Mathematics

## Research article

# Bounds of a unified integral operator for $(s, m)$-convex functions and their consequences 

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#### Abstract

The unified integral operator presented in Definition 4 produces several kinds of known fractional and conformable integral operators. The goal of this paper is to obtain bounds of this unified integral operator by using the definition of $(s, m)$-convexity. The resulting inequalities in specific cases represent the bounds of many known fractional and conformable fractional integral operators in a compact form.


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## 1. Introduction

First we give the definitions of generalized fractional integral operators which are special cases of the unified integral operators defined in (1.9), (1.10).

Definition 1.1. [1] Let $f:[a, b] \rightarrow \mathbb{R}$ be an integrable function. Also let $g$ be an increasing and positive function on $(a, b]$, having a continuous derivative $g^{\prime}$ on $(a, b)$. The left-sided and right-sided fractional integrals of a function $f$ with respect to another function $g$ on $[a, b]$ of order $\mu$ where $\mathfrak{R}(\mu)>$ 0 are defined by:

$$
\begin{align*}
& { }_{g}^{\mu} I_{a^{+}} f(x)=\frac{1}{\Gamma(\mu)} \int_{a}^{x}(g(x)-g(t))^{\mu-1} g^{\prime}(t) f(t) d t, \quad x>a,  \tag{1.1}\\
& { }_{g}^{\mu} I_{b}-f(x)=\frac{1}{\Gamma(\mu)} \int_{x}^{b}(g(t)-g(x))^{\mu-1} g^{\prime}(t) f(t) d t, \quad x<b, \tag{1.2}
\end{align*}
$$

where $\Gamma($.$) is the gamma function.$
Definition 1.2. [2] Let $f:[a, b] \rightarrow \mathbb{R}$ be an integrable function. Also let $g$ be an increasing and positive function on ( $a, b$ ], having a continuous derivative $g^{\prime}$ on $(a, b)$. The left-sided and right-sided fractional integrals of a function $f$ with respect to another function $g$ on $[a, b]$ of order $\mu$ where $\mathfrak{R}(\mu), k>0$ are defined by:

$$
\begin{align*}
& { }_{g}^{\mu} I_{a^{+}}^{k} f(x)=\frac{1}{k \Gamma_{k}(\mu)} \int_{a}^{x}(g(x)-g(t))^{\frac{\mu}{k}-1} g^{\prime}(t) f(t) d t, x>a,  \tag{1.3}\\
& { }_{g}^{\mu} I_{b^{-}}^{k} f(x)=\frac{1}{k \Gamma_{k}(\mu)} \int_{x}^{b}(g(t)-g(x))^{\frac{\mu}{k}-1} g^{\prime}(t) f(t) d t, x<b, \tag{1.4}
\end{align*}
$$

where $\Gamma_{k}($.$) is defined as follows [3]:$

$$
\begin{equation*}
\Gamma_{k}(x)=\int_{0}^{\infty} t^{x-1} e^{-\frac{k}{k}} d t, \mathfrak{R}(x)>0 \tag{1.5}
\end{equation*}
$$

A fractional integral operator containing an extended generalized Mittag-Leffler function in its kernel is defined as follows:
Definition 1.3. [4] Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}, \mathfrak{R}(\mu), \mathfrak{R}(\alpha), \mathfrak{R}(l)>0, \mathfrak{R}(c)>\mathfrak{R}(\gamma)>0$ with $p \geq 0, \delta>0$ and $0<k \leq \delta+\mathfrak{R}(\mu)$. Let $f \in L_{1}[a, b]$ and $x \in[a, b]$. Then the generalized fractional integral operators $\epsilon_{\mu, \alpha, l, \omega, a^{+}}^{\gamma, \delta, k, c} f$ and $\epsilon_{\mu, \alpha, l, \omega, b^{-}}^{\gamma, \delta, k, c} f$ are defined by:

$$
\begin{align*}
& \left(\epsilon_{\mu, \alpha, l, \omega, a^{+}}^{\gamma, \delta, k, c} f\right)(x ; p)=\int_{a}^{x}(x-t)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}\left(\omega(x-t)^{\mu} ; p\right) f(t) d t  \tag{1.6}\\
& \left(\epsilon_{\mu, \alpha, l, \omega, b^{-}}^{\gamma, \delta, k, c} f\right)(x ; p)=\int_{x}^{b}(t-x)^{\alpha-1} E_{\mu, \alpha, l}^{\gamma, \delta, k, c}\left(\omega(t-x)^{\mu} ; p\right) f(t) d t \tag{1.7}
\end{align*}
$$

where

$$
\begin{equation*}
E_{\mu, \alpha, l}^{\gamma,,, k, c}(t ; p)=\sum_{n=0}^{\infty} \frac{\beta_{p}(\gamma+n k, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{n k}}{\Gamma(\mu n+\alpha)} \frac{t^{n}}{(l)_{n \delta}} \tag{1.8}
\end{equation*}
$$

is the extended generalized Mittag-Leffler function and $(c)_{n k}$ is the Pochhammer symbol defined by $(c)_{n k}=\frac{\Gamma(c+n k)}{\Gamma(c)}$.

Recently, a unified integral operator is defined as follows:
Definition 1.4. [5] Let $f, g:[a, b] \longrightarrow \mathbb{R}, 0<a<b$, be the functions such that $f$ be positive and $f \in L_{1}[a, b]$, and $g$ be differentiable and strictly increasing. Also let $\frac{\phi}{x}$ be an increasing function on $[a, \infty)$ and $\alpha, l, \gamma, c \in \mathbb{C}, p, \mu, \delta \geq 0$ and $0<k \leq \delta+\mu$. Then for $x \in[a, b]$ the left and right integral operators are defined by

$$
\begin{align*}
& \left({ }_{g} F_{\mu, \alpha, a^{+}}^{\phi, \gamma, \delta, k, c} f\right)(x, \omega ; p)=\int_{a}^{x} K_{x}^{y}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c, c}, g ; \phi\right) f(y) d(g(y)),  \tag{1.9}\\
& \left({ }_{g} F_{\mu, \beta, l, b-}^{\phi, \gamma, \delta, c} f\right)(x, \omega ; p)=\int_{x}^{b} K_{y}^{x}\left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g ; \phi\right) f(y) d(g(y)), \tag{1.10}
\end{align*}
$$

where the involved kernel is defined by

$$
\begin{equation*}
K_{x}^{y}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right)=\frac{\phi(g(x)-g(y))}{g(x)-g(y)} E_{\mu, \alpha, l}^{\gamma, \delta, c, c}\left(\omega(g(x)-g(y))^{\mu} ; p\right) \tag{1.11}
\end{equation*}
$$

For suitable settings of functions $\phi, g$ and certain values of parameters included in Mittag-Leffler function, several recently defined known fractional and conformable fractional integrals studied in [1,6-17] can be reproduced, see [18, Remarks 6\&7].

The aim of this study is to derive the bounds of all aforementioned integral operators in a unified form for $(s, m)$-convex functions. These bounds will hold particularly for $m$-convex, $s$-convex and convex functions and for almost all fractional and conformable integrals defined in [1,6-17].

Definition 1.5. [19] A function $f:[0, b] \rightarrow \mathbb{R}, b>0$ is said to be $(s, m)$-convex, where $(s, m) \in[0,1]^{2}$ if

$$
\begin{equation*}
f(t x+m(1-t) y) \leq t^{s} f(x)+m(1-t)^{s} f(y) \tag{1.12}
\end{equation*}
$$

holds for all $x, y \in[0, b]$ and $t \in[0,1]$.
Remark 1. 1. If we take $(s, m)=(1, m)$, then (1.12) gives the definition of $m$-convex function.
2. If we take $(s, m)=(1,1)$, then (1.12) gives the definition of convex function.
3. If we take $(s, m)=(1,0)$, then (1.12) gives the definition of star-shaped function.

## 2. Properties of the kernel $K_{x}^{y}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right)$

P1: Let $g$ and $\frac{\phi}{x}$ be increasing functions. Then for $x<t<y, x, y \in[a, b]$ the kernel $K_{x}^{y}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c}, g ; \phi\right)$ satisfies the following inequality:

$$
\begin{equation*}
K_{t}^{x}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c, c}, g ; \phi\right) g^{\prime}(t) \leq K_{y}^{x}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c, c}, g ; \phi\right) g^{\prime}(t) \tag{2.1}
\end{equation*}
$$

This can be obtained from the following two straightforward inequalities:

$$
\begin{gather*}
\frac{\phi(g(t)-g(x))}{g(t)-g(x)} g^{\prime}(t) \leq \frac{\phi(g(y)-g(x))}{g(y)-g(x)} g^{\prime}(t),  \tag{2.2}\\
E_{\mu, \alpha, l}^{\gamma, \delta, c, c}\left(\omega(g(t)-g(x))^{\mu} ; p\right) \leq E_{\mu, \alpha, l}^{\gamma, \delta, c}\left(\omega(g(y)-g(x))^{\mu} ; p\right) . \tag{2.3}
\end{gather*}
$$

The reverse of inequality (2.1) holds when $g$ and $\frac{\phi}{x}$ are decreasing.
P2: Let $g$ and $\frac{\phi}{x}$ be increasing functions. If $\phi(0)=\phi^{\prime}(0)=0$, then for $x, y \in[a, b], x<y$, $K_{y}^{x}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c, c}, g ; \phi\right) \geq 0$.
P3: For $p, q \in \mathbb{R}$,
$K_{y}^{x}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; p \phi_{1}+q \phi_{2}\right)=p K_{y}^{x}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi_{1}\right)+q K_{y}^{x}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi_{2}\right)$.
The upcoming section contains the results which deal with the bounds of several integral operators in a compact form by utilizing ( $s, m$ )-convex functions. A version of the Hadamard inequality in a compact form is presented, also a modulus inequality is given for differentiable function $f$ such that $\left|f^{\prime}\right|$ is ( $s, m$ )-convex function.

## 3. Main results

In this section first we will state the main results. The following result provides upper bound of unified integral operators.

Theorem 3.1. Let $f:[a, b] \longrightarrow \mathbb{R}, 0 \leq a<b$ be a positive integrable $(s, m)$-convex function, $m \in$ $(0,1]$. Let $g:[a, b] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function, also let $\frac{\phi}{x}$ be an increasing function on $[a, b]$. If $\alpha, \beta, l, \gamma, c \in \mathbb{C}, p, \mu \geq 0, \delta \geq 0$ and $0<k \leq \delta+\mu$, then for $x \in(a, b)$ the following inequality holds for unified integral operators:

$$
\begin{align*}
& \left({ }_{g} F_{\mu, \alpha, \alpha, a^{+}}^{\phi, \gamma, k, c} f\right)(x, \omega ; p)+\left({ }_{g} F_{\mu, \beta, l, b^{-}}^{\phi, \gamma, \delta, c} f\right)(x, \omega ; p)  \tag{3.1}\\
& \leq K_{x}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right)\left(m f\left(\frac{x}{m}\right) g(x)-f(a) g(a)-\frac{\Gamma(s+1)}{(x-a)^{s}}\left(m f\left(\frac{x}{m}\right){ }^{s} I_{x^{-}} g(a)-f(a)^{s} I_{a^{+}} g(x)\right)\right) \\
& +K_{b}^{x}\left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g ; \phi\right)\left(f(b) g(b)-m f\left(\frac{x}{m}\right) g(x)-\frac{\Gamma(s+1)}{(b-x)^{s}}\left(f(b)^{s} I_{b^{-}} g(x)-m f\left(\frac{x}{m}\right)^{s} I_{x^{+}} g(b)\right)\right) .
\end{align*}
$$

Lemma 3.2. [20] Let $f:[0, \infty] \longrightarrow \mathbb{R}$, be an ( $s, m$ )-convex function, $m \in(0,1]$. If $f(x)=f\left(\frac{a+b-x}{m}\right)$, then the following inequality holds:

$$
\begin{equation*}
f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{s}}(1+m) f(x) \quad x \in[a, b] . \tag{3.2}
\end{equation*}
$$

The following result provides generalized Hadamard inequality for $(s, m)$-convex functions.
Theorem 3.3. Under the assumptions of Theorem 3.1, in addition if $f(x)=f\left(\frac{a+b-x}{m}\right), m \in(0,1]$, then the following inequality holds:

$$
\begin{align*}
& \frac{2^{s}}{(1+m)} f\left(\frac{a+b}{2}\right)\left(\left({ }_{g} F_{\mu, \alpha, l, l, b^{-}}^{\phi, \gamma, \delta, c}\right)(a, \omega ; p)+\left({ }_{g} F_{\mu, \beta, l, a^{+}}^{\phi, \gamma, \delta, c} 1\right)(b, \omega ; p)\right)  \tag{3.3}\\
& \leq\left({ }_{g} F_{\mu, \alpha, l, b, b}^{\phi, \gamma, \delta, c} f\right)(a, \omega ; p)+\left({ }_{g} F_{\mu, \beta, l, a^{+}}^{\phi, \gamma, \delta, c} f\right)(b, \omega ; p) \\
& \leq\left(K_{b}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c, c}, g ; \phi\right)+K_{b}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right)\right)\left(f(b) g(b)-m f\left(\frac{a}{m}\right) g(a)\right. \\
& \left.-\frac{\Gamma(s+1)}{(b-a)^{s}}\left(f(b)^{s} I_{b^{-}} g(a)-m f\left(\frac{a}{m}\right)^{s} I_{a^{+}} g(b)\right)\right) .
\end{align*}
$$

Theorem 3.4. Let $f:[a, b] \longrightarrow \mathbb{R}, 0 \leq a<b$ be a differentiable function. If $\left|f^{\prime}\right|$ is $(s, m)$-convex, $m \in(0,1]$ and $g:[a, b] \longrightarrow \mathbb{R}$ be differentiable and strictly increasing function, also let $\frac{\phi}{x}$ be an increasing function on $[a, b]$. If $\alpha, \beta, l, \gamma, c \in \mathbb{C}, p, \mu \geq 0, \delta \geq 0$ and $0<k \leq \delta+\mu$, then for $x \in(a, b)$ we have

$$
\begin{align*}
& \left|\left({ }_{g} F_{\mu, \alpha, l, a^{+}}^{\phi, \gamma, \delta, c} f * g\right)(x, \omega ; p)+\left({ }_{g} F_{\mu, \beta, l, b^{-}}^{\phi, \gamma, \delta, c} f * g\right)(x, \omega ; p)\right|  \tag{3.4}\\
& \leq K_{x}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right)\left(m\left|f^{\prime}\left(\frac{x}{m}\right)\right| g(x)-\left|f^{\prime}(a)\right| g(a)-\frac{\Gamma(s+1)}{(x-a)^{s}}\left(m\left|f^{\prime}\left(\frac{x}{m}\right)\right|{ }^{s} I_{x^{-}} g(a)-\left|f^{\prime}(a)\right|^{s} I_{a^{+}} g(x)\right)\right) \\
& +K_{b}^{x}\left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g ; \phi\right)\left(\left|f^{\prime}(b)\right| g(b)-m\left|f^{\prime}\left(\frac{x}{m}\right)\right| g(x)-\frac{\Gamma(s+1)}{(b-x)^{s}}\left(\left|f^{\prime}(b)\right|^{s} I_{b^{-}} g(x)-m\left|f^{\prime}\left(\frac{x}{m}\right)\right|{ }^{s} I_{x^{+}} g(b)\right)\right),
\end{align*}
$$

where

$$
\begin{aligned}
& \left({ }_{g} F_{\mu, \alpha, l, a^{+}}^{\phi, \gamma, \delta, k, c} f * g\right)(x, \omega ; p):=\int_{a}^{x} K_{x}^{t}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right) f^{\prime}(t) d(g(t)), \\
& \left({ }_{g} F_{\mu, \beta, l, b^{-}}^{\phi \gamma, \gamma, \delta, c} f * g\right)(x, \omega ; p):=\int_{x}^{b} K_{t}^{x}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right) f^{\prime}(t) d(g(t))
\end{aligned}
$$

## 4. Proofs of main results

In this section we give the proves of the results stated in aforementioned section.
Proof of Theorem 3.1. By $\left(\mathbf{P}_{\mathbf{1}}\right)$, the following inequalities hold:

$$
\begin{align*}
& K_{x}^{t}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c, c}, g ; \phi\right) g^{\prime}(t) \leq K_{x}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c, c}, g ; \phi\right) g^{\prime}(t), a<t<x,  \tag{4.1}\\
& K_{t}^{x}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right) g^{\prime}(t) \leq K_{b}^{x}\left(E_{\mu, \beta, l}^{\gamma, \delta, c, c}, g ; \phi\right) g^{\prime}(t), \quad x<t<b . \tag{4.2}
\end{align*}
$$

For ( $s, m$ )-convex function the following inequalities hold:

$$
\begin{align*}
& f(t) \leq\left(\frac{x-t}{x-a}\right)^{s} f(a)+m\left(\frac{t-a}{x-a}\right)^{s} f\left(\frac{x}{m}\right), a<t<x,  \tag{4.3}\\
& f(t) \leq\left(\frac{t-x}{b-x}\right)^{s} f(b)+m\left(\frac{b-t}{b-x}\right)^{s} f\left(\frac{x}{m}\right), \quad x<t<b . \tag{4.4}
\end{align*}
$$

From (4.1) and (4.3), the following integral inequality holds true:

$$
\begin{align*}
& \int_{a}^{x} K_{x}^{t}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right) f(t) d(g(t)) \leq f(a) K_{x}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right)  \tag{4.5}\\
& \times \int_{a}^{x}\left(\frac{x-t}{x-a}\right)^{s} d(g(t))+m f\left(\frac{x}{m}\right) K_{x}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c, c}, g ; \phi\right) \int_{a}^{x}\left(\frac{t-a}{x-a}\right)^{s} d(g(t)) .
\end{align*}
$$

Further the aforementioned inequality takes the form which involves Riemann-Liouville fractional integrals in the right hand side, provides the upper bound of unified left sided integral operator (1.1) as follows:

$$
\begin{align*}
& \left({ }_{g} F_{\mu, \alpha, l, a^{+}}^{\phi, \gamma, \delta, k, c} f\right)(x, \omega ; p) \leq K_{x}^{a}\left(E_{\mu,, l, l}^{\gamma, \delta, k, c}, g ; \phi\right)\left(m f\left(\frac{x}{m}\right) g(x)-f(a) g(a)\right.  \tag{4.6}\\
& \left.-\frac{\Gamma(s+1)}{(x-a)^{s}}\left(m f\left(\frac{x}{m}\right){ }^{s} I_{x^{-}} g(a)-f(a)^{s} I_{a^{+}} g(x)\right)\right) .
\end{align*}
$$

On the other hand from (4.2) and (4.4), the following integral inequality holds true:

$$
\begin{align*}
& \int_{x}^{b} K_{t}^{x}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right) f(t) d(g(t)) \leq f(b) K_{b}^{x}\left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g ; \phi\right)  \tag{4.7}\\
& \times \int_{x}^{b}\left(\frac{t-x}{b-x}\right)^{s} d(g(t))+m f\left(\frac{x}{m}\right) K_{x}^{b}\left(E_{\mu,, l, l}^{\gamma, \delta, k, c}, g ; \phi\right) \int_{x}^{b}\left(\frac{b-t}{b-x}\right)^{s} d(g(t)) .
\end{align*}
$$

Further the aforementioned inequality takes the form which involves Riemann-Liouville fractional integrals in the right hand side, provides the upper bound of unified right sided integral operator (1.2) as follows:

$$
\begin{align*}
& \left({ }_{g} F_{\mu, \beta, l, b^{-}}^{\phi, \gamma, \delta, c} f\right)(x, \omega ; p) \leq K_{b}^{x}\left(E_{\mu, \beta, l}^{\gamma, \delta, c, c}, g ; \phi\right)\left(f(b) g(b)-m f\left(\frac{x}{m}\right) g(x)\right.  \tag{4.8}\\
& \left.-\frac{\Gamma(s+1)}{(b-x)^{s}}\left(f(b)^{s} I_{b^{-}} g(x)-m f\left(\frac{x}{m}\right)^{s} I_{x^{+}} g(b)\right)\right) .
\end{align*}
$$

By adding (4.6) and (4.8), (3.1) can be obtained.

Remark 2. (i) If we consider $(s, m)=(1,1)$ in (3.1), $[18$, Theorem 1] is obtained.
(ii) If we consider $p=\omega=0$ in (3.1), [20, Theorem 1] is obtained.
(iii) If we consider $\phi(t)=\Gamma(\alpha) t^{\alpha}, p=\omega=0$ and $(s, m)=(1,1)$ in (3.1), [21, Theorem 1] is obtained.
(iv) If we consider $\alpha=\beta$ in the result of (iii), then [21, Corollary 1] is obtained.
(v) If we consider $\phi(t)=t^{\alpha}, g(x)=x$ and $m=1$ in (3.1), then [22, Theorem 2.1] is obtained.
(vi) If we consider $\alpha=\beta$ in the result of (v), then [22, Corollary 2.1] is obtained.
(vii) If we consider $\phi(t)=\frac{\Gamma(\alpha) \frac{\alpha}{k}}{k \Gamma_{k}(\alpha)},(s, m)=(1,1), g(x)=x$ and $p=\omega=0$ in (3.1), then [23, Theorem 1] can be obtained.
(viii) If we consider $\alpha=\beta$ in the result of (vii), then [23, Corollary 1] can be obtained.
(ix) If we consider $\phi(t)=\Gamma(\alpha) t^{\alpha}, g(x)=x$ and $p=\omega=0$ and $(s, m)=(1,1)$ in (3.1), then [24, Theorem 1] is obtained.
(x) If we consider $\alpha=\beta$ in the result of (ix), then [24, Corollary 1] can be obtained.
(xi) If we consider $\alpha=\beta=1$ and $x=a$ or $x=b$ in the result of (x), then [24, Corollary 2] can be obtained.
(xii) If we consider $\alpha=\beta=1$ and $x=\frac{a+b}{2}$ in the result of (x), then [24, Corollary 3] can be obtained.

Proof of Theorem 3.3. By $\left(\mathbf{P}_{\mathbf{1}}\right)$, the following inequalities hold:

$$
\begin{align*}
& K_{x}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c, c}, g ; \phi\right) g^{\prime}(x) \leq K_{b}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right) g^{\prime}(x), a<x<b,  \tag{4.9}\\
& K_{b}^{x}\left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g ; \phi\right) g^{\prime}(x) \leq K_{b}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c, c}, g ; \phi\right) g^{\prime}(x) a<x<b \tag{4.10}
\end{align*}
$$

For $(s, m)$-convex function $f$, the following inequality holds:

$$
\begin{equation*}
f(x) \leq\left(\frac{x-a}{b-a}\right)^{s} f(b)+m\left(\frac{b-x}{b-a}\right)^{s} f\left(\frac{a}{m}\right), \quad a<x<b . \tag{4.11}
\end{equation*}
$$

From (4.9) and (4.11), the following integral inequality holds true:

$$
\begin{aligned}
& \int_{a}^{b} K_{x}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right) f(x) d(g(x)) \\
& \leq m f\left(\frac{a}{m}\right) K_{b}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right) \int_{a}^{b}\left(\frac{b-x}{b-a}\right)^{s} d(g(x))+f(b) K_{b}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right) \int_{a}^{b}\left(\frac{x-a}{b-a}\right)^{s} d(g(x)) .
\end{aligned}
$$

Further the aforementioned inequality takes the form which involves Riemann-Liouville fractional integrals in the right hand side, provides the upper bound of unified right sided integral operator (1.1) as follows:

$$
\begin{align*}
& \left({ }_{g} F_{\mu, \alpha, l, b^{-}}^{\phi, \gamma, \delta, c, c}\right)(a, \omega ; p) \leq K_{b}^{a}\left(E_{\mu, a, l}^{\gamma, \delta, k, c}, g ; \phi\right)\left(f(b) g(b)-m f\left(\frac{a}{m}\right) g(a)\right.  \tag{4.12}\\
& \left.-\frac{\Gamma(s+1)}{(b-a)^{s}}\left(f(b)^{s} I_{b^{-}} g(a)-m f\left(\frac{a}{m}\right)^{s} I_{a^{+}} g(b)\right)\right) .
\end{align*}
$$

On the other hand from (4.9) and (4.11), the following inequality holds which involves RiemannLiouville fractional integrals on the right hand side and estimates of the integral operator (1.2):

$$
\begin{align*}
& \left({ }_{g} F_{\mu, \beta, l, a^{+}}^{\phi, \gamma, \delta, c, c} f\right)(b, \omega ; p) \leq K_{b}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right)\left(f(b) g(b)-m f\left(\frac{a}{m}\right) g(a)\right.  \tag{4.13}\\
& \left.-\frac{\Gamma(s+1)}{(b-a)^{s}}\left(f(b)^{s} I_{b^{-}} g(a)-m f\left(\frac{a}{m}\right)^{s} I_{a^{+}} g(b)\right)\right) .
\end{align*}
$$

By adding (4.12) and (4.13), following inequality can be obtained:

$$
\begin{align*}
& \left({ }_{g} F_{\mu, \alpha, l, b^{-}}^{\phi, \gamma, \delta, c}\right)(a, \omega ; p)+\left({ }_{g} F_{\mu, \beta, l, a^{+}, c}^{\phi, \gamma, \delta, c}\right)(b, \omega ; p)  \tag{4.14}\\
& \leq\left(K_{b}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right)+K_{b}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c,}, g ; \phi\right)\right)\left(f(b) g(b)-m f\left(\frac{a}{m}\right) g(a)\right. \\
& \left.-\frac{\Gamma(\alpha+1)}{(b-a)^{s}}\left(f(b)^{s} I_{b^{-}} g(b)-m f\left(\frac{a}{m}\right)^{s} I_{a^{+}} g(b)\right)\right) .
\end{align*}
$$

Multiplying both sides of (3.2) by $K_{x}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c, c}, g ; \phi\right) g^{\prime}(x)$, and integrating over $[a, b]$ we have

$$
f\left(\frac{a+b}{2}\right) \int_{a}^{b} K_{x}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right) d(g(x)) \leq\left(\frac{1}{2^{s}}\right)(1+m) \int_{a}^{b} K_{b}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right) f(x) d(g(x)) .
$$

From Definition 1.4, the following inequality is obtained:

$$
\begin{equation*}
f\left(\frac{a+b}{2}\right) \frac{2^{s}}{(1+m)}\left({ }_{g} F_{\mu, \alpha, l, b^{-}}^{\phi, \gamma, k, c}\right)(a, \omega ; p) \leq\left({ }_{g} F_{\mu, \alpha, \gamma, b^{-}}^{\phi, \gamma, \delta, c} f\right)(a, \omega ; p) . \tag{4.15}
\end{equation*}
$$

Similarly multiplying both sides of (3.2) by $K_{b}^{x}\left(E_{\mu, \beta, l}^{\gamma, \delta, c, c}, g ; \phi\right) g^{\prime}(x)$, and integrating over $[a, b]$ we have

$$
\begin{equation*}
f\left(\frac{a+b}{2}\right) \frac{2^{s}}{(1+m)}\left({ }_{g} F_{\mu, \beta, l, a^{+}}^{\phi, \gamma, \delta, c, c}\right)(b, \omega ; p) \leq\left({ }_{g} F_{\mu, \beta, l, a^{+}}^{\phi, \gamma,, k, c} f\right)(b, \omega ; p) . \tag{4.16}
\end{equation*}
$$

By adding (4.15) and (4.16) the following inequality is obtained:

$$
\begin{align*}
& f\left(\frac{a+b}{2}\right) \frac{2^{s}}{(1+m)}\left(\left({ }_{g} F_{\mu, \beta, l, a^{+}}^{\phi, \gamma, \delta, k, c} 1\right)(b, \omega ; p)+\left({ }_{g} F_{\mu, \alpha, l, b^{-}}^{\phi, \gamma, \delta, k, c} 1\right)(a, \omega ; p)\right)  \tag{4.17}\\
& \leq\left({ }_{g} F_{\mu, \gamma, \gamma, a^{+}}^{\phi, \gamma, k, c} f\right)(b, \omega ; p)+\left({ }_{g} F_{\mu, \alpha, l, b^{-}}^{\phi, \gamma, \delta, k, c} f\right)(a, \omega ; p) .
\end{align*}
$$

Using (4.14) and (4.17), inequality (3.3) can be obtained, this completes the proof.
Remark 3. (i) If we consider $(s, m)=(1,1)$ in (3.3), [18, Theorem 2$]$ is obtained.
(ii) If we consider $p=\omega=0$ in (3.3), [20, Theorem 3] is obtained.
(iii) If we consider $\phi(t)=\Gamma(\alpha) t^{\alpha+1}, p=\omega=0$ and $(s, m)=(1,1)$ in (3.3), [21, Theorem 3] is obtained.
(iv) If we consider $\alpha=\beta$ in the result of (iii), then [21, Corollary 3] is obtained.
(v) If we consider $\phi(t)=t^{\alpha+1}, g(x)=x$ and $m=1$ in (3.3), then [22, Theorem 2.4] is obtained.
(vi) If we consider $\alpha=\beta$ in the result of (v), then [22, Corollary 2.6] is obtained.
(vii) If we consider $\phi(t)=\Gamma(\alpha) t^{\frac{\alpha}{k}+1},(s, m)=(1,1), g(x)=x$ and $p=\omega=0$ in (3.3), then [23, Theorem 3] can be obtained.
(viii) If we consider $\alpha=\beta$ in the result of (vii), then [23, Corollary 6] can be obtained.
(ix) If we consider $\phi(t)=\Gamma(\alpha) t^{\alpha+1}, p=\omega=0,(s, m)=1$ and $g(x)=x$ in (3.3), [24, Theorem 3] can be obtained.
(x) If we consider $\alpha=\beta$ in the result of (ix), [24, Corrolary 6] can be obtained.

Proof of Theorem 3.4. For $(s, m)$-convex function the following inequalities hold:

$$
\begin{align*}
& \left|f^{\prime}(t)\right| \leq\left(\frac{x-t}{x-a}\right)^{s}\left|f^{\prime}(a)\right|+m\left(\frac{t-a}{x-a}\right)^{s}\left|f^{\prime}\left(\frac{x}{m}\right)\right|, a<t<x,  \tag{4.18}\\
& \left|f^{\prime}(t)\right| \leq\left(\frac{t-x}{b-x}\right)^{s}\left|f^{\prime}(b)\right|+m\left(\frac{b-t}{b-x}\right)^{s}\left|f^{\prime}\left(\frac{x}{m}\right)\right|, x<t<b . \tag{4.19}
\end{align*}
$$

From (4.1) and (4.18), the following inequality is obtained:

$$
\begin{align*}
& \left|\left({ }_{g} F_{\mu, \alpha, l, a^{+}}^{\phi, \gamma, \delta, c}(f * g)\right)(x, \omega ; p)\right| \leq \frac{K_{x}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c, c}, g ; \phi\right)}{(x-a)^{s}}  \tag{4.20}\\
& \times\left((x-a)^{s}\left(m\left|f^{\prime}\left(\frac{x}{m}\right)\right| g(x)-\left|f^{\prime}(a)\right| g(a)\right)-\Gamma(s+1)\left(m\left|f^{\prime}\left(\frac{x}{m}\right)\right|{ }^{s} I_{x^{-}} g(a)-\left|f^{\prime}(a)\right|^{s} I_{a^{+}} g(x)\right)\right) .
\end{align*}
$$

Similarly, from (4.2) and (4.19), the following inequality is obtained:

$$
\begin{align*}
& \left|\left({ }_{g} F_{\mu, \beta, l, b^{-}}^{\phi, \gamma, \delta, c, c}(f * g)\right)(x, \omega ; p)\right| \leq \frac{K_{b}^{x}\left(E_{\mu, \beta, l}^{\gamma, \delta, c, c}, g ; \phi\right)}{(b-x)^{s}}  \tag{4.21}\\
& \times\left((b-x)^{s}\left(\left|f^{\prime}(b)\right| g(b)-m f^{\prime}\left|\left(\frac{x}{m}\right)\right| g(x)\right)-\Gamma(s+1)\left(\left|f^{\prime}(b)\right|^{s} I_{b^{-}} g(x)-m f^{\prime}\left|\left(\frac{x}{m}\right)\right|{ }^{s} I_{x^{+}} g(b)\right)\right)
\end{align*}
$$

By adding (4.20) and (4.21), inequality (3.4) can be achieved.
Remark 4. (i) If we consider $(s, m)=(1,1)$ in $(3.4)$, then [18, Theorem 3$]$ is obtained.
(ii) If we consider $p=\omega=0$ in (3.4), then [20, Theorem 2] is obtained.
(iii) If we consider $\phi(t)=\Gamma(\alpha) t^{\alpha+1}, p=\omega=0$ and $(s, m)=(1,1)$ in (3.4), then [21, Theorem 2] is obtained.
(iv) If we consider $\alpha=\beta$ in the result of (iii), then [21, Corollary 2] is obtained.
(v) If we consider $\phi(t)=t^{\alpha}, g(x)=x$ and $m=1$ in (3.4), then [22, Theorem 2.3] is obtained.
(vi) If we consider $\alpha=\beta$ in the result of (v), then [22, Corollary 2.5] is obtained.
(vii) If we consider $\phi(t)=\Gamma(\alpha) t^{\frac{\alpha}{k}+1},(s, m)=(1,1), g(x)=x$ and $p=\omega=0$ in (3.4), then [23, Theorem 2] can be obtained.
(viii) If we consider $\alpha=\beta$ in the result of (vii), then [23, Corollary 4] can be obtained.
(ix) If we consider $\alpha=\beta=k=1$ and $x=\frac{a+b}{2}$, in the result of (viii), then [23, Corollary 5] can be obtained.
(x) If we consider $\phi(t)=\Gamma(\alpha) t^{\alpha+1}, g(x)=x$ and $p=\omega=0$ and $(s, m)=(1,1)$ in (3.4), then [24, Theorem 2] is obtained.
(xi) If we consider $\alpha=\beta$ in the result of (x), then [24, Corollary 5] can be obtained.

## 5. Boundedness and continuity

In this section, we have established boundedness and continuity of unified integral operators for $m$-convex and convex functions.

Theorem 5.1. Under the assumptions of Theorem 1, the following inequality holds for m-convex functions:

$$
\begin{align*}
& \left({ }_{g} F_{\mu, \alpha, l, a^{+}}^{\phi, \gamma, \gamma, c} f\right)(x, \omega ; p)+\left({ }_{g} F_{\mu, \beta, l, b^{-}}^{\phi, \gamma, c} f\right)(x, \omega ; p)  \tag{5.1}\\
& \leq K_{x}^{a}\left(E_{\mu,, l, l}^{\gamma, \delta, c, c}, g ; \phi\right)(g(x)-g(a))\left(m f\left(\frac{x}{m}\right)+f(a)\right)+K_{b}^{x}\left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g ; \phi\right)(g(b)-g(x))\left(m f\left(\frac{x}{m}\right)+f(b)\right) .
\end{align*}
$$

Proof. If we put $s=1$ in (4.5), we have

$$
\begin{align*}
& \int_{a}^{x} K_{x}^{t}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right) f(t) d(g(t)) \leq f(a) K_{x}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right)  \tag{5.2}\\
& \times \int_{a}^{x}\left(\frac{x-t}{x-a}\right) d(g(t))+m f\left(\frac{x}{m}\right) K_{x}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right) \int_{a}^{x}\left(\frac{t-a}{x-a}\right) d(g(t)) .
\end{align*}
$$

Further from simplification of (5.2), the following inequality holds:

$$
\begin{equation*}
\left({ }_{g} F_{\mu, \alpha, l, a^{+}}^{\phi, \gamma, \delta, k, c} f\right)(x, \omega ; p) \leq K_{x}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right)(g(x)-g(a))\left(m f\left(\frac{x}{m}\right)+f(a)\right) . \tag{5.3}
\end{equation*}
$$

Similarly from (4.8), the following inequality holds:

$$
\begin{equation*}
\left({ }_{g} F_{\mu, \beta, l, b^{-}}^{\phi, \gamma, \delta, k, c} f\right)(x, \omega ; p) \leq K_{b}^{x}\left(E_{\mu, \beta, l}^{\gamma, \delta, k, c}, g ; \phi\right)(g(b)-g(x))\left(m f\left(\frac{x}{m}\right)+f(b)\right) . \tag{5.4}
\end{equation*}
$$

From (5.3) and (5.4), (5.1) can be obtained.
Theorem 5.2. With assumptions of Theorem 4, if $f \in L_{\infty}[a, b]$, then unified integral operators for $m$-convex functions are bounded and continuous.

Proof. From (5.3) we have

$$
\left|\left|\left(g_{\mu, \alpha, l, a^{+}}^{\phi, \gamma, \delta, k, c}\right)(x, \omega ; p)\right| \leq K_{b}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, k, c}, g ; \phi\right)(g(b)-g(a))(m+1)\|f\|_{\infty},\right.
$$

which further gives

$$
\left|\left({ }_{g} F_{\mu, \alpha, l, a^{+}}^{\phi \phi, \gamma, \delta, c} f\right)(x, \omega ; p)\right| \leq K\|f\|_{\infty},
$$

where $K=(g(b)-g(a))(m+1) K_{b}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c, c}, g ; \phi\right)$.
Similarly, from (5.4) the following inequality holds:

$$
\left|\left({ }_{s} F_{\mu, \beta, l, b^{-}}^{\phi, \gamma, \delta, c, c}\right)(x, \omega ; p)\right| \leq K\|f\|_{\infty} .
$$

Hence the boundedness is followed, further from linearity the continuity of (1.9) and (1.10) is obtained.

Corollary 1. If we take $m=1$ in Theorem 5, then unified integral operators for convex functions are bounded and continuous and following inequalities hold:

$$
\begin{aligned}
& \left|\left({ }_{g} F_{\mu, \alpha, l, a^{+}}^{\phi, \gamma, \delta, k, c} f\right)(x, \omega ; p)\right| \leq K\|f\|_{\infty}, \\
& \left|\left({ }_{g} F_{\mu, \beta, \gamma, l^{-}}^{\phi, \gamma, \delta, c} f\right)(x, \omega ; p)\right| \leq K\|f\|_{\infty},
\end{aligned}
$$

where $K=2(g(b)-g(a)) K_{b}^{a}\left(E_{\mu, \alpha, l}^{\gamma, \delta, c, c}, g ; \phi\right)$.

## 6. Conclusions

This paper has explored bounds of a unified integral operator for $(s, m)$-convex functions. These bounds are obtained in a compact form which have further interesting consequences with respect to fractional and conformable integrals for convex, $m$-convex and $s$-convex functions. Furthermore by applying Theorems 3.1, 3.3 and 3.4 several associated results can be derived for different kinds of fractional integral operators of convex, $m$-convex and $s$-convex functions.

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## Conflict of interest

The authors declare that no competing interests exist.

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