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Research article

Mean value of the Hardy sums over short intervals

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Abstract: The main purpose of this paper is using the mean value of Dirichlet L-functions and character sums to study a kind of mean value of the Hardy sums over short intervals, and give some interesting formulae.

Keywords: Hardy sum; L-functions; short intervals; character sums; mean value
Mathematics Subject Classification: 11F20, 11L40

1. Introduction

For a positive integer q and an arbitrary integer h, the Dedekind sum $S(h,q)$ is defined by

$$S(h,q)=\sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ not be an integer;}\\ 0, & \text{if } x \text{ be an integer.}\end{cases}$$

It plays an important role in the transformation theory of the Dedekind η function. The various properties of $S(h,q)$ were investigated by many authors, related works can be founded in [2–4] and [9,10]. Berndt [1] studied the following Hardy sums:

$$H(h,q)=\sum_{j=1}^{q-1}(-1)^{j+1+[\frac{hj}{q}]}, \qquad s_1(h,q)=\sum_{j=1}^{q-1}(-1)^{[\frac{hj}{q}]}\left(\left(\frac{j}{q}\right)\right),$$

$$s_2(h,q)=\sum_{j=1}^{q-1}(-1)^{j}\left(\left(\frac{j}{q}\right)\right)\left(\left(\frac{hj}{q}\right)\right), \qquad s_3(h,q)=\sum_{j=1}^{q-1}(-1)^{j}\left(\left(\frac{hj}{q}\right)\right),$$

$$s_4(h,q)=\sum_{j=1}^{q-1}(-1)^{[\frac{hj}{q}]}, \qquad s_5(h,q)=\sum_{j=1}^{q-1}(-1)^{j+[\frac{hj}{q}]}\left(\left(\frac{j}{q}\right)\right).$$

In [6], R. Sitaramachandra Rao expressed Hardy sums in terms of Dedekind sum $S(h,q)$ as following:

$$\begin{aligned} H(h,q) &= -8S(h+q,2q)+4S(h,q), && \text{if } h{+}q \text{ is odd;}\\ s_1(h,q) &= 2S(h,q)-4S(h,2q), && \text{if } h \text{ is even;}\\ s_2(h,q) &= -S(h,q)+2S(2h,q), && \text{if } q \text{ is even;}\\ s_3(h,q) &= 2S(h,q)-4S(2h,q), && \text{if } q \text{ is odd;}\\ s_4(h,q) &= -4S(h,q)+8S(h,2q), && \text{if } h \text{ is odd;}\\ s_5(h,q) &= -10S(h,q)+4S(2h,q)+4S(h,2q), && \text{if } h{+}q \text{ is even.} \end{aligned}$$

Each one of $H(h,q)$ ($h+q$ even), $s_1(h,q)$ (h odd), $s_2(h,q)$ (q odd), $s_3(h,q)$ (q even), $s_4(h,q)$ (h even), $s_5(h,q)$ ($h+q$ odd) is zero. Z. Xu and W. Zhang [8] proved that if p is a prime and $\bar{b}$ denotes the multiplicative inverse of b mod$p$, then

$$\sum_{a<\frac{p}{4}}\sum_{b<\frac{p}{4}} H(2a\bar{b},p)=\frac{3}{16}p^2+O(p^{1+\epsilon}).$$

W. Liu [5] proved that if $p\geq 5$ is a prime and $\bar{b}$ denotes the multiplicative inverse of b mod$p$, then

$$\sum_{a<\frac{p}{3}}\sum_{b<\frac{p}{3}} H(2a\bar{b},p)=\frac{1}{5}p^2+O(p^{1+\epsilon}),$$

$$\sum_{a<\frac{p}{3}}\sum_{b<\frac{p}{4}} H(2a\bar{b},p)=\frac{27}{320}p^2+O(p^{1+\epsilon}).$$

Let $1<N<p$, is there a mean value distribution formula of the Hardy sum in the short interval $[1,N]$? In fact, it is very difficult to get a asymptotic formula for the mean value in the case of one variable as

$$\sum_{a<N} H(a,p).$$

In this paper, we use the mean value of L-functions and estimate of character sums to study the mean value

$$\sum_{a\leq N}\sum_{b\leq N} a^l b^k H(2a\overline{b},p),$$

where l,k be two non-negative integers and $\overline{b}$ denote the multiplicative inverse of b modulo p, and obtain the following main conclusion:

Theorem. *Let* $p>2$ *be a prime and* N *a positive integer with* $1<N<p$*,* l *and* k *be two non-negative integers. Then we have*

$$\begin{aligned} &\sum_{a\leq N}\sum_{b\leq N} a^l b^k H(2a\overline{b},p)\\ =&\frac{C(l,k)}{7\cdot 2^{l-2}\zeta(3)(l+k+1)}pN^{l+k+1} \end{aligned}$$

$$+\begin{cases} O\left(pN^{l+k}\log^2 N + N^{l+k+2}p^{o(1)}\right), & \text{if } l=k=0;\ l=0, k\geq 1 \text{ or } l\geq 1, k=0; \\ O\left(pN^{l+k}\log N + N^{l+k+2}p^{o(1)}\right), & \text{if } l\geq 1, k\geq 1. \end{cases}$$

here

$$C(l,k) = -\frac{7\zeta(3)}{8} - \sum_{\substack{a=1 \\ (a,2)=1,(b,2)=1}}^{\infty}\sum_{b=a+1}^{\infty}\frac{a^{l-1}}{b^{l+2}} - \sum_{\substack{b=1 \\ (a,2)=1,(b,2)=1}}^{\infty}\sum_{a=b+1}^{2b}\frac{a^{l-1}}{b^{l+2}} - 2^{l+k+1}\sum_{\substack{b=1 \\ (a,2)=1,(b,2)=1}}^{\infty}\sum_{a=2b+1}^{\infty}\frac{b^{k-1}}{a^{k+2}}$$

is a constant depending on l and k.

It is clear that the asymptotic formulas in the Theorem are non-trivial in the range $p^{\epsilon} < N < p^{1-\epsilon}$.

From the theorem we may immediately deduce the following corollaries:

Corollary 1. *Let $p > 2$ be a prime and N an integer with $1 < N < p$. Then we have*

$$\sum_{a\leq N}\sum_{b\leq N} aH(2a\overline{b}, p) = \frac{C(1,0)}{7\zeta(3)}pN^2 + O(pN\log^2 N + N^3 p^{o(1)}),$$

where

$$C(1,0) = \frac{7}{16}\zeta(3) - \frac{\pi^2}{24} - \sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{1}{(a+2b)^3} - \frac{9}{2}\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{1}{(a+2b)^2 b} + 2\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{1}{(a+4b)^2 b}$$

is a constant.

Corollary 2. *Let $p > 2$ be a prime and N an integer with $1 < N < p$. Then we have*

$$\sum_{a\leq N}\sum_{b\leq N} abH(2a\overline{b}, p) = \frac{2C(1,1)}{21\zeta(3)}pN^3 + O(pN^2\log N + N^4 p^{o(1)}),$$

where

$$C(1,1) = \frac{\pi^2}{8} - \frac{25}{16}\zeta(3) - 10\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{1}{(a+2b)^3} + 8\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\frac{1}{(a+4b)^3}$$

is a constant.

2. Some Lemmas

To complete the proof of the theorem, we need several lemmas.

Lemma 1. *Let $q \geq 3$ and h be two integers with $(h,q) = 1$. Then we have*

$$S(h,q) = \frac{1}{\pi^2 q}\sum_{d|q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}}\chi(h)|L(1,\chi)|^2,$$

where $\phi(d)$ is the Euler function.

Proof. See Lemma 1 of reference [9].

Lemma 2. *Let $q > 1$ be an odd integer and h an integer with $(h,q) = 1$. Then we have*

$$H(h,q) = \begin{cases} \frac{-16}{\pi^2 q}\sum_{d|q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}}\chi(h)|L(1,\chi\chi_2^0)|^2, & 2 \mid h; \\ 0, & 2 \nmid h. \end{cases}$$

where χ_2^0 be a principal Dirichlet character modulo 2.
Proof. From [1] , we have

$$\begin{aligned} H(h,q) =& -8S(h+q,2q)+4S(h,q) \\ =& -\frac{4}{\pi^2 q}\sum_{d|2q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(h+q)|L(1,\chi)|^2+\frac{4}{\pi^2 q}\sum_{d|q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(h)|L(1,\chi)|^2. \end{aligned}$$

Note that q be an odd integer, we can get

$$\sum_{d|2q} f(d)=\sum_{d|q} f(d)+\sum_{d|q} f(2d)$$

Hence we can write

$$\begin{aligned} H(h,q) =& -\frac{4}{\pi^2 q}\left(\sum_{d|q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(h+q)|L(1,\chi)|^2+\sum_{d|q}\frac{(2d)^2}{\phi(2d)}\sum_{\substack{\chi \bmod 2d\\ \chi(-1)=-1}}\chi(h+q)|L(1,\chi)|^2\right) \\ &+\frac{4}{\pi^2 q}\sum_{d|q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(h)|L(1,\chi)|^2. \\ =& -\frac{16}{\pi^2 q}\sum_{d|q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(h+q)\chi_2^0(h+q)|L(1,\chi\chi_2^0)|^2 \\ =& \begin{cases} \frac{-16}{\pi^2 q}\sum\limits_{d|q}\frac{d^2}{\phi(d)}\sum\limits_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(h)|L(1,\chi\chi_2^0)|^2, & 2\mid h; \\ 0, & 2\nmid h. \end{cases} \end{aligned}$$

This proves Lemma 2.
Lemma 3. *Let $p>2$ be a prime, a be a positive integer with $(a,p)=1$. Then we have the asymptotic formula*

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2a\bar{b})|L(1,\chi\chi_2^0)|^2=\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2a\bar{b})\left|\sum_{1\le u\le p^2}\frac{\chi\chi_2^0(u)}{u}\right|^2+O\left(\frac{\log^2 p}{\sqrt{p}}\right).$$

Proof. For convenience, we let

$$A(\chi,y)=\sum_{p^2<u\le y}\chi(u).$$

Then applying Abel's identity we have

$$L(1,\chi\chi_2^0)=\sum_{1\le u\le p^2}\frac{\chi\chi_2^0(u)}{u}+\int_{p^2}^{\infty}\frac{A(\chi\chi_2^0,y)}{y^2}dy.$$

Hence

$$\begin{aligned}
&\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2a\bar{b})|L(1,\chi\chi_2^0)|^2\\
=&\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2a\bar{b})\left|\sum_{1\le u\le p^2}\frac{\chi\chi_2^0(u)}{u}+\int_{p^2}^{\infty}\frac{A(\chi\chi_2^0,y)}{y^2}dy\right|^2\\
=&\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2a\bar{b})\left|\sum_{1\le u\le p^2}\frac{\chi\chi_2^0(u)}{u}\right|^2+\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2a\bar{b})\left(\sum_{1\le u\le p^2}\frac{\chi\chi_2^0(u)}{u}\right)\left(\int_{p^2}^{\infty}\frac{A(\overline{\chi\chi_2^0},y)}{y^2}dy\right)\\
&+\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2a\bar{b})\left(\sum_{1\le u\le p^2}\frac{\overline{\chi\chi_2^0}(u)}{u}\right)\left(\int_{p^2}^{\infty}\frac{A(\chi\chi_2^0,y)}{y^2}dy\right)+\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2a\bar{b})\left|\int_{p^2}^{\infty}\frac{A(\chi\chi_2^0,y)}{y^2}dy\right|^2.
\end{aligned}$$

Using the Cauchy inequality and the estimate $|\sum_{m\le n\le M}\chi(n)|\ll\sqrt{p}\log p$, we have the estimates

$$\begin{aligned}
&\left|\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2a\bar{b})\left(\sum_{1\le u\le p^2}\frac{\chi\chi_2^0(u)}{u}\right)\left(\int_{p^2}^{\infty}\frac{A(\overline{\chi\chi_2^0},y)}{y^2}dy\right)\right|\\
\ll&\left[\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|\sum_{1\le u\le p^2}\frac{\chi\chi_2^0(u)}{u}\right|^2\right]^{\frac12}\left[\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|\int_{p^2}^{\infty}\frac{A(\overline{\chi\chi_2^0},y)}{y^2}dy\right|^2\right]^{\frac12}\\
\ll&\frac{\log p}{\sqrt{p}},
\end{aligned}$$

$$\left|\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2a\bar{b})\left(\sum_{1\le u\le p^2}\frac{\overline{\chi\chi_2^0}(u)}{u}\right)\left(\int_{p^2}^{\infty}\frac{A(\chi\chi_2^0,y)}{y^2}dy\right)\right|\ll\frac{\log p}{\sqrt{p}},$$

and

$$\left|\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2a\bar{b})\left|\int_{p^2}^{\infty}\frac{A(\chi\chi_2^0,y)}{y^2}dy\right|^2\right|\ll\frac{\log^2 p}{p^2}.$$

So we may immediately get the asymptotic formula

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2a\bar{b})|L(1,\chi\chi_2^0)|^2=\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2a\bar{b})\left|\sum_{1\le u\le p^2}\frac{\chi\chi_2^0(u)}{u}\right|^2+O\left(\frac{\log^2 p}{\sqrt{p}}\right).$$

This proves Lemma 3.

Lemma 4. *Let $p > 2$ be a prime and N an integer with $1 < N < p$. Then*

$$\sum_{a\le N}\sum_{b\le N}\sum_{\chi \bmod p}\chi(-2a\overline{b})\left|\sum_{1\le u\le p^2}\frac{\chi\chi_2^0(u)}{u}\right|^2 = O\left(N^2p^{o(1)}\right).$$

Proof. From the orthogonality relation for character sums modulo p we have

$$\begin{aligned}
&\sum_{a\le N}\sum_{b\le N}\sum_{\chi \bmod p}\chi(-2a\overline{b})\left|\sum_{1\le u\le p^2}\frac{\chi\chi_2^0(u)}{u}\right|^2\\
&=\sum_{a\le N}\sum_{b\le N}\sum_{\chi \bmod p}\chi(-2a\overline{b})\sum_{1\le u\le p^2}\frac{\chi\chi_2^0(u)}{u}\sum_{1\le v\le p^2}\frac{\overline{\chi\chi_2^0}(v)}{v}\\
&=\phi(p)\sum_{\substack{a\le N\\ -2au\equiv bv(\bmod p),(u,2p)=1,(v,2p)=1}}\sum_{b\le N}\sum_{1\le u\le p^2}\sum_{1\le v\le p^2}\frac{1}{uv},
\end{aligned}$$

Let $J = \lfloor 2\log p\rfloor$. By using the similar method used in [7] we can write

$$\begin{aligned}
&\phi(p)\sum_{\substack{a\le N\\ -2au\equiv bv(\bmod p),(u,2p)=1,(v,2p)=1}}\sum_{b\le N}\sum_{1\le u\le p^2}\sum_{1\le v\le p^2}\frac{1}{uv}\\
&\le \phi(p)\sum_{i,j=0}^{J}\sum_{e^i\le u<e^{i+1}}\frac{1}{u}\sum_{e^j\le v<e^{j+1}}\frac{1}{v}\sum_{\substack{a,b=1\\ -2au\equiv bv(\bmod p)}}^{N}1\\
&\le 2\phi(p)\sum_{0\le i\le j\le J}\sum_{e^i\le u<e^{i+1}}\frac{1}{u}\sum_{e^j\le v<e^{j+1}}\frac{1}{v}\sum_{\substack{a,b=1\\ -2au\equiv bv(\bmod p)}}^{N}1\\
&\le 2\phi(p)\sum_{0\le i\le j\le J}e^{-(i+j)}\sum_{e^i\le u<e^{i+1}}\sum_{e^j\le v<e^{j+1}}\sum_{\substack{a,b=1\\ -2au\equiv bv(\bmod p)}}^{N}1
\end{aligned}$$

Thus,

$$\sum_{a\le N}\sum_{b\le N}\sum_{\chi \bmod p}\chi(-2a\overline{b})\left|\sum_{1\le u\le p^2}\frac{\chi\chi_2^0(u)}{u}\right|^2 \le 2\phi(p)\sum_{0\le i\le j\le J}e^{-(i+j)}K_{i,j},$$

where $K_{i,j}$ is the number of solutions (a, b, u, v) to the congruence $-2au \equiv bv(\mathrm{mod} p)$ with $1 \le a, b \le N, e^i \le u < e^{i+1}$ and $e^j \le v < e^{j+1}$.

For a solution (a, b, u, v), write $bv = |k|p - 2au$. We have

$$p \le |k|p \le \max\{2au + bv\} \le N(2e^{i+1} + e^{j+1}) \le 3e^{j+1}N.$$

So the product bv can take at most

$$\frac{3e^{j+1}N}{p}\cdot e^{i+1}\cdot N=\frac{3e^{i+j+2}N^2}{p}$$

values. It is clear that if a, u, k are fixed, then b and v can take at most $p^{o(1)}$ possible values. Hence,

$$K_{i,j}\leq\frac{3e^{i+j+2}N^2}{p}\cdot p^{o(1)}.$$

Noting that

$$\sum_{0\leq i\leq j\leq J}e^{-(i+j)}=O(1).$$

So we have

$$\sum_{a\leq N}\sum_{b\leq N}\sum_{\chi\bmod p}\chi(-2a\bar{b})\left|\sum_{1\leq u\leq p^2}\frac{\chi\chi_2^0(u)}{u}\right|^2=O\left(N^2p^{o(1)}\right).$$

This proves Lemma 4.

3. Proof of Theorem

In this section we complete the proof of the Theorem.

From Lemma 2 and Lemma 3 we have

$$\begin{aligned}
&\sum_{a\leq N}\sum_{b\leq N}a^lb^kH(2a\bar{b},p)\\
=&-\frac{16p}{\pi^2\phi(p)}\sum_{a\leq N}\sum_{b\leq N}a^lb^k\sum_{\substack{\chi\bmod p\\ \chi(-1)=-1}}\chi(2a\bar{b})\left|L(1,\chi\chi_2^0)\right|^2\\
=&-\frac{16p}{\pi^2\phi(p)}\sum_{a\leq N}\sum_{b\leq N}a^lb^k\left(\sum_{\substack{\chi\bmod p\\ \chi(-1)=-1}}\chi(2a\bar{b})\left|\sum_{1\leq u\leq p^2}\frac{\chi\chi_2^0(u)}{u}\right|^2+O\left(\frac{\log^2 p}{\sqrt{p}}\right)\right)\\
=&-\frac{8p}{\pi^2\phi(p)}\sum_{a\leq N}\sum_{b\leq N}a^lb^k\sum_{\chi\bmod p}(1-\chi(-1))\chi(2a\bar{b})\left|\sum_{1\leq u\leq p^2}\frac{\chi\chi_2^0(u)}{u}\right|^2\\
&+O\left(\frac{N^{l+k+2}\sqrt{p}\log^2 p}{\phi(p)}\right)\\
=&-\frac{8p}{\pi^2\phi(p)}\sum_{a\leq N}\sum_{b\leq N}a^lb^k\sum_{\chi\bmod p}\chi(2a\bar{b})\left|\sum_{1\leq u\leq p^2}\frac{\chi\chi_2^0(u)}{u}\right|^2+O\left(\frac{N^{l+k+2}\sqrt{p}\log^2 p}{\phi(p)}\right)\\
&+\frac{8p}{\pi^2\phi(p)}\sum_{a\leq N}\sum_{b\leq N}a^lb^k\sum_{\chi\bmod p}\chi(-2a\bar{b})\left|\sum_{1\leq u\leq p^2}\frac{\chi\chi_2^0(u)}{u}\right|^2
\end{aligned}$$

$$:= M_1 + M_2 + O\left(\frac{N^{l+k+2}\sqrt{p}\log^2 p}{\phi(p)}\right) \tag{1}$$

From the orthogonality relations for characters modulo p, we have

$$\begin{aligned}
M_1 &= -\frac{8p}{\pi^2\phi(p)}\sum_{a\le N}\sum_{b\le N} a^l b^k \sum_{\chi \bmod p}\chi(2a\bar{b})\left|\sum_{1\le u\le p^2}\frac{\chi\chi_2^0(u)}{u}\right|^2 \\
&= -\frac{8p}{\pi^2}\sum_{\substack{a\le N\ b\le N\ 1\le u\le p^2\ 1\le v\le p^2 \\ 2au\equiv bv(\bmod p),(u,2)=1,(v,2)=1}}\sum\sideset{}{'}\sum\sideset{}{'}\sum \frac{a^l b^k}{uv} \\
&= -\frac{8p}{\pi^2}\sum_{\substack{a\le N\ b\le N\ 1\le u\le p^2\ 1\le v\le p^2 \\ 2au=bv,(u,2)=1,(v,2)=1}}\sum\sideset{}{'}\sum\sideset{}{'}\sum \frac{a^l b^k}{uv} - \frac{8p}{\pi^2}\sum_{\substack{a\le N\ b\le N\ 1\le u\le p^2\ 1\le v\le p^2 \\ 2au\equiv bv, 2au\ne bv,(u,2)=1,(v,2)=1}}\sum\sideset{}{'}\sum\sideset{}{'}\sum \frac{a^l b^k}{uv} \\
&:= M_{11} + M_{12}
\end{aligned}$$

here $\sideset{}{'}\sum_{1\le u\le p^2}$ denotes the summation over u from 1 to p^2 with $(u,p)=1$.

Now, we calculate M_{11}. First, we write

$$\begin{aligned}
M_{11} &= -\frac{8p}{\pi^2}\sum_{\substack{a\le N\ b\le N\ 1\le u\le p^2\ 1\le v\le p^2 \\ 2au=bv,(u,2)=1,(v,2)=1}}\sum\sideset{}{'}\sum\sideset{}{'}\sum \frac{a^l b^k}{uv} \\
&= -\frac{2^{k+3}p}{\pi^2}\sum_{\substack{d\le N\ a\le \frac{N}{d}\ b\le\frac{N}{2d}\ 1\le v\le \min\left\{\frac{p^2}{a},\frac{p^2}{b}\right\} \\ (a,b)=1,(a,2)=1,(v,2)=1,(b,2)=1}}\sum\sum\sideset{}{'}\sum \frac{d^{l+k}a^{l-1}b^{k-1}}{v^2} \\
&= -\frac{2^{k+3}p}{\pi^2}\sum_{\substack{a\le N\ b\le N\ d\le\min\left\{\frac{N}{a},\frac{N}{2b}\right\}\ 1\le v\le \min\left\{\frac{p^2}{a},\frac{p^2}{b}\right\} \\ (a,b)=1,(a,2)=1,(v,2)=1,(b,2)=1}}\sum\sum\sideset{}{'}\sum \frac{d^{l+k}a^{l-1}b^{k-1}}{v^2} \\
&= -\frac{2^{k+3}p}{\pi^2}\sum_{\substack{1\le a<N\ a<b\le N\ d\le\frac{N}{2b}\ 1\le v\le\frac{p^2}{b} \\ (a,b)=1,(a,2)=1,(v,2)=1,(b,2)=1}}\sum\sum\sideset{}{'}\sum \frac{d^{l+k}a^{l-1}b^{k-1}}{v^2} - \frac{2^{k+3}p}{\pi^2}\sum_{\substack{1\le b<\frac{N}{2}\ 2b<a\le N\ d\le\frac{N}{a}\ 1\le v\le\frac{p^2}{a} \\ (a,b)=1,(a,2)=1,(v,2)=1,(b,2)=1}}\sum\sum\sideset{}{'}\sum \frac{d^{l+k}a^{l-1}b^{k-1}}{v^2} \\
&\quad - \frac{2^{k+3}p}{\pi^2}\sum_{\substack{1\le b\le\frac{N}{2}\ b<a<2b\ d\le\frac{N}{2b}\ 1\le v\le\frac{p^2}{a} \\ (a,b)=1,(a,2)=1,(v,2)=1,(b,2)=1}}\sum\sum\sideset{}{'}\sum \frac{d^{l+k}a^{l-1}b^{k-1}}{v^2} - \frac{2^{k+3}p}{\pi^2}\sum_{d\le\frac{N}{2}}\sideset{}{'}\sum_{\substack{1\le v\le p^2 \\ (v,2)=1}}\frac{d^{l+k}}{v^2}.
\end{aligned}$$

Note that

$$\sideset{}{'}\sum_{\substack{1\le v\le p^2 \\ (v,2)=1}} v^{-2} = \frac{\pi^2}{8} + O(p^{-2})$$

and $l+k \geq 0$, we can get

$$-\frac{2^{k+3}p}{\pi^2}\sum_{d\leq \frac{N}{2}}\sideset{}{'}\sum_{\substack{1\leq v\leq p^2\\(v,2)=1}}\frac{d^{l+k}}{v^2}=\frac{-pN^{l+k+1}}{2^{l+1}(l+k+1)}+O(pN^{l+k}).$$

So, we can write

$$M_{11}:=A+B+C-\frac{pN^{l+k+1}}{2^{l+1}(l+k+1)}+O(pN^{l+k}). \tag{2}$$

We shall calculate the first three terms in the expression (2). First we calculate A. Write

$$\begin{aligned}A=&-\frac{2^{k+3}p}{\pi^2}\sum_{\substack{1\leq a<N\\(a,b)=1,(a,2)=1,(v,2)=1,(b,2)=1}}\sum_{a<b\leq N}\sum_{d\leq\frac{N}{2b}}\sideset{}{'}\sum_{1\leq v\leq\frac{p^2}{b}}\frac{d^{l+k}a^{l-1}b^{k-1}}{v^2}\\=&\frac{-pN^{l+k+1}}{2^{l+1}(l+k+1)}\sum_{\substack{1\leq a<N\\(a,b)=1,(a,2)=1,(b,2)=1}}\sum_{a<b\leq N}a^{l-1}b^{-l-2}+O\left(pN^{l+k}\sum_{1\leq a<N}\sum_{a<b\leq N}a^{l-1}b^{-l-1}\right),\end{aligned}$$

note that

$$\begin{aligned}\sum_{\substack{1\leq a<N\\(a,b)=1,(a,2)=1,(b,2)=1}}\sum_{a<b\leq N}a^{l-1}b^{-l-2}&=\sum_{\substack{1\leq a<N\\(a,2)=1,(b,2)=1}}\sum_{a<b\leq N}a^{l-1}b^{-l-2}\sum_{d|(a,b)}\mu(d)\\&=\sum_{\substack{1\leq a<N\\(a,2)=1,(b,2)=1}}\sum_{a<b\leq N}a^{l-1}b^{-l-2}\sum_{\substack{d\leq\frac{N}{b}\\(d,2)=1}}\frac{\mu(d)}{d^3},\end{aligned}$$

we have

$$\begin{aligned}A=&\frac{-pN^{l+k+1}}{2^{l+1}(l+k+1)}\sum_{\substack{1\leq a<N\\(a,2)=1,(b,2)=1}}\sum_{a<b\leq N}a^{l-1}b^{-l-2}\left(\sum_{\substack{d=1\\(d,2)=1}}^{\infty}\frac{\mu(d)}{d^3}-\sum_{\substack{d>\frac{N}{b}\\(d,2)=1}}\frac{\mu(d)}{d^3}\right)\\&+O\left(pN^{l+k}\sum_{1\leq a<N}\sum_{a<b\leq N}a^{l-1}b^{-l-1}\right).\end{aligned}$$

By using the identity

$$\sum_{\substack{d=1\\(d,2)=1}}^{\infty}\frac{\mu(d)}{d^s}=\frac{2^s}{(2^s-1)\zeta(s)},$$

we have

$$A=\frac{-pN^{l+k+1}}{7\cdot 2^{l-2}\zeta(3)(l+k+1)}\sum_{\substack{1\leq a<N\\(a,2)=1,(b,2)=1}}\sum_{a<b\leq N}a^{l-1}b^{-l-2}+O\left(pN^{l+k}\sum_{1\leq a<N}\sum_{a<b\leq N}a^{l-1}b^{-l-1}\right). \tag{3}$$

For the case $l = 0$, we have

$$A = \frac{-4pN^{k+1}}{7\zeta(3)(k+1)} \sum_{\substack{1\le a<N \\ (a,2)=1,}} \sum_{\substack{a<b\le N \\ (b,2)=1}} \frac{1}{ab^2} + O\left(pN^k \sum_{1\le a<N} \sum_{a<b\le N} \frac{1}{ab}\right)$$

$$= \frac{-4pN^{k+1}}{7\zeta(3)(k+1)} \left(\sum_{\substack{a=1 \\ (a,2)=1,}}^{\infty} \sum_{\substack{b=a+1 \\ (b,2)=1}}^{\infty} \frac{1}{ab^2} - \sum_{\substack{a<N \\ (a,2)=1,}} \sum_{\substack{b>N \\ (b,2)=1}} \frac{1}{ab^2} - \sum_{\substack{a>N \\ (a,2)=1,}} \sum_{\substack{b>a \\ (b,2)=1}} \frac{1}{ab^2} \right) + O\left(pN^k \log^2 N\right)$$

$$= \frac{-4pN^{k+1}}{7\zeta(3)(k+1)} \sum_{\substack{a=1 \\ (a,2)=1,}}^{\infty} \sum_{\substack{b=a+1 \\ (b,2)=1}}^{\infty} \frac{1}{ab^2} + O(pN^k \log^2 N). \tag{4}$$

For the case $l \geq 1$, we have

$$A = \frac{-pN^{l+k+1}}{7 \cdot 2^{l-2}\zeta(3)(l+k+1)} \sum_{\substack{1\le a<N \\ (a,2)=1,}} \sum_{\substack{a<b\le N \\ (b,2)=1}} a^{l-1}b^{-l-2} + O\left(pN^{l+k} \sum_{1\le a<N} \sum_{a<b\le N} a^{l-1}b^{-l-1}\right)$$

$$= \frac{-pN^{l+k+1}}{7 \cdot 2^{l-2}\zeta(3)(l+k+1)} \left(\sum_{\substack{a=1 \\ (a,2)=1,}}^{\infty} \sum_{\substack{b=a+1 \\ (b,2)=1}}^{\infty} \frac{a^{l-1}}{b^{l+2}} - \sum_{\substack{a<N \\ (a,2)=1,}} \sum_{\substack{b>N \\ (b,2)=1}} \frac{a^{l-1}}{b^{l+2}} - \sum_{\substack{a>N \\ (a,2)=1,}} \sum_{\substack{b>a \\ (b,2)=1}} \frac{a^{l-1}}{b^{l+2}} \right)$$

$$+ O\left(pN^{l+k} \sum_{1\le a<N} \sum_{a<b\le N} a^{l-1}b^{-l-1}\right)$$

$$= \frac{-pN^{l+k+1}}{7 \cdot 2^{l-2}\zeta(3)(l+k+1)} \sum_{\substack{a=1 \\ (a,2)=1,}}^{\infty} \sum_{\substack{b=a+1 \\ (b,2)=1}}^{\infty} \frac{a^{l-1}}{b^{l+2}} + O(pN^{l+k} \log N). \tag{5}$$

By using the same method, we can calculate B:

$$B = -\frac{2^{k+3}p}{\pi^2} \sum_{\substack{1\le b<\frac{N}{2} \\ (a,b)=1,}} \sum_{\substack{2b<a\le N \\ (a,2)=1,}} \sum_{\substack{d\le \frac{N}{a} \\ (v,2)=1,}} \sum_{\substack{1\le v\le \frac{p^2}{a} \\ (b,2)=1}}{}' \frac{d^{l+k}a^{l-1}b^{k-1}}{v^2}$$

$$= \frac{-2^{k+3}pN^{l+k+1}}{7\zeta(3)(l+k+1)} \sum_{\substack{1\le b<\frac{N}{2} \\ (a,2)=1,}} \sum_{\substack{2b<a\le N \\ (b,2)=1}} a^{-k-2}b^{k-1} + O\left(pN^{l+k} \sum_{1\le b<\frac{N}{2}} \sum_{2b<a\le N} a^{-k-1}b^{k-1}\right).$$

For the case $k = 0$, we can also get

$$B = \frac{-2^3 pN^{l+1}}{7\zeta(3)(l+1)} \sum_{\substack{b=1 \\ (a,2)=1,}}^{\infty} \sum_{\substack{a=2b+1 \\ (b,2)=1}}^{\infty} \frac{1}{a^2 b} + O(pN^l \log^2 N). \tag{6}$$

For the case $k \geq 1$, we can also get

$$B = \frac{-2^{k+3}pN^{l+k+1}}{7\zeta(3)(l+k+1)} \sum_{\substack{b=1 \\ (a,2)=1,}}^{\infty} \sum_{\substack{a=2b+1 \\ (b,2)=1}}^{\infty} \frac{b^{k-1}}{a^{k+2}} + O(pN^{l+k} \log N). \tag{7}$$

By using the same method, we can also calculate C:

$$C = -\frac{2^{k+3}p}{\pi^2}\sum_{\substack{1\le b\le\frac{N}{2}\\(a,b)=1,(a,2)=1,(v,2)=1,(b,2)=1}}\sum_{b<a<2b}\sum_{d\le\frac{N}{2b}}\sum_{1\le v\le\frac{p^2}{a}}{}' \frac{d^{l+k}a^{l-1}b^{k-1}}{v^2}$$

$$= \frac{-pN^{l+k+1}}{7\cdot 2^{l-2}\zeta(3)(l+k+1)}\sum_{\substack{1\le b\le\frac{N}{2}\\(a,2)=1,(b,2)=1}}\sum_{b<a<2b}\frac{a^{l-1}}{b^{l+2}} + O\left(pN^{l+k}\sum_{1\le b\le\frac{N}{2}}\sum_{b<a<2b}\frac{a^{l-1}}{b^{l+1}}\right).$$

For the case $l = 0$, we can also get

$$C = \frac{-4pN^{k+1}}{7\zeta(3)(k+1)}\sum_{\substack{b=1\\(a,2)=1,(b,2)=1}}^{\infty}\sum_{a=b+1}^{2b}\frac{1}{ab^2} + O(pN^k\log^2 N). \tag{8}$$

For the case $l \ge 1$, we can also get

$$C = \frac{-pN^{l+k+1}}{7\cdot 2^{l-2}\zeta(3)(l+k+1)}\sum_{\substack{b=1\\(a,2)=1,(b,2)=1}}^{\infty}\sum_{a=b+1}^{2b}\frac{a^{l-1}}{b^{l+2}} + O(pN^{l+k}\log N). \tag{9}$$

Then from (2)–(9), we can get

$$M_{11} = \frac{C(l,k)pN^{l+k+1}}{7\cdot 2^{l-2}\zeta(3)(l+k+1)} + \begin{cases} O\left(pN^{l+k}\log^2 N\right), & \text{if } l=k=0;\ l=0, k\ge 1 \text{ or } l\ge 1, k=0;\\ O\left(pN^{l+k}\log N\right), & \text{if } l\ge 1, k\ge 1.\end{cases} \tag{10}$$

here

$$C(l,k) = -\frac{7\zeta(3)}{8} - \sum_{\substack{a=1\\(a,2)=1,(b,2)=1}}^{\infty}\sum_{b=a+1}^{\infty}\frac{a^{l-1}}{b^{l+2}} - \sum_{\substack{b=1\\(a,2)=1,(b,2)=1}}^{\infty}\sum_{a=b+1}^{2b}\frac{a^{l-1}}{b^{l+2}} - 2^{l+k+1}\sum_{\substack{b=1\\(a,2)=1,(b,2)=1}}^{\infty}\sum_{a=2b+1}^{\infty}\frac{b^{k-1}}{a^{k+2}}$$

is a constant related to l, k.

Now, we calculate M_{12}. Similarly to the proof of Lemma 4 we have

$$M_{12} = -\frac{8p}{\pi^2}\sum_{\substack{a\le N\\2au\equiv bv, 2au\ne bv, (u,2)=1,(v,2)=1}}\sum_{b\le N}\sum_{1\le u\le p^2}{}'\sum_{1\le v\le p^2}{}'\frac{a^l b^k}{uv}$$

$$\ll p\sum_{i,j=0}^{J}\sum_{e^i\le u<e^{i+1}}\frac{1}{u}\sum_{e^j\le v<e^{j+1}}\frac{1}{v}\sum_{\substack{a,b=1\\2au\equiv bv(\bmod p)\\2au\ne bv}}^{N} a^l b^k$$

$$\ll pN^{l+k}\sum_{0\le i\le j\le J} e^{-i-j}\sum_{e^i\le u<e^{i+1}}\sum_{e^j\le v<e^{j+1}}\sum_{\substack{a,b=1\\2au\equiv bv(\bmod p)\\2au\ne bv}}^{N} 1$$

$$\ll N^{l+k+2}p^{o(1)}. \tag{11}$$

In addition, from Lemma 4 we have

$$\begin{aligned} M_2 &= \frac{8p}{\pi^2\phi(p)}\sum_{a\le N}\sum_{b\le N}a^lb^k\sum_{\chi \bmod p}\chi(-2a\overline{b})\left|\sum_{1\le u\le p^2}\frac{\chi\chi_2^0(u)}{u}\right|^2 \\ &\ll N^{l+k}\sum_{a\le N}\sum_{b\le N}\sum_{\chi \bmod p}\chi(-2a\overline{b})\left|\sum_{1\le u\le p^2}\frac{\chi\chi_2^0(u)}{u}\right|^2 \\ &\ll N^{l+k+2}p^{o(1)}. \end{aligned} \tag{12}$$

Then combining (1) and (10)–(12), we obtain the asymptotic formula

$$\begin{aligned} &\sum_{a\le N}\sum_{b\le N}a^lb^kH(2a\overline{b},p) \\ =&\frac{C(l,k)}{7\cdot 2^{l-2}\zeta(3)(l+k+1)}pN^{l+k+1} \\ &+\begin{cases} O\left(pN^{l+k}\log^2 N + N^{l+k+2}p^{o(1)}\right), & \text{if } l=k=0;\ l=0,k\ge 1 \text{ or } l\ge 1,k=0; \\ O\left(pN^{l+k}\log N + N^{l+k+2}p^{o(1)}\right), & \text{if } l\ge 1,k\ge 1. \end{cases} \end{aligned}$$

This completes the proof of the theorem.

Taking $l = 1, k = 0$ and $l = k = 1$ in the Theorem, we may immediately get the corollaries.

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Conflict of interest

We declare that we have no conflict of interest.

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