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Research article

L-fuzzy upper approximation operators associated with L-generalized fuzzy remote neighborhood systems of L-fuzzy points

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Abstract: It is well known that the approximation operators are always primitive concepts in kinds of general rough set theories. In this paper, considering L to be a completely distributive lattice, we introduce a notion of L-fuzzy upper approximation operator based on L-generalized fuzzy remote neighborhood systems of L-fuzzy points. It is shown that the new approximation operator is a fuzzification of the upper approximation operator in the rough set theory based on general remote neighborhood systems of classical points. Then the basic properties, axiomatic characterizations and the reduction theory on the L-fuzzy upper approximation operator are presented. Furthermore, the L-fuzzy upper approximation operators corresponding to the serial, reflexive, unary and transitive L-generalized fuzzy remote neighborhood systems, are discussed and characterized respectively.

Keywords: fuzzy mathematics; fuzzy rough sets; fuzzy points; fuzzy upper approximation operator; fuzzy remote neighborhood system
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1. Introduction and background

Rough set theory was originally proposed by Pawlak [28] in 1982 as a useful mathematical tool for dealing with uncertainty. The classical Pawlak rough set model is based on equivalence relations. This heavily restricts the application scope of rough sets. So, by relaxing the equivalence relations to binary relations and coverings, many kinds of general rough set models were developed [6,17,22,23,25,42,44, 45,52,54–56]. In 2014 and 2015, Syau, Lin [35] and Zhang [50] observed that many binary relation-based rough sets and covering-based rough sets were actually defined through their neighborhood (system). Hence, they introduced a rough set model based on generalized neighborhood system, and unified the binary relation-based rough set model and covering-based rough set model into a common framework [52]. The concept of remote neighborhood system is abstracted from the geometric notion of "remote", and it is the dual concept of neighborhood system. Since general neighborhood systems has been successful used to build rough set model [35, 50, 51], so it is naturally to define a rough set model through general remote neighborhood systems [33].

Fuzzy set theory is also an important mathematical tool to study uncertainty. Nowadays, there have been many branches of fuzzy mathematics, such as fuzzy algebra, fuzzy topology and fuzzy logic, etc [5,8–12,14,34,37–39,46,49]. Particularly, fuzzy rough set theory is an important branch, which can handle more complicated uncertain problems since it has the advantages of both fuzzy set and rough set [1–3, 7, 15, 16, 18–21, 24, 40, 41, 47, 48]. Furthermore, replacing the unit interval [0, 1] with a complete lattice L as the range of the membership function, the more general L-fuzzy rough sets further extend the theoretical framework and application range of classic rough sets [13, 26, 27, 29–32, 43, 51, 53]. Fuzzy rough sets have a variety of forms due to the different approaches of fuzzification. Fuzzifying binary relations and coverings are the common methods to define fuzzy rough sets. As we have seen that general neighborhood systems and generalized remote neighborhood systems are all important tools to define general rough sets. Hence, it is naturally to establish fuzzy general rough sets by fuzzifying them, respectively. Quite recently, by fuzzifying the notion of generalized neighborhood systems, the authors [51,53] established a rough set model based on L-generalized fuzzy neighborhood systems, where L is a complete residuated lattice. It was proved that this model brought the fuzzy relation-based rough set model, fuzzy covering-based rough set model and generalized neighborhood-system based rough set models under a unified framework.

In this paper, considering L to be a completely distributive lattice, by fuzzifying the notion of generalized remote neighborhood systems of classical points, we will introduce the notion of L-fuzzy generalized remote neighborhood systems of L-fuzzy points [33], and then develop an L-fuzzy upper approximation operator derived from L-fuzzy generalized remote neighborhood systems. It should be pointed out that the notion of L-fuzzy generalized remote neighborhood systems is a relaxation of the notion of L-fuzzy remote neighborhood systems in [36], which plays a crucial role in the theory of L-fuzzy topological spaces. Moreover, it is an intrinsic way to study (fuzzy) rough sets from a topological perspective [2, 15, 26, 54]. The main difference between the new fuzzy rough sets and the previous fuzzy rough sets is that the method of fuzzification. Said precisely, we further fuzzifying the points of the universe of discourse.

The contents are arranged as follows. In Section 2, we recall some notions and notations used in this paper. In Sections 3, we introduce the concept of L-fuzzy upper approximation operators derived from L-generalized fuzzy remote neighborhood systems of fuzzy points. We further discuss the special L-fuzzy upper approximation operators correspond to serial, reflexive, unary, (strong) transitive L-generalized fuzzy remote neighborhood systems, respectively. In Sections 4, we give the axiomatic characterizations on the L-fuzzy upper approximation operators discussed in Section 3. In Sections 5, we present a theory of reduction on our L-fuzzy upper approximation operator. In Sections 6, we make a conclusion.

2. Preliminaries

For any non-empty set X, we denote 2^X as the power set of X. For each $A \in 2^X$, we denote A^c as the complement of A.

Throughout this paper, L is denoted a completely distributive lattice. The smallest element (resp., the largest element) in L is denoted by $\perp$ (resp., $\top$). An element a in L is called a prime element if $a \geqslant b \wedge c$ implies $a \geqslant b$ or $a \geqslant c$. The set of non-$\top$ prime elements in L is denoted by $P(L)$. An element a in L is called co-prime if $a \leqslant b \vee c$ implies $a \leqslant b$ or $a \leqslant c$. The set of non-$\perp$ co-prime elements in L is denoted by $J(L)$ [4].

Let X be a non-empty set and L^X be the set of all L-fuzzy sets (or L-sets for short) on X. Then $(L^X, \leq)$ also forms a completely distributive lattice, where $\leq$ is the partial order inherited from L. Precisely, for $A, B \in L^X$, that $A \leq B$ means $A(x) \leq B(x)$ for each $x \in X$. The smallest (resp., largest) element in L^X is denoted by $\perp_X$ (resp., $\top_X$). The set of non-unit prime elements in L^X is denoted by $P(L^X)$. The set of non-zero co-prime elements L^X is denoted by $J(L^X)$ [36]. For $x \in X$, $a \in J(L)$, we denote x_a as the L-fuzzy set defined by $x_a(y) = a$ if $y = x$ and $x_a(y) = \perp$ if $y \neq x$. Generally, x_a is called an L-fuzzy point in X. For an L-fuzzy set A and an L-fuzzy point x_a, by $x_a \in A$ we mean that $x_a \leq A$.

In the following, we shall recall some results about generalized remote neighborhood system-based approximation operators from [33].

Definition 1. *[33] Let X be non-empty set. Then a mapping* $\mathscr{RN} : X \longrightarrow 2^{2^X}$ *is said to be a generalized remote neighborhood system operator (GRNSO, for short) on X provided* $\mathscr{RN}(x)$ *is non-empty for each* $x \in X$. *Usually,* $\mathscr{RN}(x)$ *is said to be a generalized remote neighborhood system (GRNS, for short) of x and each* $K \in \mathscr{RN}(x)$ *is said to be a remote neighborhood of x.*

Definition 2. *[33] Let* $\mathscr{RN} : X \longrightarrow 2^{2^X}$ *be a GRNSO. Then for* $A \in 2^X$, *its lower and upper approximation operators* $\underline{\mathscr{RN}}(A)$ *and* $\overline{\mathscr{RN}}(A)$, *are defined by*

$$\underline{\mathscr{RN}}(A) = \{x \in X \mid \exists K \in \mathscr{RN}(x), A^c \subseteq K\}.$$

$$\overline{\mathscr{RN}}(A) = \{x \in X \mid \forall K \in \mathscr{RN}(x), A \nsubseteq K\}.$$

A is called a definable set if $\underline{\mathscr{RN}}(A) = \overline{\mathscr{RN}}(A)$, *otherwise, it is a rough sets .*

Let $\mathscr{RN}$ be a GRNSO on X.

- ◇ $\mathscr{RN}$ is called *serial* provided for any $x \in X$ and $A \in \mathscr{RN}(x)$, $A \neq X$.
- ◇ $\mathscr{RN}$ is called *reflexive* provided for any $x \in X$ and $A \in \mathscr{RN}(x)$, $x \notin A$.
- ◇ $\mathscr{RN}$ is called *unary* provided for any $x \in X$ and $A, B \in \mathscr{RN}(x)$, then there exists an $C \in \mathscr{RN}(x)$ such that $A \cup B \subseteq C$.
- ◇ $\mathscr{RN}$ is called *transitive* provided for any $x \in X$ and $A \in \mathscr{RN}(x)$, then there exists a $B \in \mathscr{RN}(x)$ such that for each $y \notin B$ there exists a $B_y \in \mathscr{RN}(y)$ with $A \subseteq B_y$.
- ◇ $\mathscr{RN}$ is called *strong-transitive* provided for any $x, y, z \in X, A \in \mathscr{RN}(y)$ and $B \in \mathscr{RN}(z)$, $x \notin A$ and $y \notin B \Rightarrow x \notin B$.

For detail meanings about the above concepts, please refer to [33].

3. L-fuzzy upper approximation operators derived from L-generalized fuzzy remote neighborhood systems of L-fuzzy points

In this section, we will introduce the concept of L-fuzzy upper approximation operators derived from L-generalized fuzzy remote neighborhood systems of L-fuzzy points. We further discuss the special L-fuzzy upper approximation operators correspond to serial, reflexive, unary, (strong) transitive L-generalized fuzzy remote neighborhood systems, respectively.

Definition 3. *An L-generalized fuzzy remote neighborhood system operator (LGFRNSO, for short) is a mapping* $\mathscr{FRN} : J(L^X) \longrightarrow 2^{L^X}$ *such that* $\mathscr{FRN}(x_a)$ *is non-empty for each* $x_a \in J(L^X)$.

Usually, $\mathscr{FRN}(x_a)$ *is called L-generalized fuzzy remote neighborhood system (LGFRNS, for short) of L-fuzzy point* x_a*, and each* $K \in \mathscr{FRN}(x_a)$ *is called L-generalized fuzzy remote neighborhood of* x_a.

Remark 1. *As the adjacent structure of L-fuzzy point, L-fuzzy remote neighborhood system was proposed by Wang in [36]. It is well known by the scholars familiar with fuzzy topology that L-fuzzy remote neighborhood system overcomes some shortcomings of L-fuzzy neighborhood system in L-fuzzy topological spaces, and it greatly promotes the development of L-fuzzy topological space theory. By relaxing the requirements of L-fuzzy remote neighborhood system in L-topological space, we introduce the notion of L-generalized fuzzy remote neighborhood system.*

Next, we define the L-fuzzy upper approximation operator derived from LGFRNSO.

Definition 4. *Let* $\mathscr{FRN} : J(L^X) \longrightarrow 2^{L^X}$ *be a LGFRNSO. Then for each A of* L^X*, the upper approximation operator* $\overline{\mathscr{FRN}}(A)$ *is defined as below:*

$$\overline{\mathscr{FRN}}(A) = \bigvee \{x_a \in J(L^X) | \forall K \in \mathscr{FRN}(x_a), A \nleq K\}.$$

Remark 2. *It is easy to see that when* $L = \{\bot, \top\}$*, the L-generalized fuzzy remote neighborhood system operator reduces to the generalized remote neighborhood system operator, and the L-fuzzy upper approximation in this paper reduces to the upper approximation based on generalized remote neighborhood system operator in [33].*

Now, we turn our attention to the properties of L-fuzzy upper approximation operator. The following lemma will be used frequently in the sequel.

Lemma 1. [36] *Let* $A, B \in L^X$*, then*

$$\begin{aligned} A \leq B \quad &\Leftrightarrow \quad \forall x_a \in J(L^X), x_a \in A \Rightarrow x_a \in B \\ &\Leftrightarrow \quad \forall x_a \in J(L^X), x_a \notin B \Rightarrow x_a \notin A. \end{aligned}$$

Proposition 1. *Let* $\mathscr{FRN}$ *be a* LGFRNSO. *Then*

(1) $\overline{\mathscr{FRN}}(\bot_X) = \bot_X$,

(2) *For any* $A, B \in L^X$ *and* $A \leq B \Rightarrow \overline{\mathscr{FRN}}(A) \leq \overline{\mathscr{FRN}}(B)$.

Proof. (1) For any $x_a \in J(L^X)$ and $K \in \mathscr{FRN}(x_a)$, then there exists a $K \in \mathscr{FRN}(x_a)$ such that $\bot_X \leq K$. It follows that $x_a \notin \overline{\mathscr{FRN}}(\bot_X)$. This means that there is no L-fuzzy point is contained in $\overline{\mathscr{FRN}}(\bot_X)$. Hence, $\overline{\mathscr{FRN}}(\bot_X) = \bot_X$.

(2) For each $x_a \in \overline{\mathscr{FRN}}(A)$ and $K \in \mathscr{FRN}(x_a)$, we have $A \nleq K$. Since $A \leq B$, so $B \nleq K$. This shows that $x_a \in \overline{\mathscr{FRN}}(B)$. It follows by Lemma 1 that $\overline{\mathscr{FRN}}(A) \leq \overline{\mathscr{FRN}}(B)$. ◻

Next, we present the serial, reflexive, unary, transitive and strong-transitive conditions for LGFRNSO. It is not difficult to see that these conditions are natural extensions of the corresponding conditions for GRNSO in [33]. It should be addressed that the serial, reflexive, unary, transitive conditions for neighborhood (systems), are discussed because that the approximation operators associated with them corresponds to different modal logic systems, respectively, please refer to [23, 44, 52].

Definition 5. *Let $\mathscr{FRN}$ be a* LGFRNSO*.*

(FRN1) $\mathscr{FRN}$ *is called serial provided for any* $x_a \in J(L^X)$ *and* $A \in \mathscr{FRN}(x_a)$, $A \neq \top_X$;

(FRN2) $\mathscr{FRN}$ *is called reflexive provided for any* $x_a \in J(L^X)$ *and* $A \in \mathscr{FRN}(x_a)$, $x_a \notin A$;

(FRN3) $\mathscr{FRN}$ *is called unary provided for any* $x_a \in J(L^X)$ *and* $A, B \in \mathscr{FRN}(x_a)$, *there exists an* $C \in \mathscr{FRN}(x_a)$ *such that* $A \vee B \leq C$.

(FRN4) $\mathscr{FRN}$ *is called transitive provided for any* $x_a \in J(L^X)$ *and* $A \in \mathscr{FRN}(x_a)$, *there exists a* $B \in \mathscr{FRN}(x_a)$ *such that for each* $y_\mu \notin B$ *there exists a* $B_{y_\mu} \in \mathscr{FRN}(y_\mu)$ *with* $A \leq B_{y_\mu}$;

(FRN5) $\mathscr{FRN}$ *is called strong-transitive provided for any* $x_a, y_\mu, z_\nu \in J(L^X)$, $A \in \mathscr{FRN}(y_\mu)$ *and* $C \in \mathscr{FRN}(z_\nu)$, $x_a \notin A$ *and* $y_\mu \notin C \Rightarrow x_a \notin C$.

Proposition 2. *Let $\mathscr{FRN}$ be a* LGFRNSO*. Then $\mathscr{FRN}$ is serial iff* $\overline{\mathscr{FRN}}(\top_X) = \top_X$.

Proof. ($\Rightarrow$) Let $\mathscr{FRN}$ be serial. Take any $x_a \in J(L^X)$, then for each $K \in \mathscr{FRN}(x_a)$, we have $K \neq \top_X$ (i.e.,$\top_X \nleq K$) since $\mathscr{FRN}$ is serial. It follows that $x_a \in \overline{\mathscr{FRN}}(\top_X)$, and so $\top_X \leq \overline{\mathscr{FRN}}(\top_X)$. Thus, $\overline{\mathscr{FRN}}(\top_X) = \top_X$.

($\Leftarrow$) Let $\overline{\mathscr{FRN}}(\top_X) = \top_X$. For any $x_a \in J(L^X)$, $K \in \mathscr{FRN}(x_a)$, we have $x_a \in \overline{\mathscr{FRN}}(\top_X) = \top_X$, it follows that $\top_X \nleq K$. That means $K \neq \top_X$ for any $K \in \mathscr{FRN}(x_a)$. Hence $\mathscr{FRN}$ is serial. □

Proposition 3. *Let $\mathscr{FRN}$ be a* LGFRNSO*. Then $\mathscr{FRN}$ is reflexive iff* $\forall A \in L^X$, $A \leq \overline{\mathscr{FRN}}(A)$.

Proof. ($\Rightarrow$) Let $\mathscr{FRN}$ be reflexive and $A \in L^X$, $x_a \in A$, we obtain that for each $K \in \mathscr{FRN}(x_a)$, $x_a \notin K$. Then $A \nleq K$, this tells us $x_a \in \overline{\mathscr{FRN}}(A)$. Therefore $A \leq \overline{\mathscr{FRN}}(A)$.

($\Leftarrow$) For each $x_a \in J(L^X)$ and $K \in \mathscr{FRN}(x_a)$, by $K \leq K$, then $x_a \notin \overline{\mathscr{FRN}}(K)$. Since $K \leq \overline{\mathscr{FRN}}(K)$, so $x_a \notin K$. We have $\mathscr{FRN}$ is reflexive. □

Proposition 4. *Let $\mathscr{FRN}$ be a LGFRNSO. Then $\mathscr{FRN}$ is unary iff for any* $A, B \in L^X$, $\overline{\mathscr{FRN}}(A \vee B) = \overline{\mathscr{FRN}}(A) \vee \overline{\mathscr{FRN}}(B)$.

Proof. ($\Rightarrow$) For any $A, B \in L^X$, since $\overline{\mathscr{FRN}}(A) \leq \overline{\mathscr{FRN}}(A \vee B)$ and $\overline{\mathscr{FRN}}(B) \leq \overline{\mathscr{FRN}}(A \vee B)$, so

$$\overline{\mathscr{FRN}}(A) \vee \overline{\mathscr{FRN}}(B) \leq \overline{\mathscr{FRN}}(A \vee B).$$

Then we only need to prove

$$\overline{\mathscr{FRN}}(A) \vee \overline{\mathscr{FRN}}(B) \geq \overline{\mathscr{FRN}}(A \vee B).$$

Next we prove that if for any $x_a \notin (\overline{\mathscr{FRN}}(A) \vee \overline{\mathscr{FRN}}(B))$ then $x_a \notin \overline{\mathscr{FRN}}(A \vee B)$.

For any $x_a \notin (\overline{\mathscr{FRN}}(A) \vee \overline{\mathscr{FRN}}(B))$, we have $x_a \notin \overline{\mathscr{FRN}}(A)$ and $x_a \notin \overline{\mathscr{FRN}}(B)$.Then there exists $K, V \in \mathscr{FRN}(x_a)$ such that $A \leq K$ and $B \leq V$. Therefore $A \vee B \leq K \vee V$. Because $\mathscr{FRN}$

is unary, so for $K, V \in \mathscr{FRN}(x_a)$, there exists an $M \in \mathscr{FRN}(x_a)$ such that $K \vee V \leq M$. We have $A \vee B \leq M$. Hence $x_a \notin \overline{\mathscr{FRN}}(A \vee B)$. It follows by Lemma 1 that

$$\overline{\mathscr{FRN}}(A) \vee \overline{\mathscr{FRN}}(B) \geq \overline{\mathscr{FRN}}(A \vee B).$$

Thus

$$\overline{\mathscr{FRN}}(A \vee B) = \overline{\mathscr{FRN}}(A) \vee \overline{\mathscr{FRN}}(B).$$

($\Leftarrow$) For each $x_a \in J(L^X)$ and $K, V \in \mathscr{FRN}(x_a)$. By $K \leq K$, we obtain $x_a \notin \overline{\mathscr{FRN}}(K)$. In the same way, we have $x_a \notin \overline{\mathscr{FRN}}(V)$. Thus

$$x_a \notin (\overline{\mathscr{FRN}}(K) \vee \overline{\mathscr{FRN}}(V)) = \overline{\mathscr{FRN}}(K \vee V).$$

We have there exists $M \in \mathscr{FRN}(x_a)$ such that $K \vee V \leq M$. Therefore $\mathscr{FRN}$ is unary. □

Proposition 5. *Let* $\mathscr{FRN}$ *be a* LGFRNSO. *Then the* $\mathscr{FRN}$ *is transitive iff for any* $A \in L^X$, $\overline{\mathscr{FRN}}(A) \geq \overline{\mathscr{FRN}}(\overline{\mathscr{FRN}}(A))$.

Proof. ($\Rightarrow$) For each $x_a \notin \overline{\mathscr{FRN}}(A)$, there exists a $K \in \mathscr{FRN}(x_a)$ such that $A \leq K$. By $\mathscr{FRN}$ is transition, for the $K \in \mathscr{FRN}(x_a)$, there exists a $V \in \mathscr{FRN}(x_a)$ such that for each $y_\mu \notin V$, there exists a $V_{y_\mu} \in \mathscr{FRN}(y_\mu)$ such that $K \leq V_{y_\mu}$. Thus $A \leq V_{y_\mu}$, $y_\mu \notin \overline{\mathscr{FRN}}(A)$. We obtain that $\overline{\mathscr{FRN}}(A) \leq V$. Therefore $x_a \notin \overline{\mathscr{FRN}}(\overline{\mathscr{FRN}}(A))$. Then for any

$$A \in L^X, \overline{\mathscr{FRN}}(A) \geq \overline{\mathscr{FRN}}(\overline{\mathscr{FRN}}(A)).$$

($\Leftarrow$) For each $x_a \in J(L^X)$ and $K \in \mathscr{FRN}(x_a)$. By $K \leq K$, we have

$$x_a \notin \overline{\mathscr{FRN}}(K) \geq \overline{\mathscr{FRN}}(\overline{\mathscr{FRN}}(K)),$$

and so

$$x_a \notin \overline{\mathscr{FRN}}(\overline{\mathscr{FRN}}(K)).$$

Hence, there exists a $V \in \mathscr{FRN}(x_a)$ such that $\overline{\mathscr{FRN}}(K) \leq V$. It follows by Lemma 1 that $y_\mu \notin \overline{\mathscr{FRN}}(K)$ for each $y_\mu \notin V$. Then there exists a $V_{y_\mu} \in \mathscr{FRN}(y_\mu)$ such that $K \leq V_{y_\mu}$. Therefore, $\mathscr{FRN}$ is transitive. □

Proposition 6. *Let* $\mathscr{FRN}$ *be a* LGFRNSO. *If* $\mathscr{FRN}$ *is strong-transitive then for each* $A \in L^X$, $\overline{\mathscr{FRN}}(A) \geq \overline{\mathscr{FRN}}(\overline{\mathscr{FRN}}(A))$.

Proof. For each $x_a \in \overline{\mathscr{FRN}}(\overline{\mathscr{FRN}}(A))$ and $K \in \mathscr{FRN}(x_a)$, then $\overline{\mathscr{FRN}}(A) \nleq K$. We have there exists a $y_\mu \in \overline{\mathscr{FRN}}(A)$ such that $y_\mu \notin K$. By $y_\mu \in \overline{\mathscr{FRN}}(A)$, we have for each $M \in \mathscr{FRN}(y_\mu)$, $A \nleq M$, there exists a $z_\nu \in A$ such that $z_\nu \notin M$. By $\mathscr{FRN}$ is strong-transitive, we have $z_\nu \notin K$ and so $A \nleq K$, which means $x_a \in \overline{\mathscr{FRN}}(A)$. Hence,

$$\overline{\mathscr{FRN}}(A) \geq \overline{\mathscr{FRN}}(\overline{\mathscr{FRN}}(A)). \quad \square$$

The following Example 1 shows that the converse of the Proposition 6 is not true.

Example 1. *Let* $X = \{x, y, z\}$, $L = [0, 1]$. *For any* $a \in (0, 1]$, *define*

$$\mathscr{FRN}(x_a) = \mathscr{FRN}(y_a) = \{0_X\}, \mathscr{FRN}(z_a) = \{0_X\} \text{ if } a \neq 0.8,$$

$$\mathscr{FRN}(z_{0.8}) = \{0_X, x_{0.8}, x_{0.8} \vee z_{0.8}, y_{0.8} \vee z_{0.8}, x_{0.8} \vee y_{0.8}\}.$$

Then

$$\overline{\mathscr{FRN}}(x_a) = \overline{\mathscr{FRN}}(y_a) = \overline{\mathscr{FRN}}(z_a) = 1_X.$$

Take any $A \in L^X$. *If* $A = 0_X$ *then*

$$\overline{\mathscr{FRN}}(0_X) = 0_X = \overline{\mathscr{FRN}}(\overline{\mathscr{FRN}}(0_X)).$$

If $A \neq 0_X$, *then there is* $w \in X$ *such that* $A(w) > 0$, *so* $\overline{\mathscr{FRN}}(A) \geq \overline{\mathscr{FRN}}(w_{A(w)}) = 1_X$.

Hence, $\overline{\mathscr{FRN}}(A) \geq \overline{\mathscr{FRN}}(\overline{\mathscr{FRN}}(A))$ *for any* $A \in L^X$.

However, take $A = 0_X$, $C = x_{0.8} \vee z_{0.8}$, *note that*

$$x_{0.6} \notin A \in \mathscr{FRN}(y_{0.7}), y_{0.7} \notin C \in \mathscr{FRN}(z_{0.8}),$$

but $x_{0.6} \in C$. *Therefore,* $\mathscr{FRN}$ *is not strong-transitive.*

4. An axiomatic characterization on L-fuzzy upper approximation operators derived from LGFRNSO

Using a set of axioms to characterize approximation operators is a hot topic in (fuzzy) rough set theory [13, 15, 25, 26, 32, 40]. In this section, we will give an axiomatic characterization on L-fuzzy upper approximation operators derived from LGFRNSO.

In this section, we always assume that $f : L^X \longrightarrow L^X$ to be an operator.

Theorem 1. *There is a* LGFRNSO $\mathscr{FRN}$ *s.t.* $f = \overline{\mathscr{FRN}}$ *iff f fulfills*

(F1): $f(\perp_X) = \perp_X$;

(F2): $A \leq B \Rightarrow f(A) \leq f(B)$.

Proof. (⇒) It is known from Proposition 1.

(⇐) Assume that $f : L^X \longrightarrow L^X$ satisfies (F1) and (F2). Then we define an operator $\mathscr{FRN}_f : J(L^X) \to 2^{L^X}$ as that for each $x_a \in J(L^X)$,

$$\mathscr{FRN}_f(x_a) = \{A \in L^X | \exists B \in L^X \text{ s.t. } A \leq B \text{ and } x_a \notin f(B)\}.$$

Taking any $x_a \in J(L^X)$, it follows by (F1) that $x_a \notin \perp_X = f(\perp_X)$, so $\perp_X \in \mathscr{FRN}_f$. Hence, $\mathscr{FRN}_f(x_a)$ is non-empty, and so $\mathscr{FRN}_f$ is an LGFRNSO. Next, we prove that $\overline{\mathscr{FRN}_f} = f$. From Lemma 1, we need only check that for any $A \in L^X$, $x_a \notin \overline{\mathscr{FRN}_f}(A)$ iff $x_a \notin f(A)$.

If $x_a \notin \overline{\mathscr{FRN}_f}(A)$, then there exists a $B \in \mathscr{FRN}_f(x_a)$ such that $A \leq B$. By the definition of $\mathscr{FRN}_f(x_a)$ and $B \in \mathscr{FRN}_f(x_a)$ we get that there is a $C \in L^X$ such that $B \leq C$ (and so $A \leq C$ since $A \leq B$) and $x_a \notin f(C)$. It follows by (F2) that $f(A) \leq f(C)$ and so $x_a \notin f(A)$, as desired.

Conversely, if $x_a \notin f(A)$, then $A \in \mathscr{FRN}_f(x_a)$. By $A \leq A$, then $x_a \notin \overline{\mathscr{FRN}_f}(A)$, as desired. □

Theorem 2. *There is a serial* LGFRNSO $\mathscr{FRN}$ *s.t.* $f = \overline{\mathscr{FRN}}$ *iff f fulfills* (F1), (F2) *and* (F3): $f(\top_X) = \top_X$.

Proof. (⇒) Assume that $\mathscr{FRN}$ is a serial LGFRNSO and $f = \overline{\mathscr{FRN}}$. Then it follows by Proposition 1 and Proposition 2 that $f = \overline{\mathscr{FRN}}$ fulfills (F1)- (F3).

(⇐) Assume that f fulfills (F1)- (F3) and $\mathscr{FRN}_f$ is defined as that in Theorem 1. Note that we only need to check the serial condition. In fact, for every $x_a \in J(L^X)$, by $f(\top_X) = \top_X$ we have $x_a \in f(\top_X)$, it follows by the definition of $\mathscr{FRN}_f(x_a)$ that $\top_X \notin \mathscr{FRN}_f(x_a)$. Hence, for every $K \in \mathscr{FRN}_f(x_a)$, $K \neq \top_X$, and so $\mathscr{FRN}$ is serial. □

Theorem 3. *There is a reflexive* LGFRNSO $\mathscr{FRN}$ *s.t.* $f = \overline{\mathscr{FRN}}$ *iff f fulfills* (F1)*,* (F2) *and* (F4)*:* $A \leq f(A)$*, for every* $A \in L^X$*.*

Proof. (⇒) Assume that $\mathscr{FRN}$ is a reflexive LGFRNSO and $f = \overline{\mathscr{FRN}}$. Then it follows by Proposition 1 and Proposition 3 that $f = \overline{\mathscr{FRN}_f}$ fulfills (F1), (F2) and (F4).

(⇐) Assume that f fulfill (F1), (F2) and (F4) and $\mathscr{FRN}_f$ is defined as that in Theorem 1. Note that we only need to check the reflexive condition. In fact, for every $x_a \in J(L^X)$, take $A \in \mathscr{FRN}_f(x_a)$, by the definition of $\mathscr{FRN}_f$, then there is a $B \in L^X$ such that $A \leq B$ and $x_a \notin f(B)$. It follows by $A \leq B$ and (F4) that $A \leq B \leq f(B)$, then $x_a \notin A$. Therefore, $\mathscr{FRN}_f$ is reflexive. □

Theorem 4. *There is a transitive* LGFRNSO $\mathscr{FRN}$ *s.t.* $f = \overline{\mathscr{FRN}}$ *iff f satisfies* (F1)*,* (F2) *and* (F5)*:* $f(A) \geq f(f(A))$*, for every* $A \in L^X$*.*

Proof. (⇒) Assume that $\mathscr{FRN}$ be a transitive LGFRNSO and $f = \overline{\mathscr{FRN}}$. Then it follows by Proposition 1 and Proposition 5 that $f = \overline{\mathscr{FRN}_f}$ fulfills (F1), (F2) and (F5).

(⇐) Suppose that f fulfills (F1), (F2) and (F5) and $\mathscr{FRN}_f$ is defined as that in Theorem 1. Note that we only need to check the transitive condition. Indeed, let $x_a \in J(L^X)$ and $A \in \mathscr{FRN}_f(x_a)$. Then for every $x_a \notin f(A) = \overline{\mathscr{FRN}_f}(A)$, by (F5), we have $x_a \notin \overline{\mathscr{FRN}_f}(\overline{\mathscr{FRN}_f}(A))$. It follows by Definition 4 that there exists a $B \in \mathscr{FRN}_f(x_a)$ such that $\overline{\mathscr{FRN}_f}(A) \leq B$. From Lemma 1 we conclude that for each $y_\mu \notin B$, $y_\mu \notin \overline{\mathscr{FRN}_f}(A)$. So, there exists a $V_{y_\mu} \in \mathscr{FRN}_f(y_\mu)$ such that $A \leq V_{y_\mu}$. Hence, $\mathscr{FRN}_f$ is transitive. □

Theorem 5. *There is a unary* LGFRNSO $\mathscr{FRN}$ *s.t.* $f = \overline{\mathscr{FRN}}$ *iff f fulfills* (F1)*,* (F2) *and* (F6)*:* $f(A \vee B) = f(A) \vee f(B)$*, for every* $A, B \in L^X$*.*

Proof. (⇒) Suppose that $\mathscr{FRN}$ is a unary LGFRNSO and $f = \overline{\mathscr{FRN}}$. Then it holds by Proposition 1 and Proposition 4 that $f = \overline{\mathscr{FRN}_f}$ fulfills (F1), (F2) and (F6).

(⇐) Assume that f fulfills (F1), (F2) and (F6) and $\mathscr{FRN}_f$ is defined as that in Theorem 1. Note that we only need to check the unary condition. In fact, let $x_a \in J(L^X)$ and $A, B \in \mathscr{FRN}_f(x_a)$. Then by the definition of $\mathscr{FRN}_f(x_a)$ we have $x_a \notin f(A)$ and $x_a \notin f(B)$. By (F6), we have

$$x_a \notin (f(A) \vee f(B)) = f(A \vee B),$$

otherwise we will have $x_a \in f(A)$ or $x_a \in f(B)$ since a is co-prime in L. Then there exists $C \in \mathscr{FRN}_f(x_a)$ such that $A \vee B \leq C$. Thus $\mathscr{FRN}_f$ is unary. □

Remark 3. *It is not difficult to prove that (F6) ⇒ (F2), (F4) ⇒ (F3), and (F5) can be rewritten as (F5′)* $f(A) = f(f(A))$ *in the present of (F2) and (F4).*

The following corollary give the axiomatic characterizations on L-fuzzy upper approximation operators associated with some compositions of the mentioned conditions.

Corollary 1. (1) *There is a serial and transitive* LGFRNSO $\mathscr{FRN}$ *s.t.* $f = \overline{\mathscr{FRN}}$ *iff* f *fulfills* (F1)–(F3) *and* (F5).

(2) *There is a serial and unary* LGFRNSO $\mathscr{FRN}$ *s.t.* $f = \overline{\mathscr{FRN}}$ *iff* f *satisfies* (F1)–(F3) *and* (F6) *iff* f *fulfills* (F1), (F3) *and* (F6).

(3) *There is a reflexive and transitive* LGFRNSO $\mathscr{FRN}$ *s.t.* $f = \overline{\mathscr{FRN}}$ *iff* f *satisfies* (F1), (F2) *and* (F4), (F5) *iff* f *fulfills* (F1), (F2) *and* (F4), (F5′).

(4) *There is a reflexive and unary* LGFRNSO $\mathscr{FRN}$ *s.t.* $f = \overline{\mathscr{FRN}}$ *iff* f *satisfies* (F1), (F2), (F4) *and* (F6) *iff* f *fulfills* (F1), (F4) *and* (F6).

(5) *There is a transitive and unary* LGFRNSO $\mathscr{FRN}$ *s.t.* $f = \overline{\mathscr{FRN}}$ *iff* f *satisfies* (F1), (F2) *and* (F5), (F6) *iff* f *fulfills* (F1) *and* (F5), (F6).

(6) *There is a reflexive, transitive and unary* LGFRNSO $\mathscr{FRN}$ *s.t.* $f = \overline{\mathscr{FRN}}$ *iff* f *satisfies* (F1), (F2) *and* (F4)–(F6) *iff* f *fulfills* (F1), *and* (F4)–(F6) *iff* f *satisfies* (F1), (F4), (F5′) *and* (F6).

Note 1. *An operator* $f : L^X \longrightarrow L^X$ *fulfilling* (F1), (F4), (F5′) *and* (F6) *is usually called a L-closure operator. It is known that there is a bijection between L-closure operators and L-topologies. Hence, reflexive, transitive and unary* LGFRNSO *can characterize L-topology.*

5. Reduction on L-fuzzy upper approximation operator based on LGFRNSO

As we all know, reduction theory is the foundation of the application of rough sets. In this section, we will present a theory of reduction on L-fuzzy upper approximation operator based on LGFRNSO. The core think is to get rid of the smaller redundant L-fuzzy remote neighborhoods.

Definition 6. *Let* $\mathscr{FRN}$ *be a* LGFRNSO. *It is easily seen that the following mapping* $\mathscr{MFRN} : J(L^X) \longrightarrow 2^{L^X}$ *defined by* $\forall x_a \in J(L^X)$,

$$\mathscr{MFRN}(x_a) = \{K \in \mathscr{FRN}(x_a) | \forall V \in \mathscr{FRN}(x_a), K \leq V \Rightarrow K = V\}$$

is also a LGFRNSO, *and each element of* $\mathscr{FRN}(x_a)$ *is called the maximum remote neighborhood at* x_a.

Definition 7. *Let* $\mathscr{FRN}$ *be a* LGFRNSO.

(1) For a $K \in \mathscr{FRN}(x_a)$, K *is called a reducible element of* $\mathscr{FRN}$ *at* $x_a \in J(L^X)$, *if there exists a* $V \in \mathscr{FRN}(x_a)$ *such that* $K < V$ *(i.e.,* $K \leq V$ *but* $K \neq V$*), otherwise,* K *is called an irreducible element.*

(2) $\mathscr{FRN}$ *is called irreducible if for any* $x_a \in J(L^X)$, *each* $K \in \mathscr{FRN}(x_a)$ *is irreducible at* x_a, *otherwise,* $\mathscr{FRN}$ *is called reducible.*

Proposition 7. *Let* $\mathscr{FRN}$ *be an* LGFRNSO *and* $K \in \mathscr{FRN}(x_a)$ *be reducible at* x_a. *It is observed easily that the following mapping* $\mathscr{FRN}_K : J(L^X) \longrightarrow 2^{L^X}$ *defined by*

$$\forall y_b \in J(L^X), \mathscr{FRN}_K(y_b) = \begin{cases} \mathscr{FRN}(y_b) - K, & y = x, b = a; \\ \mathscr{FRN}(y_b), & \text{otherwise.} \end{cases},$$

is also a LGFRNSO.

Proof. The proof is obviously, so we omit it. □

Proposition 8. *Let $\mathscr{FRN}$ be a* LGFRNSO *and K be a reducible element of $\mathscr{FRN}$ at a point $x_a \in J(L^X)$. Then $V \in \mathscr{FRN}_K(x_a)$ is a reducible element of $\mathscr{FRN}_K$ at x_a iff V is a reducible element of $\mathscr{FRN}$ at x_a.*

Proof. (⇒) From $\mathscr{FRN}_K(x_a) \subseteq \mathscr{FRN}(x_a)$ we conclude easily that if V is a reducible element $\mathscr{FRN}_K$ at x_a then V is a reducible element of $\mathscr{FRN}$ at x_a.

(⇐) Let V be a reducible element of $\mathscr{FRN}$ at x_a. Then there exists an $M \in \mathscr{FRN}(x_a)$ such that $V < M$. If $M \neq K$, then $M \in \mathscr{FRN}_K(x_a)$, and so V is a reducible element of $\mathscr{FRN}_K$ at x_a. If $M = K$, by K is a reducible element of $\mathscr{FRN}$ at x_a, there exists an $H \in \mathscr{FRN}(x_a)$ such that $H > K = M > V$. It follows that V is a reducible element of $\mathscr{FRN}_K$ at x_a. □

Definition 8. *Let $\mathscr{FRN}$ be a* LGFRNSO. *Then* **redu**$(\mathscr{FRN})$*, generated by eliminating all reductive elements of $\mathscr{FRN}$ at every L-fuzzy point, is called the reduction of $\mathscr{FRN}$.*

Proposition 9. *Let $\mathscr{FRN}$ be a* LGFRNSO. *Then $K \in \mathscr{FRN}(x_a)$ is a reducible element of $\mathscr{FRN}$ at $x_a \in J(L^X)$ iff $K \notin \mathscr{MFRN}(x_a)$.*

Proof. (⇒) Let $K \in \mathscr{FRN}(x_a)$ be a reducible element of $\mathscr{FRN}$ at $x_a \in J(L^X)$. Then there exists a $V \in \mathscr{FRN}(x_a)$ such that $V > K$. By Definition 6, we have $K \notin \mathscr{MFRN}(x_a)$.

(⇐) Let $K \in \mathscr{FRN}(x_a)$ and $K \notin \mathscr{MFRN}(x_a)$. Then there exists a $V \in \mathscr{FRN}(x_a)$ such that $V > K$. Therefore $K \in \mathscr{FRN}(x_a)$ is a reducible element of $\mathscr{FRN}$ at $x_a \in J(L^X)$. □

Lemma 2. *Let $\mathscr{FRN}_1$ and $\mathscr{FRN}_2$ be two* LGFRNSO. *If $\forall x_a \in J(L^X)$, $\mathscr{FRN}_1(x_a) \supseteq \mathscr{FRN}_2(x_a)$, then $\overline{\mathscr{FRN}_1}(A) \leq \overline{\mathscr{FRN}_2}(A)$ for every $A \in L^X$.*

Proof. Take $x_a \in \overline{\mathscr{FRN}_1}(A)$, then $A \nleq K$ for each $K \in \mathscr{FRN}_1(x_a)$. By $\mathscr{FRN}_1(x_a) \supseteq \mathscr{FRN}_2(x_a)$, it follows that $A \nleq V$ for any $V \in \mathscr{FRN}_2(x_a)$, that means, $x_a \in \overline{\mathscr{FRN}_2}(A)$. Hence, $\overline{\mathscr{FRN}_1}(A) \leq \overline{\mathscr{FRN}_2}(A)$ by Lemma 1. □

Proposition 10. *Let $\mathscr{FRN}$ be a* LGFRNSO *and $K \in \mathscr{FRN}(x_a)$ be a reducible element of $\mathscr{FRN}$ at $x_a \in J(L^X)$. Then $\mathscr{FRN}$ and $\mathscr{FRN}_K$ generate the same L-fuzzy upper approximation operator. That is,*

$$\overline{\mathscr{FRN}}(A) = \overline{\mathscr{FRN}_K}(A), \forall A \in L^X.$$

Proof. Let $A \in L^X$. Then for any $x_a \in J(L^X)$, by $\mathscr{FRN}(x_a) \supseteq \mathscr{FRN}_K(x_a)$ and Lemma 2, we have $\overline{\mathscr{FRN}}(A) \leq \overline{\mathscr{FRN}_K}(A)$.

Next we prove that $\overline{\mathscr{FRN}}(A) \geq \overline{\mathscr{FRN}_K}(A)$. For all $x_a \notin \overline{\mathscr{FRN}}(A)$, by Definition 4, there exists an $V \in \mathscr{FRN}(x_a)$ such that $A \leq V$.

Case1: If $V \neq K$, then $V \in \mathscr{FRN}_K(x_a)$ and so $x_a \notin \overline{\mathscr{FRN}_K}(A)$.

Case2: If $V = K$, by $K \in \mathscr{FRN}(x_a)$ is a reducible element of $\mathscr{FRN}$ at $x_a \in J(L^X)$, then there exists a $M \in \mathscr{FRN}(x_a)$ such that $A \leq V = K < M$ and so $M \in \mathscr{FRN}_K(x_a)$. Therefore, $x_a \notin \overline{\mathscr{FRN}_K}(A)$.

A combination of Case1 and Case2, it follows by Lemma 1 that $\overline{\mathscr{FRN}}(A) \geq \overline{\mathscr{FRN}_K}(A)$. □

By Proposition 10, we obtain the following corollary.

Corollary 2. *Let $\mathscr{FRN}$ be a* LGFRNSO. *Then $\mathscr{FRN}$ and* **redu**$(\mathscr{FRN})$ *generate the same L-fuzzy upper approximation operator.*

Proposition 11. *Let $\mathscr{FRN}_1$ and $\mathscr{FRN}_2$ be two irreducible* LGFRNSO. *Then $\mathscr{FRN}_1$ and $\mathscr{FRN}_2$ generate the same L-fuzzy upper approximation operator iff $\mathscr{FRN}_1 = \mathscr{FRN}_2$.*

Proof. ($\Leftarrow$) The proof is obviously, so we omit it.

($\Rightarrow$) Take any $K \in \mathscr{FRN}_1(x_a)$, then by Definition 4, we have

$$x_a \notin \overline{\mathscr{FRN}_1}(K) = \overline{\mathscr{FRN}_2}(K),$$

so there exists an $V \in \mathscr{FRN}_2(x_a)$ such that $K \le V$. Because $\mathscr{FRN}_2$ is irreducible we obtain that $K = V \in \mathscr{FRN}_2(x_a)$. Hence, $\mathscr{FRN}_1(x_a) \subseteq \mathscr{FRN}_2(x_a)$. In the same way, we can prove that $\mathscr{FRN}_2(x_a) \subseteq \mathscr{FRN}_1(x_a)$. Therefore, $\mathscr{FRN}_1 = \mathscr{FRN}_2$. □

By Corollary 2 and Proposition 11 we have the following theorem.

Theorem 6. *Let $\mathscr{FRN}_1$, $\mathscr{FRN}_2$ be two* LGFRNSO. *Then $\mathscr{FRN}_1$ and $\mathscr{FRN}_2$ generate the same L-fuzzy upper approximation operator iff*

$$\mathbf{redu}(\mathscr{FRN}_\mathbf{1}) = \mathbf{redu}(\mathscr{FRN}_\mathbf{2}).$$

At last, we say some about the L-fuzzy lower approximation based on L-generalized fuzzy remote neighborhood systems.

Remark 4. (1) *In classical set theory, it holds the law of excluded middle. That means, for $A \in 2^X$ and $x \in X$, we have*

$$A \cup A^c = X, A \cap A^c = \emptyset \text{ and } x \in A \text{ or } x \in A^c.$$

This makes that upper and lower approximations based on GRNSO *are not independent because they can also be represented by each other, precisely, for $A \in 2^X$,*

$$\underline{\mathscr{RN}}(A) \;=\; (\overline{\mathscr{RN}}(A^c))^c, \overline{\mathscr{RN}}(A) = (\underline{\mathscr{RN}}(A^c))^c,$$

which are usually called **Dual Theorem**.

(2) *In L-fuzzy set theory, to analogize the classical negative operator, we usually consider L together with an order-reversing involution $\prime : L \longrightarrow L$ [36]. Then for each $A \in L^X$, the L-fuzzy set A' can be defined pointwisely. Note that for $A \in L^X$ and $x_a \in J(L^X)$, we have no*

$$A \vee A' = \top_X, A \wedge A' = \bot_X \text{ and } x_a \in A \text{ or } x_a \in A'.$$

That means, the law of excluded middle in L-fuzzy set dose not hold. This makes that we have no the fuzzy version of Dual Theorem, so we can not define and study the L-fuzzy lower approximation through the L-fuzzy upper approximation with an order-reversing involution $\prime$.

6. Conclusions

In this paper, we constructed an L-fuzzy upper approximation operator from the LGFRNSO. Then we presented the basic properties, axiomatic characterizations and reduction theory on the new approximation operator. Furthermore, the serial, reflexive, unary and (strong) transitive conditions in

LGFRNSO were proposed, and the associated approximation operator with them were discussed, respectively.

As we have seen in Remark 4, for GRNSO, since the upper and lower approximations can be represented by each other, then we can easily define and study lower approximation through upper approximation. But for LGFRNSO, we can not define and study the *L*-fuzzy lower approximation through the *L*-fuzzy upper approximation. Therefore, the study on *L*-fuzzy lower approximation and that on *L*-fuzzy upper approximation are independent work. We will leave the research on *L*-fuzzy lower approximation based on LGFRNSO as a future work. Additionally, as to our knowledge, general neighborhood systems based rough set have important application in information systems, see [52] and it references. Note that fuzzy set can be regarded as a fuzzy information granule and general *L*-fuzzy remote neighborhood systems can be regarded as the fuzzy information associated with fuzzy point. Therefore, it seems that fuzzy remote neighborhood-based rough sets should have some applications in fuzzy information systems. We will also consider this problem in the future work.

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Conflict of interest

The authors declare that they have no conflict of interest.

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