilon,x^{1}})^{p} PU_{\varepsilon,x^{i}} + kO\left(\int_{\Omega_{1}} \left|\Phi(\frac{|x|}{\nu}) - 1\right| \sum_{i=2}^{k} U_{\varepsilon,x^{1}}^{p} U_{\varepsilon,x^{i}} + \int_{\Omega_{1}} U_{\varepsilon,x^{1}}^{(p+1)/2} (\sum_{i=2}^{k} U_{\varepsilon,x^{i}})^{(p+1)/2} \right).$$
(A.5)

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Next, we estimate each terms in (A.5). For $x \in \Omega_1$, $|x - x^i| \ge |x - x^1|$, we obtain for any $\kappa \in (1, n-2s)$,

$$\sum_{i=2}^{k} U_{\varepsilon,x^{i}} \leq C \sum_{i=2}^{k} \frac{1}{|x^{1} - x^{i}|^{\kappa}} \frac{1}{(1 + |x - x^{1}|)^{n-2s-\kappa}},$$

which implies that if κ close to n - 2s,

$$\int_{\Omega_1} U_{\varepsilon,x^1}^{p+1/2} (\sum_{i=2}^k U_{\varepsilon,x^i})^{p+1/2} = O(\frac{k}{\nu})^{\kappa \cdot \frac{n}{n-2s}} = O(\frac{1}{\nu^{1+\epsilon}}).$$

On the other hand, from (A.1) again, we have

$$\begin{split} &\int_{\Omega_{1}} \sum_{i=2}^{k} (PU_{\varepsilon,x^{1}})^{p} PU_{\varepsilon,x^{i}} \\ &= \int_{\Omega_{1}} \sum_{i=2}^{k} \left(P \frac{C_{n,s} \varepsilon^{\frac{n-2s}{2}}}{(1+\varepsilon^{2}|x-x^{i}|^{2})^{\frac{n-2s}{2}}} \right)^{p} \left(\frac{1}{\varepsilon^{\frac{n-2s}{2}} v^{n-2s}} G(\bar{x}, \bar{x}^{i}) + O(\frac{1}{(v\ell)^{n+2-2s}}) \right) \\ &= \varepsilon^{\frac{n+2s}{2}} \int_{\tilde{\Omega}_{1}} \sum_{i=2}^{k} \left(P \frac{C_{n,s}}{(1+|y|^{2})^{\frac{n-2s}{2}}} \right)^{p} \left(\frac{1}{\varepsilon^{\frac{n-2s}{2}} v^{n-2s}} G(\overline{\frac{y}{\varepsilon}+x^{i}}, \bar{x}^{i}) + O(\frac{1}{(v\ell)^{n+2-2s}}) \right) \varepsilon^{-n} dy \\ &= \int_{B_{v}(0)} U_{1,0}^{p} \sum_{i=2}^{k} \frac{G(\bar{x}, \bar{x}^{i})}{\varepsilon^{n-2s} v^{n-2s}} + O(\frac{1}{v^{1+\epsilon}}), \end{split}$$

where $\tilde{\Omega}_1 = \{\frac{\nu}{\varepsilon} + x^1 \in \Omega_1\}$. Now we estimate $\int_{\Omega_1} \left| \Phi(\frac{|x|}{\nu}) - 1 \right| U_{\varepsilon,x^1}^p \sum_{i=2}^k U_{\varepsilon,x^i}$. For $x \in \Omega_1$ and $||x| - \nu| \ge \delta \nu$, where $\delta > 0$ is a fixed constant, we have

$$||x| - |x^{1}|| \ge ||x| - \nu| - ||x^{1}| - \nu| \ge \frac{1}{2}\delta\nu$$

and from Lemma A.1,

$$\begin{split} &\int_{\Omega_{1}} \left| \Phi(\frac{|x|}{\nu}) - 1 \right| U_{\varepsilon,x^{1}}^{p} \sum_{i=2}^{k} U_{\varepsilon,x^{i}} \\ &\leq C \int_{\Omega_{1}} \sum_{i=2}^{k} \frac{1}{(1+|x-x^{1}|)^{n+2s}} \frac{1}{(1+|x-x^{i}|)^{n-2s}} \\ &\leq C(\frac{1}{\nu})^{1+s} \sum_{i=2}^{k} \frac{1}{|x^{i}-x^{1}|^{n-1-2s}} \int_{\Omega_{1}} \frac{1}{(1+|x-x^{1}|)^{n+s}} \\ &\leq C(\frac{1}{\nu})^{1+\epsilon}. \end{split}$$
(A.6)

If $x \in \Omega_1$ and $||x| - \nu| \le \delta \nu$, we have

$$\left| \Phi(\frac{|x|}{\nu}) - 1 \right| \le C \left| \frac{|x|}{\nu} - 1 \right| \le \frac{C}{\nu} ((|x| - |x^1|) + (|x^1| - \nu))$$

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$$\leq \quad \frac{C}{\nu}(|x|-|x^1|)+\frac{C}{k},$$

and $||x| - |x^1|| \le ||x| - \nu| + |\nu - |x^1|| \le 2\delta\nu$. Choosing $\kappa \in (1, n - 2s)$, we obtain that

$$\begin{split} &\int_{\Omega_{1}} \left| \Phi(\frac{|x|}{\nu}) - 1 \right| U_{\varepsilon,x^{1}}^{p} \sum_{i=2}^{k} U_{\varepsilon,x^{i}} \\ &\leq C \int_{\Omega_{1}} \left(\frac{1}{\nu} (|x| - |x^{1}|) + \frac{1}{k} \right) \sum_{i=2}^{k} \frac{1}{(1 + |x - x^{1}|)^{n+2s}} \frac{1}{(1 + |x - x^{1}|)^{n-2s}} \\ &\leq \frac{C}{\nu} \int_{\Omega_{1}} \sum_{i=2}^{k} \frac{1}{(1 + |x - x^{1}|)^{n+2s-1}} \frac{1}{(1 + |x - x^{1}|)^{n-2s}} \\ &\quad + \frac{C}{k} \sum_{i=2}^{k} \frac{1}{|x^{i} - x^{1}|^{\kappa}} \int_{\Omega_{1}} \frac{1}{(1 + |x - x^{1}|)^{2n-\kappa}} \\ &\leq C(\frac{1}{\nu})^{1+\epsilon}. \end{split}$$

This, together with (A.6), tells us that

$$\int_{\Omega_1} \left| \Phi(\frac{|x|}{\nu}) - 1 \right| U^p_{\varepsilon, x^1} \sum_{i=2}^k U_{\varepsilon, x^i} \le C(\frac{1}{\nu})^{1+\epsilon}.$$
(A.7)

Finally,

$$\begin{split} &\int_{\Omega_{1}} \Phi(\frac{|x|}{\nu}) (PU_{\varepsilon,x^{1}})^{p+1} \\ &= \int_{\Omega_{1}} (PU_{\varepsilon,x^{1}})^{p+1} + \int_{\Omega_{1}} \left(\Phi(\frac{|x|}{\nu}) - 1 \right) U_{\varepsilon,x^{1}}^{p+1} + O\left(\int_{\Omega_{1}} \left| \Phi(\frac{|x|}{\nu}) - 1 \right| U_{\varepsilon,x^{1}}^{p} \frac{H(\bar{x},\bar{x}^{1})}{\nu^{n-2s}} \right) \\ &= \int_{B_{\nu}(0)} U_{1,0}^{p+1} - \frac{(p+1)B_{0}H(\bar{x}^{1},\bar{x}^{1})}{\varepsilon^{n-2s}\nu^{n-2s}} + \int_{\Omega_{1}} \left(\Phi(\frac{|x|}{\nu}) - 1 \right) U_{\varepsilon,x^{1}}^{p+1} + O\left(\frac{1}{\nu^{1+\epsilon}} \right). \end{split}$$

But

$$\begin{split} \int_{\Omega_{1}} \left(\Phi(\frac{|x|}{\nu}) - 1 \right) U_{\varepsilon,x^{1}}^{p+1} &= \int_{\Omega_{1}} \left(\Phi(|\bar{x}^{1}|) - 1 \right) U_{\varepsilon,x^{1}}^{p+1} + \int_{\Omega_{1}} \left(\Phi(\frac{|x|}{\nu}) - \Phi(|\bar{x}^{1}|) \right) U_{\varepsilon,x^{1}}^{p+1} \\ &= \int_{\Omega_{1}} \left(\Phi(|\bar{x}^{1}|) - 1 \right) U_{\varepsilon,x^{1}}^{p+1} + O(\frac{1}{\nu^{1+\epsilon}}) \\ &= -\Phi'(1)\ell \int_{B_{\nu}(0)} U_{1,0}^{p+1} + O(\frac{1}{\nu^{1+\epsilon}}). \end{split}$$

Thus, from the estimates above, we have proved

$$\int_{B_{\nu}(0)} \Phi(\frac{|x|}{\nu}) U_{\varepsilon,r}^{p+1} = k \Big(\int_{B_{\nu}(0)} U_{1,0}^{p+1} - \Phi'(1)\ell \int_{B_{\nu}(0)} U_{1,0}^{p+1} - \frac{(p+1)B_0 H(\bar{x}^1, \bar{x}^1)}{\varepsilon^{n-2s} \nu^{n-2s}} \Big)$$

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$$+(p+1)\sum_{i=2}^{k}\frac{B_{0}G(\bar{x}^{i},\bar{x}^{1})}{\varepsilon^{n-2s}\nu^{n-2s}}+O(\frac{1}{\nu^{1+\epsilon}})\Big),$$

which, together with (A.4), implies that our desired result holds.

Similar to Proposition A.5, we also have

Proposition A.6.

$$\frac{\partial I(U_{\varepsilon,r})}{\partial \varepsilon} = k(n-2s)A_1\Big(-\frac{H(\bar{x}^1,\bar{x}^1)}{\varepsilon^{n+1-2s}\nu^{n-2s}} + \sum_{i=2}^k \frac{G(\bar{x}^i,\bar{x}^1)}{\varepsilon^{n+1-2s}\nu^{n-2s}} + O(\frac{1}{\nu^{1+\epsilon}})\Big),$$

and

$$\frac{\partial I(U_{\varepsilon,r})}{\partial r} = k \Big(A_1 \frac{\frac{\partial H(\bar{x}^1, \bar{x}^1)}{\partial r}}{\varepsilon^{n-2s} \nu^{n-2s}} - A_2 \Phi'(1) \frac{1}{\nu} - A_1 \sum_{i=2}^k \frac{\frac{\partial G(\bar{x}^i, \bar{x}^1)}{\partial r}}{\varepsilon^{n-2s} \nu^{n-2s}} + O(\frac{1}{\nu^{1+\epsilon}}) \Big)$$

where A_1, A_2 are the same constants as in Proposition A.5.



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