AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 5572–5627.
DOI:10.3934/math.2020358
Received: 20 April 2020
Accepted: 19 May 2020
Published: 01 July 2020

Research article

Numerical method for solving the continuous-time linear programming problems with time-dependent matrices and piecewise continuous functions

Hsien-Chung Wu*

Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 802, Taiwan

* **Correspondence:** Email: hcwu@nknucc.nknu.edu.tw.

Abstract: The numerical method is proposed in this paper to solve a general class of continuous-time linear programming problems in which the functions appeared in the coefficients and the time-dependent matrices are assumed to be piecewise continuous. In order to make sure that all the subintervals of time interval will not contain the discontinuities of the involved functions, a methodology for not equally partitioning the time interval is proposed. The main issue of this paper is to obtain an analytic formula of the error bound, where the strong duality theorem for the primal and dual pair of continuous-time linear programming problems with time-dependent matrices and piecewise continuous functions is a by-product. We shall propose two kinds of computational procedure to evaluate the error bounds. One needs to solve the dual problem of the discretized linear programming problem, and another one does not need to solve the dual problem. The detailed differences between these two computational procedures will be also presented. Finally we present a numerical example to demonstrate the usefulness of the numerical method.

Keywords: continuous-time linear programming problems; discretized problems; piecewise continuous functions; weak duality theorem; strong duality theorem
Mathematics Subject Classification: 90C05, 90C46, 90C90

1. Introduction

The theory of continuous-time linear programming problem has received considerable attention for a long time. Tyndall [31, 32] treated rigorously a continuous-time linear programming problem with the constant matrices, which was originated from the "bottleneck problem" proposed by Bellman [4]. Levinson [12] generalized the results of Tyndall by considering the time-dependent matrices in which the functions shown in the objective and constraints were assumed to be continuous on the time interval $[0, T]$. In this paper, we shall consider the piecewise continuous functions on $[0, T]$. Meidan and Perold [13], Papageorgiou [16] and Schechter [27] have also obtained some interesting results for continuous-time linear programming problems.

Anderson *et al.* [1–3], Fleischer and Sethuraman [8], Pullan [17–21] and Wang et al. [33] investigated a subclass of continuous-time linear programming problems, which is called the separated continuous-time linear programming problems and can be used to model certain job-shop scheduling problems. Weiss [34] proposed a simplex-like algorithm to solve the separated continuous-time linear programming problem. Shindin and Weiss [28, 34] also studied a more generalized separated continuous-time linear programming problem. However the error estimate was not studied in the above articles. One of the contribution of this paper is to obtain the error bound between the optimal solution and numerical optimal solution. This paper is also the first attempt to numerically solve the continuous-time linear programming problems with time-dependent matrices. As far as the author knows, the numerical methods for solving the problems with time-dependent matrices were seemingly not studied in the literature.

Buie and Abrham [5] initiated the numerical method for solving the continuous-time linear programming problem by considering the constant matrices. However the error of approximated optimal solution was not studied. In other words, the error of numerical optimal solution was unknown. To locate the error of approximated optimal solution is an important issue. Another issue for discretizing the continuous-time linear programming problem is the partition of time interval $[0, T]$. Buie and Abrham [5] adopted the equidistant dissections of time interval $[0, T]$. In order to discretize the continuous-time linear programming problems as a finite-dimensional linear programming problem, the methods for equally subdividing the time interval as the equal length of subintervals are frequently adopted. However, when the involved functions in the continuous-time linear programming problems are not continuous on $[0, T]$, the subdivided subinterval of $[0, T]$ may contain the discontinuities of the involved functions. Therefore, for the piecewise continuous functions, equally subdividing the time interval is not the appropriate adoption. In this paper, we propose the numerical method to solve the general class of continuous-time linear programming problems in which the functions appeared in the time-dependent matrices are allowed to be piecewise continuous. The problem solved in this paper includes the separated continuous-time linear programming problem. In order to make sure that the subintervals will not contain the discontinuities, a different methodology for not equally partitioning the time interval is proposed.

On the other hand, the nonlinear type of continuous-time optimization problems was studied by Farr and Hanson [6, 7], Grinold [9, 10], Hanson and Mond [11], Reiland [22, 23], Reiland and Hanson [24] and Singh [29]. The nonsmooth continuous-time optimization problems was studied by Rojas-Medar et al. [26] and Singh and Farr [30]. The nonsmooth continuous-time multiobjective programming problems was also studied by Nobakhtian and Pouryayevali [14, 15]. Zalmai [39–42] investigated the continuous-time fractional programming problems. However the numerical methods were not developed in the above articles.

Wen and Wu [35–37] have developed the different numerical methods to solve the continuous-time linear fractional programming problems. In order to solve the continuous-time problems, the discretized problems should be considered by dividing the time interval $[0, T]$ into many subintervals. Since the functions considered in Wen and Wu [35–37] are assumed to be continuous on $[0, T]$, we can take this advantage to equally divide the time interval $[0, T]$. In other words, each subinterval has the same length. In Wu [38], the functions are assumed to be piecewise continuous on the time interval $[0, T]$. In this case, the time interval cannot be equally divided. The reason is that, in order to develop the numerical technique, the functions should be continuous on each subinterval. Therefore a different methodology for not equally partitioning the time interval was proposed in Wu [38]. The above papers were studied based on the constant matrices. In this paper, we shall solve a more general model that considers the time-dependent matrices in continuous-time linear programming problem. We still consider the piecewise continuous functions on the time interval $[0, T]$. Therefore, the time interval cannot be equally divided.

The main issue of this paper is to obtain an analytic formula of the error bound. In this paper, we shall propose two kinds of computational procedures to evaluate the error bounds. One needs to solve the dual problem of the discretized linear programming problem, and another one does not need to solve the dual problem. Solving the dual problem is time-consuming, since it can be a large scale linear programming problem. Alternatively, we can also obtain the error bound without solving the dual problem. In this case, we may obtain a larger error bound. However we prefer to obtain a tighter error bound. The detailed differences between these two computational procedures will be described in the context of this paper. Finally, we present a numerical example to demonstrate the usefulness of the numerical method developed in this paper.

This paper is organized as follows. In Section 2, the primal and dual pair of continuous-time linear programming problems with time-dependent matrices and piecewise continuous functions are introduced. The assumptions needed in this paper are also presented. In Section 3, since the piecewise continuous functions are adopted in this paper, we propose a method to partition the time interval such that all the subintervals will not contain the discontinuities. Based on this partition, we introduce the discretized problem of the continuous-time linear programming problem with time-dependent matrices and piecewise continuous functions. In Section 4, under the desired settings, we can derive an analytic formula of the error bound. The strong duality theorem for the primal and dual pair of continuous-time linear programming problems with time-dependent matrices and piecewise continuous functions is also obtained. In Section 5, we are going to present the convergence of approximate optimal solutions that are step functions constructed from the optimal solutions of discretized linear programming problems. In Section 6, two computational procedures are proposed, and a numerical example is also provided to demonstrate the usefulness of this practical algorithm.

2. Formulation and motivation

The continuous-time linear programming problem is formulated as follows:

$$
\begin{aligned}
\text{(CLP)} \qquad \max \qquad & \int_0^T \mathbf{a}^\top(t)\mathbf{z}(t)dt \\
\text{subject to} \quad & B(t)\mathbf{z}(t) \leq \mathbf{c}(t) + \int_0^t K(t,s)\mathbf{z}(s)ds \text{ for all } t \in [0,T] \\
& \mathbf{z} \in L_q^2[0,T] \text{ and } \mathbf{z}(t) \geq \mathbf{0} \text{ for all } t \in [0,T].
\end{aligned}
$$

The dual problem of (CLP) is defined as follows:

$$
\begin{aligned}
\text{(DCLP)} \qquad \min \qquad & \int_0^T \mathbf{c}^\top(t)\mathbf{w}(t)dt \\
\text{subject to} \quad & (B(t))^\top \mathbf{w}(t) \geq \mathbf{a}(t) + \int_t^T K^\top(s,t)\mathbf{w}(s)ds \text{ for all } t \in [0,T]
\end{aligned}
$$

$$\mathbf{w} \in L_p^2[0,T] \text{ and } \mathbf{w}(t) \geq \mathbf{0} \text{ for all } t \in [0,T].$$

Assume the following conditions are satisfied.

- $B(t)$ and $K(t,s)$ are $p \times q$ time-dependent matrices, and all the entries B_{ij} and K_{ij} of B and K, respectively, are nonnegative and piecewise continuous for $i = 1, \cdots p$ and $j = 1, \cdots, q$.
- $\mathbf{a}$ and $\mathbf{c}$ are q-dimensional and p-dimensional vector-valued functions on $[0,T]$, respectively, and all the entries a_j and c_i are piecewise continuous on $[0,T]$ for $i = 1, \cdots p$ and $j = 1, \cdots, q$, respectively.
- for each $t \in [0,T]$ and $j = 1, \cdots, q$, the following inequality is satisfied:

$$\sum_{i=1}^{p} B_{ij}(t) > 0; \tag{2.1}$$

- the following inequality is satisfied:

$$\min_{i=1,\cdots,p} \min_{j=1,\cdots,q} \inf_{t\in[0,T]} \left\{ B_{ij}(t) : B_{ij}(t) > 0 \right\} = \sigma > 0. \tag{2.2}$$

  In other words, if $B_{ij}(t) \neq 0$, then $B_{ij}(t) \geq \sigma$ for all $i = 1, \cdots, p$ and $j = 1, \cdots, q$.

We have the following observations.

- If the vector-valued function $\mathbf{c}(t)$ is nonnegative, then it is obvious that the zero vector-valued function is a feasible solution of problem (CLP).
- The dual problem (DCLP) is always feasible that can be realized in Proposition 4.1 below.

The real-valued functions a_j and c_i will not be assumed to be nonnegative on $[0,T]$ for $i = 1, \cdots, p$ and $j = 1, \cdots, q$ in this paper. However, in the literature regarding the topic of continuous-time linear programming problems, the real-valued functions c_i were always assumed to be nonnegative on $[0,T]$ for $i = 1, \cdots, p$. In this paper, we can just assume that the primal problem (CLP) is feasible without considering the real-valued functions c_i to be nonnegative on $[0,T]$ for $i = 1, \cdots, p$. In other words, when the primal problem (CLP) is feasible, the numerical method developed in this paper does not rely on the nonnegativity of the real-valued functions c_i for $i = 1, \cdots, p$.

As a matter of fact, the feasibility of primal problem (CLP) can be guaranteed by the feasibility of its discretization problem (P_n) that will be shown below. Of course, if the real-valued functions c_i are assumed to be nonnegative on $[0,T]$ for $i = 1, \cdots, p$, then we can show that the discretization problem (P_n) is feasible for each integer $n \in \mathbb{N}$. Therefore, in order to avoid the nonnegativity of the real-valued functions c_i for $i = 1, \cdots, p$, we can simply assume that the discretization problem (P_n) is feasible for each integer $n \in \mathbb{N}$. This assumption is reasonable, unless we can prove that the discretization problem (P_n) is infeasible for some integer $n \in \mathbb{N}$ when some of the real-valued functions c_i are not nonnegative for some $i = 1, \cdots, p$. Here we also raise an open question. Whether we can prove that the discretization problem (P_n) is infeasible for some integer $n \in \mathbb{N}$ when some of the real-valued functions c_i are not nonnegative for some $i = 1, \cdots, p$.

3. Discretization

Let $\mathfrak{A}_j$, $\mathfrak{C}_i$, $\mathfrak{B}_{ij}$ and $\mathfrak{K}_{ij}$ denote the set of discontinuities of the real-valued functions a_j, c_i, B_{ij} and K_{ij}, respectively. Then $\mathfrak{A}_j$, $\mathfrak{C}_i$ and $\mathfrak{B}_{ij}$ are finite subsets of $[0,T]$ and $\mathfrak{K}_{ij}$ is a finite subset of $[0,T] \times [0,T]$.

We also write

$$\mathfrak{R}_{ij} = \mathfrak{R}_{ij}^{(1)} \times \mathfrak{R}_{ij}^{(2)},$$

where $\mathfrak{R}_{ij}^{(1)}$ and $\mathfrak{R}_{ij}^{(2)}$ are finite subset of $[0, T]$. In order to determine the partition of the time interval $[0, T]$, we consider the following set

$$\mathcal{D} = \left(\bigcup_{j=1}^{q} \mathfrak{A}_j\right) \bigcup \left(\bigcup_{i=1}^{p} \mathfrak{S}_i\right) \bigcup \left(\bigcup_{i=1}^{p}\bigcup_{j=1}^{q} \mathfrak{B}_{ij}\right) \bigcup \left(\bigcup_{i=1}^{p}\bigcup_{j=1}^{q} \mathfrak{R}_{ij}^{(1)}\right) \bigcup \left(\bigcup_{i=1}^{p}\bigcup_{j=1}^{q} \mathfrak{R}_{ij}^{(2)}\right) \bigcup \{0, T\}.$$

Then, $\mathcal{D}$ is a finite subset of $[0, T]$ written by

$$\mathcal{D} = \{d_0, d_1, d_2, \cdots, d_r\},$$

where, for convenience, we set $d_0 = 0$ and $d_r = T$. It means that d_0 and d_r may be the end-point continuities of functions **a**, **c**, B and K. Let $\mathcal{P}_n$ be a partition of $[0, T]$ such that $\mathcal{D} \subseteq \mathcal{P}_n$, which means that each closed $[d_v, d_{v+1}]$ is also divided into many closed subintervals. In this case, the time interval $[0, T]$ is not necessarily equally divided into n closed subintervals. Let

$$\mathcal{P}_n = \left\{e_0^{(n)}, e_1^{(n)}, \cdots, e_n^{(n)}\right\},$$

where $e_0^{(n)} = 0$ and $e_n^{(n)} = T$. Then, the n closed subintervals are denoted by

$$\bar{E}_l^{(n)} = \left[e_{l-1}^{(n)}, e_l^{(n)}\right] \text{ for } l = 1, \cdots, n.$$

We also write

$$E_l^{(n)} = \left(e_{l-1}^{(n)}, e_l^{(n)}\right) \text{ and } F_l^{(n)} = \left[e_{l-1}^{(n)}, e_l^{(n)}\right).$$

Let $\mathfrak{d}_l^{(n)}$ denote the length of closed interval $\bar{E}_l^{(n)}$, and let

$$\| \mathcal{P}_n \| = \max_{l=1,\cdots,n} \mathfrak{d}_l^{(n)}.$$

In the limiting case, we shall assume that

$$\| \mathcal{P}_n \| \to 0 \text{ as } n \to \infty.$$

In this paper, we assume that there exists $n_*, n^* \in \mathbb{N}$ such that

$$n_* \cdot r \leq n \leq n^* \cdot r \text{ and } \| \mathcal{P}_n \| \leq \frac{T}{n^*}. \tag{3.1}$$

Therefore, in the limiting case, we assume that n_* is sufficiently large, which also implies that n is sufficiently large. In the sequel, when we say that $n \to \infty$, it implicitly means that $n_* \to \infty$.

For example, suppose that the length of closed interval $[d_v, d_{v+1}]$ is $\mathfrak{l}_v$ for $v = 1, \cdots, r$. We consider the following types of partitions of $[0, T]$.

Example 3.1. Each closed interval $[d_v, d_{v+1}]$ is equally divided by n_* subintervals for $v = 1, \cdots, r$. In this case, the total subintervals are $n = n_* \cdot r$. We also see that $n_* = n^*$ and

$$\| \mathcal{P}_n \| = \frac{1}{n^*} \cdot \max_{v=1,\cdots,r} \mathfrak{l}_v \leq \frac{T}{n^*}, \text{ and } n \to \infty \text{ if and only if } n_* \to \infty.$$

Example 3.2. Let

$$\mathfrak{l}^* = \max_{v=1,\cdots,r} \mathfrak{l}_v.$$

Suppose that each closed subinterval $[d_v, d_{v+1}]$ is equally divided by n_v subintervals for $v = 1, \cdots, r$. In this case, the total subintervals are

$$n = \sum_{v=1}^{r} n_v.$$

Let

$$n^* = \max_{v=1,\cdots,r} n_v \text{ and } n_* = \min_{v=1,\cdots,r} n_v.$$

We also assume that the partition satisfies

$$\frac{n_*}{n^*} \geq \frac{\mathfrak{l}^*}{T}.$$

Then we have

$$n_* \cdot r \leq n \leq n^* \cdot r \text{ and } \parallel \mathcal{P}_n \parallel = \max_{v=1,\cdots,r} \frac{\mathfrak{l}_v}{n_v} \leq \frac{\mathfrak{l}^*}{n_*} \leq \frac{T}{n^*}.$$

In the limiting case, we shall assume $n_* \to \infty$, which also means that $n \to \infty$.

Under the above construction for the partition $\mathcal{P}_n$, we see that the real-valued functions a_j, c_i and B_{ij} are continuous on the open interval $E_l^{(n)}$ and the real-valued function K_{ij} is continuous on the open rectangle $E_l^{(n)} \times E_k^{(n)}$ for $l, k = 1, \cdots, n$. Now we define

$$a_{lj}^{(n)} = \inf_{t \in \bar{E}_l^{(n)}} a_j(t) \text{ and } c_{li}^{(n)} = \inf_{t \in \bar{E}_l^{(n)}} c_i(t) \tag{3.2}$$

and the vectors

$$\mathbf{a}_l^{(n)} = \left(a_{l1}^{(n)}, a_{l2}^{(n)}, \cdots, a_{lq}^{(n)}\right)^\top \in \mathbb{R}^q \text{ and } \mathbf{c}_l^{(n)} = \left(c_{l1}^{(n)}, c_{l2}^{(n)}, \cdots, c_{lp}^{(n)}\right)^\top \in \mathbb{R}^p.$$

Then we see that

$$\mathbf{a}(t) \geq \mathbf{a}_l^{(n)} \text{ and } \mathbf{c}(t) \geq \mathbf{c}_l^{(n)} \tag{3.3}$$

for all $t \in \bar{E}_l^{(n)}$ and $l = 1, \cdots, n$.

For the time-dependent matrices $B(t)$ and $K(t,s)$, the (i,j)-th entries of constant matrices $B_l^{(n)}$ and $K_{lk}^{(n)}$ are defined and denoted by

$$B_{lij}^{(n)} = \sup_{t \in \bar{E}_l^{(n)}} B_{ij}(t) \text{ and } K_{lkij}^{(n)} = \inf_{(t,s) \in \bar{E}_l^{(n)} \times \bar{E}_k^{(n)}} K_{ij}(t,s). \tag{3.4}$$

We see that

$$B(t) \leq B_l^{(n)} \text{ and } K(t,s) \geq K_{lk}^{(n)} \tag{3.5}$$

for all $t \in \bar{E}_l^{(n)}$ and $(t,s) \in \bar{E}_l^{(n)} \times \bar{E}_k^{(n)}$, respectively, for $l, k = 1, \cdots, n$.

Remark 3.1. From (2.2), it follows that if $B_{lij}^{(n)} \neq 0$, then $B_{lij}^{(n)} \geq \sigma > 0$ for all $i = 1, \cdots, p$, $j = 1, \cdots, q$ and $l = 1, \cdots, n$. Given any fixed $t \in \bar{E}_l^{(n)}$, from (2.1), for any $j = 1, \cdots, q$, there exists $i_j \in \{1, \cdots, p\}$ such that $B_{i_j j}(t) > 0$, which says that $B_{l i_j j}^{(n)} \neq 0$, i.e., $B_{l i_j j}^{(n)} \geq \sigma > 0$. In other words, for each j and l, there exists $i_{lj} \in \{1, 2, \cdots, p\}$ such that $B_{l i_{lj} j}^{(n)} \geq \sigma > 0$.

For each $n \in \mathbb{N}$ and $l = 1, \cdots, n$, we define the following linear programming problem:

$$
\begin{aligned}
(\mathrm{P}_n) \qquad \max \qquad & \sum_{l=1}^{n} \mathfrak{d}_l^{(n)} \cdot \left(\mathbf{a}_l^{(n)}\right)^\top \mathbf{z}_l \\
\text{subject to} \quad & B_1^{(n)} \mathbf{z}_1 \leq \mathbf{c}_1^{(n)} \\
& B_l^{(n)} \mathbf{z}_l \leq \mathbf{c}_l^{(n)} + \sum_{k=1}^{l-1} \mathfrak{d}_k^{(n)} K_{lk}^{(n)} \mathbf{z}_k \text{ for } l = 2, \cdots, n \\
& \mathbf{z}_l \in \mathbb{R}_+^q \text{ for } l = 1, \cdots, n.
\end{aligned}
$$

In order to formulate the dual problem of (P_n), we need to write it as the following standard form:

$$
\begin{aligned}
(\mathrm{P}_n) \qquad \max \qquad & \mathbf{b}^\top \mathbf{z} \\
\text{subject to} \quad & M\mathbf{z} \leq \bar{\mathbf{b}} \text{ and } \mathbf{z} \geq \mathbf{0},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{z} &= (\mathbf{z}_1, \cdots, \mathbf{z}_l, \cdots, \mathbf{z}_n)^\top \\
\bar{\mathbf{b}} &= \left(\mathbf{c}_1^{(n)}, \cdots, \mathbf{c}_l^{(n)}, \cdots, \mathbf{c}_n^{(n)}\right)^\top \\
\mathbf{b} &= \left(\mathfrak{d}_1^{(n)} \mathbf{a}_1^{(n)}, \cdots, \mathfrak{d}_l^{(n)} \mathbf{a}_l^{(n)}, \cdots, \mathfrak{d}_n^{(n)} \mathbf{a}_n^{(n)}\right)^\top \\
M &= \begin{bmatrix}
B_1^{(n)} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
-\mathfrak{d}_1^{(n)} K_{21}^{(n)} & B_2^{(n)} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
-\mathfrak{d}_1^{(n)} K_{31}^{(n)} & -\mathfrak{d}_2^{(n)} K_{32}^{(n)} & B_3^{(n)} & \cdots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
-\mathfrak{d}_1^{(n)} K_{n1}^{(n)} & -\mathfrak{d}_2^{(n)} K_{n2}^{(n)} & -\mathfrak{d}_3^{(n)} K_{n3}^{(n)} & \cdots & -\mathfrak{d}_{n-1}^{(n)} K_{n,n-1}^{(n)} & B_n^{(n)}
\end{bmatrix}
\end{aligned}
$$

The dual problem of (P_n) is given by

$$
\begin{aligned}
(\widehat{\mathrm{D}}_n) \qquad \min \qquad & \bar{\mathbf{b}}^\top \widehat{\mathbf{w}} \\
\text{subject to} \quad & M^\top \widehat{\mathbf{w}} \geq \mathbf{b} \text{ and } \widehat{\mathbf{w}} \geq \mathbf{0},
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{\mathbf{w}} &= (\widehat{\mathbf{w}}_1, \cdots, \widehat{\mathbf{w}}_l, \cdots, \widehat{\mathbf{w}}_n)^\top \\
M^\top &= \begin{bmatrix}
\left(B_1^{(n)}\right)^\top & -\mathfrak{d}_1^{(n)} \left(K_{21}^{(n)}\right)^\top & -\mathfrak{d}_1^{(n)} \left(K_{31}^{(n)}\right)^\top & \cdots & -\mathfrak{d}_1^{(n)} \left(K_{n1}^{(n)}\right)^\top \\
\mathbf{0} & \left(B_2^{(n)}\right)^\top & -\mathfrak{d}_2^{(n)} \left(K_{32}^{(n)}\right)^\top & \cdots & -\mathfrak{d}_2^{(n)} \left(K_{n2}^{(n)}\right)^\top \\
\mathbf{0} & \mathbf{0} & \left(B_3^{(n)}\right)^\top & \cdots & -\mathfrak{d}_3^{(n)} \left(K_{n3}^{(n)}\right)^\top \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -\mathfrak{d}_{n-1}^{(n)} \left(K_{n,n-1}^{(n)}\right)^\top \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \left(B_n^{(n)}\right)^\top
\end{bmatrix}
\end{aligned}
$$

More precisely, the dual problem $(\widehat{\mathrm{D}}_n)$ is written by

$$(\widehat{\mathrm{D}}_n) \qquad \min \qquad \sum_{l=1}^{n} \left(\mathbf{c}_l^{(n)}\right)^\top \widehat{\mathbf{w}}_l$$

$$\text{subject to} \quad \left(B_l^{(n)}\right)^\top \widehat{\mathbf{w}}_l \geq \mathfrak{d}_l^{(n)} \mathbf{a}_l^{(n)} + \mathfrak{d}_l^{(n)} \sum_{k=l+1}^{n} \left(K_{kl}^{(n)}\right)^\top \widehat{\mathbf{w}}_k \text{ for } l = 1, \cdots, n-1$$

$$\left(B_n^{(n)}\right)^\top \widehat{\mathbf{w}}_n \geq \mathfrak{d}_n^{(n)} \mathbf{a}_n^{(n)}$$

$$\widehat{\mathbf{w}}_l \in \mathbb{R}_+^p \text{ for } l = 1, \cdots, n,$$

Now, let

$$\mathbf{w}_l = \frac{\widehat{\mathbf{w}}_l}{\mathfrak{d}_l^{(n)}}.$$

Then, by dividing $\mathfrak{d}_l^{(n)}$ on both sides of the constraints, the dual problem $(\widehat{\mathrm{D}}_n)$ can be equivalently written by

$$(\mathrm{D}_n) \qquad \min \qquad \sum_{l=1}^{n} \mathfrak{d}_l^{(n)} \cdot \left(\mathbf{c}_l^{(n)}\right)^\top \mathbf{w}_l$$

$$\text{subject to} \quad \left(B_l^{(n)}\right)^\top \mathbf{w}_l \geq \mathbf{a}_l^{(n)} + \sum_{k=l+1}^{n} \mathfrak{d}_k^{(n)} \left(K_{kl}^{(n)}\right)^\top \mathbf{w}_k \text{ for } l = 1, \cdots, n-1$$

$$\left(B_n^{(n)}\right)^\top \mathbf{w}_n \geq \mathbf{a}_n^{(n)}$$

$$\mathbf{w}_l \in \mathbb{R}_+^p \text{ for } l = 1, \cdots, n.$$

Remark 3.2. We have the following observations.

- If $\mathbf{c}_l^{(n)} \geq \mathbf{0}$ for all $l = 1, \cdots, n$, then the problem (P_n) is feasible, since $(\mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_n) = \mathbf{0}$ is a feasible solution of (P_n). If the vector-valued function $\mathbf{c}$ is nonnegative, then $\mathbf{c}_l^{(n)} \geq \mathbf{0}$ for all $l = 1, \cdots, n$, which says that the primal problem (P_n) is feasible.
- The dual problem (D_n) is always feasible for each $n \in \mathbb{N}$, which can be realized from part (i) of Proposition 3.1 given below.

Recall that $\mathfrak{d}_l^{(n)}$ denote the length of closed interval $\bar{E}_l^{(n)}$. We also define

$$\mathfrak{s}_l^{(n)} = \max_{k=l,\cdots,n} \mathfrak{d}_k^{(n)} \tag{3.6}$$

Then we have

$$\mathfrak{s}_l^{(n)} = \max\left\{\mathfrak{d}_l^{(n)}, \mathfrak{d}_{l+1}^{(n)}, \cdots, \mathfrak{d}_n^{(n)}\right\} = \max\left\{\mathfrak{d}_l^{(n)}, \mathfrak{s}_{l+1}^{(n)}\right\}$$

which say that

$$\mathfrak{s}_l^{(n)} \geq \mathfrak{d}_l^{(n)} \text{ and } \parallel \mathcal{P}_n \parallel \geq \mathfrak{s}_l^{(n)} \geq \mathfrak{s}_{l+1}^{(n)} \tag{3.7}$$

for $l = 1, \cdots, n-1$. For further discussion, we adopt the following notations:

$$\bar{\tau}_l^{(n)} = \max_{j=1,\cdots,q} a_{lj}^{(n)} \text{ and } \tau_l^{(n)} = \max_{k=l,\cdots,n} \bar{\tau}_k^{(n)} \tag{3.8}$$

$$\bar{\sigma}_l^{(n)} = \min_{i=1,\cdots,p} \min_{j=1,\cdots,q} \left\{ B_{lij}^{(n)} : B_{lij}^{(n)} > 0 \right\} \text{ and } \sigma_l^{(n)} = \min_{k=l,\cdots,n} \bar{\sigma}_k^{(n)} \tag{3.9}$$

$$\bar{\nu}_l^{(n)} = \max_{k=1,\cdots,n} \max_{j=1,\cdots,q} \left\{ \sum_{i=1}^{p} K_{klij}^{(n)} \right\} \text{ and } \nu_l^{(n)} = \max_{k=l,\cdots,n} \bar{\nu}_k^{(n)} \tag{3.10}$$

$$\bar{\phi}_l^{(n)} = \max_{k=1,\cdots,n} \max_{i=1,\cdots,p} \left\{ \sum_{j=1}^{q} K_{klij}^{(n)} \right\} \text{ and } \phi_l^{(n)} = \max_{k=l,\cdots,n} \bar{\phi}_k^{(n)}$$

$$\tau = \max_{j=1,\cdots,q} \sup_{t\in[0,T]} a_j(t)$$

$$\zeta = \max_{i=1,\cdots,p} \sup_{t\in[0,T]} c_i(t)$$

$$\nu = \max_{j=1,\cdots,q} \sup_{(t,s)\in[0,T]\times[0,T]} \sum_{i=1}^{p} K_{ij}(t,s) \tag{3.11}$$

$$\phi = \max_{i=1,\cdots,p} \sup_{(t,s)\in[0,T]\times[0,T]} \sum_{j=1}^{q} K_{ij}(t,s). \tag{3.12}$$

For each $l = 1, \cdots, n$, by Remark 3.1, since $B_{lij}^{(n)} \geq \sigma > 0$ for $B_{lij}^{(n)} \neq 0$ and there exists i_{lj} such that $B_{li_{lj}j}^{(n)} \geq \sigma$, it follows that $\sigma_l^{(n)} \geq \sigma$. We also have the following inequalities:

$$\sigma_l^{(n)} \leq \sigma_{l+1}^{(n)}, \quad \tau_l^{(n)} \geq \tau_{l+1}^{(n)} \text{ and } \nu_l^{(n)} \geq \nu_{l+1}^{(n)} \tag{3.13}$$

and

$$\bar{\tau}_l^{(n)} \leq \tau_l^{(n)} \leq \tau, \quad \bar{\nu}_l^{(n)} \leq \nu_l^{(n)} \leq \nu \text{ and } \bar{\sigma}_l^{(n)} \geq \sigma_l^{(n)} \geq \sigma > 0 \tag{3.14}$$

for any $n \in \mathbb{N}$.

Proposition 3.1. *The following statements hold true.*

(i) *Let*

$$\mathfrak{w}_l^{(n)} = \frac{\tau_l^{(n)}}{\sigma_l^{(n)}} \cdot \left(1 + \mathfrak{s}_l^{(n)} \cdot \frac{\nu_l^{(n)}}{\sigma_l^{(n)}}\right)^{n-l} \geq 0. \tag{3.15}$$

We define $\breve{w}_{li}^{(n)} = \mathfrak{w}_l^{(n)}$ for $i = 1, \cdots, p$ and $l = 1, \cdots, n$, and consider the following vector

$$\breve{\mathbf{w}}^{(n)} = \left(\breve{\mathbf{w}}_1^{(n)}, \breve{\mathbf{w}}_2^{(n)}, \cdots, \breve{\mathbf{w}}_n^{(n)}\right)^\top \text{ with } \breve{\mathbf{w}}_l^{(n)} = \left(\breve{w}_{l1}^{(n)}, \breve{w}_{l2}^{(n)}, \cdots, \breve{w}_{lp}^{(n)}\right)^\top.$$

Then $\breve{\mathbf{w}}^{(n)}$ is a feasible solution of problem (D_n). *Moreover, we have*

$$\breve{w}_{li}^{(n)} \leq \frac{\tau}{\sigma} \cdot \exp\left(r \cdot T \cdot \frac{\nu}{\sigma}\right) \tag{3.16}$$

for all $n \in \mathbb{N}$, $i = 1, \cdots, p$ and $l = 1, \cdots, n$.

(ii) *Given a feasible solution $\mathbf{w}^{(n)}$ of problem* (D_n)*, we define*

$$\bar{w}_{li}^{(n)} = \min\left\{w_{li}^{(n)}, \mathfrak{w}_l^{(n)}\right\}$$

for $i = 1, \cdots, p$ and $l = 1, \cdots, n$, and consider the following vector

$$\bar{\mathbf{w}}^{(n)} = \left(\bar{\mathbf{w}}_1^{(n)}, \bar{\mathbf{w}}_2^{(n)}, \cdots, \bar{\mathbf{w}}_n^{(n)}\right)^\top \text{ with } \bar{\mathbf{w}}_l^{(n)} = \left(\bar{w}_{l1}^{(n)}, \bar{w}_{l2}^{(n)}, \cdots, \bar{w}_{lp}^{(n)}\right)^\top.$$

Then $\bar{\mathbf{w}}^{(n)}$ is a feasible solution of problem (D_n) *satisfying the following inequalities*

$$\bar{w}_{li}^{(n)} \leq \mathfrak{w}_l^{(n)} \leq \frac{\tau}{\sigma} \cdot \exp\left(r \cdot T \cdot \frac{\nu}{\sigma}\right)$$

for all $n \in \mathbb{N}$, $i = 1, \cdots, p$ and $l = 1, \cdots, n$. Moreover, if each $\mathbf{c}_l^{(n)}$ is nonnegative and $\mathbf{w}^{(n)}$ is an optimal solution of problem (D_n)*, then $\bar{\mathbf{w}}^{(n)}$ is also an optimal solution of problem* (D_n).

Proof. To prove part (i), by Remark 3.1, for each j and l, there exists $i_{lj} \in \{1, 2, \cdots, p\}$ such that $B_{li_{lj}j}^{(n)} > 0$. By (3.9), it follows that $B_{li_{lj}j}^{(n)} \geq \sigma_l^{(n)} > 0$. Therefore we have

$$\sum_{i=1}^{p} B_{lij}^{(n)} \cdot \breve{w}_{li}^{(n)} \geq B_{li_{lj}j}^{(n)} \cdot \breve{w}_{li_{lj}}^{(n)} = B_{li_{lj}j}^{(n)} \cdot \frac{\tau_l^{(n)}}{\sigma_l^{(n)}} \cdot \left(1 + \mathfrak{s}_l^{(n)} \cdot \frac{\nu_l^{(n)}}{\sigma_l^{(n)}}\right)^{n-l} \geq \tau_l^{(n)} \cdot \left(1 + \mathfrak{s}_l^{(n)} \cdot \frac{\nu_l^{(n)}}{\sigma_l^{(n)}}\right)^{n-l}. \tag{3.17}$$

Since

$$\begin{aligned}\sum_{i=1}^{p} \mathfrak{d}_k^{(n)} \cdot K_{klij}^{(n)} \cdot \breve{w}_{ki}^{(n)} &\leq \sum_{i=1}^{p} \mathfrak{s}_k^{(n)} \cdot K_{klij}^{(n)} \cdot \frac{\tau_k^{(n)}}{\sigma_k^{(n)}} \cdot \left(1 + \mathfrak{s}_k^{(n)} \cdot \frac{\nu_k^{(n)}}{\sigma_k^{(n)}}\right)^{n-k} \text{ (by (3.7))} \\ &\leq \mathfrak{s}_k^{(n)} \cdot \frac{\nu_l^{(n)} \cdot \tau_k^{(n)}}{\sigma_k^{(n)}} \left(1 + \mathfrak{s}_k^{(n)} \cdot \frac{\nu_k^{(n)}}{\sigma_k^{(n)}}\right)^{n-k} \text{ (by (3.10)),}\end{aligned}$$

it follows that, for $l = 1, \cdots, n-1$,

$$\begin{aligned}& a_{lj}^{(n)} + \sum_{k=l+1}^{n} \sum_{i=1}^{p} \mathfrak{d}_k^{(n)} \cdot K_{klij}^{(n)} \cdot \breve{w}_{ki}^{(n)} \\ &\leq \tau_l^{(n)} + \sum_{k=l+1}^{n} \mathfrak{s}_k^{(n)} \cdot \frac{\nu_l^{(n)} \cdot \tau_k^{(n)}}{\sigma_k^{(n)}} \left(1 + \mathfrak{s}_k^{(n)} \cdot \frac{\nu_k^{(n)}}{\sigma_k^{(n)}}\right)^{n-k} \\ &\leq \tau_l^{(n)} + \sum_{k=l+1}^{n} \mathfrak{s}_l^{(n)} \cdot \frac{\nu_l^{(n)} \cdot \tau_l^{(n)}}{\sigma_l^{(n)}} \left(1 + \mathfrak{s}_l^{(n)} \cdot \frac{\nu_l^{(n)}}{\sigma_l^{(n)}}\right)^{n-k} \text{ (by (3.7) and (3.13))} \\ &= \tau_l^{(n)} \cdot \left[1 + \sum_{k=l+1}^{n} \mathfrak{s}_l^{(n)} \cdot \frac{\nu_l^{(n)}}{\sigma_l^{(n)}} \cdot \left(1 + \mathfrak{s}_l^{(n)} \cdot \frac{\nu_l^{(n)}}{\sigma_l^{(n)}}\right)^{n-k}\right] \\ &= \tau_l^{(n)} \cdot \left(1 + \mathfrak{s}_l^{(n)} \cdot \frac{\nu_l^{(n)}}{\sigma_l^{(n)}}\right)^{n-l}.\end{aligned} \tag{3.18}$$

Therefore, from (3.17) and (3.18), we obtain

$$\sum_{i=1}^{p} B_{lij}^{(n)} \cdot \breve{w}_{li}^{(n)} - \sum_{k=l+1}^{n} \sum_{i=1}^{p} \mathfrak{d}_l^{(n)} \cdot K_{klij}^{(n)} \cdot \breve{w}_{ki}^{(n)} \geq a_{lj}^{(n)} \text{ for } l = 1, \cdots, n-1$$

and

$$\sum_{i=1}^{p} B_{nij}^{(n)} \cdot \breve{w}_{ni}^{(n)} \geq \tau_n^{(n)} \geq a_{nj}^{(n)},$$

which show that $\breve{\mathbf{w}}^{(n)}$ is indeed a feasible solution of problem (D_n). Moreover, for $l = 1, \cdots, n$, from (3.7), (3.1) and (3.14), we have

$$\frac{\tau_l^{(n)}}{\sigma_l^{(n)}} \cdot \left(1 + \mathfrak{s}_l^{(n)} \cdot \frac{\nu_l^{(n)}}{\sigma_l^{(n)}}\right)^{n-l} \leq \frac{\tau}{\sigma} \cdot \left(1 + \parallel \mathcal{P}_n \parallel \cdot \frac{\nu}{\sigma}\right)^n \leq \frac{\tau}{\sigma} \cdot \left(1 + \frac{T}{n^*} \cdot \frac{\nu}{\sigma}\right)^n \leq \frac{\tau}{\sigma} \cdot \left(1 + \frac{r \cdot T}{n} \cdot \frac{\nu}{\sigma}\right)^n.$$

Since

$$\left(1 + \frac{r \cdot T}{n} \cdot \frac{\nu}{\sigma}\right)^n \uparrow \exp\left(r \cdot T \cdot \frac{\nu}{\sigma}\right) \text{ as } n \to \infty,$$

i.e., $n^* \to \infty$ in the limiting case, this proves (3.16).

To prove part (ii), for each $j = 1, \cdots, q$ and $l = 1, \cdots, n$, we define the index set

$$I_{lj} = \left\{ i : B_{lij}^{(n)} > 0 \right\}.$$

It is clear that $I_{lj} \neq \emptyset$ by Remark 3.1. For each $l = 1, \cdots, n$, we see that

$$\sum_{i=1}^{p} B_{lij}^{(n)} \bar{w}_{li}^{(n)} = \sum_{i \in I_{lj}} B_{lij}^{(n)} \bar{w}_{li}^{(n)}. \tag{3.19}$$

For each fixed $l = 1, \cdots, n$, we also define the index set

$$\bar{I}_{lj} = \left\{ i \in I_{lj} : \bar{w}_{li}^{(n)} = \mathfrak{w}_l^{(n)} \right\}. \tag{3.20}$$

For each fixed $j = 1, \cdots, q$ and $l = 1, \cdots, n$, we consider the following two cases.

- Suppose that $\bar{I}_{lj} \neq \emptyset$, i.e., there exists i_{lj} such that $B_{li_{lj}j}^{(n)} > 0$ and $\bar{w}_{li_{lj}}^{(n)} = \mathfrak{w}_l^{(n)} = \breve{w}_{li_{lj}}^{(n)}$. In this case, by (3.17), for $l = 1, \cdots, n$, we have

$$\sum_{i=1}^{p} B_{lij}^{(n)} \cdot \bar{w}_{li}^{(n)} \geq B_{li_{lj}j}^{(n)} \cdot \bar{w}_{li_{lj}}^{(n)} = B_{li_{lj}j}^{(n)} \cdot \breve{w}_{li_{lj}}^{(n)} \geq \tau_l^{(n)} \cdot \left(1 + \mathfrak{s}_l^{(n)} \cdot \frac{\nu_l^{(n)}}{\sigma_l^{(n)}}\right)^{n-l} \tag{3.21}$$

Since $\bar{w}_{ki}^{(n)} \leq \mathfrak{w}_l^{(n)} = \breve{w}_{ki}^{(n)}$ for each k, by referring to (3.18), for $l = 1, \cdots, n-1$, we also have

$$a_{lj}^{(n)} + \sum_{i=1}^{p} \sum_{k=l+1}^{n} \mathfrak{d}_k^{(n)} \cdot K_{klij}^{(n)} \cdot \bar{w}_{ki}^{(n)} \leq a_{lj}^{(n)} + \sum_{i=1}^{p} \sum_{k=l+1}^{n} \mathfrak{d}_k^{(n)} \cdot K_{klij}^{(n)} \cdot \breve{w}_{ki}^{(n)} \leq \tau_l^{(n)} \cdot \left(1 + \mathfrak{s}_l^{(n)} \cdot \frac{\nu_l^{(n)}}{\sigma_l^{(n)}}\right)^{n-l},$$

which implies, by (3.21),

$$a_{lj}^{(n)} + \sum_{i=1}^{p} \sum_{k=l+1}^{n} \mathfrak{d}_k^{(n)} \cdot K_{klij}^{(n)} \cdot \bar{w}_{ki}^{(n)} \leq \sum_{i=1}^{p} B_{lij}^{(n)} \cdot \bar{w}_{li}^{(n)}.$$

For $l = n$, from (3.21) again, we also have

$$\sum_{i=1}^{p} B_{nij}^{(n)} \cdot \bar{w}_{ni}^{(n)} \geq \tau_n^{(n)} \geq a_{nj}^{(n)}.$$

- Suppose that $\bar{I}_{lj} = \emptyset$, i.e., if $i \in I_{lj}$ then $\bar{w}_{li}^{(n)} = w_{li}^{(n)}$ for $l = 1, \cdots, n$. In this case, for $l = 1, \cdots, n-1$, we have

$$\begin{aligned}
&\sum_{i=1}^{p} B_{lij}^{(n)} \bar{w}_{li}^{(n)} - \sum_{k=l+1}^{n} \sum_{i=1}^{p} \mathfrak{d}_k^{(n)} \cdot K_{klij}^{(n)} \cdot \bar{w}_{ki}^{(n)} \\
&= \sum_{i \in I_{lj}} B_{lij}^{(n)} \bar{w}_{li}^{(n)} - \sum_{k=l+1}^{n} \sum_{i=1}^{p} \mathfrak{d}_k^{(n)} \cdot K_{klij}^{(n)} \cdot \bar{w}_{ki}^{(n)} \text{ (by (3.19))} \\
&= \sum_{i \in I_{lj}} B_{lij}^{(n)} w_{li}^{(n)} - \sum_{k=l+1}^{n} \sum_{i=1}^{p} \mathfrak{d}_k^{(n)} \cdot K_{klij}^{(n)} \cdot \bar{w}_{ki}^{(n)} \\
&= \sum_{i=1}^{p} B_{lij}^{(n)} w_{li}^{(n)} - \sum_{k=l+1}^{n} \sum_{i=1}^{p} \mathfrak{d}_k^{(n)} \cdot K_{klij}^{(n)} \cdot \bar{w}_{ki}^{(n)} \\
&\geq \sum_{i=1}^{p} B_{lij}^{(n)} w_{li}^{(n)} - \sum_{k=l+1}^{n} \sum_{i=1}^{p} \mathfrak{d}_k^{(n)} \cdot K_{klij}^{(n)} \cdot w_{ki}^{(n)} \text{ (since } w_{ki}^{(n)} \geq \bar{w}_{ki}^{(n)} \text{ and } \mathfrak{d}_k^{(n)} \cdot K_{klij}^{(n)} \geq 0) \\
&\geq a_{lj}^{(n)} \text{ (since } \mathbf{w}^{(n)} \text{ is a feasible solution of problem (D}_n)).
\end{aligned}$$

For $l = n$, by (3.19) again, we also have

$$\begin{aligned}
\sum_{i=1}^{p} B_{nij}^{(n)} \bar{w}_{ni}^{(n)} &= \sum_{i \in I_{nj}} B_{nij}^{(n)} \bar{w}_{ni}^{(n)} = \sum_{i \in I_{nj}} B_{nij}^{(n)} w_{ni}^{(n)} = \sum_{i=1}^{p} B_{nij}^{(n)} w_{ni}^{(n)} \\
&\geq a_{nj}^{(n)} \text{ (by the feasibility of } \mathbf{w}^{(n)}).
\end{aligned}$$

From the above two cases, we conclude that $\bar{\mathbf{w}}^{(n)}$ is indeed a feasible solution of problem (D_n).

Finally, since the objective values satisfy

$$\sum_{l=1}^{n} \mathfrak{d}_l^{(n)} \cdot \left(\mathbf{c}_l^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} \leq \sum_{l=1}^{n} \mathfrak{d}_l^{(n)} \cdot \left(\mathbf{c}_l^{(n)}\right)^\top \mathbf{w}_l^{(n)},$$

it says that if $\mathbf{w}^{(n)}$ is an optimal solution of problem (D_n), then $\bar{\mathbf{w}}^{(n)}$ is also an optimal solution of problem (D_n). This completes the proof. ■

Proposition 3.2. *Suppose that the primal problem* (P_n) *is feasible with a feasible solution* $\mathbf{z}^{(n)} = (\mathbf{z}_1^{(n)}, \mathbf{z}_2^{(n)}, \cdots, \mathbf{z}_n^{(n)})$, *where* $\mathbf{z}_l^{(n)} = (z_{l1}^{(n)}, z_{l2}^{(n)}, \cdots, z_{lq}^{(n)})^\top$ *for* $l = 1, \cdots, n$. *Then*

$$0 \leq z_{lj}^{(n)} \leq \frac{\zeta}{\sigma} \cdot \left(1 + \| \mathcal{P}_n \| \cdot \frac{\phi}{\sigma}\right)^{l-1} \leq \frac{\zeta}{\sigma} \cdot \exp\left(r \cdot T \cdot \frac{\phi}{\sigma}\right) \tag{3.22}$$

for all $j = 1, \cdots, q$, $l = 1, \cdots, n$ *and* $n \in \mathbb{N}$.

Proof. By Remark 3.1, for each j and l, there exists $i_{lj} \in \{1, 2, \cdots, p\}$ such that $B_{li_{lj}j}^{(n)} > 0$, which implies

$$B_{li_{lj}j}^{(n)} \geq \sigma_l^{(n)} \geq \sigma > 0.$$

If $\phi = 0$, then $K_{ij}(t, s) = 0$ for all $(t, s) \in [0, T] \times [0, T]$, which imply that each $K_{lk}^{(n)}$ is a zero matrix for $l, k = 1, \cdots, n$. In this case, using the feasibility of $\mathbf{z}^{(n)}$, we have

$$0 \leq \sigma \cdot z_{lj}^{(n)} \leq B_{li_lj j}^{(n)} \cdot z_{lj}^{(n)} \leq \sum_{s=1}^{q} B_{li_lj s}^{(n)} \cdot z_{ls}^{(n)} \leq c_{li_lj}^{(n)} \leq \zeta$$

which implies

$$0 \leq z_{lj}^{(n)} \leq \frac{\zeta}{\sigma}.$$

For the case of $\phi \neq 0$, we want to show that

$$z_{lj}^{(n)} \leq \frac{\zeta}{\sigma} \cdot \left(1 + \parallel \mathcal{P}_n \parallel \cdot \frac{\phi}{\sigma}\right)^{l-1}$$

for all $j = 1, \cdots, q$ and $l = 1, \cdots, n$. We shall prove it by induction on l. Since $\mathbf{z}^{(n)}$ is a feasible solution of problem (P_n), we have $B_1 \mathbf{z}_1^{(n)} \leq \mathbf{c}_1^{(n)}$ for $l = 1$, which says that

$$B_{1i_1j j}^{(n)} \cdot z_{1j}^{(n)} \leq \sum_{h=1}^{q} B_{1i_1j h}^{(n)} \cdot z_{1h}^{(n)} \leq c_{1i_1j}^{(n)},$$

Therefore, for each j, we obtain

$$z_{1j}^{(n)} \leq \frac{c_{1i_lj}^{(n)}}{B_{1i_lj j}^{(n)}} \leq \frac{\zeta}{\sigma}.$$

Suppose that

$$z_{lj}^{(n)} \leq \frac{\zeta}{\sigma} \cdot \left(1 + \parallel \mathcal{P}_n \parallel \cdot \frac{\phi}{\sigma}\right)^{l-1}$$

for $l = 1, 2, \cdots, n-1$. Then, for each j, we have

$$\sum_{l=1}^{n-1} z_{lj}^{(n)} \leq \sum_{l=1}^{n-1} \frac{\zeta}{\sigma} \cdot \left(1 + \parallel \mathcal{P}_n \parallel \cdot \frac{\phi}{\sigma}\right)^{l-1} = \frac{\zeta}{\phi \cdot \parallel \mathcal{P}_n \parallel} \cdot \left[\left(1 + \parallel \mathcal{P}_n \parallel \cdot \frac{\phi}{\sigma}\right)^{n-1} - 1\right]. \tag{3.23}$$

By the feasibility of $\mathbf{z}^{(n)}$, we have

$$B_n^{(n)} \mathbf{z}_n^{(n)} \leq \mathbf{c}_n^{(n)} + \sum_{k=1}^{n-1} \mathfrak{d}_k^{(n)} K_{nk}^{(n)} \mathbf{z}_k^{(n)}.$$

Therefore, for each j, we obtain

$$\begin{aligned} B_{ni_nj j}^{(n)} \cdot z_{nj}^{(n)} &\leq \sum_{h=1}^{q} B_{ni_nj h}^{(n)} \cdot z_{nh}^{(n)} \leq c_{ni_nj}^{(n)} + \sum_{k=1}^{n-1} \sum_{h=1}^{q} \mathfrak{d}_k^{(n)} \cdot K_{nki_nj h}^{(n)} \cdot z_{kh}^{(n)} \\ &\leq c_{ni_nj}^{(n)} + \sum_{h=1}^{q} \sum_{k=1}^{n-1} \mathfrak{d}_k^{(n)} \cdot K_{i_nj h}(t, s) \cdot z_{kh}^{(n)} \text{ for some } (t, s) \in \bar{E}_n \times \bar{E}_k \\ &\leq c_{ni_nj}^{(n)} + \parallel \mathcal{P}_n \parallel \cdot \sum_{h=1}^{q} K_{i_nj h}(t, s) \cdot \frac{\zeta}{\phi \cdot \parallel \mathcal{P}_n \parallel} \cdot \left[\left(1 + \parallel \mathcal{P}_n \parallel \cdot \frac{\phi}{\sigma}\right)^{n-1} - 1\right] \text{ (by (3.23))} \end{aligned}$$

$$\leq \zeta + \zeta \cdot \left[\left(1 + \| \mathcal{P}_n \| \cdot \frac{\phi}{\sigma}\right)^{n-1} - 1 \right] \text{ (by (3.12))},$$

which implies

$$z_{nj}^{(n)} \leq \frac{\zeta}{\sigma} \cdot \left(1 + \| \mathcal{P}_n \| \cdot \frac{\phi}{\sigma}\right)^{n-1}.$$

Therefore, by induction, we obtain

$$z_{lj}^{(n)} \leq \frac{\zeta}{\sigma} \cdot \left(1 + \| \mathcal{P}_n \| \cdot \frac{\phi}{\sigma}\right)^{l-1} \leq \frac{\zeta}{\sigma} \cdot \left(1 + \| \mathcal{P}_n \| \cdot \frac{\phi}{\sigma}\right)^{n} \tag{3.24}$$

for all $j = 1, \cdots, q$ and $l = 1, \cdots, n$. From (3.1), since

$$\frac{\zeta}{\sigma} \cdot \left(1 + \| \mathcal{P}_n \| \cdot \frac{\phi}{\sigma}\right)^{n} \leq \frac{\zeta}{\sigma} \cdot \left(1 + \frac{T}{n^*} \cdot \frac{\phi}{\sigma}\right)^{n} \leq \frac{\zeta}{\sigma} \cdot \left(1 + \frac{r \cdot T}{n} \cdot \frac{\phi}{\sigma}\right)^{n}$$

and

$$\frac{\zeta}{\sigma} \cdot \left(1 + \frac{r \cdot T}{n} \cdot \frac{\phi}{\sigma}\right)^{n} \uparrow \frac{\zeta}{\sigma} \cdot \exp\left(r \cdot T \cdot \frac{\phi}{\sigma}\right) \text{ as } n^* \to \infty \text{ (i.e., } n \to \infty),$$

using (3.24), we complete the proof. ■

Let $\bar{\mathbf{z}}^{(n)} = (\bar{\mathbf{z}}_1^{(n)}, \bar{\mathbf{z}}_2^{(n)}, \cdots, \bar{\mathbf{z}}_n^{(n)})$ with $\bar{\mathbf{z}}_l^{(n)} = (\bar{z}_{l1}^{(n)}, \bar{z}_{l2}^{(n)}, \cdots, \bar{z}_{lq}^{(n)})^\top$ be an optimal solution of problem (P_n). We construct a vector-valued step function $\widehat{\mathbf{z}}^{(n)} : [0, T] \to \mathbb{R}^q$ as follows:

$$\widehat{\mathbf{z}}^{(n)}(t) = \begin{cases} \bar{\mathbf{z}}_l^{(n)} & \text{if } t \in F_l^{(n)} = \left[e_{l-1}^{(n)}, e_l^{(n)}\right) \text{ for } l = 1, \cdots, n \\ \bar{\mathbf{z}}_n^{(n)} & \text{if } t = T. \end{cases} \tag{3.25}$$

Then we have the following result.

Proposition 3.3. *Suppose that the primal problem* (P_n) *is feasible with a feasible solution* $\bar{\mathbf{z}}^{(n)} = (\bar{\mathbf{z}}_1^{(n)}, \bar{\mathbf{z}}_2^{(n)}, \cdots, \bar{\mathbf{z}}_n^{(n)})$, *where* $\bar{\mathbf{z}}_l^{(n)} = (\bar{z}_{l1}^{(n)}, \bar{z}_{l2}^{(n)}, \cdots, \bar{z}_{lq}^{(n)})^\top$ *for* $l = 1, \cdots, n$. *Then the vector-valued step function* $\widehat{\mathbf{z}}^{(n)}$ *defined in* (3.25) *is a feasible solution of problem* (CLP).

Proof. Since $\bar{\mathbf{z}}^{(n)}$ is a feasible solution of problem (P_n), it follows that

$$B_1^{(n)} \bar{\mathbf{z}}_1^{(n)} \leq \mathbf{c}_1^{(n)} \text{ and } B_l^{(n)} \bar{\mathbf{z}}_l^{(n)} \leq \mathbf{c}_l^{(n)} + \sum_{k=1}^{l-1} \mathfrak{d}_k^{(n)} K_{lk}^{(n)} \bar{\mathbf{z}}_k^{(n)} \text{ for } l = 2, \cdots, n. \tag{3.26}$$

We consider the following two cases.

- Suppose that $t \in F_l^{(n)}$ for $l = 2, \cdots, n$. Recall that $e_{l-1}^{(n)}$ is the left-end point of closed interval $\bar{E}_l^{(n)}$. Then

$$B(t)\widehat{\mathbf{z}}^{(n)}(t) - \int_0^t K(t, s)\widehat{\mathbf{z}}^{(n)}(s)ds$$
$$= B(t)\widehat{\mathbf{z}}^{(n)}(t) - \sum_{k=1}^{l-1} \int_{\bar{E}_k^{(n)}} K(t, s)\widehat{\mathbf{z}}^{(n)}(s)ds - \int_{e_{l-1}^{(n)}}^t K(t, s)\widehat{\mathbf{z}}^{(n)}(s)ds$$

$$
\begin{aligned}
&= B(t)\bar{\mathbf{z}}_l^{(n)} - \sum_{k=1}^{l-1} \int_{\bar{E}_k^{(n)}} K(t,s)\bar{\mathbf{z}}_k^{(n)} ds - \int_{e_{l-1}^{(n)}}^{t} K(t,s)\bar{\mathbf{z}}_l^{(n)} ds \\
&\leq B_l^{(n)}\bar{\mathbf{z}}_l^{(n)} - \sum_{k=1}^{l-1} \int_{\bar{E}_k^{(n)}} K_{lk}^{(n)}\bar{\mathbf{z}}_k^{(n)} ds - \int_{e_{l-1}^{(n)}}^{t} K_{ll}^{(n)}\bar{\mathbf{z}}_l^{(n)} ds \text{ (by (3.5))} \\
&\leq B_l^{(n)}\bar{\mathbf{z}}_l^{(n)} - \sum_{k=1}^{l-1} \int_{\bar{E}_k^{(n)}} K_{lk}^{(n)}\bar{\mathbf{z}}_k^{(n)} ds \text{ (since the last term is nonnegative)} \\
&= B_l^{(n)}\bar{\mathbf{z}}_l^{(n)} - \sum_{k=1}^{l-1} \mathfrak{d}_k^{(n)} K_{lk}^{(n)}\bar{\mathbf{z}}_k^{(n)} \leq \mathbf{c}_l^{(n)} \text{ (by (3.26))} \\
&\leq \mathbf{c}(t) \text{ (by (3.3)).}
\end{aligned}
$$

For $l = 1$, the desired inequality can be similarly obtained.

- Suppose that $t = T$. Then

$$
\begin{aligned}
B(T)\widehat{\mathbf{z}}^{(n)}(T) - \int_0^T K(T,s)\widehat{\mathbf{z}}^{(n)}(s) ds &= B(T)\bar{\mathbf{z}}_n^{(n)} - \sum_{k=1}^{n} \int_{\bar{E}_k^{(n)}} K(T,s)\bar{\mathbf{z}}_k^{(n)} ds \\
&\leq B_n^{(n)}\bar{\mathbf{z}}_n^{(n)} - \sum_{k=1}^{n} \int_{\bar{E}_k^{(n)}} K_{nk}^{(n)}\bar{\mathbf{z}}_k^{(n)} ds = B_n^{(n)}\bar{\mathbf{z}}_n^{(n)} - \sum_{k=1}^{n} \mathfrak{d}_k^{(n)} K_{nk}^{(n)}\bar{\mathbf{z}}_k^{(n)} \\
&\leq B_n^{(n)}\bar{\mathbf{z}}_n^{(n)} - \sum_{k=1}^{n-1} \mathfrak{d}_k^{(n)} K_{nk}^{(n)}\bar{\mathbf{z}}_k^{(n)} \text{ (since } K_{nn}^{(n)} \geq \mathbf{0}) \\
&\leq \mathbf{c}_n^{(n)} \text{ (by (3.26))} \\
&\leq \mathbf{c}(T) \text{ (by (3.3)).}
\end{aligned}
$$

Therefore we conclude that $\widehat{\mathbf{z}}^{(n)}$ is indeed a feasible solution of problem (CLP). This completes the proof. ■

4. Analytic formula of the error estimation

Given an optimization problem (P), if problem (P) is a maximization problems, then $V(\mathrm{P})$ denotes the supremum of the objective function, and if problem (P) is a minimization problems, then $V(\mathrm{P})$ denotes the infimum of the objective function. We have to mention that the supremum or infimum is attained when the optimal solution of problem (P) exists. In other words, $V(\mathrm{P})$ denotes the optimal objective value of problem (P) when there exists a feasible solution x^* of problem (P) such that the supremum or infimum is equal to the objective value of x^*. For example, the supremum of problem (CLP) is denoted by $V(\mathrm{CLP})$.

Let $\bar{\mathbf{z}}^{(n)}$ be an optimal solution of problem (P_n). Then, using (3.3), we have

$$
\int_0^T (\mathbf{a}(t))^\top \widehat{\mathbf{z}}^{(n)}(t) dt \geq \sum_{l=1}^{n} \int_{\bar{E}_l^{(n)}} \left(\mathbf{a}_l^{(n)}\right)^\top \bar{\mathbf{z}}_l^{(n)} dt = \sum_{l=1}^{n} \mathfrak{d}_l^{(n)} \cdot \left(\mathbf{a}_l^{(n)}\right)^\top \bar{\mathbf{z}}_l^{(n)} = V(\mathrm{P}_n). \tag{4.1}
$$

Therefore we have

$$\begin{aligned} V(\mathrm{CLP}) &\geq \int_0^T (\mathbf{a}(t))^\top \widehat{\mathbf{z}}^{(n)}(t)dt \text{ (by Proposition 3.3)} \\ &\geq V(\mathrm{P}_n) \text{ (by (4.1)).} \end{aligned}$$

Using the weak duality theorem for the primal-dual pair of problems (DCLP) and (CLP), it follows that

$$V(\mathrm{DCLP}) \geq V(\mathrm{CLP}) \geq V(\mathrm{P}_n) = V(\mathrm{D}_n). \tag{4.2}$$

In the sequel, we want to show that

$$\lim_{n\to\infty} V(\mathrm{D}_n) = V(\mathrm{DCLP}). \tag{4.3}$$

Let $\mathbf{w}^{(n)} = (\mathbf{w}_1^{(n)}, \mathbf{w}_2^{(n)}, \cdots, \mathbf{w}_n^{(n)})$ with $\mathbf{w}_l^{(n)} = (w_{l1}^{(n)}, w_{l2}^{(n)}, \cdots, w_{lp}^{(n)})^\top$ be an optimal solution of problem (D_n). We define

$$\bar{w}_{li}^{(n)} = \min\left\{w_{li}^{(n)}, \mathfrak{w}_l^{(n)}\right\},$$

where $\mathfrak{w}_l^{(n)}$ is defined in (3.15), for $i = 1, \cdots, p$ and $l = 1, \cdots, n$, and consider the following vector

$$\bar{\mathbf{w}}^{(n)} = \left(\bar{\mathbf{w}}_1^{(n)}, \bar{\mathbf{w}}_2^{(n)}, \cdots, \bar{\mathbf{w}}_n^{(n)}\right)^\top \text{ with } \bar{\mathbf{w}}_l^{(n)} = \left(\bar{w}_{l1}^{(n)}, \bar{w}_{l2}^{(n)}, \cdots, \bar{w}_{lp}^{(n)}\right)^\top$$

Then, according to part (ii) of Proposition 3.1, we see that $\bar{\mathbf{w}}^{(n)}$ is an optimal solution of problem (D_n) satisfying the following inequalities

$$\bar{w}_{li}^{(n)} \leq \mathfrak{w}_l^{(n)} \leq \frac{\tau}{\sigma} \cdot \exp\left(r \cdot T \cdot \frac{\nu}{\sigma}\right) \tag{4.4}$$

for all $n \in \mathbb{N}$, $i = 1, \cdots, p$ and $l = 1, \cdots, n$.

For each $l = 1, \cdots, n$, we define a vector-valued function $\bar{\mathbf{h}}_l^{(n)}$ on $F_l^{(n)} = [e_{l-1}^{(n)}, e_l^{(n)})$ by

$$\begin{aligned} \bar{\mathbf{h}}_l^{(n)}(t) &= \left(e_l^{(n)} - t\right)\left(K_{ll}^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} + \left(B_l^{(n)} - B(t)\right)^\top \bar{\mathbf{w}}_l^{(n)} \\ &\quad + \int_t^{e_l^{(n)}} \left(K(s,t) - K_{ll}^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} ds + \sum_{k=l+1}^{n} \int_{\bar{E}_k^{(n)}} \left(K(s,t) - K_{kl}^{(n)}\right)^\top \bar{\mathbf{w}}_k^{(n)} ds. \end{aligned} \tag{4.5}$$

We also define a vector

$$\mathbf{r}^{(n)} = \left(B_n^{(n)} - B(T)\right)^\top \bar{\mathbf{w}}_n^{(n)}. \tag{4.6}$$

For $l = 1, \cdots, n$, let

$$\bar{\pi}_l^{(n)} = \max_{j=1,\cdots,q} \sup_{t\in E_l^{(n)}} \left[\bar{h}_{lj}^{(n)}(t) + a_j(t) - a_{lj}^{(n)}\right], \tag{4.7}$$

where $\bar{h}_{lj}^{(n)}$ is the jth component of $\bar{\mathbf{h}}_l^{(n)}$, and let

$$\pi_l^{(n)} = \max_{k=l,\cdots,n} \bar{\pi}_k^{(n)}.$$

Then we have

$$\pi_l^{(n)} = \max\left\{\bar{\pi}_l^{(n)}, \bar{\pi}_{l+1}^{(n)}, \cdots, \bar{\pi}_n^{(n)}\right\} = \max\left\{\bar{\pi}_l^{(n)}, \pi_{l+1}^{(n)}\right\} \tag{4.8}$$

which says that

$$\pi_l^{(n)} \geq \pi_{l+1}^{(n)} \tag{4.9}$$

and, for any $t \in E_l^{(n)}$,

$$\pi_l^{(n)} \geq \bar{\pi}_l^{(n)} \geq \bar{h}_{lj}^{(n)}(t) + a_j(t) - a_{lj}^{(n)} \tag{4.10}$$

for $l = 1, \cdots, n-1$ and $j = 1, \cdots, q$. We want to prove

$$\lim_{n\to\infty} \bar{\pi}_l^{(n)} = 0 = \lim_{n\to\infty} \pi_l^{(n)}.$$

We first provide some useful lemmas.

Lemma 4.1. *For $i = 1, \cdots, p$, $j = 1, \cdots, q$ and $l = 1, \cdots, n$, we have*

$$\sup_{t\in E_l^{(n)}} \left[a_j(t) - a_{lj}^{(n)}\right] \to 0, \sup_{t\in E_l^{(n)}} \left[c_i(t) - c_{li}^{(n)}\right] \to 0 \text{ and } \sup_{t\in E_l^{(n)}} \left[B_{lij}^{(n)} - B_{ij}(t)\right] \to 0 \text{ as } n \to \infty.$$

Proof. According to the construction of partition $\mathcal{P}_n$, we see that a_j is continuous on the open interval $E_l^{(n)} = (e_{l-1}^{(n)}, e_l^{(n)})$. Let $\{\delta_m\}_{m=1}^{\infty}$ be a decreasing sequence and be convergent to zero such that $\delta_m > 0$ for all m, where δ_1 is defined by

$$\delta_1 = \frac{1}{2} \cdot \left(e_l^{(n)} - e_{l-1}^{(n)}\right).$$

Therefore we can define the compact interval

$$E_{lm}^{(n)} = \left[e_{l-1}^{(n)} + \delta_m, e_l^{(n)} - \delta_m\right].$$

Then we have

$$E_l^{(n)} = \bigcup_{m=1}^{\infty} E_{lm}^{(n)} \text{ and } E_{lm_1}^{(n)} \subseteq E_{lm_2}^{(n)} \text{ for } m_2 > m_1 \tag{4.11}$$

Since $E_{lm}^{(n)} \subset E_l^{(n)}$, it follows that a_j is continuous on each compact interval $E_{lm}^{(n)}$, which also means that a_j is uniformly continuous on each compact interval $E_{lm}^{(n)}$. Therefore, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies

$$\left|a_j(t_1) - a_j(t_2)\right| < \frac{\epsilon}{2} \text{ for any } t_1, t_2 \in E_{lm}^{(n)}. \tag{4.12}$$

Since the length of $E_l^{(n)}$ is less than or equal to $\| \mathcal{P}_n \| \leq T/n^*$ with $n^* \to \infty$ by (3.1), we can consider a sufficiently large $n_0 \in \mathbb{N}$ such that $T/n_0 < \delta$. In this case, each length of $E_l^{(n)}$ for $l = 1, \cdots, n$ is less than δ for $n \geq n_0$. In other words, if $n \geq n_0$, then (4.12) is satisfied for any $t_1, t_2 \in E_{lm}^{(n)}$. We consider the following cases.

- Suppose that the infimum $a_{lj}^{(n)}$ is attained at $t^{(n*)} \in E_l^{(n)}$. From (4.11), there exists m^* such that $t^{(n*)} \in E_{lm^*}^{(n)}$. Now, given any $t \in E_l^{(n)}$, we see that $t \in E_{lm_0}^{(n)}$ for some m_0. Let $m = \max\{m_0, m^*\}$. From (4.11), it follows that $t, t^{(n*)} \in E_{lm}^{(n)}$. Then we have

$$\left|a_j(t) - a_{lj}^{(n)}\right| = \left|a_j(t) - a_j\left(t^{(n*)}\right)\right| < \frac{\epsilon}{2}$$

since the length of $E_{lm}^{(n)}$ is less than δ, where ϵ is independent of t because of the uniform continuity.

- Suppose that the infimum $a_{lj}^{(n)}$ is not attained at any point in $E_l^{(n)}$. Since a_j is continuous on the open interval $E_l^{(n)}$, it follows that the infimum $a_{lj}^{(n)}$ is either the righthand limit or lefthand limit given by
$$a_{lj}^{(n)} = \lim_{t\to e_{l-1}^{(n)}+} a_j(t) \text{ or } a_{lj}^{(n)} = \lim_{t\to e_{l}^{(n)}-} a_j(t).$$
Therefore, for sufficiently large n, i.e., the open interval $E_l^{(n)}$ is sufficiently small such that its length is less than δ, we have
$$\left|a_j(t) - a_{lj}^{(n)}\right| < \frac{\epsilon}{2}$$
for all $t \in E_l^{(n)}$.

From the above two cases, since $a_j(t) \geq a_{lj}^{(n)}$ for all $t \in E_l^{(n)}$, we conclude that

$$0 \leq \sup_{t\in E_l^{(n)}} \left[a_j(t) - a_{lj}^{(n)}\right] \leq \frac{\epsilon}{2} < \epsilon \text{ for } l = 1, \cdots, n.$$

The other cases can be similarly obtained. This completes the proof. ■

Lemma 4.2. *For $i = 1, \cdots, p$, $j = 1, \cdots, q$ and $l, k = 1, \cdots, n$, we have*

$$\sup_{(s,t)\in E_k^{(n)}\times E_l^{(n)}} \left[K_{ij}(s,t) - K_{klij}^{(n)}\right] \to 0 \text{ as } n \to \infty$$

and

$$\sup_{t\in E_l^{(n)}} \left[\int_{\bar{E}_k^{(n)}} \left(K_{ij}(s,t) - K_{klij}^{(n)}\right) \cdot \bar{w}_{ki}^{(n)} ds\right] \to 0 \text{ as } n \to \infty.$$

Proof. According to the construction of partition $\mathcal{P}_n$, we see that K_{ij} is continuous on the open rectangle

$$E_k^{(n)} \times E_l^{(n)} = \left(e_{k-1}^{(n)}, e_k^{(n)}\right) \times \left(e_{l-1}^{(n)}, e_l^{(n)}\right).$$

Let $\{\delta_m\}_{m=1}^{\infty}$ be a decreasing sequence and be convergent to zero such that $\delta_m > 0$ for all m, where δ_1 is defined by
$$\delta_1 = \frac{1}{2} \cdot \min\left\{\left(e_k^{(n)} - e_{k-1}^{(n)}\right), \left(e_l^{(n)} - e_{l-1}^{(n)}\right)\right\}.$$
Therefore we can define the compact rectangle

$$E_{km}^{(n)} \times E_{lm}^{(n)} = \left[e_{k-1}^{(n)} + \delta_m, e_k^{(n)} - \delta_m\right] \times \left[e_{l-1}^{(n)} + \delta_m, e_l^{(n)} - \delta_m\right].$$

The following inclusion
$$\bigcup_{m=1}^{\infty} E_{km}^{(n)} \times E_{lm}^{(n)} \subseteq E_k^{(n)} \times E_l^{(n)}$$
is obvious. For $(s,t) \in E_k^{(n)} \times E_l^{(n)}$, there exists m_1 and m_2 such that $s \in E_{km_1}^{(n)}$ and $t \in E_{lm_2}^{(n)}$, respectively. Let $m = \max\{m_1, m_2\}$. Then we have $E_{km_1}^{(n)} \subseteq E_{km}^{(n)}$ and $E_{lm_2}^{(n)} \subseteq E_{lm}^{(n)}$. Therefore we obtain

$$(s,t) \in E_{km_1}^{(n)} \times E_{lm_2}^{(n)} \subseteq E_{km}^{(n)} \times E_{lm}^{(n)},$$

which proves

$$E_k^{(n)} \times E_l^{(n)} = \bigcup_{m=1}^{\infty} E_{km}^{(n)} \times E_{lm}^{(n)}. \tag{4.13}$$

We also see that

$$E_{km_1}^{(n)} \times E_{lm_1}^{(n)} \subseteq E_{km_2}^{(n)} \times E_{lm_2}^{(n)} \text{ for } m_2 > m_1. \tag{4.14}$$

Since $E_{km}^{(n)} \times E_{lm}^{(n)} \subset E_k^{(n)} \times E_l^{(n)}$, it follows that K_{ij} is continuous on each compact rectangle $E_{km}^{(n)} \times E_{lm}^{(n)}$, which also means that K_{ij} is uniformly continuous on each compact rectangle $E_{km}^{(n)} \times E_{lm}^{(n)}$. Therefore, given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|t_1 - t_2| < \delta \text{ and } |s_1 - s_2| < \delta$$

implies

$$\left|K_{ij}(s_1, t_1) - K_{ij}(s_2, t_2)\right| < \frac{\epsilon}{2} \tag{4.15}$$

for $(s_1, t_1), (s_2, t_2) \in E_{km}^{(n)} \times E_{lm}^{(n)}$. Since the length of $E_k^{(n)}$ is less than or equal to $\| \mathcal{P}_n \| \leq T/n^*$ with $n^* \to \infty$ by (3.1), we can consider a sufficiently large $n_0 \in \mathbb{N}$ such that $T/n_0 < \delta$. In this case, each length of $E_k^{(n)}$ for $k = 1, \cdots, n$ is less than δ. In other words, if $n \geq n_0$, then (4.15) is satisfied for any $(s_1, t_1), (s_2, t_2) \in E_{km}^{(n)} \times E_{lm}^{(n)}$. We consider the following cases.

- Suppose that the infimum $K_{klij}^{(n)}$ is attained at $(s^{(n*)}, t^{(n*)}) \in E_k^{(n)} \times E_l^{(n)}$. From (4.13), there exists m^* such that $(s^{(n*)}, t^{(n*)}) \in E_{km^*}^{(n)} \times E_{lm^*}^{(n)}$. Given any $(s, t) \in E_k^{(n)} \times E_l^{(n)}$, we see that $(s, t) \in E_{km_0}^{(n)} \times E_{lm_0}^{(n)}$ for some m_0. Let $m = \max\{m^*, m_0\}$. From (4.14), it follows that $(s, t), (s^{(n*)}, t^{(n*)}) \in E_{km}^{(n)} \times E_{lm}^{(n)}$. Then we have

$$\left|K_{ij}(s, t) - K_{klij}^{(n)}\right| = \left|K_{ij}(s, t) - K_{ij}\left(s^{(n*)}, t^{(n*)}\right)\right| < \frac{\epsilon}{2},$$

  since the lengths of $E_{km}^{(n)}$ and $E_{lm}^{(n)}$ are less than δ, where ϵ is independent of (s, t) in $E_k^{(n)} \times E_l^{(n)}$ because of the uniform continuity.
- Suppose that the infimum $K_{klij}^{(n)}$ is not attained at any point in $E_k^{(n)} \times E_l^{(n)}$. Let

$$\mathcal{K}_{ij} = \left\{K_{ij}(s, t) : (s, t) \in E_k^{(n)} \times E_l^{(n)}\right\}.$$

  Since K_{ij} is continuous on the open rectangle $E_k^{(n)} \times E_l^{(n)}$, it follows that the infimum $K_{klij}^{(n)}$ is in the boundary of the closure of $\mathcal{K}_{ij}$ and is the limit of the function K_{ij} on $E_k^{(n)} \times E_l^{(n)}$. Therefore, for sufficiently large n, i.e., the open rectangle $E_k^{(n)} \times E_l^{(n)}$ is sufficiently small such that the lengths of $E_k^{(n)}$ and $E_l^{(n)}$ are less than δ, we have

$$\left|K_{ij}(s, t) - K_{klij}^{(n)}\right| < \frac{\epsilon}{2}$$

  for all $(s, t) \in E_k^{(n)} \times E_l^{(n)}$.

From the above two cases, we conclude that

$$0 \leq \sup_{(s,t) \in E_k^{(n)} \times E_l^{(n)}} \left[K_{ij}(s, t) - K_{klij}^{(n)}\right] \leq \frac{\epsilon}{2} < \epsilon$$

and

$$0 \le \sup_{t\in E_l^{(n)}} \left[\int_{\bar{E}_k^{(n)}} \left(K_{ij}(s,t) - K_{klij}^{(n)} \right) \cdot \bar{w}_{ki}^{(n)} ds \right]$$
$$\le \frac{\epsilon}{2} \cdot \mathfrak{d}_k^{(n)} \cdot \bar{w}_{ki}^{(n)} < \epsilon \cdot T \cdot \frac{\tau}{\sigma} \cdot \exp\left(r \cdot T \cdot \frac{\nu}{\sigma} \right) \text{ (using (4.4))},$$

which implies

$$\sup_{t\in E_l^{(n)}} \left[\int_{\bar{E}_k^{(n)}} \left(K_{ij}(s,t) - K_{klij}^{(n)} \right) \cdot \bar{w}_{ki}^{(n)} ds \right] \to 0 \text{ as } n \to \infty.$$

This completes the proof. ■

Lemma 4.3. *For each $l = 1, \cdots, n$, we have*

$$\lim_{n\to\infty} \bar{\pi}_l^{(n)} = 0 = \lim_{n\to\infty} \pi_l^{(n)}.$$

Proof. It suffices to prove

$$\sup_{t\in E_l^{(n)}} \left[\bar{h}_{lj}^{(n)}(t) + a_j(t) - a_{lj}^{(n)} \right] \to 0 \text{ as } n \to \infty.$$

From (3.1), since

$$\mathfrak{d}_l^{(n)} \le \| \mathcal{P}_n \| \le \frac{r \cdot T}{n} \to 0 \text{ as } n \to \infty$$

and $\bar{w}_{li}^{(n)}$ is bounded according to (4.4), it follows that

$$\left(e_l^{(n)} - t \right) \cdot K_{llij}^{(n)} \cdot \bar{w}_{li}^{(n)} \le \mathfrak{d}_l^{(n)} \cdot K_{llij}^{(n)} \cdot \bar{w}_{li}^{(n)} \to 0 \text{ as } n \to \infty.$$

Now, we have

$$0 \le \sup_{t\in E_l^{(n)}} \left[\bar{h}_{lj}^{(n)}(t) + a_j(t) - a_{lj}^{(n)} \right]$$
$$\le \mathfrak{d}_l^{(n)} \cdot \sum_{i=1}^{p} K_{llij}^{(n)} \cdot \bar{w}_{li}^{(n)} + \sum_{i=1}^{p} \bar{w}_{li}^{(n)} \cdot \sup_{t\in E_l^{(n)}} \left[B_{lij}^{(n)} - B_{ij}(t) \right]$$
$$+ \sum_{k=l}^{n} \sum_{i=1}^{p} \sup_{t\in E_l^{(n)}} \left[\int_{\bar{E}_k^{(n)}} \left(K_{ij}(s,t) - K_{klij}^{(n)} \right) \cdot \bar{w}_{ki}^{(n)} ds \right] + \sup_{t\in E_l^{(n)}} \left[a_j(t) - a_{lj}^{(n)} \right]$$

Using Lemmas 4.1 and 4.2, we complete the proof. ■

We define the following notations:

$$\bar{\mathfrak{k}}_l^{(n)} = \max_{j=1,\cdots,q} \left\{ \sup_{(s,t)\in[e_{l-1}^{(n)},T]\times\bar{E}_l^{(n)}} \sum_{i=1}^{p} K_{ij}(s,t) \right\} \tag{4.16}$$

and

$$\bar{\mathfrak{b}}_l^{(n)} = \min_{j=1,\cdots,q} \left\{ \inf_{t\in\bar{E}_l^{(n)}} \sum_{i=1}^{p} B_{ij}(t) \right\}.$$

From (2.2), (2.1) and (3.11), we see that

$$\bar{\mathfrak{b}}_l^{(n)} \geq \min_{j=1,\cdots,q} \left\{ \inf_{t\in[0,T]} \sum_{i=1}^{p} B_{ij}(t) \right\} \geq \sigma > 0 \text{ and } \bar{\mathfrak{k}}_l^{(n)} \leq \nu.$$

Let

$$\mathfrak{k}_l^{(n)} = \max_{k=l,\cdots,n} \bar{\mathfrak{k}}_k^{(n)} \leq \nu \text{ and } \mathfrak{b}_l^{(n)} = \min_{k=l,\cdots,n} \bar{\mathfrak{b}}_k^{(n)} \geq \sigma. \tag{4.17}$$

Then we see that $0 < \mathfrak{b}_l^{(n)}$ and

$$\mathfrak{k}_l^{(n)} \geq \mathfrak{k}_{l+1}^{(n)} \text{ and } \mathfrak{b}_l^{(n)} \leq \mathfrak{b}_{l+1}^{(n)}. \tag{4.18}$$

Now we define the real-valued functions $\mathfrak{u}^{(n)}$ and $\mathfrak{v}^{(n)}$ on $[0, T]$ by

$$\mathfrak{u}^{(n)}(t) = \begin{cases} \mathfrak{k}_l^{(n)} & \text{if } t \in F_l^{(n)} \text{ for } l = 1, \cdots, n \\ \mathfrak{k}_n^{(n)} & \text{if } t = T \end{cases}$$

and

$$\mathfrak{v}^{(n)}(t) = \begin{cases} \mathfrak{b}_l^{(n)} & \text{if } t \in F_l^{(n)} \text{ for } l = 1, \cdots, n \\ \mathfrak{b}_n^{(n)} & \text{if } t = T. \end{cases}$$

Then we have

$$\mathfrak{u}^{(n)}(t) \leq \nu \text{ and } \mathfrak{v}^{(n)}(t) \geq \sigma \text{ for all } t \in [0, T] \tag{4.19}$$

From (4.4) and Lemmas 4.1 and 4.2, we see that the sequence $\{\bar{h}_{lj}^{(n)}\}_{n=1}^{\infty}$ is uniformly bounded, which also says that $\{\pi_l^{(n)}\}_{n=1}^{\infty}$ is uniformly bounded. Therefore there exists a constant $\mathfrak{x}$ such that $\pi_l^{(n)} \leq \mathfrak{x}$ for all $n \in \mathbb{N}$ and $l = 1, \cdots, n$. We also define a real-valued function $\mathfrak{p}^{(n)}$ on $[0, T]$ by

$$\mathfrak{p}^{(n)}(t) = \begin{cases} \mathfrak{x} & \text{if } t = e_{l-1}^{(n)} \text{ for } l = 1, \cdots, n \\ \pi_l^{(n)} & \text{if } t \in E_l^{(n)} \text{ for } l = 1, \cdots, n \\ \max\limits_{j=1,\cdots,q} \left\{ r_j^{(n)} + a_j(T) - a_{nj}^{(n)} \right\} & \text{if } t = e_n^{(n)} = T, \end{cases}$$

where $r_j^{(n)}$ is the jth component of $\mathbf{r}^{(n)}$ in (4.6). Then we have

$$\mathfrak{p}^{(n)}(t) \leq \mathfrak{x} \text{ for all } n \in \mathbb{N} \text{ and } t \in [0, T) \tag{4.20}$$

Let $\mathbf{1}_p = (1, 1, \cdots, 1)^\top \in \mathbb{R}^p$ denote a p-dimensional vector such that each component of $\mathbf{1}_p$ is 1. Let the real-valued function $\mathfrak{f}^{(n)} : [0, T] \to \mathbb{R}_+$ be defined by

$$\mathfrak{f}^{(n)}(t) = \frac{\mathfrak{p}^{(n)}(t)}{\mathfrak{v}^{(n)}(t)} \cdot \exp\left[\frac{\mathfrak{u}^{(n)}(t) \cdot (T - t)}{\mathfrak{v}^{(n)}(t)} \right] \tag{4.21}$$

The following observations will be adopted for further discussion.

Remark 4.1. Suppose that f is piecewise continuous on $[0,T]$ with the finite discontinuities $\mathcal{D} = \{d_1, \cdots, d_r\}$. Let G_v be the open intervals with end-points consisting of the discontinuities in $\mathcal{D}$ for $v = 0, 1, \cdots, r$, where $G_0 = (0, d_1)$, $G_v = (d_v, d_{v+1})$ for $v = 1, \cdots, r-1$ and $G_r = (d_r, T)$. Then we see that f is continuous on the open intervals G_v, which says that f is Riemann-integrable on $[0,T]$. We also have the following observations.

- Let $\bar{f}$ be a function defined on $[0,T]$ such that $\bar{f}(t) = f(t)$ for $t \notin \mathcal{D}$, i.e., their function values are different only for the discontinuities of f. Then we see that
$$\int_0^T \bar{f}(x)dx = \int_0^T f(x)dx.$$
- Let $E = (a,b)$ be an open interval, let $F = [a,b)$ be an helf-open interval and let $\bar{E} = [a,b]$ be a closed interval. Suppose that g is defined on $[a,b]$ and is continuous on (a,b). Then g is Riemann-integrable. We also see that the following improper Riemann integrals
$$\int_E g(t)dt = \int_{a+}^{b-} g(t)dt \text{ and } \int_F g(t)dt = \int_a^{b-} g(t)dt$$
are identical with
$$\int_{\bar{E}} g(t)dt = \int_a^b g(t)dt = \int_E g(t)dt = \int_F g(t)dt.$$

The following lemma will be used for further discussion.

Lemma 4.4. *We have*
$$\mathfrak{f}^{(n)}(t)\,(B(t))^\top\,\mathbf{1}_p \geq \bar{\mathbf{h}}_l^{(n)}(t) + \mathbf{a}(t) - \mathbf{a}_l^{(n)} + \int_t^T \mathfrak{f}^{(n)}(s)\,(K(s,t))^\top\,\mathbf{1}_p ds$$
for $t \in F_l^{(n)}$ and $l = 1, \cdots, n$, and
$$\mathfrak{f}^{(n)}(T)\,(B(T))^\top\,\mathbf{1}_p \geq \mathbf{r}^{(n)} + \mathbf{a}(T) - \mathbf{a}_n^{(n)}.$$
Moreover the sequence of real-valued functions $\{\mathfrak{f}^{(n)}\}_{n=1}^\infty$ is uniformly bounded.

Proof. According to Remark 4.1, for $t \in F_l^{(n)}$, from (4.21), we have
$$\begin{aligned}\int_t^T \mathfrak{f}^{(n)}(s)ds &= \int_t^T \frac{\mathfrak{p}^{(n)}(s)}{\mathfrak{v}^{(n)}(s)} \cdot \exp\left[\frac{\mathfrak{u}^{(n)}(s)\cdot(T-s)}{\mathfrak{v}^{(n)}(s)}\right]ds\\ &= \int_t^{e_l^{(n)}} \frac{\pi_l^{(n)}}{\mathfrak{b}_l^{(n)}} \cdot \exp\left[\frac{\mathfrak{t}_l^{(n)}\cdot(T-s)}{\mathfrak{b}_l^{(n)}}\right]ds + \sum_{k=l+1}^n \int_{\bar{E}_k^{(n)}} \frac{\pi_k^{(n)}}{\mathfrak{b}_k^{(n)}} \cdot \exp\left[\frac{\mathfrak{t}_k^{(n)}\cdot(T-s)}{\mathfrak{b}_k^{(n)}}\right]ds\\ &\leq \int_t^{e_l^{(n)}} \frac{\pi_l^{(n)}}{\mathfrak{b}_l^{(n)}} \cdot \exp\left[\frac{\mathfrak{t}_l^{(n)}\cdot(T-s)}{\mathfrak{b}_l^{(n)}}\right]ds + \sum_{k=l+1}^n \int_{\bar{E}_k^{(n)}} \frac{\pi_l^{(n)}}{\mathfrak{b}_l^{(n)}} \cdot \exp\left[\frac{\mathfrak{t}_l^{(n)}\cdot(T-s)}{\mathfrak{b}_l^{(n)}}\right]ds\\ &\quad\text{(by (4.9) and (4.18))}\\ &= \int_t^T \frac{\pi_l^{(n)}}{\mathfrak{b}_l^{(n)}} \cdot \exp\left[\frac{\mathfrak{t}_l^{(n)}\cdot(T-s)}{\mathfrak{b}_l^{(n)}}\right]ds = \frac{\pi_l^{(n)}}{\mathfrak{t}_l^{(n)}} \cdot \left(\exp\left[\frac{\mathfrak{t}_l^{(n)}\cdot(T-t)}{\mathfrak{b}_l^{(n)}}\right] - 1\right) \end{aligned} \tag{4.22}$$

Since

$$\mathfrak{b}_l^{(n)} \cdot \mathfrak{f}^{(n)}(t) = \begin{cases} \mathfrak{x} \cdot \exp\left[\dfrac{\mathfrak{k}_l^{(n)} \cdot (T-t)}{\mathfrak{b}_l^{(n)}}\right] & \text{if } t = e_{l-1}^{(n)} \text{ for } l = 1, \cdots, n \\ \pi_l^{(n)} \cdot \exp\left[\dfrac{\mathfrak{k}_l^{(n)} \cdot (T-t)}{\mathfrak{b}_l^{(n)}}\right] & \text{if } t \in E_l^{(n)} \text{ for } l = 1, \cdots, n, \end{cases}$$

using (4.22), it follows that, for $t \in F_l^{(n)}$,

$$\mathfrak{b}_l^{(n)} \cdot \mathfrak{f}^{(n)}(t) \geq \begin{cases} \mathfrak{x} \cdot \left(1 + \dfrac{\mathfrak{k}_l^{(n)}}{\pi_l^{(n)}} \cdot \displaystyle\int_t^T \mathfrak{f}^{(n)}(s)ds\right) & \text{if } t = e_{l-1}^{(n)} \text{ for } l = 1, \cdots, n \\ \pi_l^{(n)} + \mathfrak{k}_l^{(n)} \cdot \displaystyle\int_t^T \mathfrak{f}^{(n)}(s)ds & \text{if } t \in E_l^{(n)} \text{ for } l = 1, \cdots, n. \end{cases}$$

$$\geq \pi_l^{(n)} + \mathfrak{k}_l^{(n)} \cdot \int_t^T \mathfrak{f}^{(n)}(s)ds \text{ (since } \pi_l^{(n)} \leq \mathfrak{x} \text{ for all } l = 1, \cdots, n). \tag{4.23}$$

For $t = e_n^{(n)} = T$, we also have,

$$\mathfrak{b}_n^{(n)} \cdot \mathfrak{f}^{(n)}(T) = \max_{j=1,\cdots,q} \left\{ r_j^{(n)} + a_j(T) - a_{nj}^{(n)} \right\}. \tag{4.24}$$

For each $j = 1, \cdots, q$ and $l = 1, \cdots, n$, we consider the following cases.

- For $t = e_n^{(n)} = T$, from (4.24), we have

$$\sum_{i=1}^{p} B_{ij}(T) \cdot \mathfrak{f}^{(n)}(T) \geq \bar{\mathfrak{b}}_n^{(n)} \cdot \mathfrak{f}^{(n)}(T) = \mathfrak{b}_n^{(n)} \cdot \mathfrak{f}^{(n)}(T) \geq r_j^{(n)} + a_j(T) - a_{nj}^{(n)}.$$

- For $t \in F_l^{(n)}$, by (4.23) and (4.10), we have

$$\sum_{i=1}^{p} B_{ij}(t) \cdot \mathfrak{f}^{(n)}(t) \geq \bar{\mathfrak{b}}_l^{(n)} \cdot \mathfrak{f}^{(n)}(t) \geq \mathfrak{b}_l^{(n)} \cdot \mathfrak{f}^{(n)}(t)$$

$$\geq \bar{h}_{lj}^{(n)}(t) + a_j(t) - a_{lj}^{(n)} + \int_t^T \sum_{i=1}^{p} K_{ij}(s,t) \cdot \mathfrak{f}^{(n)}(s)ds.$$

Finally, from (4.19) and (4.20), it is obvious that the sequence of real-valued functions $\{\mathfrak{f}^{(n)}\}_{n=1}^{\infty}$ is uniformly bounded. This completes the proof. ■

We define a vector-valued function $\widehat{\mathbf{w}}^{(n)}(t) : [0,T] \to \mathbb{R}^p$ by

$$\widehat{\mathbf{w}}^{(n)}(t) = \begin{cases} \bar{\mathbf{w}}_l^{(n)} + \mathfrak{f}^{(n)}(t)\mathbf{1}_p & \text{if } t \in F_l^{(n)} \text{ for } l = 1, \cdots, n \\ \bar{\mathbf{w}}_n^{(n)} + \mathfrak{f}^{(n)}(T)\mathbf{1}_p & \text{if } t = T. \end{cases} \tag{4.25}$$

Remark 4.2. Since the sequence of real-valued functions $\{\mathfrak{f}^{(n)}\}_{n=1}^{\infty}$ is uniformly bounded by Lemma 4.4, using (4.4), it follows that the family of vector-valued functions $\{\widehat{\mathbf{w}}^{(n)}\}_{n\in\mathbb{N}}$ is also uniformly bounded.

Proposition 4.1. *For any $n \in \mathbb{N}$, $\widehat{\mathbf{w}}^{(n)}$ is a feasible solution of problem* (DCLP).

Proof. For $l = 1, \cdots, n$, we define a vector-valued function $\mathbf{b}^{(n)}$ on $F_l^{(n)}$ by

$$\begin{aligned}\mathbf{b}^{(n)}(t) = &\left(B(t) - B_l^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} \\ &- \int_t^{e_l^{(n)}} \left(K(s,t) - K_{ll}^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} ds - \sum_{k=l+1}^{n} \int_{\bar{E}_k^{(n)}} \left(K(s,t) - K_{kl}^{(n)}\right)^\top \bar{\mathbf{w}}_k^{(n)} ds \\ &+ \mathfrak{f}^{(n)}(t)\,(B(t))^\top \mathbf{1}_p - \int_t^T \mathfrak{f}^{(n)}(s)\,(K(s,t))^\top \mathbf{1}_p ds, \end{aligned} \tag{4.26}$$

which implies

$$\begin{aligned}&- \mathbf{b}^{(n)}(t) + \mathfrak{f}^{(n)}(t)\,(B(t))^\top \mathbf{1}_p - \int_t^T \mathfrak{f}^{(n)}(s)\,(K(s,t))^\top \mathbf{1}_p ds \\ &= \left(B_l^{(n)} - B(t)\right)^\top \bar{\mathbf{w}}_l^{(n)} + \int_t^{e_l^{(n)}} \left(K(s,t) - K_{ll}^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} ds \\ &+ \sum_{k=l+1}^{n} \int_{\bar{E}_k^{(n)}} \left(K(s,t) - K_{kl}^{(n)}\right)^\top \bar{\mathbf{w}}_k^{(n)} ds.\end{aligned}$$

Therefore, by adding the term $(e_l^{(n)} - t)(K_{ll}^{(n)})^\top \bar{\mathbf{w}}_l^{(n)}$ on both sides, we obtain

$$\begin{aligned}&\left(e_l^{(n)} - t\right)\left(K_{ll}^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} - \mathbf{b}^{(n)}(t) + \mathfrak{f}^{(n)}(t)\,(B(t))^\top \mathbf{1}_p - \int_t^T \mathfrak{f}^{(n)}(s)\,(K(s,t))^\top \mathbf{1}_p ds \\ &= \left(e_l^{(n)} - t\right)\left(K_{ll}^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} + \left(B_l^{(n)} - B(t)\right)^\top \bar{\mathbf{w}}_l^{(n)} \\ &\quad + \int_t^{e_l^{(n)}} \left(K(s,t) - K_{ll}^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} ds + \sum_{k=l+1}^{n} \int_{\bar{E}_k^{(n)}} \left(K(s,t) - K_{kl}^{(n)}\right)^\top \bar{\mathbf{w}}_k^{(n)} ds \\ &= \bar{\mathbf{h}}_l^{(n)}(t),\end{aligned}$$

which implies

$$\begin{aligned}&\mathbf{b}^{(n)}(t) - \left(e_l^{(n)} - t\right)\left(K_{ll}^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} \\ &= -\bar{\mathbf{h}}_l^{(n)}(t) + \mathfrak{f}^{(n)}(t)\,(B(t))^\top \mathbf{1}_p - \int_t^T \mathfrak{f}^{(n)}(s)\,(K(s,t))^\top \mathbf{1}_p ds \\ &\geq \mathbf{a}(t) - \mathbf{a}_l^{(n)} \text{ (by Lemma 4.4)}\end{aligned} \tag{4.27}$$

Now we obtain

$$\begin{aligned}&(B(t))^\top \widehat{\mathbf{w}}^{(n)}(t) - \int_t^T (K(s,t))^\top \widehat{\mathbf{w}}^{(n)}(s) ds \\ &= (B(t))^\top \left(\bar{\mathbf{w}}_l^{(n)} + \mathfrak{f}^{(n)}(t)\mathbf{1}_p\right) - \int_t^T (K(s,t))^\top \left(\bar{\mathbf{w}}_l^{(n)} + \mathfrak{f}^{(n)}(s)\mathbf{1}_p\right) ds \text{ (by (4.25))} \\ &= \left(\left(B_l^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} - \int_t^{e_l^{(n)}} \left(K_{ll}^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} ds - \sum_{k=l+1}^{n} \int_{\bar{E}_k^{(n)}} \left(K_{kl}^{(n)}\right)^\top \bar{\mathbf{w}}_k^{(n)} ds\right) + \mathbf{b}^{(n)}(t) \text{ (by (4.26))}\end{aligned}$$

$$= \left(\left(B_l^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} - \sum_{k=l+1}^{n} \mathfrak{d}_k^{(n)} \left(K_{kl}^{(n)}\right)^\top \bar{\mathbf{w}}_k^{(n)} \right) + \mathbf{b}^{(n)}(t) - \left(e_l^{(n)} - t\right)\left(K_{ll}^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)}$$
$$\geq \mathbf{a}_l^{(n)} + \mathbf{b}^{(n)}(t) - \left(e_l^{(n)} - t\right)\left(K_{ll}^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} \quad \text{(by the feasibility of } \bar{\mathbf{w}}^{(n)} \text{ for problem } (\mathrm{D}_n))$$
$$\geq \mathbf{a}_l^{(n)} + \mathbf{a}(t) - \mathbf{a}_l^{(n)} \text{ (by (4.27))}$$
$$= \mathbf{a}(t)$$

Suppose that $t = T$. We define

$$\widehat{\mathbf{b}}^{(n)} = \left(B(T) - B_n^{(n)}\right)^\top \bar{\mathbf{w}}_n^{(n)} + \mathfrak{f}^{(n)}(T)\,(B(T))^\top \mathbf{1}_p.$$

Then we have

$$-\widehat{\mathbf{b}}^{(n)} + \mathfrak{f}^{(n)}(T)\,(B(T))^\top \mathbf{1}_p = \left(B_n^{(n)} - B(T)\right)^\top \bar{\mathbf{w}}_n^{(n)} = \mathbf{r}^{(n)}$$

which implies that

$$\widehat{\mathbf{b}}^{(n)} = -\mathbf{r}^{(n)} + \mathfrak{f}^{(n)}(T)\,(B(T))^\top \mathbf{1}_p \geq \mathbf{a}(T) - \mathbf{a}_n^{(n)} \text{ (by Lemma 4.4)} \tag{4.28}$$

Now we obtain

$$(B(T))^\top \widehat{\mathbf{w}}^{(n)}(T) = \left(B_n^{(n)}\right)^\top \bar{\mathbf{w}}_n^{(n)} + \widehat{\mathbf{b}}^{(n)} \geq \mathbf{a}_n^{(n)} + \widehat{\mathbf{b}}^{(n)} \text{ (by the feasibility of } \bar{\mathbf{w}}^{(n)})$$
$$\geq \mathbf{a}_n^{(n)} + \mathbf{a}(T) - \mathbf{a}_n^{(n)} \text{ (using (4.28))}$$
$$= \mathbf{a}(T).$$

Therefore we conclude that $\widehat{\mathbf{w}}^{(n)}$ is indeed a feasible solution of problem (DCLP). This completes this proof. ■

For $i = 1, \cdots, p$ and $j = 1, \cdots, q$, we define the step functions $\bar{a}_j^{(n)} : [0, T] \to \mathbb{R}$ and $\bar{c}_i^{(n)} : [0, T] \to \mathbb{R}$ as follows:

$$\bar{a}_j^{(n)}(t) = \begin{cases} a_{lj}^{(n)} & \text{if } t \in F_l^{(n)} \text{ for } l = 1, \cdots, n \\ a_{nj}^{(n)} & \text{if } t = T. \end{cases}$$

and

$$\bar{c}_i^{(n)}(t) = \begin{cases} c_{li}^{(n)} & \text{if } t \in F_l^{(n)} \text{ for } l = 1, \cdots, n \\ c_{ni}^{(n)} & \text{if } t = T, \end{cases}$$

respectively. For $i = 1, \cdots, p$, we also define step function $\bar{w}_i^{(n)}(t) : [0, T] \to \mathbb{R}$ by

$$\bar{w}_i^{(n)}(t) = \begin{cases} \bar{w}_{li}^{(n)} & \text{if } t \in F_l^{(n)} \text{ for } l = 1, \cdots, n \\ \bar{w}_{ni}^{(n)} & \text{if } t = T. \end{cases}$$

Lemma 4.5. *For $i = 1, \cdots, p$ and $j = 1, \cdots, q$, we have*

$$\int_0^T \left[a_j(t) - \bar{a}_j^{(n)}(t)\right] \cdot \widehat{z}_j^{(n)}(t)dt \to 0 \text{ as } n \to \infty \tag{4.29}$$

and

$$\int_0^T \left[c_i(t) - \bar{c}_i^{(n)}(t)\right] \cdot \bar{w}_i^{(n)}(t)dt \to 0 \text{ as } n \to \infty. \tag{4.30}$$

Proof. It is obvious that the following functions

$$\left[a_j(t) - \bar{a}_j^{(n)}(t)\right] \cdot \widehat{z}_j^{(n)}(t) \text{ and } \left[c_i(t) - \bar{c}_i^{(n)}(t)\right] \cdot \bar{w}_i^{(n)}(t)$$

are continuous a.e. on $[0, T]$, i.e., they are Riemann-integrable on $[0, T]$. In other words, their Riemann integral and Lebesgue integral are identical. For $i = 1, \cdots, n$ and $t \in E_l^{(n)}$, Lemma 4.1 says that

$$0 \leq a_j(t) - \bar{a}_j^{(n)}(t) = a_j(t) - a_{lj}^{(n)} \leq \sup_{t \in E_l^{(n)}} \left[a_j(t) - a_{lj}^{(n)}\right] \to 0 \text{ as } n \to \infty$$

and

$$0 \leq c_i(t) - \bar{c}_i^{(n)}(t) = c_i(t) - c_{li}^{(n)} \leq \sup_{t \in E_l^{(n)}} \left[c_i(t) - c_{li}^{(n)}\right] \to 0 \text{ as } n \to \infty,$$

which implies

$$a_j(t) - \bar{a}_j^{(n)}(t) \to 0 \text{ and } c_i(t) - \bar{c}_i^{(n)}(t) \to 0 \text{ as } n \to \infty \text{ a.e. on } [0, T].$$

Since the sequence $\{\widehat{z}_j^{(n)}\}_{n=1}^{\infty}$ is uniformly bounded from Proposition 3.2, using the Lebesgue bounded convergence theorem for integrals, we obtain (4.29). On the other hand, since the sequence $\{\bar{w}_i^{(n)}\}_{n=1}^{\infty}$ is also uniformly bounded according to (4.4), using the Lebesgue bounded convergence theorem, we can also obtain (4.30). This completes the proof. ■

Theorem 4.1. *The following statements hold true.*

(i) *We have*

$$\limsup_{n \to \infty} V(\mathrm{D}_n) = V(\mathrm{DCLP}) \text{ and } 0 \leq V(\mathrm{DCLP}) - V(\mathrm{D}_n) \leq \varepsilon_n,$$

where

$$\varepsilon_n = -V(\mathrm{D}_n) + \sum_{l=1}^{n} \int_{\bar{E}_l^{(n)}} (\mathbf{c}(t))^\top \, \bar{\mathbf{w}}_l^{(n)} dt + \sum_{l=1}^{n} \int_{\bar{E}_l^{(n)}} \frac{\pi_l^{(n)}}{\mathfrak{b}_l^{(n)}} \cdot \exp\left[\frac{\mathfrak{f}_l^{(n)} \cdot (T - t)}{\mathfrak{b}_l^{(n)}}\right] (\mathbf{c}(t))^\top \, \mathbf{1}_p dt \tag{4.31}$$

satisfying $\varepsilon_n \to 0$ as $n \to \infty$. Moreover there exists a convergent subsequence $\{V(\mathrm{D}_{n_k})\}_{k=1}^{\infty}$ of $\{V(\mathrm{D}_n)\}_{n=1}^{\infty}$ such that

$$\lim_{k \to \infty} V(\mathrm{D}_{n_k}) = V(\mathrm{DCLP}). \tag{4.32}$$

(ii) (**No Duality Gap**). *We have*

$$V(\mathrm{DCLP}) = V(\mathrm{CLP}) = \limsup_{n \to \infty} V(\mathrm{D}_n) = \limsup_{n \to \infty} V(\mathrm{P}_n)$$

and

$$0 \leq V(\mathrm{CLP}) - V(\mathrm{P}_n) = V(\mathrm{DCLP}) - V(\mathrm{D}_n) \leq \varepsilon_n.$$

Proof. To prove part (i), we have

$$\begin{aligned} 0 &\le V(\text{DCLP}) - V(\text{D}_n) \text{ (by (4.2))} \\ &= V(\text{DCLP}) - \sum_{l=1}^{n} \int_{\bar{E}_l^{(n)}} \left(\mathbf{c}_l^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} dt \\ &\le \int_0^T (\mathbf{c}(t))^\top \widehat{\mathbf{w}}^{(n)}(t)dt - \sum_{l=1}^{n} \int_{\bar{E}_l^{(n)}} \left(\mathbf{c}_l^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} dt \quad \text{(by Proposition 4.1)} \\ &= \sum_{l=1}^{n} \int_{\bar{E}_l^{(n)}} \left(\mathbf{c}(t) - \mathbf{c}_l^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} dt + \int_0^T \mathfrak{f}^{(n)}(t) \, (\mathbf{c}(t))^\top \mathbf{1}_p dt \\ &\equiv \varepsilon_n, \end{aligned} \tag{4.33}$$

which implies

$$V(\text{D}_n) \le V(\text{DCLP}) \le V(\text{D}_n) + \varepsilon_n. \tag{4.34}$$

Using Lemma 4.5, we obtain

$$0 \le \sum_{l=1}^{n} \int_{\bar{E}_l^{(n)}} \left(c_i(t) - c_{li}^{(n)}\right) \cdot \bar{w}_{li}^{(n)} dt = \int_0^T \left[c_i(t) - \bar{c}_i^{(n)}(t)\right] \cdot \bar{w}_i^{(n)}(t)dt \to 0 \text{ as } n \to \infty. \tag{4.35}$$

Since $\mathfrak{k}_l^{(n)} \le \nu$ and $\mathfrak{b}_l^{(n)} \ge \sigma$ for all n by (4.17), and $\pi_l^{(n)} \to 0$ as $n \to \infty$ by Lemma 4.3, it follows that $\mathfrak{p}^{(n)} \to 0$ as $n \to \infty$ a.e. on $[0, T]$, which implies that $\mathfrak{f}^{(n)} \to 0$ as $n \to \infty$ a.e. on $[0, T]$. Applying the Lebesgue bounded convergence theorem for integrals, we obtain

$$\int_0^T \mathfrak{f}^{(n)}(t) \, (\mathbf{c}(t))^\top \mathbf{1}_p dt \to 0 \text{ as } n \to \infty. \tag{4.36}$$

From (4.35) and (4.36), we conclude that $\varepsilon_n \to 0$ as $n \to \infty$. From (4.34), we also obtain

$$\limsup_{n\to\infty} V(\text{D}_n) \le V(\text{DCLP}) \le \limsup_{n\to\infty} V(\text{D}_n) + \limsup_{n\to\infty} \varepsilon_n = \limsup_{n\to\infty} V(\text{D}_n).$$

From part (ii) of Proposition 3.1, we see that $\{V(\text{D}_n)\}_{n=1}^\infty$ is a bounded sequence. Therefore there exists a convergent subsequence $\{V(\text{D}_{n_k})\}_{k=1}^\infty$ of $\{V(\text{D}_n)\}_{n=1}^\infty$. Using (4.34), we obtain the equality (4.32). On the other hand, it is easy to see that ε_n can be written as (4.31), which proves part (i).

To prove part (ii), by part (i) and inequality (4.2), we obtain

$$V(\text{DCLP}) \ge V(\text{CLP}) \ge \limsup_{n\to\infty} V(\text{D}_n) = V(\text{DCLP}).$$

Since $V(\text{D}_n) = V(\text{P}_n)$ for each $n \in \mathbb{N}$, we also have

$$V(\text{DCLP}) = V(\text{CLP}) = \limsup_{n\to\infty} V(\text{D}_n) = \limsup_{n\to\infty} V(\text{P}_n)$$

and

$$0 \le V(\text{CLP}) - V(\text{P}_n) = V(\text{DCLP}) - V(\text{D}_n) \le \varepsilon_n.$$

This completes the proof. ■

From Remark 3.2 and Theorem 4.1, if the vector-valued function **c** is nonnegative, i.e., the primal problem (P$_n$) is feasible, then the strong duality holds for the primal and dual pair of continuou-time linear programming problems (CLP) and (DCLP).

Proposition 4.2. *The following statements hold true.*

(i) *Let* $\widehat{\mathbf{z}}^{(n)}$ *be defined in* (3.25). *Then, the error between* $V(\text{CLP})$ *and the objective value of* $\widehat{\mathbf{z}}^{(n)}$ *is less than or equal to* ε_n *defined in* (4.31), *i.e.,*

$$0 \le V(\text{CLP}) - \int_0^T \mathbf{a}(t)^\top \widehat{\mathbf{z}}^{(n)}(t)dt \le \varepsilon_n.$$

(ii) *Let* $\widehat{\mathbf{w}}^{(n)}$ *be defined in* (4.25). *Then, the error between* $V(\text{DCLP})$ *and the objective value of* $\widehat{\mathbf{w}}^{(n)}$ *is less than or equal to* ε_n, *i.e.,*

$$0 \le \int_0^T (\mathbf{c}(t))^\top \widehat{\mathbf{w}}^{(n)}(t)dt - V(\text{DCLP}) \le \varepsilon_n$$

Proof. To prove part (i), Proposition 3.3 says that $\widehat{\mathbf{z}}^{(n)}$ is a feasible solution of problem (CLP). Since

$$\sum_{l=1}^{n} \int_{\bar{E}_l^{(n)}} \left(\mathbf{a}_l^{(n)}\right)^\top \widehat{\mathbf{z}}^{(n)}(t)dt = \sum_{l=1}^{n} \mathfrak{d}_l^{(n)} \left(\mathbf{a}_l^{(n)}\right)^\top \bar{\mathbf{z}}_l^{(n)} = V(\text{P}_n) = V(\text{D}_n) \tag{4.37}$$

and $\mathbf{a}_l^{(n)} \le \mathbf{a}(t)$ for all $t \in \bar{E}_l^{(n)}$ and $l = 1, \cdots, n$, it follows that

$$\begin{aligned} 0 \le V(\text{CLP}) - \int_0^T (\mathbf{a}(t))^\top \widehat{\mathbf{z}}^{(n)}(t)dt &\le V(\text{CLP}) - \sum_{l=1}^{n} \int_{\bar{E}_l^{(n)}} \left(\mathbf{a}_l^{(n)}\right)^\top \widehat{\mathbf{z}}^{(n)}(t)dt \\ &= V(\text{DCLP}) - V(\text{D}_n) \text{ (by (4.37) and part (ii) of Theorem 4.1)} \\ &\le \varepsilon_n \text{ (by part (i) of Theorem 4.1).} \end{aligned}$$

To prove part (ii), we have

$$\begin{aligned} 0 &\le \int_0^T (\mathbf{c}(t))^\top \widehat{\mathbf{w}}^{(n)}(t)dt - V(\text{DCLP}) \text{ (by Proposition 4.1)} \\ &\le \int_0^T (\mathbf{c}(t))^\top \widehat{\mathbf{w}}^{(n)}(t)dt - V(\text{D}_n) \text{ (since } V(\text{D}_n) \le V(\text{DCLP}) \text{ by part (i) of Theorem 4.1)} \\ &= \varepsilon_n \text{ (by (4.33) and (4.34))} \end{aligned}$$

This completes the proof. ■

Definition 4.1. Given any $\epsilon > 0$, we say that the feasible solution $\mathbf{z}^{(\epsilon)}$ of problem (CLP) is an ϵ-*optimal solution* if and only if

$$0 \le V(\text{CLP}) - \int_0^T (\mathbf{a}(t))^\top \mathbf{z}^{(\epsilon)}(t)dt < \epsilon.$$

We say that the feasible solution $\mathbf{w}^{(\epsilon)}$ of problem $V(\text{DCLP})$ is an ϵ-*optimal solution* if and only if

$$0 \le \int_0^T (\mathbf{c}(t))^\top \mathbf{w}^{(\epsilon)}(t)dt - V(\text{DCLP}) < \epsilon.$$

Theorem 4.2. *Given any* $\epsilon > 0$, *the following statements hold true.*

(i) *The ϵ-optimal solution of problem* (CLP) *exists in the following sense: there exists $n \in \mathbb{N}$ such that $\mathbf{z}^{(\epsilon)} = \widehat{\mathbf{z}}^{(n)}$, where $\widehat{\mathbf{z}}^{(n)}$ is obtained from Proposition 4.2 satisfying $\varepsilon_n < \epsilon$.*

(ii) *The ϵ-optimal solution of problem* (DCLP) *exists in the following sense: there exists $n \in \mathbb{N}$ such that $\mathbf{w}^{(\epsilon)} = \widehat{\mathbf{w}}^{(n)}$, where $\widehat{\mathbf{w}}^{(n)}$ is obtained from Proposition 4.2 satisfying $\varepsilon_n < \epsilon$.*

Proof. Given any $\epsilon > 0$, from Proposition 4.2, since $\varepsilon_n \to 0$ as $n \to \infty$, there exists $n \in \mathbb{N}$ such that $\varepsilon_n < \epsilon$. Then the result follows immediately. ∎

5. Convergence of approximate solutions

In the sequel, by referring to (3.25) and (4.25), we shall present the convergent properties of the sequences $\{\widehat{\mathbf{z}}^{(n)}\}_{n=1}^{\infty}$ and $\{\widehat{\mathbf{w}}^{(n)}\}_{n=1}^{\infty}$ that are constructed from the optimal solutions $\bar{\mathbf{z}}^{(n)}$ of problem (P_n) and the optimal solution $\bar{\mathbf{w}}^{(n)}$ of problem (D_n), respectively. We first provide a useful lemma.

Lemma 5.1. *Let the real-valued function η be defined by*

$$\eta(t) = \frac{\tau}{\sigma} \cdot \exp\left[\frac{\nu \cdot (T - t)}{\sigma}\right] \tag{5.1}$$

on $[0, T]$*, and let* $\mathbf{w}^{(0)}$ *be a feasible solution of dual problem* (DCLP). *We define*

$$w_i^{(1)}(t) = \min\left\{w_i^{(0)}(t), \eta(t)\right\} \tag{5.2}$$

for all $i = 1, \cdots, p$ and $t \in [0, T]$. Then $\mathbf{w}^{(1)}$ is a feasible solution of dual problem (DCLP) *satisfying $\mathbf{w}^{(1)}(t) \leq \mathbf{w}^{(0)}(t)$ and $w_i^{(1)}(t) \leq \eta(t)$ for all $i = 1, \cdots, p$ and $t \in [0, T]$.*

Proof. By the feasibility of $\mathbf{w}^{(0)}$ for problem (DCLP), we have

$$\sum_{i=1}^{p} B_{ij}(t) \cdot w_i^{(0)}(t) \geq a_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}(s, t) \cdot w_i^{(0)}(s) ds \tag{5.3}$$

Since $K_{ij}(s, t) \geq 0$ and $w_i^{(1)}(t) \leq w_i^{(0)}(t)$, from (5.3), we obtain

$$\sum_{i=1}^{p} B_{ij}(t) \cdot w_i^{(0)}(t) \geq a_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}(s, t) \cdot w_i^{(1)}(s) ds. \tag{5.4}$$

For any fixed $t \in [0, T]$, we define the index sets

$$I_{\leq} = \left\{i : w_i^{(0)}(t) \leq \eta(t)\right\} \text{ and } I_{>} = \left\{i : w_i^{(0)}(t) > \eta(t)\right\}.$$

Then

$$w_i^{(1)}(t) = \begin{cases} w_i^{(0)}(t) & \text{if } i \in I_{\leq} \\ \eta(t) & \text{if } i \in I_{>}. \end{cases}$$

For each fixed j, we consider the following three cases:

- Suppose that $I_> = \emptyset$ (i.e., the second sum is zero). Then $w_i^{(0)}(t) = w_i^{(1)}(t)$ for all i. Therefore, from (5.4), we have

$$\sum_{i=1}^{p} B_{ij}(t) \cdot w_i^{(1)}(t) = \sum_{i=1}^{p} B_{ij}(t) \cdot w_i^{(0)}(t) \geq a_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}(s,t) \cdot w_i^{(1)}(s)ds.$$

- Suppose that $I_> \neq \emptyset$ and $B_{ij}(t) = 0$ for all $i \in I_>$. Then

$$\begin{aligned}\sum_{i=1}^{p} B_{ij}(t) \cdot w_i^{(1)}(t) &= \sum_{i \in I_\leq} B_{ij}(t) \cdot w_i^{(1)}(t) + \sum_{i \in I_>} B_{ij}(t) \cdot w_i^{(1)}(t) \\ &= \sum_{i \in I_\leq} B_{ij}(t) \cdot w_i^{(0)}(t) + \sum_{i \in I_>} B_{ij}(t) \cdot w_i^{(0)}(t) \\ &= \sum_{i=1}^{p} B_{ij}(t) \cdot w_i^{(0)}(t) \geq a_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}(s,t) \cdot w_i^{(1)}(s)ds \text{ (by (5.4))}.\end{aligned}$$

- Suppose that $I_> \neq \emptyset$ and there exists $i^* \in I_>$ with $B_{i^*j}(t) \neq 0$, i.e., $B_{i^*j}(t) \geq \sigma$. From (5.1), we see that

$$\sigma \cdot \eta(t) = \tau + \nu \cdot \int_t^T \eta(s)ds \tag{5.5}$$

for all $t \in [0,T]$. By (5.5), for each $t \in [0,T]$, we have

$$\sigma \cdot \eta(t) \geq a_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}(s,t) \cdot \eta(s)ds. \tag{5.6}$$

Therefore we obtain

$$\sum_{i=1}^{p} B_{ij}(t) \cdot w_i^{(1)}(t) \geq \sum_{i \in I_>} B_{ij}(t) \cdot w_i^{(1)}(t) = \sum_{i \in I_>} B_{ij}(t) \cdot \eta(t) \geq \sigma \cdot \eta(t). \tag{5.7}$$

Using (5.6), (5.7) and the fact of $w_i^{(1)}(t) \leq \eta(t)$, we also have

$$\sum_{i=1}^{p} B_{ij}(t) \cdot w_i^{(1)}(t) \geq a_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}(s,t) \cdot \eta(s)ds \geq a_j(t) + \sum_{i=1}^{p} \int_t^T K_{ij}(s,t) \cdot w_i^{(1)}(s)ds.$$

This concludes that $\mathbf{w}^{(1)}$ is a feasible solution of (DCLP), and the proof is complete. ■

We also need the following useful lemmas.

Lemma 5.2. (Riesz and Sz.-Nagy [25, p.64]) *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $L^2[0,T]$. If the sequence $\{f_k\}_{k=1}^{\infty}$ is uniformly bounded with respect to $\|\cdot\|_2$, then exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ that weakly converges to $f \in L^2[0,T]$. In other words, for any $g \in L^2[0,T]$, we have*

$$\lim_{j\to\infty} \int_0^T f_{k_j}(t)g(t)dt = \int_0^T f(t)g(t)dt.$$

Lemma 5.3. (Levinson [12]) *If the sequence* $\{f_k\}_{k=1}^{\infty}$ *is uniformly bounded on* $[0,T]$ *with respect to* $\|\cdot\|_2$ *and weakly converges to* $f \in L^2[0,T]$*, then*

$$f(t) \leq \limsup_{k\to\infty} f_k(t) \text{ and } f(t) \geq \liminf_{k\to\infty} f_k(t) \text{ a.e. on } [0,T].$$

Lemma 5.4. *Let* $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty}$ *be two sequences in* $L^2[0,T]$ *that weakly converge to* f_0 *and* g_0 *in* $L^2[0,T]$*, respectively.*

(i) *If the function* η *defined on* $[0,T]$ *is bounded, then the sequence* $\{\eta\cdot f_k\}_{k=1}^{\infty}$ *weakly converges to* $\eta\cdot f_0$*.*

(ii) *The sequence* $\{f_k+g_k\}_{k=1}^{\infty}$ *weakly converges to* f_0+g_0*.*

Proof. To prove part (i), for any $h \in L^2[0,T]$, we see that $h\cdot\eta \in L^2[0,T]$. Therefore the weak convergence says that

$$\lim_{k\to\infty}\int_0^T h\cdot(\eta\cdot f_k)dt = \lim_{k\to\infty}\int_0^T (h\cdot\eta)\cdot f_k dt = \int_0^T (h\cdot\eta)\cdot f_0 dt = \int_0^T h\cdot(\eta\cdot f_0)dt,$$

which says that the sequence $\{\eta\cdot f_k\}_{k=1}^{\infty}$ weakly converges to $\eta\cdot f_0$.

To prove part (ii), for any $h\in L^2[0,T]$, we have

$$\begin{aligned}
\lim_{k\to\infty}\int_0^T (f_k(t)+g_k(t))\cdot h(t)dt &= \lim_{k\to\infty}\left(\int_0^T f_k(t)\cdot h(t)dt + \int_0^T g_k(t)\cdot h(t)dt\right)\\
&= \int_0^T f_0(t)\cdot h(t)dt + \int_0^T g_0(t)\cdot h(t)dt\\
&\quad \text{(by the weak convergence for the sequences } \{f_k\}_{k=1}^{\infty} \text{ and } \{g_k\}_{k=1}^{\infty})\\
&= \int_0^T (f_0(t)+g_0(t))\cdot h(t)dt.
\end{aligned}$$

This completes the proof. ■

Let $\{\mathbf{f}^{(n)}\}_{n=1}^{\infty}$ be a sequence of vector-valued functions in $L^2[0,T]$. Then we say that $\{\mathbf{f}^{(n)}\}_{n=1}^{\infty}$ weakly converges to a vector-valued function $\mathbf{f}$ in $L^2[0,T]$ if and only if each sequence $\{f_j^{(n)}\}_{n=1}^{\infty}$ of component functions weakly converges to f_j.

Theorem 5.1. (**Strong Duality Theorem**) *Let* $\{\widehat{\mathbf{z}}^{(n)}\}_{n=1}^{\infty}$ *and* $\{\widehat{\mathbf{w}}^{(n)}\}_{n=1}^{\infty}$ *be the sequences that are constructed from the optimal solutions* $\bar{\mathbf{z}}^{(n)}$ *of problem* (P$_n$) *and the optimal solution* $\bar{\mathbf{w}}^{(n)}$ *of problem* (D$_n$) *according to* (3.25) *and* (4.25)*, respectively. Then the following statements hold true.*

(i) *There is a subsequence* $\{\widehat{\mathbf{z}}^{(n_k)}\}_{k=1}^{\infty}$ *of* $\{\widehat{\mathbf{z}}^{(n)}\}_{n=1}^{\infty}$ *such that* $\{\widehat{\mathbf{z}}^{(n_k)}\}_{k=1}^{\infty}$ *is weakly convergent to an optimal solution* $\widehat{\mathbf{z}}^*$ *of primal problem* (CLP).

(ii) *For each n, we define*

$$\breve{w}_i^{(n)}(t) = \min\left\{\widehat{w}_i^{(n)}(t), \eta(t)\right\}.$$

Then there is a subsequence $\{\breve{\mathbf{w}}^{(n_k)}\}_{k=1}^{\infty}$ *of* $\{\breve{\mathbf{w}}^{(n)}\}_{n=1}^{\infty}$ *such that* $\{\breve{\mathbf{w}}^{(n_k)}\}_{k=1}^{\infty}$ *is weakly convergent to an optimal solution* $\widehat{\mathbf{w}}^*$ *of dual problem* (DCLP).

Moreover we have $V(\text{DCLP}) = V(\text{CLP})$.

Proof. From Proposition 3.2, we see that the sequence of functions $\{\widehat{\mathbf{z}}^{(n)}\}_{n=1}^{\infty}$ is uniformly bounded with respect to $\|\cdot\|_2$. We denote by $\widehat{z}_j^{(n)}$ the jth component of $\widehat{\mathbf{z}}^{(n)}$. Using Lemma 5.2, there exists a subsequence $\left\{\widehat{z}_1^{(n_k^{(1)})}\right\}_{k=1}^{\infty}$ of $\left\{\widehat{z}_1^{(n)}\right\}_{n=1}^{\infty}$ that weakly converges to some $\widehat{z}_1^{(0)} \in L^2[0,T]$. Using Lemma 5.2 again, there exists a subsequence $\left\{\widehat{z}_2^{(n_k^{(2)})}\right\}_{k=1}^{\infty}$ of $\left\{\widehat{z}_2^{(n_k^{(1)})}\right\}_{k=1}^{\infty}$ that weakly converges to some $\widehat{z}_2^{(0)} \in L^2[0,T]$. By induction, there exists a subsequence $\left\{\widehat{z}_j^{(n_k^{(j)})}\right\}_{k=1}^{\infty}$ of $\left\{\widehat{z}_j^{(n_k^{(j-1)})}\right\}_{k=1}^{\infty}$ that weakly converges to some $\widehat{z}_j^{(0)} \in L^2[0,T]$ for each j. Therefore we can construct a subsequence $\{\widehat{\mathbf{z}}^{(n_k)}\}_{k=1}^{\infty}$ that weakly converges to $\widehat{\mathbf{z}}^{(0)}$. Since $\widehat{\mathbf{z}}^{(n_k)}$ is a feasible solution of problem (CLP) for each n_k, we have

$$\widehat{\mathbf{z}}^{(n_k)}(t) \geq \mathbf{0} \text{ and } B(t)\widehat{\mathbf{z}}^{(n_k)}(t) \leq \mathbf{c}(t) + \int_0^t K(t,s)\widehat{\mathbf{z}}^{(n_k)}(s)ds \text{ for all } t \in [0,T]. \tag{5.8}$$

Using Lemma 5.3 and (5.8), for each j, we have

$$\widehat{z}_j^{(0)}(t) \geq \liminf_{k\to\infty} \widehat{z}_j^{(n_k)}(t) \geq 0 \text{ a.e. in } [0,T],$$

which says that $\widehat{\mathbf{z}}^{(0)}(t) \geq \mathbf{0}$ a.e. in $[0,T]$. Using Lemma 5.4, it is clear to see that the sequence $\{\sum_{j=1}^{q} B_{ij}(t)\cdot\widehat{z}_j^{(n_k)}\}_{k=1}^{\infty}$ weakly converges to $\sum_{j=1}^{q} B_{ij}(t)\cdot\widehat{z}_j^{(0)}$ for $i = 1,\cdots,p$. Therefore we obtain

$$\begin{aligned}
B(t)\widehat{\mathbf{z}}^{(0)}(t) &\leq \limsup_{k\to\infty} B(t)\widehat{\mathbf{z}}^{(n_k)}(t) \text{ (using Lemma 5.3)}\\
&\leq \mathbf{c}(t) + \limsup_{k\to\infty} \int_0^t K(t,s)\widehat{\mathbf{z}}^{(n_k)}(s)ds \text{ (by (5.8))}\\
&= \mathbf{c}(t) + \int_0^t K(t,s)\widehat{\mathbf{z}}^{(0)}(s)ds \text{ a.e. in } [0,T] \text{ (by the weak convergence)}
\end{aligned} \tag{5.9}$$

Let $\mathcal{N}_0$ be the subset of $[0,T]$ on which the inequality (5.9) is violated, let $\mathcal{N}_1$ be the subset of $[0,T]$ on which $\widehat{\mathbf{z}}^{(0)}(t) \not\geq \mathbf{0}$, let $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1$, and define

$$\widehat{\mathbf{z}}^*(t) = \begin{cases} \widehat{\mathbf{z}}^{(0)}(t) & \text{if } t \notin \mathcal{N} \\ \mathbf{0} & \text{if } t \in \mathcal{N}, \end{cases}$$

where the set $\mathcal{N}$ has measure zero. We see that $\widehat{\mathbf{z}}^*(t) \geq \mathbf{0}$ for all $t \in [0,T]$ and $\widehat{\mathbf{z}}^*(t) = \widehat{\mathbf{z}}^{(0)}(t)$ a.e. on $[0,T]$.

- For $t \notin \mathcal{N}$, from (5.9), we have

$$B(t)\widehat{\mathbf{z}}^*(t) = B(t)\widehat{\mathbf{z}}^{(0)}(t) \leq \mathbf{c}(t) + \int_0^t K(t,s)\widehat{\mathbf{z}}^{(0)}(s)ds = \mathbf{c}(t) + \int_0^t K(t,s)\widehat{\mathbf{z}}^*(s)ds.$$

- For $t \in \mathcal{N}$, we have

$$B(t)\widehat{\mathbf{z}}^*(t) = \mathbf{0} \leq \mathbf{c}(t) + \int_0^t K(t,s)\widehat{\mathbf{z}}^{(0)}(s)ds = \mathbf{c}(t) + \int_0^t K(t,s)\widehat{\mathbf{z}}^*(s)ds.$$

This shows that $\widehat{\mathbf{z}}^*$ is a feasible solution of problem (CLP). Since $\widehat{\mathbf{z}}^*(t) = \widehat{\mathbf{z}}^{(0)}(t)$ a.e. on $[0,T]$, it follows that the subsequence $\{\widehat{\mathbf{z}}^{(n_k)}\}_{k=1}^\infty$ is also weakly convergent to $\widehat{\mathbf{z}}^*$.

On the other hand, since $\widehat{\mathbf{w}}^{(n)}$ is a feasible solution of problem (DCLP) for each n, Lemma 5.1 says that $\breve{\mathbf{w}}^{(n)}$ is also a feasible solution of problem (DCLP) for each n satisfying $\breve{w}_i^{(n)}(t) \leq \widehat{w}_i^{(n)}(t)$ for each $i = 1, \cdots, p$ and $t \in [0,T]$. From Remark 4.2, it follows that the sequence $\{\widehat{\mathbf{w}}^{(n)}\}_{n=1}^\infty$ is uniformly bounded. Since

$$\eta(t) = \frac{\tau}{\sigma} \cdot \exp\left[\frac{\nu \cdot (T-t)}{\sigma}\right] \leq \frac{\tau}{\sigma} \cdot \exp\left(\frac{\nu \cdot T}{\sigma}\right) \text{ for all } t \in [0,T],$$

we see that the sequence $\{\breve{\mathbf{w}}^{(n)}\}_{n=1}^\infty$ is also uniformly bounded, which implies that the sequence $\{\breve{\mathbf{w}}^{(n)}\}_{n=1}^\infty$ is uniformly bounded with respect to $\|\cdot\|_2$. Using Lemma 5.2, we can similarly show that there is a subsequence $\{\breve{\mathbf{w}}^{(n_k)}\}_{k=1}^\infty$ of $\{\breve{\mathbf{w}}^{(n)}\}_{n=1}^\infty$ that weakly converges to some $\breve{\mathbf{w}}^{(0)}$. Since $\breve{\mathbf{w}}^{(n_k)}$ is a feasible solution of problem (DCLP) for each n_k, we have

$$\breve{\mathbf{w}}^{(n_k)}(t) \geq \mathbf{0} \text{ and } (B(t))^\top \breve{\mathbf{w}}^{(n_k)}(t) \geq \mathbf{a}(t) + \int_t^T (K(s,t))^\top \breve{\mathbf{w}}^{(n_k)}(s)ds \text{ for all } t \in [0,T]. \tag{5.10}$$

Using Lemma 5.3 and , for each i, we have

$$\breve{w}_i^{(0)}(t) \geq \liminf_{k\to\infty} \breve{w}_i^{(n_k)}(t) \geq 0 \text{ a.e. in } [0,T],$$

which says that $\breve{\mathbf{w}}^{(0)}(t) \geq \mathbf{0}$ a.e. in $[0,T]$. Using Lemma 5.4, it is clear to see that the sequence $\{\sum_{i=1}^p B_{ij}(t) \cdot \breve{w}_i^{(n_k)}\}_{k=1}^\infty$ weakly converges to $\sum_{i=1}^p B_{ij}(t) \cdot \breve{w}_i^{(0)}$ for $j = 1, \cdots, q$. Therefore we obtain

$$\begin{aligned}
(B(t))^\top \breve{\mathbf{w}}^{(0)}(t) &\geq \liminf_{k\to\infty} (B(t))^\top \breve{\mathbf{w}}^{(n_k)}(t) \text{ (using Lemma 5.3)} \\
&\geq \mathbf{a}(t) + \liminf_{k\to\infty} \int_t^T (K(s,t))^\top \breve{\mathbf{w}}^{(n_k)}(s)ds \text{ (by (5.10))} \\
&= \mathbf{a}(t) + \int_t^T (K(s,t))^\top \breve{\mathbf{w}}^{(0)}(s)ds \text{ a.e. in } [0,T] \text{ (by the weak convergence)}
\end{aligned} \tag{5.11}$$

We define $\boldsymbol{\eta}(t) = \eta(t)\mathbf{1}_p$. Then we see that $\breve{\mathbf{w}}^{(n_k)}(t) \leq \boldsymbol{\eta}(t)$ for each k and for all $t \in [0,T]$. Let $\widehat{\mathcal{N}}_0$ be the subset of $[0,T]$ on which the inequality (5.11) is violated, let $\widehat{\mathcal{N}}_1$ be the subset of $[0,T]$ on which $\breve{\mathbf{w}}^{(0)}(t) \ngeq \mathbf{0}$, let $\widehat{\mathcal{N}} = \widehat{\mathcal{N}}_0 \cup \widehat{\mathcal{N}}_1$, and define

$$\widehat{\mathbf{w}}^*(t) = \begin{cases} \breve{\mathbf{w}}^{(0)}(t) & \text{if } t \notin \widehat{\mathcal{N}} \\ \boldsymbol{\eta}(t) & \text{if } t \in \widehat{\mathcal{N}}, \end{cases}$$

where the set $\widehat{\mathcal{N}}$ has measure zero. Then we see that $\widehat{\mathbf{w}}^*(t) \geq \mathbf{0}$ for all $t \in [0,T]$ and $\widehat{\mathbf{w}}^*(t) = \breve{\mathbf{w}}^{(0)}(t)$ a.e. on $[0,T]$. Now, we are going to claim that $\widehat{\mathbf{w}}^*$ is a feasible solution of (DCLP).

- Suppose that $t \notin \widehat{\mathcal{N}}$. From (5.11), we have

$$\begin{aligned}
(B(t))^\top \widehat{\mathbf{w}}^*(t) &= (B(t))^\top \breve{\mathbf{w}}^{(0)}(t) \\
&\geq \mathbf{a}(t) + \int_t^T (K(s,t))^\top \breve{\mathbf{w}}^{(0)}(s)ds = \mathbf{a}(t) + \int_t^T (K(s,t))^\top \widehat{\mathbf{w}}^*(s)ds.
\end{aligned}$$

- Suppose that $t \in \widehat{\mathcal{N}}$. Since $\breve{\mathbf{w}}^{(n_k)}(t) \leq \boldsymbol{\eta}(t)$ for all $t \in [0, T]$, using Lemma 5.3, we have

$$\breve{\mathbf{w}}^{(0)}(t) \leq \limsup_{k\to\infty} \breve{\mathbf{w}}^{(n_k)}(t) \leq \boldsymbol{\eta}(t) \text{ a.e. on } [0, T].$$

Since $\widehat{\mathbf{w}}^*(t) = \breve{\mathbf{w}}^{(0)}(t)$ a.e. on $[0, T]$, it follows that

$$\widehat{\mathbf{w}}^*(t) \leq \boldsymbol{\eta}(t) \text{ a.e. on } [0, T]. \tag{5.12}$$

Therefore we obtain

$$\begin{aligned}(B(t))^\top \widehat{\mathbf{w}}^*(t) = (B(t))^\top \boldsymbol{\eta}(t) &\geq \mathbf{a}(t) + \int_t^T (K(s,t))^\top \boldsymbol{\eta}(s)ds \text{ (by (5.6))}\\ &\geq \mathbf{a}(t) + \int_t^T (K(s,t))^\top \widehat{\mathbf{w}}^*(s)ds \text{ (by (5.12))}.\end{aligned}$$

Therefore we conclude that $\widehat{\mathbf{w}}^*$ is a feasible solution of (DCLP). Since $\widehat{\mathbf{w}}^*(t) = \breve{\mathbf{w}}^{(0)}(t)$ a.e. on $[0, T]$, it follows that the subsequence $\{\breve{\mathbf{w}}^{(n_k)}\}_{k=1}^{\infty}$ is also weakly convergent to $\widehat{\mathbf{w}}^*$.

Finally, we want to prove the optimality. Now, we have

$$\begin{aligned}\int_0^T (\mathbf{a}(t))^\top \widehat{\mathbf{z}}^{(n_k)}(t)dt &= \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left[\mathbf{a}(t) - \mathbf{a}_l^{(n_k)}\right]^\top \widehat{\mathbf{z}}^{(n_k)}(t)dt + \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left(\mathbf{a}_l^{(n_k)}\right)^\top \widehat{\mathbf{z}}^{(n_k)}(t)dt\\ &= \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left[\mathbf{a}(t) - \mathbf{a}_l^{(n_k)}\right]^\top \bar{\mathbf{z}}_l^{(n_k)}dt + \sum_{l=1}^{n_k} \mathfrak{d}_l^{(n_k)} \left(\mathbf{a}_l^{(n_k)}\right)^\top \bar{\mathbf{z}}_l^{(n_k)} \text{ (by Remark 4.1)}\\ &= \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left[\mathbf{a}(t) - \mathbf{a}_l^{(n_k)}\right]^\top \bar{\mathbf{z}}_l^{(n_k)}dt + V(\mathrm{P}_{n_k})\end{aligned}$$

and

$$\begin{aligned}\int_0^T (\mathbf{c}(t))^\top \widehat{\mathbf{w}}^{(n_k)}(t)dt &= \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} (\mathbf{c}(t))^\top \bar{\mathbf{w}}_l^{(n_k)}dt + \int_0^T (\mathbf{c}(t))^\top \mathfrak{f}^{(n_k)}(t)\mathbf{1}_p dt\\ &= \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left[\mathbf{c}(t) - \mathbf{c}_l^{(n_k)}\right]^\top \bar{\mathbf{w}}_l^{(n_k)}dt + \sum_{l=1}^{n_k} \mathfrak{d}_l^{(n_k)} \left(\mathbf{c}_l^{(n_k)}\right)^\top \bar{\mathbf{w}}_l^{(n_k)} + \int_0^T (\mathbf{c}(t))^\top \mathfrak{f}^{(n_k)}(t)\mathbf{1}_p dt\\ &= \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left[\mathbf{c}(t) - \mathbf{c}_l^{(n_k)}\right]^\top \bar{\mathbf{w}}_l^{(n_k)}dt + V(\mathrm{D}_{n_k}) + \int_0^T (\mathbf{c}(t))^\top \mathfrak{f}^{(n_k)}(t)\mathbf{1}_p dt\end{aligned}$$

which imply

$$V(\mathrm{P}_{n_k}) = \int_0^T (\mathbf{a}(t))^\top \widehat{\mathbf{z}}^{(n_k)}(t)dt - \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left[\mathbf{a}(t) - \mathbf{a}_l^{(n_k)}\right]^\top \bar{\mathbf{z}}_l^{(n_k)}dt. \tag{5.13}$$

and

$$V(\mathrm{D}_{n_k}) = \int_0^T (\mathbf{c}(t))^\top \widehat{\mathbf{w}}^{(n_k)}(t)dt - \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left[\mathbf{c}(t) - \mathbf{c}_l^{(n_k)}\right]^\top \bar{\mathbf{w}}_l^{(n_k)}dt$$

$$-\int_0^T (\mathbf{c}(t))^\top \mathfrak{f}^{(n_k)}(t)\mathbf{1}_p dt. \tag{5.14}$$

Since $V(\mathrm{P}_{n_k}) = V(\mathrm{D}_{n_k})$ and $\breve{\mathbf{w}}^{(n_k)} \leq \widehat{\mathbf{w}}^{(n_k)}$ for each n_k, from (5.13) and (5.14), we have

$$\begin{aligned}&\int_0^T (\mathbf{a}(t))^\top \widehat{\mathbf{z}}^{(n_k)}(t)dt - \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left[\mathbf{a}(t) - \mathbf{a}_l^{(n_k)}\right]^\top \bar{\mathbf{z}}_l^{(n_k)} dt \\ &\quad \geq \int_0^T (\mathbf{c}(t))^\top \breve{\mathbf{w}}^{(n_k)}(t)dt - \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left[\mathbf{c}(t) - \mathbf{c}_l^{(n_k)}\right]^\top \bar{\mathbf{w}}_l^{(n_k)} dt - \int_0^T (\mathbf{c}(t))^\top \mathfrak{f}^{(n_k)}(t)\mathbf{1}_p dt \end{aligned} \tag{5.15}$$

Using Lemma 4.5, we have

$$\begin{aligned}0 &\leq \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left[a_j(t) - a_{lj}^{(n_k)}\right] \cdot \bar{z}_{lj}^{(n_k)} dt \\ &= \int_0^T \left[a_j(t) - \bar{a}_j^{(n_k)}(t)\right] \cdot \widehat{z}_j^{(n_k)}(t)dt \to 0 \text{ as } k \to \infty \end{aligned} \tag{5.16}$$

and

$$\begin{aligned}0 &\leq \sum_{l=1}^{n_k} \int_{\bar{E}_l^{(n_k)}} \left[c_i(t) - c_{li}^{(n_k)}\right] \cdot \bar{w}_{li}^{(n_k)} dt \\ &= \int_0^T \left[c_i(t) - \bar{c}_i^{(n_k)}(t)\right] \cdot \bar{w}_i^{(n_k)}(t)dt \to 0 \text{ as } k \to \infty. \end{aligned} \tag{5.17}$$

By taking limit on both sides of (5.15), and using (4.36), (5.16) and (5.17), we obtain

$$\lim_{k\to\infty} \int_0^T (\mathbf{a}(t))^\top \widehat{\mathbf{z}}^{(n_k)}(t)dt \geq \lim_{k\to\infty} \int_0^T (\mathbf{c}(t))^\top \breve{\mathbf{w}}^{(n_k)}(t)dt.$$

Using the weak convergence, we also obtain

$$\int_0^T (\mathbf{a}(t))^\top \widehat{\mathbf{z}}^*(t)dt \geq \int_0^T (\mathbf{c}(t))^\top \widehat{\mathbf{w}}^*(t)dt.$$

According to the weak duality theorem between problems (CLP) and (DCLP), we have that

$$\int_0^T (\mathbf{a}(t))^\top \widehat{\mathbf{z}}^*(t)dt = \int_0^T (\mathbf{c}(t))^\top \widehat{\mathbf{w}}^*(t)dt,$$

and conclude that $\widehat{\mathbf{z}}^*$ and $\widehat{\mathbf{w}}^*$ are the optimal solutions of problems (CLP) and (DCLP), respectively. Theorem 4.1 also says that $V(\mathrm{DCLP}) = V(\mathrm{CLP})$. This completes the proof. ■

6. Computational procedure and numerical example

In the sequel, we are going to provide the computational procedure to obtain the approximate solutions of the continuous-time linear programming problem (CLP). Of course, the approximate solutions will be the step functions. According to Proposition 4.2, it is possible to obtain the appropriate step functions so that the corresponding objective function value is close enough to the optimal objective function value when n is taken to be sufficiently large.

Recall that, from Theorem 4.1 and Proposition 4.2, the error between the approximate objective value and the optimal objective value is given by

$$\varepsilon_n = -V(\mathrm{D}_n) + \sum_{l=1}^{n}\left[\int_{\bar{E}_l^{(n)}} (\mathbf{c}(t))^\top \bar{\mathbf{w}}_l^{(n)} dt + \int_{\bar{E}_l^{(n)}} \frac{\pi_l^{(n)}}{\mathfrak{b}_l^{(n)}} \cdot \exp\left[\frac{\mathfrak{k}_l^{(n)} \cdot (T-t)}{\mathfrak{b}_l^{(n)}}\right] (\mathbf{c}(t))^\top \mathbf{1}_p dt\right].$$

We shall propose two kinds of computational procedure. One needs to solve the dual problem (D_n) for many times in order to attain the pre-determined error, and another one does not need to solve the dual problem (D_n).

6.1. Computational procedure without solving (D_n)

We first rewrite the error ε_n and $\bar{\mathbf{h}}_l^{(n)}$ as follows:

$$\begin{aligned}\varepsilon_n = &-\sum_{l=1}^{n}\sum_{i=1}^{p} \mathfrak{d}_l^{(n)} c_{li}^{(n)} \cdot \bar{w}_{li}^{(n)} + \sum_{l=1}^{n}\sum_{i=1}^{p} \bar{w}_{li}^{(n)} \cdot \int_{\bar{E}_l^{(n)}} c_i(t)dt \\ &+ \sum_{l=1}^{n}\sum_{i=1}^{p} \int_{\bar{E}_l^{(n)}} \frac{\pi_l^{(n)}}{\mathfrak{b}_l^{(n)}} \cdot \exp\left[\frac{\mathfrak{k}_l^{(n)} \cdot (T-t)}{\mathfrak{b}_l^{(n)}}\right] \cdot c_i(t)dt\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}\bar{\mathbf{h}}_l^{(n)}(t) = &\left(B_l^{(n)} - B(t)\right)^\top \bar{\mathbf{w}}_l^{(n)} + \int_t^{e_l^{(n)}} (K(s,t))^\top \bar{\mathbf{w}}_l^{(n)} ds \\ &+ \sum_{k=l+1}^{n} \int_{\bar{E}_k^{(n)}} \left(K(s,t) - K_{kl}^{(n)}\right)^\top \bar{\mathbf{w}}_k^{(n)} ds.\end{aligned} \tag{6.2}$$

We define

$$\begin{aligned}\mathfrak{z}_l^{(n)} = &\sum_{i=1}^{p} \mathfrak{w}_l^{(n)} \cdot \left[\max_{j=1,\cdots,q} \sup_{t\in\bar{E}_l^{(n)}} \left(B_{lij}^{(n)} - B_{ij}(t)\right)\right] + \sum_{i=1}^{p} \mathfrak{w}_l^{(n)} \cdot \left[\max_{j=1,\cdots,q} \sup_{t\in\bar{E}_l^{(n)}} \int_t^{e_l^{(n)}} K_{ij}(s,t)ds\right] \\ &+ \sum_{k=l+1}^{n}\sum_{i=1}^{p} \mathfrak{w}_k^{(n)} \cdot \max_{j=1,\cdots,q} \sup_{t\in\bar{E}_l^{(n)}} \int_{\bar{E}_k^{(n)}} \left(K_{ij}(s,t) - K_{klij}^{(n)}\right) ds + \max_{j=1,\cdots,q} \sup_{t\in\bar{E}_l^{(n)}} \left[a_j(t) - a_{lj}^{(n)}\right],\end{aligned}$$

where $\mathfrak{w}_l^{(n)}$ is given by (3.15). Using (4.7) and (6.2), we have

$$\bar{\pi}_l^{(n)} \leq \sum_{i=1}^{p} \bar{w}_{li}^{(n)} \cdot \left[\max_{j=1,\cdots,q} \sup_{t\in\bar{E}_l^{(n)}} \left(B_{lij}^{(n)} - B_{ij}(t)\right)\right] + \sum_{i=1}^{p} \max_{j=1,\cdots,q} \sup_{t\in\bar{E}_l^{(n)}} \int_t^{e_l^{(n)}} K_{ij}(s,t) \cdot \bar{w}_{li}^{(n)} ds$$

$$+\sum_{k=l+1}^{n}\sum_{i=1}^{p}\max_{j=1,\cdots,q}\sup_{t\in\bar{E}_l^{(n)}}\int_{\bar{E}_k^{(n)}}\left(K_{ij}(s,t)-K_{klij}^{(n)}\right)\cdot\bar{w}_{ki}^{(n)}ds+\max_{j=1,\cdots,q}\sup_{t\in\bar{E}_l^{(n)}}\left[a_j(t)-a_{lj}^{(n)}\right]$$
$$\leq\mathfrak{z}_l^{(n)}\text{ (by (4.4) and the fact of } K_{ij}(s,t)-K_{klij}^{(n)}\geq 0\text{ by (3.5)).}$$

By (4.8), since

$$\pi_l^{(n)}=\max\left\{\bar{\pi}_l^{(n)},\bar{\pi}_{l+1}^{(n)},\cdots,\bar{\pi}_n^{(n)}\right\},$$

it follows that

$$\pi_l^{(n)}\leq\max_{k=l,\cdots,n}\mathfrak{z}_k^{(n)}.\tag{6.3}$$

For any fixed $t\in\bar{E}_l^{(n)}$, let

$$\sup_{s\in\bar{E}_k^{(n)}}K_{ij}(s,t)=\hat{v}_{klij}^{(n)}(t).$$

We also define

$$\widehat{\mathfrak{z}}_l^{(n)}=\sum_{i=1}^{p}\mathfrak{w}_l^{(n)}\cdot\left[\max_{j=1,\cdots,q}\sup_{t\in\bar{E}_l^{(n)}}\left(B_{lij}^{(n)}-B_{ij}(t)\right)\right]+\sum_{i=1}^{p}\mathfrak{w}_l^{(n)}\cdot\mathfrak{d}_l^{(n)}\cdot\max_{j=1,\cdots,q}\sup_{(s,t)\in\bar{E}_l^{(n)}\times\bar{E}_l^{(n)}}K_{ij}(s,t)$$
$$+\sum_{k=l+1}^{n}\sum_{i=1}^{p}\mathfrak{w}_k^{(n)}\cdot\mathfrak{d}_l^{(n)}\cdot\left[\max_{j=1,\cdots,q}\sup_{(s,t)\in\bar{E}_k^{(n)}\times\bar{E}_l^{(n)}}\left(K_{ij}(s,t)-K_{klij}^{(n)}\right)\right]+\max_{j=1,\cdots,q}\sup_{t\in\bar{E}_l^{(n)}}\left[a_j(t)-a_{lj}^{(n)}\right]$$

From Lemmas 4.1 and 4.2, we see that $\widehat{\mathfrak{z}}_l^{(n)}\to 0$ as $n\to\infty$. Then we have

$$\mathfrak{z}_l^{(n)}\leq\sum_{i=1}^{p}\mathfrak{w}_l^{(n)}\cdot\left[\max_{j=1,\cdots,q}\sup_{t\in\bar{E}_l^{(n)}}\left(B_{lij}^{(n)}-B_{ij}(t)\right)\right]+\sum_{i=1}^{p}\mathfrak{w}_l^{(n)}\cdot\max_{j=1,\cdots,q}\sup_{t\in\bar{E}_l^{(n)}}\left[\left(e_l^{(n)}-t\right)\cdot\hat{v}_{llij}^{(n)}(t)\right]$$
$$+\sum_{k=l+1}^{n}\sum_{i=1}^{p}\mathfrak{w}_k^{(n)}\cdot\left[\max_{j=1,\cdots,q}\sup_{t\in\bar{E}_l^{(n)}}\mathfrak{d}_l^{(n)}\cdot\left(\hat{v}_{klij}^{(n)}(t)-K_{klij}^{(n)}\right)\right]+\max_{j=1,\cdots,q}\sup_{t\in\bar{E}_l^{(n)}}\left[a_j(t)-a_{lj}^{(n)}\right]$$
$$\leq\widehat{\mathfrak{z}}_l^{(n)},\tag{6.4}$$

Let

$$\widehat{\pi}_l^{(n)}=\max_{k=l,\cdots,n}\widehat{\mathfrak{z}}_k^{(n)}\text{ and }\bar{K}_{klij}^{(n)}=\sup_{(s,t)\in\bar{E}_k^{(n)}\times\bar{E}_l^{(n)}}K_{ij}(s,t).$$

Using (6.3) and (6.4), we obtain

$$\pi_l^{(n)}\leq\widehat{\pi}_l^{(n)}.\tag{6.5}$$

Let

$$\bar{B}_{lij}^{(n)}=\inf_{t\in\bar{E}_l^{(n)}}B_{ij}(t).$$

We also have

$$\widehat{\mathfrak{z}}_l^{(n)}=\mathfrak{w}_l^{(n)}\cdot\sum_{i=1}^{p}\left[\max_{j=1,\cdots,q}\left(B_{lij}^{(n)}-\bar{B}_{lij}^{(n)}\right)\right]+\mathfrak{w}_l^{(n)}\cdot\mathfrak{d}_l^{(n)}\cdot\sum_{i=1}^{p}\left(\max_{j=1,\cdots,q}K_{llij}^{(\bar{n})}\right)$$

$$+\mathfrak{d}_l^{(n)}\cdot\sum_{k=l+1}^{n}\left\{\mathfrak{w}_k^{(n)}\cdot\left[\sum_{i=1}^{p}\max_{j=1,\cdots,q}\left(\bar{K}_{klij}^{(n)}-K_{klij}^{(n)}\right)\right]\right\}+\max_{j=1,\cdots,q}\sup_{t\in\bar{E}_l^{(n)}}\left[a_j(t)-a_{lj}^{(n)}\right]$$

We define the real-valued function

$$\widehat{\pi}^{(n)}(t)=\begin{cases}\widehat{\pi}_l^{(n)} & \text{if } t\in F_l^{(n)} \text{ for } l=1,\cdots,n\\ \widehat{\pi}_n^{(n)} & \text{if } t=T.\end{cases}$$

Since $\widehat{\mathfrak{z}}_l^{(n)}\to 0$ as $n\to\infty$, it follows that $\widehat{\pi}_l^{(n)}\to 0$ as $n\to\infty$. Using the Lebesgue bounded convergence theorem for integrals, we have

$$\int_0^T \frac{\widehat{\pi}^{(n)}(t)}{\mathfrak{v}^{(n)}(t)}\cdot\exp\left[\frac{\mathfrak{u}^{(n)}(t)\cdot(T-t)}{\mathfrak{v}^{(n)}(t)}\right]\cdot c_i(t)dt\to 0 \text{ as } n\to\infty \tag{6.6}$$

From (4.4), (6.1) and (6.5), we obtain

$$\begin{aligned}\varepsilon_n &\leq \sum_{l=1}^{n}\sum_{i=1}^{p}\mathfrak{w}_l^{(n)}\cdot\int_{\bar{E}_l^{(n)}}\left(c_i(t)-c_{li}^{(n)}\right)dt+\sum_{l=1}^{n}\sum_{i=1}^{p}\int_{\bar{E}_l^{(n)}}\frac{\widehat{\pi}_l^{(n)}}{\mathfrak{b}_l^{(n)}}\cdot\exp\left[\frac{\mathfrak{k}_l^{(n)}\cdot(T-t)}{\mathfrak{b}_l^{(n)}}\right]\cdot c_i(t)dt \qquad (6.7)\\ &=\sum_{l=1}^{n}\sum_{i=1}^{p}\int_{\bar{E}_l^{(n)}}\left(c_i(t)-c_{li}^{(n)}\right)\cdot\mathfrak{w}_l^{(n)}dt+\sum_{i=1}^{p}\int_0^T\frac{\widehat{\pi}^{(n)}(t)}{\mathfrak{v}^{(n)}(t)}\cdot\exp\left[\frac{\mathfrak{u}^{(n)}(t)\cdot(T-t)}{\mathfrak{v}^{(n)}(t)}\right]\cdot c_i(t)dt\\ &\to 0 \text{ as } n\to\infty \text{ (by (6.6) and the similar argument in the proof of Lemma 4.2)}\end{aligned}$$

For $i=1,\cdots,p$ and $l=1,\cdots,n$, we define

$$\widehat{\varepsilon}_{li}^{(n)}=\mathfrak{w}_l^{(n)}\cdot\left(\int_{\bar{E}_l^{(n)}}c_i(t)dt-\mathfrak{d}_l^{(n)}\cdot c_{li}^{(n)}\right)+\frac{\widehat{\pi}_l^{(n)}}{\mathfrak{b}_l^{(n)}}\cdot\int_{\bar{E}_l^{(n)}}\exp\left[\frac{\mathfrak{k}_l^{(n)}\cdot(T-t)}{\mathfrak{b}_l^{(n)}}\right]\cdot c_i(t)dt \tag{6.8}$$

and

$$\widehat{\varepsilon}_n=\sum_{l=1}^{n}\sum_{i=1}^{p}\widehat{\varepsilon}_{li}^{(n)}.$$

Then, from (6.7), we see that $\varepsilon_n\leq\widehat{\varepsilon}_n$ and $\widehat{\varepsilon}_n\to 0$ as $n\to\infty$. Given a pre-determined error tolerance ϵ, if $\widehat{\varepsilon}_n<\epsilon$, then the error ε_n between the approximate objective function value and the optimal objective function value will be less than ϵ. The numerical integrals in (6.8) can be calculated by applying the Simpson's rule.

In most of cases, the first integral in (6.8) can be obtained analytically, and the second integral in (6.8) cannot be obtained analytically. For $i=1,\cdots,p$ and $l=1,\cdots,n$, let

$$\mathfrak{o}_l^{(n)}=\exp\left[\frac{\mathfrak{k}_l^{(n)}\cdot\left(T-e_{l-1}^{(n)}\right)}{\mathfrak{b}_l^{(n)}}\right]-\exp\left[\frac{\mathfrak{k}_l^{(n)}\cdot\left(T-e_l^{(n)}\right)}{\mathfrak{b}_l^{(n)}}\right].$$

Since

$$\int_{\bar{E}_l^{(n)}}\exp\left[\frac{\mathfrak{k}_l^{(n)}\cdot(T-t)}{\mathfrak{b}_l^{(n)}}\right]dt=\frac{\mathfrak{b}_l^{(n)}}{\mathfrak{k}_l^{(n)}}\cdot\mathfrak{o}_l^{(n)},$$

we obtain

$$\widehat{\varepsilon}_n \leq \sum_{l=1}^{n}\sum_{i=1}^{p} \mathfrak{w}_l^{(n)} \cdot \left(\int_{\bar{E}_l^{(n)}} c_i(t)dt - \mathfrak{d}_l^{(n)} \cdot c_{li}^{(n)}\right) + \sum_{l=1}^{n}\sum_{i=1}^{p} \left(\sup_{t\in\bar{E}_l^{(n)}} c_i(t)\right) \cdot \frac{\widehat{\pi}_l^{(n)}}{\mathfrak{b}_l^{(n)}} \cdot \frac{\mathfrak{b}_l^{(n)}}{\mathfrak{k}_l^{(n)}} \cdot \mathfrak{v}_l^{(n)} \tag{6.9}$$

$$\leq \sum_{l=1}^{n}\sum_{i=1}^{p} \int_{\bar{E}_l^{(n)}} \left(c_i(t) - c_{li}^{(n)}\right) \cdot \mathfrak{w}_l^{(n)} dt$$

$$+ \sum_{i=1}^{p} \left(\sup_{t\in[0,T]} c_i(t)\right) \cdot \int_0^T \frac{\widehat{\pi}^{(n)}(t)}{\mathfrak{v}^{(n)}(t)} \cdot \exp\left[\frac{\mathfrak{u}^{(n)}(t)\cdot(T-t)}{\mathfrak{v}^{(n)}(t)}\right] dt \to 0 \text{ as } n \to \infty$$

For $i = 1, \cdots, p$ and $l = 1, \cdots, n$, we define

$$\bar{c}_{li}^{(n)} = \left(\sup_{t\in\bar{E}_l^{(n)}} c_i(t)\right),$$

$$\widetilde{\varepsilon}_{li}^{(n)} = \mathfrak{w}_l^{(n)} \cdot \left(\int_{\bar{E}_l^{(n)}} c_i(t)dt - \mathfrak{d}_l^{(n)} \cdot c_{li}^{(n)}\right) + \frac{\mathfrak{v}_l^{(n)} \cdot \widehat{\pi}_l^{(n)} \cdot \bar{c}_{li}^{(n)}}{\mathfrak{k}_l^{(n)}}, \tag{6.10}$$

$$\widetilde{\varepsilon}_n = \sum_{l=1}^{n}\sum_{i=1}^{p} \widetilde{\varepsilon}_{li}^{(n)}.$$

Then, from (6.9), we see that $\widetilde{\varepsilon}_n \to 0$ as $n \to \infty$ and

$$\varepsilon_n \leq \widehat{\varepsilon}_n \leq \widetilde{\varepsilon}_n. \tag{6.11}$$

Now the computational procedure is given below.

- **Step 1**. Set the error tolerance ϵ and the initial value of natural number $n \in \mathbb{N}$.
- **Step 2**. Evaluate the values $\widehat{\varepsilon}_{li}^{(n)}$ according to (6.8) (resp. $\widetilde{\varepsilon}_{li}^{(n)}$ according to (6.10)) for $i = 1, \cdots, p$ and $l = 1, \cdots, n$.
- **Step 3**. If $\widehat{\varepsilon}_n < \epsilon$ (resp. $\widetilde{\varepsilon}_n < \epsilon$), then go to Step 4; otherwise, consider one more subdivision for each closed subinterval and set $n \leftarrow n + \widehat{n}$ for some integer $\widehat{n}$ and go to Step 2, where $\widehat{n}$ is the number of new points of subdivisions for all the closed subintervals. For example, in Example 3.1, we can set $n^* \leftarrow n^*+1$ (one more subdivision for closed subinterval is considered). In this case, we have $\widehat{n} = r$ (total r new points of subdivisions for all the closed subintervals). In Example 3.2, we can set $n_\nu \leftarrow n_\nu + 1$ for $\nu = 1, \cdots, r$ (one more subdivision for closed subinterval is considered). In this case, we also have $\widehat{n} = r$ (total r new points of subdivisions for all the closed subintervals).
- **Step 4**. Find the optimal solution $\bar{\mathbf{z}}^{(n)}$ of primal problem (P_n).
- **Step 5**. Set the step function $\widehat{\mathbf{z}}^{(n)}(t)$ defined in (3.25), which will be the approximate solution of problem (CLP). The actual error between V(CLP) and the objective value of $\widehat{\mathbf{z}}^{(n)}(t)$ is less than $\varepsilon_n < \epsilon$ by Proposition 4.2, where the error tolerance ϵ is reached for this partition $\mathcal{P}_n$.

Suppose that the integrals mentioned above can be calculated analytically. Using the similar argument presented above, we can obtain a tighter error bound without considering the supremum of c_i and $K_{ij}(\cdot, t)$ on $E_l^{(n)}$ for any fixed t.

6.2. Computational procedure for solving (D_n)

In order to obtain $\pi_l^{(n)}$, by referring to (4.7), we need to solve

$$\sup_{t\in E_l^{(n)}} \left\{\bar{h}_{lj}^{(n)}(t) + a_j(t)\right\}. \tag{6.12}$$

For $t \in F_l^{(n)}$ and $l = 1, \cdots, n$, we define

$$\widehat{\mathbf{h}}_l^{(n)} = \left(B_l^{(n)}\right)^\top \bar{\mathbf{w}}_l^{(n)} - \sum_{k=l+1}^{n} \int_{\bar{E}_k^{(n)}} \left(K_{kl}^{(n)}\right)^\top \bar{\mathbf{w}}_k^{(n)} ds$$

and

$$\widetilde{\mathbf{h}}_l^{(n)}(t) = -\left(B(t)\right)^\top \bar{\mathbf{w}}_l^{(n)} + \int_t^{e_l^{(n)}} \left(K(s,t)\right)^\top \bar{\mathbf{w}}_l^{(n)} ds + \sum_{k=l+1}^{n} \int_{\bar{E}_k^{(n)}} \left(K(s,t)\right)^\top \bar{\mathbf{w}}_k^{(n)} ds \tag{6.13}$$

Then, from (6.2), we see that the vector-valued function $\bar{\mathbf{h}}_l^{(n)}$ can be rewritten as

$$\bar{\mathbf{h}}_l^{(n)}(t) = \widehat{\mathbf{h}}_l^{(n)} + \widetilde{\mathbf{h}}_l^{(n)}(t) \text{ for } t \in F_l^{(n)}. \tag{6.14}$$

Now we define the real-valued function $h_{lj}^{(n)}$ on $\bar{E}_l^{(n)}$ by

$$h_{lj}^{(n)}(t) = \begin{cases} \bar{h}_{lj}^{(n)}(t) + a_j(t), & \text{if } t \in E_l^{(n)} \\ \lim\limits_{t\to e_{l-1}^{(n)}+} \left(\bar{h}_{lj}^{(n)}(t) + a_j(t)\right), & \text{if } t = e_{l-1}^{(n)} \\ \lim\limits_{t\to e_l^{(n)}-} \left(\bar{h}_{lj}^{(n)}(t) + a_j(t)\right), & \text{if } t = e_l^{(n)} \end{cases}$$

Since $\mathbf{a}$ and B are continuous on $E_l^{(n)}$ and K is continuous on $E_k^{(n)} \times E_l^{(n)}$, respectively, for all $l, k = 1, \cdots, n$, it follows that $\bar{h}_{lj}^{(n)} + a_j$ is also continuous on $E_l^{(n)}$. This also says that $h_{lj}^{(n)}$ is continuous on the compact interval $\bar{E}_l^{(n)}$. In other words, the supremum in (6.12) can be obtained below

$$\sup_{t\in E_l^{(n)}} \left\{\bar{h}_{lj}^{(n)}(t) + a_j(t)\right\} = \sup_{t\in \bar{E}_l^{(n)}} h_{lj}^{(n)}(t) = \max_{t\in \bar{E}_l^{(n)}} h_{lj}^{(n)}(t). \tag{6.15}$$

In order to further design the computational procedure, we need to assume that $\mathbf{a}$, B and K are twice-differentiable on $[0, T]$ and $[0, T]\times[0, T]$, respectively, for the purpose of applying the Newton's method, which also says that $\mathbf{a}$, B and K are twice-differentiable on the open interval $E_l^{(n)}$ and open rectangle $E_k^{(n)} \times E_l^{(n)}$, respectively, for all $l, k = 1, \cdots, n$. From (6.15), we need to solve the following simple type of optimization problem

$$\max_{e_{l-1}^{(n)} \le t \le e_l^{(n)}} h_{lj}^{(n)}(t). \tag{6.16}$$

Then we can see that the optimal solution is

$$t^* = e_{l-1}^{(n)} \text{ or } t^* = e_l^{(n)} \text{ or satisfying } \frac{d}{dt}\left(h_{lj}^{(n)}(t)\right)\Big|_{t=t^*} = 0.$$

According to (6.14), it follows that the optimal solution of problem (6.16) is

$$t^* = e_{l-1}^{(n)} \text{ or } t^* = e_l^{(n)} \text{ or satisfying } \frac{d}{dt}\left(\widetilde{h}_{lj}^{(n)}(t) + a_j(t)\right)\Big|_{t=t^*} = 0.$$

Let $Z_{lj}^{(n)}$ denote the set of all zeros of the real-valued function $\frac{d}{dt}(\widetilde{h}_{lj}^{(n)}(t) + a_j(t))$. Then

$$\max_{t\in\bar{E}_l^{(n)}} h_{lj}^{(n)}(t) = \begin{cases} \max\left\{h_{lj}^{(n)}\left(e_{l-1}^{(n)}\right), h_{lj}^{(n)}\left(e_l^{(n)}\right), \sup\limits_{t^*\in Z_{lj}^{(n)}} h_{lj}^{(n)}(t^*)\right\}, & \text{if } Z_{lj}^{(n)} \neq \emptyset \\ \max\left\{h_{lj}^{(n)}\left(e_{l-1}^{(n)}\right), h_{lj}^{(n)}\left(e_l^{(n)}\right)\right\}, & \text{if } Z_{lj}^{(n)} = \emptyset. \end{cases} \tag{6.17}$$

Therefore, using (6.15) and (6.17), we can obtain the desired supremum (6.12).

From (6.13), the jth entry of $\widetilde{\mathbf{h}}_l^{(n)}$ is given by

$$\widetilde{h}_{lj}^{(n)}(t) = -\sum_{i=1}^{p} B_{ij}(t)\cdot\bar{w}_{li}^{(n)} + \sum_{i=1}^{p}\int_t^{e_l^{(n)}} K_{ij}(s,t)\cdot\bar{w}_{li}^{(n)}ds + \sum_{k=l+1}^{n}\sum_{i=1}^{p}\int_{\bar{E}_k^{(n)}} K_{ij}(s,t)\cdot\bar{w}_{ki}^{(n)}ds. \tag{6.18}$$

Then, for $t \in E_l^{(n)}$, we have

$$\begin{aligned}\frac{d}{dt}\widetilde{h}_{lj}^{(n)}(t) = &-\sum_{i=1}^{p}\frac{d}{dt}B_{ij}(t)\cdot\bar{w}_{li}^{(n)} + \sum_{i=1}^{p}\left[\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}K_{ij}(s,t)\cdot\bar{w}_{li}^{(n)}ds - K_{ij}(t,t)\cdot\bar{w}_{li}^{(n)}\right] \\ &+ \sum_{k=l+1}^{n}\sum_{i=1}^{p}\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}K_{ij}(s,t)\cdot\bar{w}_{ki}^{(n)}ds. \end{aligned} \tag{6.19}$$

and

$$\begin{aligned}\frac{d^2}{dt^2}\widetilde{h}_{lj}^{(n)}(t) = &-\sum_{i=1}^{p}\frac{d^2}{dt^2}B_{ij}(t)\cdot\bar{w}_{li}^{(n)} - \sum_{i=1}^{p}\frac{d}{dt}K_{ij}(t,t)\cdot\bar{w}_{li}^{(n)} \\ &+ \sum_{i=1}^{p}\left[\int_t^{e_l^{(n)}}\frac{\partial^2}{\partial t^2}K_{ij}(s,t)\cdot\bar{w}_{li}^{(n)}ds - \frac{\partial}{\partial t}K_{ij}(s,t)\Big|_{s=t}\cdot\bar{w}_{li}^{(n)}\right] \\ &+ \sum_{k=l+1}^{n}\sum_{i=1}^{p}\int_{\bar{E}_k^{(n)}}\frac{\partial^2}{\partial t^2}K_{ij}(s,t)\cdot\bar{w}_{ki}^{(n)}ds \end{aligned}$$

We consider the following cases.

- Suppose that $h_{lj}^{(n)}$ is a linear function of t on $E_l^{(n)}$ assumed by

$$h_{lj}^{(n)}(t) = \bar{h}_{lj}^{(n)}(t) + a_j(t) = \mathfrak{a}_{lj}\cdot t + \mathfrak{b}_{lj}$$

for $j = 1,\cdots,q$. Using (6.15), we obtain

$$\max_{t\in\bar{E}_l^{(n)}} h_{lj}^{(n)}(t) = \begin{cases} \max\left\{h_{lj}^{(n)}\left(e_{l-1}^{(n)}\right), h_{lj}^{(n)}\left(e_l^{(n)}\right), \mathfrak{b}_{lj}\right\}, & \text{if } \mathfrak{a}_{lj} = 0 \\ \max\left\{h_{lj}^{(n)}\left(e_{l-1}^{(n)}\right), h_{lj}^{(n)}\left(e_l^{(n)}\right), \mathfrak{a}_{lj}\cdot e_l^{(n)} + \mathfrak{b}_{lj}\right\}, & \text{if } \mathfrak{a}_{lj} > 0 \\ \max\left\{h_{lj}^{(n)}\left(e_{l-1}^{(n)}\right), h_{lj}^{(n)}\left(e_l^{(n)}\right), \mathfrak{a}_{lj}\cdot e_{l-1}^{(n)} + \mathfrak{b}_{lj}\right\}, & \text{if } \mathfrak{a}_{lj} < 0 \end{cases} \tag{6.20}$$

- Suppose that $h_{lj}^{(n)}$ is not a linear function of t on $E_l^{(n)}$. In order to obtain the zero t^* of $\frac{d}{dt}(\widetilde{h}_{lj}^{(n)}(t) + a_j(t))$, we can apply the Newton's method to generate a sequence $\{t_m\}_{m=1}^{\infty}$ such that $t_m \to t^*$ as $m \to \infty$. The iteration is given by

$$t_{m+1} = t_m - \frac{\left.\frac{d}{dt}\widetilde{h}_{lj}^{(n)}(t)\right|_{t=t_m} + \left.\frac{d}{dt}a_j(t)\right|_{t=t_m}}{\left.\frac{d^2}{dt^2}\widetilde{h}_{lj}^{(n)}(t)\right|_{t=t_m} + \left.\frac{d^2}{dt^2}a_j(t)\right|_{t=t_m}} \tag{6.21}$$

  for $m = 0, 1, 2, \cdots$. The initial guess is t_0. Since the real-valued function $\frac{d}{dt}(\widetilde{h}_{lj}^{(n)}(t) + a_j(t))$ may have more than one zero, we need to apply the Newton's method by taking as many as possible for the initial guesses t_0's.

Now the computational procedure is given below.

- **Step 1**. Set the error tolerance ϵ and the initial value of natural number $n \in \mathbb{N}$.
- **Step 2**. Find the optimal objective value $V(\mathrm{D}_n)$ and optimal solution $\bar{\mathbf{w}}$ of dual problem (D_n).
- **Step 3**. Find the set $Z_{lj}^{(n)}$ of all zeros of the real-valued function $\frac{d}{dt}(\widetilde{h}_{lj}^{(n)}(t) + a_j(t))$ by applying the Newton method given in (6.21).
- **Step 4**. Evaluate the maximum (6.16) according to (6.17), and evaluate the supremum (6.12) according to (6.15).
- **Step 5**. Obtain $\bar{\pi}_l^{(n)}$ using (4.7) and the supremum obtained in Step 4. Also, using the values of $\bar{\pi}_l^{(n)}$ to obtain $\pi_l^{(n)}$ according to (4.8).
- **Step 6**. Evaluate the error bound ε_n according to (4.31). If $\varepsilon_n < \epsilon$, then go to Step 7; otherwise, consider one more subdivision for each closed subinterval and set $n \leftarrow n + \widehat{n}$ for some integer $\widehat{n}$ and go to Step 2, where $\widehat{n}$ is the number of new points of subdivisions for all the closed subintervals. For example, in Example 3.1, we can set $n^* \leftarrow n^* + 1$. In this case, we have $\widehat{n} = r$. In Example 3.2, we can set $n_v \leftarrow n_v + 1$ for $v = 1, \cdots, r$. In this case, we also have $\widehat{n} = r$.
- **Step 7**. Find the optimal solution $\bar{\mathbf{z}}^{(n)}$ of primal problem (P_n).
- **Step 8**. Set the step function $\widehat{\mathbf{z}}^{(n)}(t)$ defined in (3.25), which will be the approximate solution of problem (CLP). The actual error between the optimal objective value $V(\mathrm{CLP})$ and the objective value at $\widehat{\mathbf{z}}^{(n)}(t)$ is less than ε_n by Proposition 4.2, where the error tolerance ϵ is reached for this partition $\mathcal{P}_n$.

6.3. *Differences*

For the computational procedure without solving the dual problem, we can obtain two error bounds $\widehat{\varepsilon}_n$ and $\widetilde{\varepsilon}_n$. For the computational procedure for solving the dual problem, we can obtain the error bound ε_n in which (6.11) is satisfied. The differences between the above two computational procedures are presented below.

- The computational procedure without solving (D_n) will save the CPU time. However, the integer n regarding the partition $\mathcal{P}_n$ may be very large. In this case, in order to obtain the approximate solution $\widehat{\mathbf{z}}^{(n)}(t)$ of problem (CLP), we may need to solve a large scale problem (P_n), which may need a lot of memory for the computer and may result in out of memory at running time. The reason is that, in order to reach the pre-determined error tolerance ϵ, we may not need so large integer n that satisfies $\widehat{\varepsilon}_n < \epsilon$ or $\widetilde{\varepsilon}_n < \epsilon$ as shown in (6.11). However, a much small integer n may satisfy $\varepsilon_n < \epsilon$, where ε_n is the tighter error bound between the approximate objective function value and the optimal objective function value.

- The computational procedure for solving (D_n) can obtain the tighter error bound ε_n. However, we will need more CPU time to obtain ε_n, since the dual problem (D_n) must be solved for many times. In other words, we can obtain a reasonable integer n regarding the partition $\mathcal{P}_n$ such that the error bound $\varepsilon_n < \epsilon$.

6.4. Numerical example

In the sequel, we present a numerical example that considers the piecewise continuous functions on the time interval $[0, T]$. We consider $T = 1$ and the following problem

$$\begin{aligned}
&\text{maximize} \quad \int_0^1 [a_1(t) \cdot z_1(t) + a_2(t) \cdot z_2(t)]\, dt \\
&\text{subject to} \quad b_1(t) \cdot z_1(t) \le c_1(t) + \int_0^t [k_1(t,s) \cdot z_1(s) + k_2(t,s) \cdot z_2(s)]\, ds \text{ for all } t \in [0,1] \\
&\qquad\qquad\quad b_2(t) \cdot z_2(t) \le c_2(t) + \int_0^t [k_3(t,s) \cdot z_1(s) + k_4(t,s) \cdot z_2(s)]\, ds \text{ for all } t \in [0,1] \\
&\qquad\qquad\quad \mathbf{z} = (z_1, z_2)^\top \in L_2^2[0,1],
\end{aligned}$$

where

$$a_1(t) = \begin{cases} e^t, & \text{if } 0 \le t \le 0.2 \\ \sin t, & \text{if } 0.2 < t \le 0.6 \\ t^2, & \text{if } 0.6 < t \le 1 \end{cases} \text{ and } a_2(t) = \begin{cases} 2t, & \text{if } 0 \le t \le 0.5 \\ t, & \text{if } 0.5 < t \le 0.7 \\ t^2, & \text{if } 0.7 < t \le 1 \end{cases}$$

$$c_1(t) = \begin{cases} t^3, & \text{if } 0 \le t \le 0.3 \\ (\ln t)^2, & \text{if } 0.3 < t \le 0.5 \\ t^2, & \text{if } 0.5 < t \le 0.8 \\ \cos t, & \text{if } 0.8 < t \le 1 \end{cases} \text{ and } c_2(t) = \begin{cases} t, & \text{if } 0 \le t \le 0.4 \\ 5t, & \text{if } 0.4 < t \le 0.5 \\ t^3, & \text{if } 0.5 < t \le 0.8 \\ t^2, & \text{if } 0.8 < t \le 1. \end{cases}$$

$$b_1(t) = \begin{cases} b_1^{(1)}(t) = 20\cos t, & \text{if } 0 \le t \le 0.2 \\ b_1^{(2)}(t) = 25\sin t, & \text{if } 0.2 < t \le 0.6 \\ b_1^{(3)}(t) = 27t^2, & \text{if } 0.6 < t \le 1 \end{cases} \text{ and } b_2(t) = \begin{cases} b_2^{(1)}(t) = 25\cos t, & \text{if } 0 \le t \le 0.5 \\ b_2^{(2)}(t) = 22t, & \text{if } 0.5 < t \le 0.7 \\ b_2^{(3)}(t) = 25t^2, & \text{if } 0.7 < t \le 1 \end{cases}$$

and

$$k_1(t,s) = \begin{cases} k_1^{(1)}(t,s) = t^3 + s^2, & \text{if } 0 \le t \le 0.8 \text{ and } 0 \le s \le 0.5 \\ k_1^{(2)}(t,s) = t^2 + \sin s, & \text{if } 0 \le t \le 0.8 \text{ and } 0.5 < s \le 1 \\ k_1^{(3)}(t,s) = (\ln t)^2 + 3e^{-s}, & \text{if } 0.8 < t \le 1 \text{ and } 0 \le s \le 0.5 \\ k_1^{(4)}(t,s) = \cos t + 5e^{-s}, & \text{if } 0.8 < t \le 1 \text{ and } 0.5 < s \le 1 \end{cases}$$

$$k_2(t,s) = \begin{cases} k_2^{(1)}(t,s) = t^3 \cdot s^2, & \text{if } 0 \le t \le 0.6 \text{ and } 0 \le s \le 0.7 \\ k_2^{(2)}(t,s) = t^2 \cdot \sin s, & \text{if } 0 \le t \le 0.6 \text{ and } 0.7 < s \le 1 \\ k_2^{(3)}(t,s) = (\ln t)^2 \cdot e^{-s}, & \text{if } 0.6 < t \le 1 \text{ and } 0 \le s \le 0.7 \\ k_2^{(4)}(t,s) = 3t^2 \cdot \sin s, & \text{if } 0.6 < t \le 1 \text{ and } 0.7 < s \le 1 \end{cases}$$

$$k_3(t,s)=\begin{cases} k_3^{(1)}(t,s)=3t^2\cdot\sin s, & \text{if } 0\le t\le 0.3 \text{ and } 0\le s\le 0.6\\ k_3^{(2)}(t,s)=2t\cdot s^2, & \text{if } 0\le t\le 0.3 \text{ and } 0.6< s\le 1\\ k_3^{(3)}(t,s)=(\ln t)^2+(\cos s)^2, & \text{if } 0.3< t\le 1 \text{ and } 0\le s\le 0.6\\ k_3^{(4)}(t,s)=t^3\cdot s^2, & \text{if } 0.3< t\le 1 \text{ and } 0.6< s\le 1\end{cases}$$

$$k_4(t,s)=\begin{cases} k_4^{(1)}(t,s)=t^2+s^2, & \text{if } 0\le t\le 0.5 \text{ and } 0\le s\le 0.3\\ k_4^{(2)}(t,s)=\sin t+s^2, & \text{if } 0\le t\le 0.5 \text{ and } 0.3< s\le 1\\ k_4^{(3)}(t,s)=(\cos t)^2+3e^{-s}, & \text{if } 0.5< t\le 1 \text{ and } 0\le s\le 0.3\\ k_4^{(4)}(t,s)=2t^3\cdot s^2, & \text{if } 0.5< t\le 1 \text{ and } 0.3< s\le 1.\end{cases}$$

The time-dependent matrices $B(t)$ and $K(t,s)$ are given below:

$$B(t)=\begin{bmatrix} B_{11}(t) & B_{12}(t)\\ B_{21}(t) & B_{22}(t)\end{bmatrix}=\begin{bmatrix} b_1(t) & 0\\ 0 & b_2(t)\end{bmatrix}$$

and

$$K(t,s)=\begin{bmatrix} K_{11}(t,s) & K_{12}(t,s)\\ K_{21}(t,s) & K_{22}(t,s)\end{bmatrix}=\begin{bmatrix} k_1(t,s) & k_2(t,s)\\ k_3(t,s) & k_4(t,s)\end{bmatrix}.$$

We see that $B(t)$ satisfies conditions (2.1) and (2.2). From the discontinuities of $a_1, a_2, c_1, c_2, b_1, b_2, k_1, k_2, k_3, k_4$, according to the setting of partition $\mathcal{P}_n$, we see that $r=8$ and

$$\mathcal{D}=\{d_0=0, d_1=0.2, d_2=0.3, d_3=0.4, d_4=0.5, d_5=0.6, d_6=0.7, d_7=0.8, d_8=1\}.$$

For $n^*=2$, it means that each closed interval $[d_v, d_{v+1}]$ is equally divided by two subintervals for $v=0,1,\cdots 7$. In this case, we have $n=2\cdot 8=16$. Therefore we obtain a partition $\mathcal{P}_{16}$.

For $t\in F_l^{(n)}$, we can rewrite (6.19) and (6.18) as follows:

$$\widetilde{h}_{lj}^{(n)}(t)=\sum_{i=1}^{p}\bar{w}_{li}^{(n)}\cdot\left[-B_{ij}(t)+\int_t^{e_l^{(n)}}K_{ij}(s,t)ds\right]+\sum_{k=l+1}^{n}\sum_{i=1}^{p}\bar{w}_{ki}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}K_{ij}(s,t)ds$$

and

$$\frac{d}{dt}\left(\widetilde{h}_{lj}^{(n)}(t)\right)=\sum_{i=1}^{p}\bar{w}_{li}^{(n)}\cdot\left[-\frac{d}{dt}B_{ij}(t)-K_{ij}(t,t)+\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}K_{ij}(s,t)ds\right]$$
$$+\sum_{k=l+1}^{n}\sum_{i=1}^{p}\bar{w}_{ki}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}K_{ij}(s,t)ds. \tag{6.22}$$

Let us recall

$$\int(\ln s)^2ds=s(\ln s)^2-2s\ln s+2s+C \text{ and } \int(\cos s)^2ds=\frac{1}{2}s+\frac{1}{4}\sin 2s+C.$$

Then we also need the following integrations:

$$\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}k_1(s,t)ds=\begin{cases}\int_t^{e_l^{(n)}}s^3ds+2t\cdot\left(e_l^{(n)}-t\right), & \text{if } 0<t<0.5 \text{ and } [t,e_l^{(n)}]\subseteq[0,0.8]\\ \int_t^{e_l^{(n)}}s^2ds+\cos t\cdot\left(e_l^{(n)}-t\right), & \text{if } 0.5<t<1 \text{ and } [t,e_l^{(n)}]\subseteq[0,0.8]\\ \int_t^{e_l^{(n)}}(\ln s)^2ds-3e^{-t}\cdot\left(e_l^{(n)}-t\right), & \text{if } 0<t<0.5 \text{ and } [t,e_l^{(n)}]\subseteq[0.8,1]\\ \int_t^{e_l^{(n)}}\cos s ds-5e^{-t}\cdot\left(e_l^{(n)}-t\right), & \text{if } 0.5<t<1 \text{ and } [t,e_l^{(n)}]\subseteq[0.8,1]\end{cases}$$

$$\int_t^{e_l^{(n)}} \frac{\partial}{\partial t}k_2(s,t)ds = \begin{cases} 2t \cdot \int_t^{e_l^{(n)}} s^3 ds, & \text{if } 0 < t < 0.7 \text{ and } [t, e_l^{(n)}] \subseteq [0, 0.6] \\ \cos t \cdot \int_t^{e_l^{(n)}} s^2 ds, & \text{if } 0.7 < t < 1 \text{ and } [t, e_l^{(n)}] \subseteq [0, 0.6] \\ -e^{-t} \cdot \int_t^{e_l^{(n)}} (\ln s)^2 ds, & \text{if } 0 < t < 0.7 \text{ and } [t, e_l^{(n)}] \subseteq [0.6, 1] \\ \cos t \cdot \int_t^{e_l^{(n)}} 3s^2 ds, & \text{if } 0.7 < t < 1 \text{ and } [t, e_l^{(n)}] \subseteq [0.6, 1] \end{cases}$$

$$\int_t^{e_l^{(n)}} \frac{\partial}{\partial t}k_3(s,t)ds = \begin{cases} 3\cos t \cdot \int_t^{e_l^{(n)}} s^2 ds, & \text{if } 0 < t < 0.6 \text{ and } [t, e_l^{(n)}] \subseteq [0, 0.3] \\ 4t \cdot \int_t^{e_l^{(n)}} s ds, & \text{if } 0.6 < t < 1 \text{ and } [t, e_l^{(n)}] \subseteq [0, 0.3] \\ \int_t^{e_l^{(n)}} (\ln s)^2 ds - \sin 2t \cdot \left(e_l^{(n)} - t\right), & \text{if } 0 < t < 0.6 \text{ and } [t, e_l^{(n)}] \subseteq [0.3, 1] \\ 2t \cdot \int_t^{e_l^{(n)}} s^3 ds, & \text{if } 0.6 < t < 1 \text{ and } [t, e_l^{(n)}] \subseteq [0.3, 1] \end{cases}$$

$$\int_t^{e_l^{(n)}} \frac{\partial}{\partial t}k_4(s,t)ds = \begin{cases} \int_t^{e_l^{(n)}} s^2 ds + 2t \cdot \left(e_l^{(n)} - t\right), & \text{if } 0 < t < 0.3 \text{ and } [t, e_l^{(n)}] \subseteq [0, 0.5] \\ \int_t^{e_l^{(n)}} \sin s ds + 2t \cdot \left(e_l^{(n)} - t\right), & \text{if } 0.3 < t < 1 \text{ and } [t, e_l^{(n)}] \subseteq [0, 0.5] \\ \int_t^{e_l^{(n)}} (\cos s)^2 ds - 3e^{-t} \cdot \left(e_l^{(n)} - t\right), & \text{if } 0 < t < 0.3 \text{ and } [t, e_l^{(n)}] \subseteq [0.5, 1] \\ 4t \cdot \int_t^{e_l^{(n)}} s^3 ds, & \text{if } 0.3 < t < 1 \text{ and } [t, e_l^{(n)}] \subseteq [0.5, 1]. \end{cases}$$

and

$$\int_t^{e_l^{(n)}} k_1(s,t)ds = \begin{cases} \int_t^{e_l^{(n)}} s^3 ds + t^2 \cdot \left(e_l^{(n)} - t\right), & \text{if } 0 < t < 0.5 \text{ and } [t, e_l^{(n)}] \subseteq [0, 0.8] \\ \int_t^{e_l^{(n)}} s^2 ds + \sin t \cdot \left(e_l^{(n)} - t\right), & \text{if } 0.5 < t < 1 \text{ and } [t, e_l^{(n)}] \subseteq [0, 0.8] \\ \int_t^{e_l^{(n)}} (\ln s)^2 ds + 3e^{-t} \cdot \left(e_l^{(n)} - t\right), & \text{if } 0 < t < 0.5 \text{ and } [t, e_l^{(n)}] \subseteq [0.8, 1] \\ \int_t^{e_l^{(n)}} \cos s ds + 5e^{-t} \cdot \left(e_l^{(n)} - t\right), & \text{if } 0.5 < t < 1 \text{ and } [t, e_l^{(n)}] \subseteq [0.8, 1] \end{cases}$$

$$\int_t^{e_l^{(n)}} k_2(s,t)ds = \begin{cases} t^2 \cdot \int_t^{e_l^{(n)}} s^3 ds, & \text{if } 0 < t < 0.7 \text{ and } [t, e_l^{(n)}] \subseteq [0, 0.6] \\ \sin t \cdot \int_t^{e_l^{(n)}} s^2 ds, & \text{if } 0.7 < t < 1 \text{ and } [t, e_l^{(n)}] \subseteq [0, 0.6] \\ e^{-t} \cdot \int_t^{e_l^{(n)}} (\ln s)^2 ds, & \text{if } 0 < t < 0.7 \text{ and } [t, e_l^{(n)}] \subseteq [0.6, 1] \\ \sin t \cdot \int_t^{e_l^{(n)}} 3s^2 ds, & \text{if } 0.7 < t < 1 \text{ and } [t, e_l^{(n)}] \subseteq [0.6, 1] \end{cases}$$

$$\int_t^{e_l^{(n)}} k_3(s,t)ds = \begin{cases} 3\sin t \cdot \int_t^{e_l^{(n)}} s^2 ds, & \text{if } 0 < t < 0.6 \text{ and } [t, e_l^{(n)}] \subseteq [0, 0.3] \\ t^2 \cdot \int_t^{e_l^{(n)}} 2s ds, & \text{if } 0.6 < t < 1 \text{ and } [t, e_l^{(n)}] \subseteq [0, 0.3] \\ \int_t^{e_l^{(n)}} (\ln s)^2 ds + (\cos t)^2 \cdot \left(e_l^{(n)} - t\right), & \text{if } 0 < t < 0.6 \text{ and } [t, e_l^{(n)}] \subseteq [0.3, 1] \\ t^2 \cdot \int_t^{e_l^{(n)}} s^3 ds, & \text{if } 0.6 < t < 1 \text{ and } [t, e_l^{(n)}] \subseteq [0.3, 1] \end{cases}$$

$$\int_t^{e_l^{(n)}} k_4(s,t)ds = \begin{cases} \int_t^{e_l^{(n)}} s^2 ds + t^2 \cdot \left(e_l^{(n)} - t\right), & \text{if } 0 < t < 0.3 \text{ and } [t, e_l^{(n)}] \subseteq [0, 0.5] \\ \int_t^{e_l^{(n)}} \sin s ds + t^2 \cdot \left(e_l^{(n)} - t\right), & \text{if } 0.3 < t < 1 \text{ and } [t, e_l^{(n)}] \subseteq [0, 0.5] \\ \int_t^{e_l^{(n)}} (\cos s)^2 ds + 3e^{-t} \cdot \left(e_l^{(n)} - t\right), & \text{if } 0 < t < 0.3 \text{ and } [t, e_l^{(n)}] \subseteq [0.5, 1] \\ 2t^2 \cdot \int_t^{e_l^{(n)}} s^3 ds, & \text{if } 0.3 < t < 1 \text{ and } [t, e_l^{(n)}] \subseteq [0.5, 1]. \end{cases}$$

Now, for $k \geq l+1$, we need

$$\int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_1(s,t)ds = \begin{cases} \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_1^{(1)}(s,t)ds = \int_{\bar{E}_k^{(n)}} s^3 ds + 2t \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0, 0.8] \text{ and } 0 < t < 0.5 \\ \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_1^{(2)}(s,t)ds = \int_{\bar{E}_k^{(n)}} s^2 ds + \cos t \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0, 0.8] \text{ and } 0.5 < t < 1 \\ \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_1^{(3)}(s,t)ds = \int_{\bar{E}_k^{(n)}} (\ln s)^2 ds - 3e^{-t} \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0.8, 1] \text{ and } 0 < t < 0.5 \\ \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_1^{(4)}(s,t)ds = \int_{\bar{E}_k^{(n)}} \cos s ds - 5e^{-t} \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0.8, 1] \text{ and } 0.5 < t < 1 \end{cases}$$

$$\int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_2(s,t)ds = \begin{cases} \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_2^{(1)}(s,t)ds = 2t \cdot \int_{\bar{E}_k^{(n)}} s^3 ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0, 0.6] \text{ and } 0 < t < 0.7 \\ \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_2^{(2)}(s,t)ds = \cos t \cdot \int_{\bar{E}_k^{(n)}} s^2 ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0, 0.6] \text{ and } 0.7 < t < 1 \\ \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_2^{(3)}(s,t)ds = -e^{-t} \cdot \int_{\bar{E}_k^{(n)}} (\ln s)^2 ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0.6, 1] \text{ and } 0 < t < 0.7 \\ \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_2^{(4)}(s,t)ds = \cos t \cdot \int_{\bar{E}_k^{(n)}} 3s^2 ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0.6, 1] \text{ and } 0.7 < t < 1 \end{cases}$$

$$\int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_3(s,t)ds = \begin{cases} \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_3^{(1)}(s,t)ds = 3\cos t \cdot \int_{\bar{E}_k^{(n)}} s^2 ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0, 0.3] \text{ and } 0 < t < 0.6 \\ \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_3^{(2)}(s,t)ds = 4t \cdot \int_{\bar{E}_k^{(n)}} s ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0, 0.3] \text{ and } 0.6 < t < 1 \\ \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_3^{(3)}(s,t)ds = \int_{\bar{E}_k^{(n)}} (\ln s)^2 ds - \sin(2t) \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0.3, 1] \text{ and } 0 < t < 0.6 \\ \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_3^{(4)}(s,t)ds = 2t \cdot \int_{\bar{E}_k^{(n)}} s^3 ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0.3, 1] \text{ and } 0.6 < t < 1 \end{cases}$$

$$\int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_4(s,t)ds = \begin{cases} \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_4^{(1)}(s,t)ds = \int_{\bar{E}_k^{(n)}} s^2 ds + 2t \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0,0.5] \text{ and } 0 < t < 0.3 \\ \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_4^{(2)}(s,t)ds = \int_{\bar{E}_k^{(n)}} \sin s ds + 2t \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0,0.5] \text{ and } 0.3 < t < 1 \\ \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_4^{(3)}(s,t)ds = \int_{\bar{E}_k^{(n)}} (\cos s)^2 ds - 3e^{-t} \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0.5,1] \text{ and } 0 < t < 0.3 \\ \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_4^{(4)}(s,t)ds = 4t \cdot \int_{\bar{E}_k^{(n)}} s^3 ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0.5,1] \text{ and } 0.3 < t < 1. \end{cases}$$

and

$$\int_{\bar{E}_k^{(n)}} k_1(s,t)ds = \begin{cases} \int_{\bar{E}_k^{(n)}} k_1^{(1)}(s,t)ds = \int_{\bar{E}_k^{(n)}} s^3 ds + t^2 \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0,0.8] \text{ and } 0 < t < 0.5 \\ \int_{\bar{E}_k^{(n)}} k_1^{(2)}(s,t)ds = \int_{\bar{E}_k^{(n)}} s^2 ds + \sin t \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0,0.8] \text{ and } 0.5 < t < 1 \\ \int_{\bar{E}_k^{(n)}} k_1^{(3)}(s,t)ds = \int_{\bar{E}_k^{(n)}} (\ln s)^2 ds + 3e^{-t} \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0.8,1] \text{ and } 0 < t < 0.5 \\ \int_{\bar{E}_k^{(n)}} k_1^{(4)}(s,t)ds = \int_{\bar{E}_k^{(n)}} \cos s ds + 5e^{-t} \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0.8,1] \text{ and } 0.5 < t < 1 \end{cases}$$

$$\int_{\bar{E}_k^{(n)}} k_2(s,t)ds = \begin{cases} \int_{\bar{E}_k^{(n)}} k_2^{(1)}(s,t)ds = t^2 \cdot \int_{\bar{E}_k^{(n)}} s^3 ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0,0.6] \text{ and } 0 < t < 0.7 \\ \int_{\bar{E}_k^{(n)}} k_2^{(2)}(s,t)ds = \sin t \cdot \int_{\bar{E}_k^{(n)}} s^2 ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0,0.6] \text{ and } 0.7 < t < 1 \\ \int_{\bar{E}_k^{(n)}} k_2^{(3)}(s,t)ds = e^{-t} \cdot \int_{\bar{E}_k^{(n)}} (\ln s)^2 ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0.6,1] \text{ and } 0 < t < 0.7 \\ \int_{\bar{E}_k^{(n)}} k_2^{(4)}(s,t)ds = \sin t \cdot \int_{\bar{E}_k^{(n)}} 3s^2 ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0.6,1] \text{ and } 0.7 < t < 1 \end{cases}$$

$$\int_{\bar{E}_k^{(n)}} k_3(s,t)ds = \begin{cases} \int_{\bar{E}_k^{(n)}} k_3^{(1)}(s,t)ds = 3\sin t \cdot \int_{\bar{E}_k^{(n)}} s^2 ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0,0.3] \text{ and } 0 < t < 0.6 \\ \int_{\bar{E}_k^{(n)}} k_3^{(2)}(s,t)ds = 2t^2 \cdot \int_{\bar{E}_k^{(n)}} s ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0,0.3] \text{ and } 0.6 < t < 1 \\ \int_{\bar{E}_k^{(n)}} k_3^{(3)}(s,t)ds = \int_{\bar{E}_k^{(n)}} (\ln s)^2 ds + (\cos t)^2 \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0.3,1] \text{ and } 0 < t < 0.6 \\ \int_{\bar{E}_k^{(n)}} k_3^{(4)}(s,t)ds = t^2 \cdot \int_{\bar{E}_k^{(n)}} s^3 ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0.3,1] \text{ and } 0.6 < t < 1 \end{cases}$$

$$\int_{\bar{E}_k^{(n)}} k_4(s,t)ds = \begin{cases} \int_{\bar{E}_k^{(n)}} k_4^{(1)}(s,t)ds = \int_{\bar{E}_k^{(n)}} s^2 ds + t^2 \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0,0.5] \text{ and } 0 < t < 0.3 \\ \int_{\bar{E}_k^{(n)}} k_4^{(2)}(s,t)ds = \int_{\bar{E}_k^{(n)}} \sin s ds + t^2 \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0,0.5] \text{ and } 0.3 < t < 1 \\ \int_{\bar{E}_k^{(n)}} k_4^{(3)}(s,t)ds = \int_{\bar{E}_k^{(n)}} (\cos s)^2 ds + 3e^{-t} \cdot \mathfrak{d}_k^{(n)}, & \text{if } \bar{E}_k^{(n)} \subseteq [0.5,1] \text{ and } 0 < t < 0.3 \\ \int_{\bar{E}_k^{(n)}} k_4^{(4)}(s,t)ds = 2t^2 \cdot \int_{\bar{E}_k^{(n)}} s^3 ds, & \text{if } \bar{E}_k^{(n)} \subseteq [0.5,1] \text{ and } 0.3 < t < 1. \end{cases}$$

Since $e_2^{(n)} = 0.3$ and $e_7^{(n)} = 0.8$, let $l_1 = 2$ and $l_2 = 7$. We have the following cases.

- For $0 < t < 0.2$ and $t \in E_l^{(n)}$, we have

$$\frac{d}{dt}\left(\widetilde{h}_{l1}^{(n)}(t)\right) = \bar{w}_{l1}^{(n)} \cdot \left(-\frac{d}{dt} b_1^{(1)}(t) - k_1^{(1)}(t,t) + \int_t^{e_l^{(n)}} \frac{\partial}{\partial t} k_1^{(1)}(s,t)ds\right)$$

$$+ \bar{w}_{l2}^{(n)} \cdot \left(-k_3^{(1)}(t,t) + \int_t^{e_l^{(n)}} \frac{\partial}{\partial t} k_3^{(1)}(s,t) ds \right)$$
$$+ \sum_{k=l+1}^{l_1} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_1^{(1)}(s,t) ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_3^{(1)}(s,t) ds \right)$$
$$+ \sum_{k=l_1+1}^{l_2} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_1^{(1)}(s,t) ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_3^{(3)}(s,t) ds \right)$$
$$+ \sum_{k=l_2+1}^{n} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_1^{(3)}(s,t) ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_3^{(3)}(s,t) ds \right)$$

and

$$\widetilde{h}_{l1}^{(n)}(t) = \bar{w}_{l1}^{(n)} \cdot \left(-b_1^{(1)}(t) + \int_t^{e_l^{(n)}} k_1^{(1)}(s,t) ds \right) + \bar{w}_{l2}^{(n)} \cdot \int_t^{e_l^{(n)}} k_3^{(1)}(s,t) ds$$
$$+ \sum_{k=l+1}^{l_1} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_1^{(1)}(s,t) ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_3^{(1)}(s,t) ds \right)$$
$$+ \sum_{k=l_1+1}^{l_2} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_1^{(1)}(s,t) ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_3^{(3)}(s,t) ds \right)$$
$$+ \sum_{k=l_2+1}^{n} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_1^{(3)}(s,t) ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_3^{(3)}(s,t) ds \right)$$

- For $0.2 < t < 0.3$ and $t \in E_l^{(n)}$, we have

$$\frac{d}{dt}\left(\widetilde{h}_{l1}^{(n)}(t)\right) = \bar{w}_{l1}^{(n)} \cdot \left(-\frac{d}{dt} b_1^{(2)}(t) - k_1^{(1)}(t,t) + \int_t^{e_l^{(n)}} \frac{\partial}{\partial t} k_1^{(1)}(s,t) ds \right)$$
$$+ \bar{w}_{l2}^{(n)} \cdot \left(-k_3^{(1)}(t,t) + \int_t^{e_l^{(n)}} \frac{\partial}{\partial t} k_3^{(1)}(s,t) ds \right)$$
$$+ \sum_{k=l+1}^{l_1} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_1^{(1)}(s,t) ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_3^{(1)}(s,t) ds \right)$$
$$+ \sum_{k=l_1+1}^{l_2} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_1^{(1)}(s,t) ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_3^{(3)}(s,t) ds \right)$$
$$+ \sum_{k=l_2+1}^{n} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_1^{(3)}(s,t) ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t} k_3^{(3)}(s,t) ds \right)$$

and

$$\widetilde{h}_{l1}^{(n)}(t) = \bar{w}_{l1}^{(n)} \cdot \left(-b_1^{(2)}(t) + \int_t^{e_l^{(n)}} k_1^{(1)}(s,t) ds \right) + \bar{w}_{l2}^{(n)} \cdot \int_t^{e_l^{(n)}} k_3^{(1)}(s,t) ds$$

$$+\sum_{k=l+1}^{l_1}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_1^{(1)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_3^{(1)}(s,t)ds\right)$$
$$+\sum_{k=l_1+1}^{l_2}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_1^{(1)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_3^{(3)}(s,t)ds\right)$$
$$+\sum_{k=l_2+1}^{n}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_1^{(3)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_3^{(3)}(s,t)ds\right)$$

- For $0.3 < t < 0.5$ and $t \in E_l^{(n)}$, we have

$$\frac{d}{dt}\left(\widetilde{h}_{l1}^{(n)}(t)\right)=\bar{w}_{l1}^{(n)}\cdot\left(-\frac{d}{dt}b_1^{(2)}(t)-k_1^{(1)}(t,t)+\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}k_1^{(1)}(s,t)ds\right)$$
$$+\bar{w}_{l2}^{(n)}\cdot\left(-k_3^{(3)}(t,t)+\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}k_3^{(3)}(s,t)ds\right)$$
$$+\sum_{k=l+1}^{l_2}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_1^{(1)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_3^{(3)}(s,t)ds\right)$$
$$+\sum_{k=l_2+1}^{n}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_1^{(3)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_3^{(3)}(s,t)ds\right)$$

and

$$\widetilde{h}_{l1}^{(n)}(t)=\bar{w}_{l1}^{(n)}\cdot\left(-b_1^{(2)}(t)+\int_t^{e_l^{(n)}}k_1^{(1)}(s,t)ds\right)+\bar{w}_{l2}^{(n)}\cdot\int_t^{e_l^{(n)}}k_3^{(3)}(s,t)ds$$
$$+\sum_{k=l+1}^{l_2}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_1^{(1)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_3^{(3)}(s,t)ds\right)$$
$$+\sum_{k=l_2+1}^{n}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_1^{(3)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_3^{(3)}(s,t)ds\right)$$

- For $0.5 < t < 0.6$ and $t \in E_l^{(n)}$, we have

$$\frac{d}{dt}\left(\widetilde{h}_{l1}^{(n)}(t)\right)=\bar{w}_{l1}^{(n)}\cdot\left(-\frac{d}{dt}b_1^{(2)}(t)-k_1^{(2)}(t,t)+\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}k_1^{(2)}(s,t)ds\right)$$
$$+\bar{w}_{l2}^{(n)}\cdot\left(-k_3^{(3)}(t,t)+\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}k_3^{(3)}(s,t)ds\right)$$
$$+\sum_{k=l+1}^{l_2}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_1^{(2)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_3^{(3)}(s,t)ds\right)$$
$$+\sum_{k=l_2+1}^{n}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_1^{(4)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_3^{(3)}(s,t)ds\right)$$

and

$$\widetilde{h}_{l1}^{(n)}(t) = \bar{w}_{l1}^{(n)} \cdot \left(-b_1^{(2)}(t) + \int_t^{e_l^{(n)}} k_1^{(2)}(s,t)ds \right) + \bar{w}_{l2}^{(n)} \cdot \int_t^{e_l^{(n)}} k_3^{(3)}(s,t)ds$$
$$+ \sum_{k=l+1}^{l_2} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_1^{(2)}(s,t)ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_3^{(3)}(s,t)ds \right)$$
$$+ \sum_{k=l_2+1}^{n} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_1^{(4)}(s,t)ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_3^{(3)}(s,t)ds \right)$$

- For $0.6 < t < 0.8$ and $t \in E_l^{(n)}$, we have

$$\frac{d}{dt}\left(\widetilde{h}_{l1}^{(n)}(t)\right) = \bar{w}_{l1}^{(n)} \cdot \left(-\frac{d}{dt}b_1^{(3)}(t) - k_1^{(2)}(t,t) + \int_t^{e_l^{(n)}} \frac{\partial}{\partial t}k_1^{(2)}(s,t)ds \right)$$
$$+ \bar{w}_{l2}^{(n)} \cdot \left(-k_3^{(4)}(t,t) + \int_t^{e_l^{(n)}} \frac{\partial}{\partial t}k_3^{(4)}(s,t)ds \right)$$
$$+ \sum_{k=l+1}^{l_2} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t}k_1^{(2)}(s,t)ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t}k_3^{(4)}(s,t)ds \right)$$
$$+ \sum_{k=l_2+1}^{n} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t}k_1^{(4)}(s,t)ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t}k_3^{(4)}(s,t)ds \right)$$

and

$$\widetilde{h}_{l1}^{(n)}(t) = \bar{w}_{l1}^{(n)} \cdot \left(-b_1^{(3)}(t) + \int_t^{e_l^{(n)}} k_1^{(2)}(s,t)ds \right) + \bar{w}_{l2}^{(n)} \cdot \int_t^{e_l^{(n)}} k_3^{(4)}(s,t)ds$$
$$+ \sum_{k=l+1}^{l_2} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_1^{(2)}(s,t)ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_3^{(4)}(s,t)ds \right)$$
$$+ \sum_{k=l_2+1}^{n} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_1^{(4)}(s,t)ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_3^{(4)}(s,t)ds \right)$$

- For $0.8 < t < 1$ and $t \in E_l^{(n)}$, we have

$$\frac{d}{dt}\left(\widetilde{h}_{l1}^{(n)}(t)\right) = \bar{w}_{l1}^{(n)} \cdot \left(-\frac{d}{dt}b_1^{(3)}(t) - k_1^{(4)}(t,t) + \int_t^{e_l^{(n)}} \frac{\partial}{\partial t}k_1^{(4)}(s,t)ds \right)$$
$$+ \bar{w}_{l2}^{(n)} \cdot \left(-k_3^{(4)}(t,t) + \int_t^{e_l^{(n)}} \frac{\partial}{\partial t}k_3^{(4)}(s,t)ds \right)$$
$$+ \sum_{k=l+1}^{n} \left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t}k_1^{(4)}(s,t)ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t}k_3^{(4)}(s,t)ds \right)$$

and

$$\widetilde{h}_{l1}^{(n)}(t) = \bar{w}_{l1}^{(n)} \cdot \left(-b_1^{(3)}(t) + \int_t^{e_l^{(n)}} k_1^{(4)}(s,t)ds \right) + \bar{w}_{l2}^{(n)} \cdot \int_t^{e_l^{(n)}} k_3^{(4)}(s,t)ds$$

$$+\sum_{k=l+1}^{n}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_1^{(4)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_3^{(4)}(s,t)ds\right)$$

Since $e_4^{(n)}=0.5$ and $e_5^{(n)}=0.6$, let $l_3=4$ and $l_4=5$. We have the following cases.

- For $0<t<0.3$ and $t\in E_l^{(n)}$, we have

$$\begin{aligned}\frac{d}{dt}\left(\widetilde{h}_{l2}^{(n)}(t)\right)=&\ \bar{w}_{l1}^{(n)}\cdot\left(-k_2^{(1)}(t,t)+\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}k_2^{(1)}(s,t)ds\right)\\&+\bar{w}_{l2}^{(n)}\cdot\left(-\frac{d}{dt}b_2^{(1)}(t)-k_4^{(1)}(t,t)+\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}k_4^{(1)}(s,t)ds\right)\\&+\sum_{k=l+1}^{l_3}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_2^{(1)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_4^{(1)}(s,t)ds\right)\\&+\sum_{k=l_3+1}^{l_4}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_2^{(1)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_4^{(3)}(s,t)ds\right)\\&+\sum_{k=l_4+1}^{n}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_2^{(3)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_4^{(3)}(s,t)ds\right)\end{aligned}$$

  and

$$\begin{aligned}\widetilde{h}_{l2}^{(n)}(t)=&\ \bar{w}_{l1}^{(n)}\cdot\int_t^{e_l^{(n)}}k_2^{(1)}(s,t)ds+\bar{w}_{l2}^{(n)}\cdot\left(-b_2^{(1)}(t)+\int_t^{e_l^{(n)}}k_4^{(1)}(s,t)ds\right)\\&+\sum_{k=l+1}^{l_3}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_2^{(1)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_4^{(1)}(s,t)ds\right)\\&+\sum_{k=l_3+1}^{l_4}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_2^{(1)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_4^{(3)}(s,t)ds\right)\\&+\sum_{k=l_4+1}^{n}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_2^{(3)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_4^{(3)}(s,t)ds\right)\end{aligned}$$

- For $0.3<t<0.5$ and $t\in E_l^{(n)}$, we have

$$\begin{aligned}\frac{d}{dt}\left(\widetilde{h}_{l2}^{(n)}(t)\right)=&\ \bar{w}_{l1}^{(n)}\cdot\left(-k_2^{(1)}(t,t)+\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}k_2^{(1)}(s,t)ds\right)\\&+\bar{w}_{l2}^{(n)}\cdot\left(-\frac{d}{dt}b_2^{(1)}(t)-k_4^{(2)}(t,t)+\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}k_4^{(2)}(s,t)ds\right)\\&+\sum_{k=l+1}^{l_3}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_2^{(1)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_4^{(2)}(s,t)ds\right)\\&+\sum_{k=l_3+1}^{l_4}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_2^{(1)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_4^{(4)}(s,t)ds\right)\end{aligned}$$

$$+\sum_{k=l_4+1}^{n}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_2^{(3)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_4^{(4)}(s,t)ds\right)$$

and

$$\begin{aligned}\widetilde{h}_{l2}^{(n)}(t)=&\ \bar{w}_{l1}^{(n)}\cdot\int_t^{e_l^{(n)}}k_2^{(1)}(s,t)ds+\bar{w}_{l2}^{(n)}\cdot\left(-b_2^{(1)}(t)+\int_t^{e_l^{(n)}}k_4^{(2)}(s,t)ds\right)\\&+\sum_{k=l+1}^{l_3}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_2^{(1)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_4^{(2)}(s,t)ds\right)\\&+\sum_{k=l_3+1}^{l_4}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_2^{(1)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_4^{(4)}(s,t)ds\right)\\&+\sum_{k=l_4+1}^{n}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_2^{(3)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_4^{(4)}(s,t)ds\right)\end{aligned}$$

- For $0.5<t<0.6$ and $t\in E_l^{(n)}$, we have

$$\begin{aligned}\frac{d}{dt}\left(\widetilde{h}_{l2}^{(n)}(t)\right)=&\ \bar{w}_{l1}^{(n)}\cdot\left(-k_2^{(1)}(t,t)+\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}k_2^{(1)}(s,t)ds\right)\\&+\bar{w}_{l2}^{(n)}\cdot\left(-\frac{d}{dt}b_2^{(2)}(t)-k_4^{(4)}(t,t)+\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}k_4^{(4)}(s,t)ds\right)\\&+\sum_{k=l+1}^{l_4}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_2^{(1)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_4^{(4)}(s,t)ds\right)\\&+\sum_{k=l_4+1}^{n}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_2^{(3)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}\frac{\partial}{\partial t}k_4^{(4)}(s,t)ds\right)\end{aligned}$$

and

$$\begin{aligned}\widetilde{h}_{l2}^{(n)}(t)=&\ \bar{w}_{l1}^{(n)}\cdot\int_t^{e_l^{(n)}}k_2^{(1)}(s,t)ds+\bar{w}_{l2}^{(n)}\cdot\left(-b_2^{(2)}(t)+\int_t^{e_l^{(n)}}k_4^{(4)}(s,t)ds\right)\\&+\sum_{k=l+1}^{l_4}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_2^{(1)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_4^{(4)}(s,t)ds\right)\\&+\sum_{k=l_4+1}^{n}\left(\bar{w}_{k1}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_2^{(3)}(s,t)ds+\bar{w}_{k2}^{(n)}\cdot\int_{\bar{E}_k^{(n)}}k_4^{(4)}(s,t)ds\right)\end{aligned}$$

- For $0.6<t<0.7$ and $t\in E_l^{(n)}$, we have

$$\frac{d}{dt}\left(\widetilde{h}_{l2}^{(n)}(t)\right)=\bar{w}_{l1}^{(n)}\cdot\left(-k_2^{(3)}(t,t)+\int_t^{e_l^{(n)}}\frac{\partial}{\partial t}k_2^{(3)}(s,t)ds\right)$$

$$+\bar{w}_{l2}^{(n)} \cdot \left(-\frac{d}{dt}b_2^{(2)}(t) - k_4^{(4)}(t,t) + \int_t^{e_l^{(n)}} \frac{\partial}{\partial t}k_4^{(4)}(s,t)ds\right)$$
$$+\sum_{k=l+1}^{n}\left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t}k_2^{(3)}(s,t)ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t}k_4^{(4)}(s,t)ds\right)$$

and

$$\widetilde{h}_{l2}^{(n)}(t) = \bar{w}_{l1}^{(n)} \cdot \int_t^{e_l^{(n)}} k_2^{(3)}(s,t)ds + \bar{w}_{l2}^{(n)} \cdot \left(-b_2^{(2)}(t) + \int_t^{e_l^{(n)}} k_4^{(4)}(s,t)ds\right)$$
$$+\sum_{k=l+1}^{n}\left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_2^{(3)}(s,t)ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_4^{(4)}(s,t)ds\right)$$

- For $0.7 < t < 1$ and $t \in E_l^{(n)}$, we have

$$\frac{d}{dt}\left(\widetilde{h}_{l2}^{(n)}(t)\right) = \bar{w}_{l1}^{(n)} \cdot \left(-k_2^{(4)}(t,t) + \int_t^{e_l^{(n)}} \frac{\partial}{\partial t}k_2^{(4)}(s,t)ds\right)$$
$$+\bar{w}_{l2}^{(n)} \cdot \left(-\frac{d}{dt}b_2^{(3)}(t) - k_4^{(4)}(t,t) + \int_t^{e_l^{(n)}} \frac{\partial}{\partial t}k_4^{(4)}(s,t)ds\right)$$
$$+\sum_{k=l+1}^{n}\left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t}k_2^{(4)}(s,t)ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} \frac{\partial}{\partial t}k_4^{(4)}(s,t)ds\right)$$

and

$$\widetilde{h}_{l2}^{(n)}(t) = \bar{w}_{l1}^{(n)} \cdot \int_t^{e_l^{(n)}} k_2^{(4)}(s,t)ds + \bar{w}_{l2}^{(n)} \cdot \left(-b_2^{(3)}(t) + \int_t^{e_l^{(n)}} k_4^{(4)}(s,t)ds\right)$$
$$+\sum_{k=l+1}^{n}\left(\bar{w}_{k1}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_2^{(4)}(s,t)ds + \bar{w}_{k2}^{(n)} \cdot \int_{\bar{E}_k^{(n)}} k_4^{(4)}(s,t)ds\right)$$

Now, in the following table, for different values of n^*, we present the error bounds $\widehat{\varepsilon}_n$ and $\widetilde{\varepsilon}_n$ by using the computational procedure that does not need to solve the dual problem (D_n). We also present the tighter error bound ε_n by using the computational procedure that needs to solve the dual problem (D_n). We denote by

$$V(\text{CLP}_n) = \int_0^T \mathbf{a}(t)^\top \widehat{\mathbf{z}}^{(n)}(t)dt$$

the approximate optimal objective value of problem (CLP). Theorem 4.1 and Proposition 4.2 say that

$$0 \le V(\text{CLP}) - V(\text{CLP}_n) \le \varepsilon_n$$

and

$$0 \le V(\text{CLP}_n) - V(\text{P}_n) \le V(\text{CLP}) - V(\text{P}_n) \le \varepsilon_n.$$

n^*	$n = n^* \cdot 8$	$\widetilde{\varepsilon}_n$	$\widehat{\varepsilon}_n$	ε_n	$V(\mathrm{P}_n)$	$V(\mathrm{CLP}_n)$
2	16	0.0903604	0.0812054	0.0264411	0.0309126	0.0334230
10	80	0.0212548	0.0207730	0.0055320	0.0376025	0.0381911
50	400	0.0044324	0.0044117	0.0011428	0.0391086	0.0392302
100	800	0.0022285	0.0022232	0.0005738	0.0393012	0.0393622
200	1600	0.0011173	0.0011160	0.0002875	0.0393979	0.0394284
300	2400	0.0007456	0.0007450	0.0001918	0.0394302	0.0394505
400	3200	0.0005594	0.0005591	0.0001439	0.0394463	0.0394616
500	4000	0.0004477	0.0004475	0.0001151	0.0394560	0.0394682

The numerical results are obtained by using MATLAB in which the simplex method is used to solve the primal problem (P_n) and the active set method is used to solve the dual problem (D_n). The reason is that there are some warning messages from MATLAB when the active set method is used to solve the primal problem (P_n) and the simplex method is used to solve the dual problem (D_n) for large n.

As we can see that the error bounds $\widetilde{\varepsilon}_n$, $\widehat{\varepsilon}_n$ and ε_n satisfies (6.11), the computational procedure for obtaining the tighter upper error bound ε_n needs much CPU time, and the computational procedure for obtaining the upper error bound $\widetilde{\varepsilon}_n$ saves the CPU time.

Suppose that the decision-maker can tolerate the error $\epsilon = 0.0005$. Then we see that $n^* = 100$ is sufficient to achieve this error ϵ by referring to the tighter upper error bound ε_n. However, if we consider the error bounds $\widetilde{\varepsilon}_n$ or $\widehat{\varepsilon}_n$, we need to take $n^* = 400$ in order to achieve this error ϵ. As a matter of fact, taking $n^* = 400$ is over-estimating the error. As we can see from the tighter upper error bound ε_n for $n^* = 400$, the error bound can be reduced to 0.0001439.

In order to achieve the error $\epsilon = 0.0005$, it is sufficient to solve problem (P_n) for $n^* = 100$ (i.e., $n = 800$) rather than solving problem (P_n) for taking $n^* = 400$ (i.e., $n = 3200$). The trade-off between CPU time and error bound should be carefully evaluated by the decision-makers based on their actually working environment.

Conflict of interest

The author declares no conflict of interest in this manuscript.

References

1. E. J. Anderson, P. Nash and A. F. Perold, *Some properties of a class of continuous linear programs*, SIAM J. Control Optim., **21** (1983), 758–765.
2. E. J. Anderson and A. B. Philpott, *On the solutions of a class of continuous linear programs*, SIAM J. Control Optimi., **32** (1994), 1289–1296.
3. E. J. Anderson and M. C. Pullan, *Purification for separated continuous linear programs*, Math. Methods Oper. Res., **43** (1996), 9–33.
4. R. E. Bellman, *Dynamic Programming*, Princeton University Press, Princeton, 1957.
5. R. N. Buie and J. Abrham, *Numerical solutions to continuous linear programming problems*, Math. Methods Oper. Res., **17** (1973), 107–117.

6. W. H. Farr, M. A. Hanson, *Continuous Time Programming with Nonlinear Constraints*, J. Math. Anal. Appl., **45** (1974), 96–115.
7. W. H. Farr, M. A. Hanson, *Continuous Time Programming with Nonlinear Time-Delayed Constraints*, J. Math. Anal. Appl., **46** (1974), 41–61.
8. L. Fleischer and J. Sethuraman, *Efficient algorithms for separated continuous linear programs: the multicommodity flow problem with holding costs and extensions*, Math. Oper. Res., **30** (2005), 916–938.
9. R. C. Grinold, *Continuous Programming Part One: Linear Objectives,* J. Math. Anal. Appl., **28** (1969), 32–51.
10. R. C. Grinold, *Continuous Programming Part Two: Nonlinear Objectives,* J. Math. Anal. Appl., **27** (1969), 639–655.
11. M. A. Hanson and B. Mond, *A Class of Continuous Convex Programming Problems,* J. Math. Anal. Appl., **22** (1968), 427–437.
12. N. Levinson, *A class of continuous linear programming problems*, J. Math. Anal. Appl., **16** (1966), 73–83.
13. R. Meidan and A. F. Perold, *Optimality conditions and strong duality in abstract and continuous-time linear programming*, J. Optimiz. Theory App., **40** (1983), 61–77.
14. S. Nobakhtian, M. R. Pouryayevali, *Optimality Criteria for Nonsmooth Continuous-Time Problems of Multiobjective Optimization,* J. Optimiz. Theory App., **136** (2008), 69–76.
15. S. Nobakhtian, M. R. Pouryayevali, *Duality for Nonsmooth Continuous-Time Problems of Vector Optimization,* J. Optimiz. Theory App., **136** (2008), 77–85.
16. N. S. Papageorgiou, *A class of infinite dimensional linear programming problems*, J. Math. Anal. Appl., **87** (1982), 228–245.
17. M. C. Pullan, *An algorithm for a class of continuous linear programs*, SIAM J. Control Optim., **31** (1993), 1558–1577.
18. M. C. Pullan, *Forms of optimal solutions for separated continuous linear programs*, SIAM J. Control Optim., **33** (1995), 1952–1977.
19. M.C. Pullan, *A duality theory for separated continuous linear programs*, SIAM J. Control Optim., **34** (1996), 931–965.
20. M. C. Pullan, *Convergence of a general class of algorithms for separated continuous linear programs*, SIAM J. Optimiz., **10** (2000), 722–731.
21. M. C. Pullan, *An extended algorithm for separated continuous linear programs*, Math. Program., **93** (2002), 415–451.
22. T. W. Reiland, *Optimality Conditions and Duality in Continuous Programming I: Convex Programs and a Theorem of the Alternative,* J. Math. Anal. Appl., **77** (1980), 297–325.
23. T. W. Reiland, *Optimality Conditions and Duality in Continuous Programming II: The Linear Problem Revisited,* J. Math. Anal. Appl., **77** (1980), 329–343.
24. T. W. Reiland, M. A. Hanson, *Generalized Kuhn-Tucker Conditions and Duality for Continuous Nonlinear Programming Problems,* J. Math. Anal. Appl., **74** (1980), 578–598.

25. F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Frederick Ungar Publishing Co., New York, 1955.
26. M. A. Rojas-Medar, J. V. Brandao, G. N. Silva, *Nonsmooth Continuous-Time Optimization Problems: Sufficient Conditions*, J. Math. Anal. Appl., **227** (1998), 305–318.
27. M. Schechter, *Duality in continuous linear programming*, J. Math. Anal. Appl., **37** (1972), 130–141.
28. E. Shindin, G. Weiss, *Structure of Solutions for Continuous Linear Programs with Constant Coefficients,* SIAM J. Optimiz., **25** (2015), 1276–1297.
29. C. Singh, *A Sufficient Optimality Criterion in Continuous Time Programming for Generalized Convex Functions,* J. Math. Anal. Appl., **62** (1978), 506–511.
30. C. Singh, W. H. Farr, *Saddle-Point Optimality Criteria of Continuous Time Programming without Differentiability,* J. Math. Anal. Appl., **59** (1977), 442–453.
31. W. F. Tyndall, *A duality theorem for a class of continuous linear programming problems*, SIAM J. Appl. Math., **15** (1965), 644–666.
32. W. F. Tyndall, *An extended duality theorem for continuous linear programming problems*, SIAM J. Appl. Math., **15** (1967), 1294–1298.
33. X. Q. Wang, S. Zhang, D. D. Yao, *Separated Continuous Conic Programming: Strong Duality and an Approximation Algorithm*, SIAM J. Control Optim., **48** (2009), 2118–2138.
34. G. Weiss, *A simplex based algorithm to solve separated continuous linear programs*, Math. Program., **115** (2008), 151–198.
35. C.-F. Wen, H.-C. Wu, *Using the Dinkelbach-Type Algorithm to Solve the Continuous-Time Linear Fractional Programming Problems,* J. Global Optim., **49** (2011), 237–263.
36. C.-F. Wen, H.-C. Wu, *Using the Parametric Approach to Solve the Continuous-Time Linear Fractional Max-Min Problems,* J. Global Optim., **54** (2012), 129–153.
37. C.-F. Wen, H.-C. Wu, *The Approximate Solutions and Duality Theorems for the Continuous-Time Linear Fractional Programming Problems,* Numer. Func. Anal. Opt., **33** (2012), 80–129.
38. H.-C. Wu, *Solving the Continuous-Time Linear Programming Problems Based on the Piecewise Continuous Functions,* Numer. Func. Anal. Opt., **37** (2016), 1168–1201.
39. G. J. Zalmai, *Duality for a Class of Continuous-Time Homogeneous Fractional Programming Problems,* Zeitschrift für Operations Research, **30** (1986), A43–A48.
40. G. J. Zalmai, *Duality for a Class of Continuous-Time Fractional Programming Problems,* Utilitas Mathematica, **31** (1987), 209–218.
41. G. J. Zalmai, *Optimality Conditions and Duality for a Class of Continuous-Time Generalized Fractional Programming Problems,* J. Math. Anal. Appl., **153** (1990), 365–371.
42. G. J. Zalmai, *Optimality Conditions and Duality models for a Class of Nonsmooth Constrained Fractional Optimal Control Problems,* J. Math. Anal. Appl., **210** (1997), 114–149.