Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 5716–5723.
DOI:10.3934/math.2020366
Received: 30 April 2020
Accepted: 30 June 2020
Published: 09 July 2020

Research article

On Opial-Traple type inequalities for β-partial derivatives

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Abstract: In the paper, we introduce a new partial derivative call it *β-partial derivatives* as the most natural extensions of the limit definitions of the partial derivative and the β-derivative, which obeys classical properties including: continuity, linearity, product rule, quotient rule, power rule, chain rule and vanishing derivatives for constant functions. As applications, we establish some new Opial-Traple type inequalities for the β-partial derivatives.

Keywords: β-derivative; partial derivative; β-partial derivative; Cauchy-Schwarz inequality
Mathematics Subject Classification: 26A33, 26D125

1. Introduction

The fractional derivative of a function to order a is often now defined by means of the Fourier or Mellin integral transforms. Various types of fractional derivatives were introduced: Riemann-Liouville, Caputo, Hadamard, Erdelyi-Kober, Grunwald-Letnikov, Marchaud and Riesz are just a few to name [1–4]. For instance in more recent times a new local, limit-based definition of a conformable derivative has been introduced in [5–7], with several follow-up papers [8–11]. Recently a new local, limit-based definition of a so-called α-conformable derivative has been formulated in [4, 12–17].

In the paper, we give a new concept of β-partial derivatives as the most natural extension of the familiar limit definition of the partial derivative. We show also that the β-partial derivatives obeys classical properties including: continuity, linearity, product rule, quotient rule, power rule, chain rule and vanishing derivatives for constant functions. As applications, we establish some new Opial-Traple type inequalities for the β-partial derivatives.

2. The β-partial derivatives

There exist a quite few definitions of fractional derivatives in the literatures, we will present one definition. Given a function $f : [0, \infty] \to \mathbb{R}$. Then for all $\beta \in (0, 1]$ and $x \in (0, \infty)$, the β-derivative, defined by (see [18])

$$ {}_0^A\mathcal{D}_x^\beta(f(x)) = \lim_{\varepsilon\to 0} \frac{f\left(x+\varepsilon(x+\frac{1}{\Gamma(\beta)})^{1-\beta}\right)-f(x)}{\varepsilon}, \tag{2.1} $$

provided the limits exist, where $\Gamma(\cdot)$ is the usual Γ function. A function f is β-differentiable at a point $x \geq 0$, if the limits in (2.1) exist and are finite.

In this section, we give a new definition as the most natural extensions of the limit definitions of the partial derivative and the β-derivative. To this end, we start with the following definition which is a generalization of the classical partial derivative and β-derivative, respectively.

Definition 2.1 (β-partial derivatives) Let $f(x,y)$ be a function, such that $f(x,y):[a,\infty)\times[b,\infty)\to\mathbb{R}$.

(i) the β-partial x derivative of a function $f(x,y)$ is defined as

$$ {}_a^A\mathcal{P}_x^\beta(f(x,y)) = \lim_{\varepsilon\to 0} \frac{f\left(x+\varepsilon(x+\frac{1}{\Gamma(\beta)})^{1-\beta},y\right)-f(x,y)}{\varepsilon(1-ax^{1-\beta})}, \tag{2.2} $$

for all $x \geq a$ and $\beta \in (0,1]$. If the limit of the above exists, then $f(x,y)$ is said to be β-partial x differentiable and call ${}_a^A\mathcal{P}_x^\beta(f(x,y))$ as β-partial x derivatives of $f(x,y)$.

(ii) the beta-partial y derivative of a function $f(x,y)$ is defined as

$$ {}_b^A\mathcal{P}_y^\beta(f(x,y)) = \lim_{\varepsilon\to 0} \frac{f\left(x,y+\varepsilon(y+\frac{1}{\Gamma(\beta)})^{1-\beta}\right)-f(x,y)}{\varepsilon(1-by^{1-\beta})}, \tag{2.3} $$

for all $y \geq b$ and $\beta \in (0,1]$. If the limit of the above exists, then $f(x,y)$ is said to be β-partial y differentiable and call ${}_b^A\mathcal{P}_y^\beta(f(x,y))$ as β-partial y derivatives of $f(x,y)$. β-partial x, and β-partial y differentiable are collectively called β-partial differentiable.

Remark 2.2 Putting $\beta = 1$ and $a = b = 0$ in (2.2) and (2.3), the β-partial derivatives ${}_a^A\mathcal{P}_x^1(f(x,y))$ and ${}_b^A\mathcal{P}_y^1(f(x,y))$ just are the usual partial derivatives $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$, respectively.

Let $f(x,y)$ become $f(x)$ and with a proper transformation in (2.2), and let $a = 0$, the β-partial x derivatives ${}_a^A\mathcal{P}_x^\beta(f(x,y))$ reduces to the well-known β-derivatives ${}_0^A\mathcal{D}_x^\beta(f(x))$.

3. Properties for β-partial derivatives

In this section, we give several results for the β-partial derivatives such as the continuity, linearity, product rule, quotient rule, power rule, chain rule and vanishing derivatives for constant functions.

Theorem 3.1 (Continuity) *If $f(x,y):[a,\infty)\times[b,\infty)\to\mathbb{R}$ is β-partial differentiable and $\beta\in(0,1]$, then $f(x,y)$ is continuous at (x_0,y_0).*

Proof Since $f(s,t)$ is β-partial differentiable, so

$$ \lim_{\varepsilon\to 0} \frac{f\left(x_0+\varepsilon(x_0+\frac{1}{\Gamma(\beta)})^{1-\beta},y_0\right)-f(x_0,y_0)}{\varepsilon(1-ax_0^{1-\beta})} = {}_a^A\mathcal{P}_x^\beta(f(x,y))\Big|_{(x_0,y_0)}, \tag{3.1} $$

and

$$ \lim_{\varepsilon\to 0} \frac{f\left(x_0,y_0+\varepsilon(y_0+\frac{1}{\Gamma(\beta)})^{1-\beta}\right)-f(x_0,y_0)}{\varepsilon(1-by_0^{1-\beta})} = {}_b^A\mathcal{P}_y^\beta(f(x,y))\Big|_{(x_0,y_0)}. \tag{3.2} $$

From (3.1), (3.2), and let $h = \varepsilon(x_0 + \frac{1}{\Gamma(\beta)})^{1-\beta}$ and $k = \varepsilon(y_0 + \frac{1}{\Gamma(\beta)})^{1-\beta}$, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}[f(x_0+h, y_0+k) - f(x_0,y_0)] &= \lim_{h\to 0, k\to 0}[f(x_0+h, y_0+k) - f(x_0,y_0)] \\
&= \lim_{\varepsilon\to 0}\left[\frac{f(x_0+h, y_0+k) - f(x_0, y_0+k)}{\varepsilon(1-ax_0^{1-\beta})}\cdot \varepsilon(1-ax_0^{1-\beta})\right] \\
&+ \lim_{\varepsilon\to 0}\left[\frac{f(x_0, y_0+k) - f(x_0,y_0)}{\varepsilon(1-by_0^{1-\beta})}\cdot \varepsilon(1-by_0^{1-\beta})\right] \\
&= {}_a^A\mathcal{P}_x^\beta(f(x,y))\Big|_{(x_0,y_0)} \lim_{\varepsilon\to 0}\varepsilon(1-ax_0^{1-\beta}) \\
&+ {}_b^A\mathcal{P}_y^\beta(f(x,y))\Big|_{(x_0,y_0)} \lim_{\varepsilon\to 0}\varepsilon(1-by_0^{1-\beta}) \\
&= 0,
\end{aligned}$$

which implies that $f(x, y)$ is continuous at (x_0, y_0).

This completes the proof. □

Theorem 3.2 *Assuming that $f(x, y)$ and $g(x, y)$ are two β-partial x differentiable functions with $\beta \in (0, 1]$, then the following relations can be satisfied:*

(i) ${}_a^A\mathcal{P}_x^\beta(af(x,y) \pm bg(x,y)) = a \cdot {}_a^A\mathcal{P}_x^\beta(f(x,y)) + b \cdot {}_a^A\mathcal{P}_x^\beta(g(x,y))$, *for all a and b real number.*

(ii) ${}_a^A\mathcal{P}_x^\beta(f(x,y)) = \frac{(x+\frac{1}{\Gamma(\beta)})^{1-\beta}}{1-ax^{1-\beta}} \cdot \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon,y)-f(x,y)}{\varepsilon}$.

(iii) ${}_a^A\mathcal{P}_x^\beta(f(x,y)g(x.y)) = g(x,y) \cdot {}_a^A\mathcal{P}_x^\beta(f(x,y) + f(x,y) \cdot {}_a^A\mathcal{P}_x^\beta(g(x.y))$.

(iv) ${}_a^A\mathcal{P}_x^\beta\left(\frac{f(x,y)}{g(x.y)}\right) = \frac{f(s,t)\cdot {}_a^A\mathcal{P}_x^\beta(g(x.y)) - g(x,y)\cdot {}_a^A\mathcal{P}_x^\beta(f(x,y))}{g(x,y)^2}$, *where* $g(x,y) \neq 0$.

(v) ${}_a^A\mathcal{P}_x^\beta(\lambda) = 0$ for λ *any given constant.*

Proof Obviously, the (i) and (v) follow immediately from Definition 2.1.

Let

$$u = x + \varepsilon(x + \frac{1}{\Gamma(\beta)})^{1-\beta}.$$

Noting that $f(x, y)$ is continuous on $[a, \infty)$ at $x \geq a$, we have

$$\lim_{\varepsilon\to 0}\frac{\partial f(u,y)}{\partial u} = \frac{\partial f(x,y)}{\partial x} \quad\text{and}\quad \lim_{\varepsilon\to 0}\frac{\partial u}{\partial \varepsilon} = (x + \frac{1}{\Gamma(\beta)})^{1-\beta}.$$

Since $f(x, y)$ is β-partial x differentiable at $x \geq a$, and by using L'Hospital rule, we obtain

$$\begin{aligned}
{}_a^A\mathcal{P}_x^\beta(f(x,y)) &= \lim_{\varepsilon\to 0}\frac{f\left(x+\varepsilon(x+\frac{1}{\Gamma(\beta)})^{1-\beta}, y\right) - f(x,y)}{\varepsilon(1-ax^{1-\beta})}, \\
&= (1-ax^{1-\beta})^{-1}\lim_{\varepsilon\to 0}\frac{\partial f(u.y)}{\partial u}\cdot\frac{\partial u}{\partial \varepsilon} \\
&= \frac{(x+\frac{1}{\Gamma(\beta)})^{1-\beta}}{1-ax^{1-\beta}}\cdot\lim_{\varepsilon\to 0}\frac{f(x+\varepsilon,y)-f(x,y)}{\varepsilon}.
\end{aligned}$$

This completes the proof of (ii).

On the other hand, from (ii), we have

$$ {}_a^A\mathcal{P}_x^\beta(f(x,y)\cdot g(x,y)) = \frac{(x+\frac{1}{\Gamma(\beta)})^{1-\beta}}{1-ax^{1-\beta}}\cdot\frac{\partial(f(x,y)\cdot g(x,y))}{\partial x}$$

$$= f(x,y)\cdot\frac{(x+\frac{1}{\Gamma(\beta)})^{1-\beta}}{1-ax^{1-\beta}}\cdot\frac{\partial g(x,y)}{\partial x}+g(x,y)\cdot\frac{(x+\frac{1}{\Gamma(\beta)})^{1-\beta}}{1-ax^{1-\beta}}\cdot\frac{\partial f(x,y)}{\partial x}$$
$$= f(x,y)\cdot{}_a^A\mathcal{P}_x^\beta(g(x,y))+g(x,y)\cdot{}_a^A\mathcal{P}_x^\beta(f(x,y)).$$

This completes the proof of (iii).

The proof of the (iv) is similar to (iii). Here, we omit this details.

This completes the proof.

Theorem 3.3 *Assuming that $f(x,y)$ and $g(x,y)$ are two β-partial y differentiable functions with $\beta\in(0,1]$, then the following relations can be satisfied:*

(i) ${}_b^A\mathcal{P}_y^\beta(af(x,y)\pm bg(x,y)) = a\cdot{}_b^A\mathcal{P}_y^\beta(f(x,y))+b\cdot{}_b^A\mathcal{P}_y^\beta(g(x,y))$, *for all a and b real number.*

(ii) ${}_b^A\mathcal{P}_y^\beta(f(x,y)) = \frac{(y+\frac{1}{\Gamma(\beta)})^{1-\beta}}{1-ay^{1-\beta}}\cdot\lim_{\varepsilon\to0}\frac{f(x,y+\varepsilon)-f(x,y)}{\varepsilon}$

(iii) ${}_b^A\mathcal{P}_y^\beta(f(x,y)g(x.y)) = g(x,y)\cdot{}_b^A\mathcal{P}_y^\beta(f(x,y)+f(x,y)\cdot{}_b^A\mathcal{P}_y^\beta(g(x.y))$.

(iv) ${}_b^A\mathcal{P}_y^\beta\left(\frac{f(x,y)}{g(x.y)}\right) = \frac{f(s,t)\cdot{}_b^A\mathcal{P}_y^\beta(g(x.y))-g(x,y)\cdot{}_b^A\mathcal{P}_y^\beta(f(x,y))}{g(x,y)^2}$, *where* $g(x,y)\neq0$.

(v) ${}_b^A\mathcal{P}_y^\beta(\lambda)=0$, for λ *any given constant.*

Proof This follows immediately from the proof of Theorem 3.2 with a proper transformation.

Theorem 3.4 *Let $f(x,y):[a,\infty)\times[b,\infty)\to\mathbb{R}$ be a function such that $f(x,y)$ is β-partial x differentiable. If $g(x,y)$ is a function defined in the range of $f(x,y)$ and also β-partial x differentiable, then*

$${}_a^A\mathcal{P}_x^\beta(f\circ g(x,y)) = f_g'(g(x,y))\cdot{}_a^A\mathcal{P}_x^\beta(g(x,y)), \tag{3.3}$$

where $f_g'(g(x,y))$ denotes the derivative of function f to $g(x,y)$.

Proof Let

$$v = g\left(x+\varepsilon(x+\frac{1}{\Gamma(\beta)})^{1-\beta},y\right),$$

and

$$u=\varepsilon(x+\frac{1}{\Gamma(\beta)})^{1-\beta}.$$

Hence

$$\lim_{\varepsilon\to0}f'(v)=f_g'(g(x,y)),\quad \lim_{\varepsilon\to0}\frac{\partial v}{\partial u}=\frac{\partial g(x,y)}{\partial x},\quad\text{and}\quad \lim_{\varepsilon\to0}\frac{\partial u}{\partial\varepsilon}=(x+\frac{1}{\Gamma(\beta)})^{1-\beta}. \tag{3.4}$$

Since $f(x,y)$ and $g(x,y)$ are two β-partial x differentiable, so $f\circ g$ is β-partial x differentiable, from (3.4) and by using L'Hospital rule, we obtain

$$\begin{aligned}
{}_a^A\mathcal{P}_x^\beta(f\circ g(x,y)) &= \lim_{\varepsilon\to0}\frac{f\circ g\left(x+\varepsilon(x+\frac{1}{\Gamma(\beta)})^{1-\beta},y\right)-f\circ g(x,y)}{\varepsilon(1-ax^{1-\beta})}\\
&= \lim_{\varepsilon\to0}\frac{f\left(g\left(x+\varepsilon(x+\frac{1}{\Gamma(\beta)})^{1-\beta},y\right)\right)-f(g(x,y))}{\varepsilon(1-ax^{1-\beta})}\\
&= (1-ax^{1-\beta})^{-1}\lim_{\varepsilon\to0}f'(v)\cdot\frac{\partial v}{\partial u}\cdot\frac{\partial u}{\partial\varepsilon}\\
&= f_g'(g(x,y))\cdot\left(\frac{(x+\frac{1}{\Gamma(\beta)})^{1-\beta}}{1-ax^{1-\beta}}\cdot\frac{\partial g(x,y)}{\partial x}\right)\\
&= f_g'(g(x,y))\cdot{}_a^A\mathcal{P}_x^\beta(g(x,y)).
\end{aligned}$$

This completes the proof.

This chain rule theorem is important, but it is also understood. In order for the reader to better understand this theorem, we give another proof below.

Second proof Let

$$\delta = g\left(x + \varepsilon(x + \frac{1}{\Gamma(\beta)})^{1-\beta}, y\right) - g(x, y).$$

Obviously, if $\varepsilon \to 0$, then $\delta \to 0$. From the hypotheses, we obtain

$$\begin{aligned}
{}^{A}_{a}\mathcal{P}^{\beta}_{x}(f \circ g(x,y)) &= \lim_{\varepsilon\to 0} \frac{f\left(g\left(x + \varepsilon(x + \frac{1}{\Gamma(\beta)})^{1-\beta}, y\right)\right) - f(g(x,y))}{\varepsilon(1 - ax^{1-\beta})} \\
&= \lim_{\varepsilon\to 0} \frac{f\left(g\left(x + \varepsilon(x + \frac{1}{\Gamma(\beta)})^{1-\beta}, y\right)\right) - f(g(x,y))}{g\left(x + \varepsilon(x + \frac{1}{\Gamma(\beta)})^{1-\beta}, y\right) - g(x,y)} \\
&\times \frac{g\left(x + \varepsilon(x + \frac{1}{\Gamma(\beta)})^{1-\beta}, y\right) - g(x,y)}{\varepsilon(1 - ax^{1-\beta})} \\
&= \lim_{\delta\to 0} \frac{f(g(x,y) + \delta) - f(g(x,y))}{\delta} \cdot \lim_{\varepsilon\to 0} \frac{g\left(x + \varepsilon(x + \frac{1}{\Gamma(\beta)})^{1-\beta}, y\right) - g(x,y)}{\varepsilon(1 - ax^{1-\beta})} \\
&= f'_g(g(x,y)) \cdot {}^{A}_{a}\mathcal{P}^{\beta}_{x}(g(x,y)).
\end{aligned}$$

This completes the proof.

Let $f(x, y)$ and $g(x, y)$ change $f(x)$ and $g(x)$ with a proper transformation in Theorem 3.4, it becomes the following result, which was established in [18].

Corollary 3.5 *Let $f(x) : [0, \infty) \to \mathbb{R}$ be a function such that $f(x)$ is β-differentiable. If $g(x, y)$ is a function defined in the range of $f(x, y)$ and also differentiable, then*

$$ {}^{A}_{0}\mathcal{D}^{\beta}_{x}(f \circ g(x)) = (x + \frac{1}{\Gamma(\beta)})^{1-\beta} f'(x) g'(f(x)), \tag{3.5}$$

where, ${}^{A}_{0}\mathcal{D}^{\beta}_{x}(f(x))$ denotes the β-derivatives of $f(x)$.

Theorem 3.6 *Let $f(x, y) : [a, \infty) \times [b, \infty) \to \mathbb{R}$ be a function such that $f(x, y)$ is β-partial y differentiable. If $g(x, y)$ is a function defined in the range of $f(x, y)$ and also β-partial y differentiable, then*

$$ {}^{A}_{b}\mathcal{P}^{\beta}_{y}(f \circ g(x,y)) = f'_g(g(x,y)) \cdot {}^{A}_{b}\mathcal{P}^{\beta}_{y}(g(x,y)), \tag{3.6}$$

where $f'_g(g(x, y))$ denotes the derivative of function f to $g(x, y)$.

Proof This follows immediately from the proof of Theorem 3.4 with a proper transformation.

4. Opial-Traple type inequalities for β-partial derivatives

In the section, we establish Opial-Traple type inequalities for the β-partial derivatives.

Definition 4.1 (β-conformable integrals) Let $\beta \in (0, 1]$, $0 \le a < b$ and $0 \le c < d$. A function $f(x, y) : [a, b] \times [c, d] \to \mathbb{R}$ is said β-conformable integrable on $[a, b] \times [c, d]$, if the integral

$$\int_a^b f(x,y) d_\beta x := \int_a^b (1 - ax^{1-\beta}) \left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} f(x,y) dx \tag{4.1}$$

exists and is finite.

Theorem 4.2 (Opial-Traple inequality for β-partial derivatives) *Let $\alpha \in (0,1]$, $a \le s \le b$, and $c \le t \le d$, and $p(s,t)$ be nonnegative and continuous function on $[a,b]\times[c,d]$. Let $u(s,t)$ and $p(s,t)$ be β-partial differentiable on $[a,b]\times[c,d]$ with $u(a,t)=u(b,t)=0$, then*

$$\int_a^b p(s,t)\,|u(s,t)|^2\, d_\beta s \le M(a,b,\beta)\cdot \int_a^b p(s,t) d_\beta s \cdot \int_a^b \left| {}_a^A\mathcal{P}_s^\beta(u(s,t))\right|^2 d_\beta s, \tag{4.2}$$

where

$$M(a,b,\beta) = \frac{1}{4}\int_a^b (1-as^{1-\beta})\left(s+\frac{1}{\Gamma(\beta)}\right)^{\beta-1} ds.$$

Proof Let

$$y(s,t) = \int_a^s \left| {}_a^A\mathcal{P}_\sigma^\beta(u(\sigma,t))\right| d_\beta\sigma,$$

and

$$z(s,t) = \int_s^b \left| {}_a^A\mathcal{P}_\sigma^\beta(u(\sigma,t))\right| d_\beta\sigma.$$

From (4.1) and Theorem 3.3, we obtain

$${}_a^A\mathcal{P}_s^\beta(y(s,t)) = \left| {}_a^A\mathcal{P}_s^\beta(u(s,t))\right| = -{}_a^A\mathcal{P}_s^\beta(z(s,t)), \tag{4.3}$$

and for all $(s,t)\in[a,b]\times[c,d]$,

$$u(s,t)\le y(s,t),\ u(s,t)\le z(s,t). \tag{4.4}$$

Hence

$$u(s,t) \le \frac{y(s,t)+z(s,t)}{2} = \frac{1}{2}\int_a^b \left| {}_a^A\mathcal{P}_\sigma^\beta(u(\sigma,t))\right| d_\beta\sigma. \tag{4.5}$$

From (4.5) and in view of Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\int_a^b p(s,t)\,|u(s,t)|^2\, d_\beta s\\
\le\ & \frac{1}{4}\int_a^b p(s,t)\left(\int_a^b \left| {}_a^A\mathcal{P}_\sigma^\beta(u(\sigma,t))\right| d_\beta\sigma\right)^2 d_\beta s\\
\le\ & \frac{1}{4}\left(\int_a^b p(s,t)d_\beta s\right)\left(\int_a^b d_\beta\sigma\right)\left(\int_a^b \left| {}_a^A\mathcal{P}_\sigma^\beta(u(\sigma,t))\right|^2 d_\beta\sigma\right)\\
=\ & \frac{1}{4}\int_a^b (1-as^{1-\beta})\left(s+\frac{1}{\Gamma(\beta)}\right)^{\beta-1} ds\cdot\left(\int_a^b p(s,t)d_\beta s\right)\left(\int_a^b \left| {}_a^A\mathcal{P}_\sigma^\beta(u(\sigma,t))\right|^2 d_\beta\sigma\right)\\
=\ & M(a,b,\beta)\cdot\left(\int_a^b p(s,t)d_\beta s\right)\left(\int_a^b \left| {}_a^A\mathcal{P}_s^\beta(u(s,t))\right|^2 d_\beta s\right).
\end{aligned}$$

This completes the proof. □

Theorem 4.3 *Let $\alpha \in (0,1]$, $0 \le s \le b$, and $p(s)$ be nonnegative and continuous function on $[0,b]$. Let $u(s)$ and $p(s)$ be β-differentiable on $[0,b]$ with $u(0) = u(b) = 0$, then*

$$\int_0^b p(s)\,|u(s)|^2\, d_\beta s \le N(b,\beta)\cdot\left(\int_0^b p(s) d_\beta s\right)\left(\int_0^b \left|{}_0^A\mathcal{D}_s^\beta(u(s))\right|^2 d_\beta s\right), \tag{4.6}$$

where ${}_0^A\mathcal{D}_s^\beta$ is as in (2.1), *and*

$$N(b,\beta) = \frac{1}{\beta}\left[\left(b + \frac{1}{\Gamma(\beta)}\right)^\beta - \left(\frac{1}{\Gamma(\beta)}\right)^\beta\right].$$

Proof Let $u(s,t)$ and $p(s,t)$ change to $u(s)$ and $p(s)$, respectively, and with a proper transformation, and let $a = 0$, (4.6) follows immediately from (4.2).

Theorem 4.4 *Let $a \le s \le b$, and $c \le t \le d$, and $p(s,t)$ be nonnegative and continuous function on $[a,b]\times[c,d]$. Let $u(s,t)$ and $p(s,t)$ be partial differentiable on $[a,b]\times[c,d]$ with $u(a,t) = u(b,t) = 0$, then*

$$\int_a^b p(s,t)\,|u(s,t)|^2\, ds \le \frac{b-a}{4}\left(\int_a^b p(s,t) ds\right)\left(\int_a^b \left|\frac{\partial u(s,t)}{\partial s}\right|^2 ds\right). \tag{4.7}$$

Proof This follows immediately from Theorem 4.2 with $\beta = 1$.

Let $p(s,t)$ and $u(s,t)$ reduce to $p(t)$ and $u(t)$, respectively, and with suitable modifications, and let $a = 0$ and $b = h$, (4.7) becomes the following result.

Corollary 4.5 *Let $p(t)$ be a nonnegative and continuous function on $[0,h]$. Let $u(t)$ be an absolutely continuous function on $[0,h]$ with $u(0) = u(h) = 0$, then*

$$\int_0^h p(t)\,|u(t)|^2\, dsdt \le \frac{h}{4}\left(\int_0^h p(t) dt\right)\left(\int_0^h |u'(t)|^2\, dt\right).$$

This is just an inequality which was established in [14]. Here, we call it Opial-Traple's inequality.

Acknowledgments

Research is supported by National Natural Science Foundation of China (11371334, 10971205). Research is partially supported by the Research Grants Council of the Hong Kong SAR, China (Project No.: HKU7017/05P).

Conflict of interest

The authors declare that they have no competing interests.

References

1. U. N. Katugampola, *New approach to a generalized fractional integral*, Appl. Math. Comput., **218** (2011), 860–865.
2. U. N. Katugampola, *A New approach to generalized fractional derivatives*, B. Math. Anal. App., **6** (2014), 1–15.

3. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier B. V., Amsterdam, Netherlands, 2006.
4. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordonand Breach, Yverdon et alibi, 1993.
5. T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2015), 57–66.
6. U. Katugampola, *A new fractional derivative with classical properties*, J. Amer. Math. Soc., in press.
7. R. Khalil, M. Al horani, A. Yousef, et al. *A new denition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65–70.
8. D. R. Anderson, D. J. Ulness, *Results for conformable differential equations*, preprint, 2016.
9. A. Atangana, D. Baleanu, A. Alsaedi, *New properties of conformable derivative*, Open Math., **13** (2015), 889–898.
10. O. S. Iyiola, E. R. Nwaeze, *Some new results on the new conformable fractional calculus with application using D'Alambert approach*, Progr. Fract. Differ. Appl., **2** (2016), 115–122.
11. A. Zheng, Y. Feng, W. Wang, *The Hyers-Ulam stability of the conformable fractional differential equation*, Mathematica Aeterna, **5** (2015), 485–492.
12. T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2015), 57–66.
13. M. Z. Sarikaya, H. Budak, *New inequalities of Opial type for conformable fractionalintegrals*, Turkish J. Math., **41** (2017), 1164–1173.
14. J. Traple, *On a boundary value problem for systems of ordinary differential equations of second order*, Zeszyty Nauk. Univ. Jagiello. Prace Mat., **5** (1971), 159–168.
15. M. Z. Sarikaya, C. C. Bilisik, *Some Opial type inequalities for conformable fractionalintegrals*, AIP Conference Proceedings, **1991** (2018), 020013.
16. M. Z. Sarikaya, H. Budak, *Opial type inequalities for conformable fractional integrals*, J. Appl. Anal.,, **25** (2019), 155–163.
17. M. Z. Sarikaya, H. Budak, F. Usta, *On generalized the conformable fractional calculus*, preprint, 2016.
18. A. Atangana, E. F. D. Goufo, *Extension of matched asymptotic method to fractional boundary layers problems*, Math. Probl. Eng., **2014** (2014), 1–7.

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