Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(6): 5402–5421.
DOI:10.3934/math.2020347
Received: 17 April 2020
Accepted: 12 June 2020
Published: 23 June 2020

Research article

Anti-periodic dynamics on high-order inertial Hopfield neural networks involving time-varying delays

Qian Cao[1] **and Xiaojin Guo**[2,*]

[1] College of Mathematics and Physics, Hunan University of Arts and Science, Changde, Hunan 415000, P. R. China

[2] Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science and Technology, Changsha, 410114, China

* **Correspondence:** Email: xiaojinguo0302@163.com; Tel: +86073188689249; Fax: +86073188689249.

Abstract: Taking into accounting time-varying delays and anti-periodic environments, this paper deals with the global convergence dynamics on a class of anti-periodic high-order inertial Hopfield neural networks. First of all, with the help of Lyapunov function method, we prove that the global solutions are exponentially attractive to each other. Secondly, by using analytical techniques in uniform convergence functions sequence, the existence of the anti-periodic solution and its global exponential stability are established. Finally, two examples are arranged to illustrate the effectiveness and feasibility of the obtained results.

Keywords: high-order inertial neural networks; anti-periodic solution; global exponential stability; time-varying delay
Mathematics Subject Classification: 34C25, 34K13, 34K25

1. Introduction

Due to the engineering backgrounds and strong biological significance, Babcock and Westervelt [1, 2] introduced an inertial term into the traditional multidirectional associative memory neural networks, and established a class of second order delay differential equations, which was called as the famous delayed inertial neural networks model. Arising from problems in different applied sciences such as mathematical physics, control theory, biology in different situations, nonlinear vibration, mechanics, electromagnetic theory and other related fields, the periodic oscillation is an important qualitative property of nonlinear differential equations [3–9]. Consequently, assuming that the activation functions are bounded and employing reduced-order variable substitution which convert the inertial systems into the first order differential equations, the authors in [10, 11] and [12] have respectively gained the existence and stability of anti-periodic solution and periodic solution for addressed inertial neural networks models. Manifestly, the above transformation will raise the dimension in the inertial neural networks system, then some new parameters need to be introduced. This will increase huge amount of computation and be attained hard in practice [13, 14]. For the above reasons, most recently, avoiding the reduced order method, the authors in [15] and [16] respectively developed some non-reduced order methods to establish the existence and stability of periodic solutions for inertial neural networks with time-varying delays.

It has been recognized that, in neural networks dynamics touching the communication, economics, biology or ecology areas, the relevant state variables are often considered as proteins and molecules, light intensity levels or electric charge, and they are naturally anti-periodic [17–19]. Such neural networks systems are often regarded as anti-periodic systems. Therefore, the convergence analysis and stability on the anti-periodic solutions in various neural networks systems with delays have attracted the interest of many researchers and some excellent results are reported in [20–27]. In particular, the anti-periodicity on inertial quaternion-valued high-order Hopfield neural networks with state-dependent delays has been established in [28] by employing reduced-order variable substitution. However, few researchers have utilized the non-reduced order methods to explore such topics on the following high-order inertial Hopfield neural networks involving time-varying delays:

$$
\begin{aligned}
&x_i''(t)\\
=\;& -\bar{a}_i(t)x_i'(t) - \bar{b}_i(t)x_i(t) + \sum_{j=1}^{n} \bar{c}_{ij}(t)A_j(x_j(t)) + \sum_{j=1}^{n} \bar{d}_{ij}(t)B_j(x_j(t-q_{ij}(t)))\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n} \theta_{ijl}(t)Q_j(x_j(t-\eta_{ijl}(t)))Q_l(x_l(t-\xi_{ijl}(t))) + J_i(t),\ t \geq 0,
\end{aligned}
\tag{1.1}
$$

associating with initial value conditions:

$$
x_i(s) = \varphi_i(s),\ x_i'(s) = \psi_i(s),\ -\tau_i \leq s \leq 0,\ \varphi_i,\ \psi_i \in C([-\tau_i, 0], \mathbb{R}), \tag{1.2}
$$

where $\sum_{j=1}^{n} \bar{c}_{ij}(t)A_j(x_j(t))$, $\sum_{j=1}^{n} \bar{d}_{ij}(t)B_j(x_j(t-q_{ij}(t)))$ and $\sum_{j=1}^{n}\sum_{l=1}^{n} \theta_{ijl}(t)Q_j(x_j(t-\eta_{ijl}(t)))Q_l(x_l(t-\xi_{ijl}(t)))$ are respectively the first-order term and the second-order term of the neural network, A_j, B_j and Q_j are the nonlinear activation functions, $\tau_i = \max_{1\leq l,j\leq n}\{\sup_{t\in\mathbb{R}} q_{ij}(t),\ \sup_{t\in\mathbb{R}} \eta_{ijl}(t), \sup_{t\in\mathbb{R}} \xi_{ijl}(t)\}$, $J_i, \bar{c}_{ij}, \bar{d}_{ij}, \theta_{ijl}, \bar{a}_i, \bar{b}_i : \mathbb{R} \to \mathbb{R}$ and $q_{ij},\ \eta_{ijl},\ \xi_{ijl} : \mathbb{R} \to \mathbb{R}^+$ are bounded and continuous functions, $\bar{a}_i, \bar{b}_i, q_{ij},\ \eta_{ijl},\ \xi_{ijl}$ are periodic functions with period $T > 0$, the input term J_i is T-anti-periodic ($J_i(t+T) = -J_i(t)$ for all $t \in \mathbb{R}$), and $i, j, l \in D := \{1, 2, \cdots, n\}$.

Motivated by the above arguments, in this paper, without adopting the reduced order method, we propose a novel approach involving differential inequality techniques coupled with Lyapunov function method to demonstrate the existence and global exponential stability of anti-periodic solutions for system (1.1). Particularly, our results are new and supplement some corresponding ones of the existing literature [19–28]. In a nutshell, the contributions of this paper can be summarized as follows. 1) A class of anti-periodic high-order inertial Hopfield neural networks involving time-varying delays are proposed; 2) Under some appropriate anti-periodic assumptions, all solutions and their derivatives in the proposed neural networks model are guaranteed to converge to the anti-periodic solution and its derivative, respectively; 3) Numerical results including comparisons are presented to verify the obtained theoretical results.

The remaining parts of this paper are organized as follows. In Section 2, we make some preparations. In Section 3, the existence and the global exponential stability of the anti-periodic solution are stated and demonstrated. Section 4 shows numerical examples. Conclusions are drawn in Section 5.

2. Preliminaries

To study the existence and uniqueness of anti-periodic solutions to system (1.1), we first require the following assumptions and some key lemmas:

Assumptions:

(F_1) For $i, j, l \in D$, $A_j(u), B_j(u), Q_j(u)$ are all non-decreasing functions with $A_j(0) = B_j(0) = Q_j(0) = 0$, and there are nonnegative constants L_j^A, L_j^B, L_j^Q and M_j^Q such that

$$|A_j(u) - A_j(v)| \leq L_j^A|u - v|,\ |B_j(u) - B_j(v)| \leq L_j^B|u - v|,\ |Q_j(u) - Q_j(v)| \leq L_j^Q|u - v|,$$

$$|Q_j(u)| \leq M_j^Q,\ \bar{c}_{ij}(t + T)A_j(u) = -\bar{c}_{ij}(t)A_j(-u),\ \ \bar{d}_{ij}(t + T)B_j(u) = -\bar{d}_{ij}(t)B_j(-u),$$

and

$$\theta_{ijl}(t + T)Q_j(u)Q_l(v) = -\theta_{ijl}(t)Q_j(-u)Q_l(-v),$$

for all $u,\ v \in \mathbb{R}$.

(F_2) There are constants $\beta_i > 0$ and $\alpha_i \geq 0, \gamma_i \geq 0$ obeying

$$E_i(t) < 0, \quad 4E_i(t)G_i(t) > H_i^2(t),\ \forall t \in \mathbb{R},\ i \in D, \tag{2.1}$$

where

$$\begin{aligned}
E_i(t) &= \alpha_i\gamma_i - \bar{a}_i(t)\alpha_i^2 + \tfrac{1}{2}\alpha_i^2 \sum_{j=1}^n (|\bar{c}_{ij}(t)|L_j^A + |\bar{d}_{ij}(t)|L_j^B) \\
&\quad + \tfrac{1}{2}\alpha_i^2 \sum_{j=1}^n \sum_{l=1}^n |\theta_{ijl}(t)|(M_j^Q L_l^Q + L_j^Q M_l^Q), \\
G_i(t) &= -\bar{b}_i(t)\alpha_i\gamma_i + \tfrac{1}{2} \sum_{j=1}^n (|\bar{c}_{ij}(t)|L_j^A + |\bar{d}_{ij}(t)|L_j^B)|\alpha_i\gamma_i| \\
&\quad + \tfrac{1}{2} \sum_{j=1}^n \alpha_j^2(|\bar{c}_{ji}(t)|L_i^A + \bar{d}_{ji}^+ L_i^B \tfrac{1}{1-\dot{q}_{ji}^+}) \\
&\quad + \tfrac{1}{2} \sum_{j=1}^n (|\bar{c}_{ji}(t)|L_i^A + \bar{d}_{ji}^+ L_i^B \tfrac{1}{1-\dot{q}_{ji}^+})|\alpha_j\gamma_j| \\
&\quad + \tfrac{1}{2} \sum_{j=1}^n \sum_{l=1}^n |\alpha_i\gamma_i||\theta_{ijl}(t)|(M_j^Q L_l^Q + L_j^Q M_l^Q) \\
&\quad + \tfrac{1}{2} \sum_{l=1}^n \sum_{j=1}^n (\alpha_l^2 + |\alpha_l\gamma_l|)\theta_{lji}^+ M_j^Q L_i^Q \tfrac{1}{1-\dot{\xi}_{lji}^+} \\
&\quad + \tfrac{1}{2} \sum_{j=1}^n \sum_{l=1}^n (\alpha_j^2 + |\alpha_j\gamma_j|)\theta_{jil}^+ L_i^Q M_l^Q \tfrac{1}{1-\dot{\eta}_{jil}^+}),
\end{aligned}$$

$$H_i(t) = \beta_i + \gamma_i^2 - \bar{a}_i(t)\alpha_i\gamma_i - \bar{b}_i(t)\alpha_i^2,\ \dot{q}_{ij}^+ = \sup_{t\in\mathbb{R}} q'_{ij}(t),$$
$$\dot{\eta}_{ijl}^+ = \sup_{t\in\mathbb{R}} \eta'_{ijl}(t),\ \dot{\xi}_{ijl}^+ = \sup_{t\in\mathbb{R}} \xi'_{ijl}(t),\ \ q_{ij}^+ = \sup_{t\in\mathbb{R}} q_{ij}(t),\ \eta_{ijl}^+ = \sup_{t\in\mathbb{R}} \eta_{ijl}(t),$$
$$\xi_{ijl}^+ = \sup_{t\in\mathbb{R}} \xi_{ijl}(t),\ \bar{c}_{ij}^+ = \sup_{t\in\mathbb{R}} |\bar{c}_{ij}(t)|,\ \bar{d}_{ij}^+ = \sup_{t\in\mathbb{R}} |\bar{d}_{ij}(t)|,\ i, j, l \in D.$$

(F_3) For $i, j, l \in D$, q_{ij}, η_{ijl} and ξ_{ijl} are continuously differentiable, $q'_{ij}(t) = \dot{q_{ij}}(t) < 1$, $\eta'_{ijl}(t) = \dot{\eta_{ijl}}(t) < 1$ and $\xi'_{ijl}(t) = \dot{\xi_{ijl}}(t) < 1$ for all $t \in \mathbb{R}$.

We will adopt the following notations:

$$\theta_{ijl}^+ = \max_{t\in[0,T]} |\theta_{ijl}(t)|, i, j, l \in D.$$

Remark 2.1. Since (1.1) can be converted into the first order functional differential equations. In view of (F_1) and ([29], p176, Theorem 5.4), one can see that all solutions of (1.1) and (1.2) exist on $[0,\ +\infty)$.

Lemma 2.1. Under (F_1), (F_2) and (F_3), label $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ and $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))$ as two solutions of system (1.1) satisfying

$$x_i(s) = \varphi_i^x(s),\ x'_i(s) = \psi_i^x(s),\ y_i(s) = \varphi_i^y(s),\ y'_i(s) = \psi_i^y(s), \tag{2.2}$$

where $-\tau_i \le s \le 0,\ i \in D,\ \varphi_i^x, \psi_i^x, \varphi_i^y, \psi_i^y \in C([-\tau_i, 0], \mathbb{R})$. Then, there are two positive constants λ and $M = M(\varphi^x, \psi^x, \varphi^y, \psi^y)$ such that

$$|x_i(t) - y_i(t)| \le Me^{-\lambda t},\quad |x'_i(t) - y'_i(t)| \le Me^{-\lambda t},\ \text{for all } t \ge 0,\ i \in D.$$

Proof. Denote $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$ and $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))$ as two solutions of (1.1) and (1.2). Let $w_i(t) = y_i(t) - x_i(t)$, then

$$\begin{aligned}
w''_i(t) &= -\bar{a}_i(t)w'_i(t) - \bar{b}_i(t)w_i(t) \\
&\quad + \sum_{j=1}^n \bar{c}_{ij}(t)\widetilde{A}_j(w_j(t)) + \sum_{j=1}^n \bar{d}_{ij}(t)\widetilde{B}_j(w_j(t - q_{ij}(t))) \\
&\quad + \sum_{j=1}^n \sum_{l=1}^n \theta_{ijl}(t)[Q_j(y_j(t - \eta_{ijl}(t)))Q_l(y_l(t - \xi_{ijl}(t))) - Q_j(y_j(t - \eta_{ijl}(t))) \\
&\quad \times Q_l(x_l(t - \xi_{ijl}(t))) + Q_j(y_j(t - \eta_{ijl}(t)))Q_l(x_l(t - \xi_{ijl}(t))) \\
&\quad - Q_j(x_j(t - \eta_{ijl}(t)))Q_l(x_l(t - \xi_{ijl}(t)))],
\end{aligned} \tag{2.3}$$

where $i, j \in D$, $\widetilde{A}_j(w_j(t)) = A_j(y_j(t)) - A_j(x_j(t))$ and

$$\widetilde{B}_j(w_j(t - q_{ij}(t))) = B_j(y_j(t - q_{ij}(t))) - B_j(x_j(t - q_{ij}(t))).$$

According to (F_2) and the periodicity in (1.1), one can select a constant $\lambda > 0$ such that

$$E_i^\lambda(t) < 0,\quad 4E_i^\lambda(t)G_i^\lambda(t) > (H_i^\lambda(t))^2,\ \forall t \in \mathbb{R}, \tag{2.4}$$

where

$$
\begin{cases}
E_i^{\lambda}(t) = \lambda\alpha_i^2 + \alpha_i\gamma_i - \bar{a}_i(t)\alpha_i^2 + \frac{1}{2}\alpha_i^2 \sum\limits_{j=1}^{n}(|\bar{c}_{ij}(t)|L_j^A + |\bar{d}_{ij}(t)|L_j^B) \\
\qquad +\frac{1}{2}\alpha_i^2 \sum\limits_{j=1}^{n}\sum\limits_{l=1}^{n}|\theta_{ijl}(t)|(M_j^Q L_l^Q + L_j^Q M_l^Q), \\
G_i^{\lambda}(t) = -\bar{b}_i(t)\alpha_i\gamma_i + \lambda\beta_i + \lambda\gamma_i^2 + \frac{1}{2}\sum\limits_{j=1}^{n}(|\bar{c}_{ij}(t)|L_j^A + |\bar{d}_{ij}(t)|L_j^B)|\alpha_i\gamma_i| \\
\qquad +\frac{1}{2}\sum\limits_{j=1}^{n}\alpha_j^2(|\bar{c}_{ji}(t)|L_i^A + \bar{d}_{ji}^{+}L_i^B\frac{1}{1-\dot{q}_{ji}^{+}}e^{2\lambda q_{ji}^{+}}) \\
\qquad +\frac{1}{2}\sum\limits_{j=1}^{n}(|\bar{c}_{ji}(t)|L_i^A + \bar{d}_{ji}^{+}L_i^B\frac{1}{1-\dot{q}_{ji}^{+}}e^{2\lambda q_{ji}^{+}})|\alpha_j\gamma_j| \\
\qquad +\frac{1}{2}\sum\limits_{j=1}^{n}\sum\limits_{l=1}^{n}|\alpha_i\gamma_i||\theta_{ijl}(t)|(M_j^Q L_l^Q + L_j^Q M_l^Q) \\
\qquad +\frac{1}{2}\sum\limits_{l=1}^{n}\sum\limits_{j=1}^{n}(\alpha_l^2 + |\alpha_l\gamma_l|)\theta_{lji}^{+}M_j^Q L_i^Q e^{2\lambda\xi_{lji}^{+}}\frac{1}{1-\dot{\xi}_{lji}^{+}} \\
\qquad +\frac{1}{2}\sum\limits_{j=1}^{n}\sum\limits_{l=1}^{n}(\alpha_j^2 + |\alpha_j\gamma_j|)\theta_{jil}^{+}L_i^Q M_l^Q e^{2\lambda\eta_{jil}^{+}}\frac{1}{1-\dot{\eta}_{jil}^{+}}), \\
H_i^{\lambda}(t) = \beta_i + \gamma_i^2 + 2\lambda\alpha_i\gamma_i - \bar{a}_i(t)\alpha_i\gamma_i - \bar{b}_i(t)\alpha_i^2, \quad i, j \in D.
\end{cases}
$$

Define the Lyapunov function by setting

$$
\begin{aligned}
K(t) \;=\;& \frac{1}{2}\sum_{i=1}^{n}\beta_i w_i^2(t)e^{2\lambda t} + \frac{1}{2}\sum_{i=1}^{n}(\alpha_i w_i'(t) + \gamma_i w_i(t))^2 e^{2\lambda t} \\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i^2\bar{d}_{ij}^{+} + |\alpha_i\gamma_i|\bar{d}_{ij}^{+})e^{2\lambda q_{ij}^{+}}L_j^B\int_{t-q_{ij}(t)}^{t} w_j^2(s)\frac{1}{1-\dot{q}_{ij}^{+}}e^{2\lambda s}ds \\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{j=1}^{n}(\alpha_l^2 + |\alpha_l\gamma_l|)\theta_{lji}^{+}M_j^Q L_i^Q e^{2\lambda\xi_{lji}^{+}}\int_{t-\xi_{lji}(t)}^{t} w_i^2(s)\frac{1}{1-\dot{\xi}_{lji}^{+}}e^{2\lambda s}ds \\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}(\alpha_j^2 + |\alpha_j\gamma_j|)\theta_{jil}^{+}L_i^Q M_l^Q e^{2\lambda\eta_{jil}^{+}}\int_{t-\eta_{jil}(t)}^{t} w_i^2(s)\frac{1}{1-\dot{\eta}_{jil}^{+}}e^{2\lambda s}ds.
\end{aligned}
$$

Straightforward computation yields that

$$
\begin{aligned}
K'(t) \;=\;& 2\lambda[\frac{1}{2}\sum_{i=1}^{n}\beta_i w_i^2(t)e^{2\lambda t} + \frac{1}{2}\sum_{i=1}^{n}(\alpha_i w_i'(t) + \gamma_i w_i(t))^2 e^{2\lambda t}] \\
&+\sum_{i=1}^{n}\beta_i w_i(t)w_i'(t)e^{2\lambda t} + \sum_{i=1}^{n}(\alpha_i w_i'(t) + \gamma_i w_i(t))(\alpha_i w_i''(t) + \gamma_i w_i'(t))e^{2\lambda t} \\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i^2\bar{d}_{ij}^{+} + |\alpha_i\gamma_i|\bar{d}_{ij}^{+})e^{2\lambda q_{ij}^{+}}\frac{1}{1-\dot{q}_{ij}^{+}}L_j^B \\
&\times[w_j^2(t)e^{2\lambda t} - w_j^2(t-q_{ij}(t))e^{2\lambda(t-q_{ij}(t))}(1-q_{ij}'(t))] \\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{j=1}^{n}(\alpha_l^2 + |\alpha_l\gamma_l|)\theta_{lji}^{+}M_j^Q L_i^Q e^{2\lambda\xi_{lji}^{+}}\frac{1}{1-\dot{\xi}_{lji}^{+}} \\
&\times[w_i^2(t)e^{2\lambda t} - w_i^2(t-\xi_{lji}(t))e^{2\lambda(t-\xi_{lji}(t))}(1-\xi_{lji}'(t))]
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}(\alpha_j^2+|\alpha_j\gamma_j|)\theta_{jil}^{+}L_i^{Q}M_l^{Q}e^{2\lambda\eta_{jil}^{+}}\frac{1}{1-\dot{\eta}_{jil}^{+}} \\
& \times[w_i^2(t)e^{2\lambda t}-w_i^2(t-\eta_{jil}(t))e^{2\lambda(t-\eta_{jil}(t))}(1-\eta_{jil}'(t))] \\
= \; & 2\lambda[\frac{1}{2}\sum_{i=1}^{n}\beta_i w_i^2(t)e^{2\lambda t}+\frac{1}{2}\sum_{i=1}^{n}(\alpha_i w_i'(t)+\gamma_i w_i(t))^2e^{2\lambda t}] \\
& +\sum_{i=1}^{n}(\beta_i+\gamma_i^2)w_i(t)w_i'(t)e^{2\lambda t}+\sum_{i=1}^{n}\alpha_i(\alpha_i w_i'(t)+\gamma_i w_i(t))e^{2\lambda t} \\
& \times[-\bar{a}_i(t)w_i'(t)-\bar{b}_i(t)w_i(t)+\sum_{j=1}^{n}\bar{c}_{ij}(t)\widetilde{A}_j(w_j(t))+\sum_{j=1}^{n}\bar{d}_{ij}(t)\widetilde{B}_j(w_j(t-q_{ij}(t))) \\
& +\sum_{j=1}^{n}\sum_{l=1}^{n}\theta_{ijl}(t)(Q_j(y_j(t-\eta_{ijl}(t)))Q_l(y_l(t-\xi_{ijl}(t))) \\
& -Q_j(y_j(t-\eta_{ijl}(t)))Q_l(x_l(t-\xi_{ijl}(t)))+Q_j(y_j(t-\eta_{ijl}(t)))Q_l(x_l(t-\xi_{ijl}(t))) \\
& -Q_j(x_j(t-\eta_{ijl}(t)))Q_l(x_l(t-\xi_{ijl}(t))))]+\sum_{i=1}^{n}\alpha_i\gamma_i(w_i'(t))^2e^{2\lambda t} \\
& +\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i^2\bar{d}_{ij}^{+}+|\alpha_i\gamma_i|\bar{d}_{ij}^{+})e^{2\lambda q_{ij}^{+}}L_j^{B} \\
& \times[w_j^2(t)\frac{1}{1-\dot{q}_{ij}^{+}}e^{2\lambda t}-w_j^2(t-q_{ij}(t))e^{2\lambda(t-q_{ij}(t))}\frac{1-q_{ij}'(t)}{1-\dot{q}_{ij}^{+}}] \\
& +\frac{1}{2}\sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{j=1}^{n}(\alpha_l^2+|\alpha_l\gamma_l|)\theta_{lji}^{+}M_j^{Q}L_i^{Q}e^{2\lambda\xi_{lji}^{+}} \\
& \times[w_i^2(t)\frac{1}{1-\dot{\xi}_{lji}^{+}}e^{2\lambda t}-w_i^2(t-\xi_{lji}(t))e^{2\lambda(t-\xi_{lji}(t))}\frac{1-\xi_{lji}'(t)}{1-\dot{\xi}_{lji}^{+}}] \\
& +\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}(\alpha_j^2+|\alpha_j\gamma_j|)\theta_{jil}^{+}L_i^{Q}M_l^{Q}e^{2\lambda\eta_{jil}^{+}} \\
& \times[w_i^2(t)\frac{1}{1-\dot{\eta}_{jil}^{+}}e^{2\lambda t}-w_i^2(t-\eta_{jil}(t))e^{2\lambda(t-\eta_{jil}(t))}\frac{1-\eta_{jil}'(t)}{1-\dot{\eta}_{jil}^{+}}] \\
\leq \; & e^{2\lambda t}\{\sum_{i=1}^{n}(\beta_i+\gamma_i^2+2\lambda\alpha_i\gamma_i-\bar{a}_i(t)\alpha_i\gamma_i-\bar{b}_i(t)\alpha_i^2)w_i(t)w_i'(t) \\
& +\sum_{i=1}^{n}(\lambda\alpha_i^2+\alpha_i\gamma_i-\bar{a}_i(t)\alpha_i^2)(w_i'(t))^2-\sum_{i=1}^{n}(\bar{b}_i(t)\alpha_i\gamma_i-\lambda\beta_i-\lambda\gamma_i^2)w_i^2(t) \\
& +\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i^2\bar{d}_{ij}^{+}+|\alpha_i\gamma_i|\bar{d}_{ij}^{+})\frac{1}{1-\dot{q}_{ij}^{+}}e^{2\lambda q_{ij}^{+}}L_j^{B}w_j^2(t) \\
& +\frac{1}{2}\sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{j=1}^{n}(\alpha_l^2+|\alpha_l\gamma_l|)\theta_{lji}^{+}M_j^{Q}L_i^{Q}e^{2\lambda\xi_{lji}^{+}}w_i^2(t)\frac{1}{1-\dot{\xi}_{lji}^{+}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}(\alpha_j^2+|\alpha_j\gamma_j|)\theta_{jil}^{+}L_i^{Q}M_l^{Q}e^{2\lambda\eta_{jil}^{+}}w_i^2(t)\frac{1}{1-\dot{\eta}_{jil}^{+}}\\
&-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i^2\bar{d}_{ij}^{+}+|\alpha_i\gamma_i|\bar{d}_{ij}^{+})L_j^{B}w_j^2(t-q_{ij}(t))\\
&-\frac{1}{2}\sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{j=1}^{n}(\alpha_l^2+|\alpha_l\gamma_l|)\theta_{lji}^{+}M_j^{Q}L_i^{Q}w_i^2(t-\xi_{lji}(t))\\
&-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}(\alpha_j^2+|\alpha_j\gamma_j|)\theta_{jil}^{+}L_i^{Q}M_l^{Q}w_i^2(t-\eta_{jil}(t))\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i^2|w_i'(t)|+|\alpha_i\gamma_i||w_i(t)|)|\bar{c}_{ij}(t)||\widetilde{A}_j(w_j(t))|\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i^2|w_i'(t)|+|\alpha_i\gamma_i||w_i(t)|)|\bar{d}_{ij}(t)||\widetilde{B}_j(w_j(t-q_{ij}(t)))|\}\\
&+\sum_{i=1}^{n}(\alpha_i^2|w_i'(t)|+|\alpha_i\gamma_i||w_i(t)|)\\
&\times\sum_{j=1}^{n}\sum_{l=1}^{n}|\theta_{ijl}(t)|(M_j^{Q}L_l^{Q}|w_l(t-\xi_{ijl}(t))|+L_j^{Q}|w_j(t-\eta_{ijl}(t))|M_l^{Q})\}\\
=\;&e^{2\lambda t}\{\sum_{i=1}^{n}(\beta_i+\gamma_i^2+2\lambda\alpha_i\gamma_i-\bar{a}_i(t)\alpha_i\gamma_i-\bar{b}_i(t)\alpha_i^2)w_i(t)w_i'(t)\\
&+\sum_{i=1}^{n}(\lambda\alpha_i^2+\alpha_i\gamma_i-\bar{a}_i(t)\alpha_i^2)(w_i'(t))^2\\
&+\sum_{i=1}^{n}[-\bar{b}_i(t)\alpha_i\gamma_i+\lambda\beta_i+\lambda\gamma_i^2+\frac{1}{2}\sum_{j=1}^{n}(\alpha_j^2\bar{d}_{ji}^{+}+|\alpha_j\gamma_j|\bar{d}_{ji}^{+})\frac{1}{1-\dot{q}_{ji}^{+}}e^{2\lambda q_{ji}^{+}}L_i^{B}\\
&+\frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{n}(\alpha_l^2+|\alpha_l\gamma_l|)\theta_{lji}^{+}M_j^{Q}L_i^{Q}e^{2\lambda\xi_{lji}^{+}}\frac{1}{1-\dot{\xi}_{lji}^{+}}\\
&+\frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}(\alpha_j^2+|\alpha_j\gamma_j|)\theta_{jil}^{+}L_i^{Q}M_l^{Q}e^{2\lambda\eta_{jil}^{+}}\frac{1}{1-\dot{\eta}_{jil}^{+}}]w_i^2(t)\\
&-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i^2\bar{d}_{ij}^{+}+|\alpha_i\gamma_i|\bar{d}_{ij}^{+})L_j^{B}w_j^2(t-q_{ij}(t))\\
&-\frac{1}{2}\sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{j=1}^{n}(\alpha_l^2+|\alpha_l\gamma_l|)\theta_{lji}^{+}M_j^{Q}L_i^{Q}w_i^2(t-\xi_{lji}(t))\\
&-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}(\alpha_j^2+|\alpha_j\gamma_j|)\theta_{jil}^{+}L_i^{Q}M_l^{Q}w_i^2(t-\eta_{jil}(t))
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i^2|w_i'(t)|+|\alpha_i\gamma_i||w_i(t)|)|\bar{c}_{ij}(t)||\widetilde{A}_j(w_j(t))|\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i^2|w_i'(t)|+|\alpha_i\gamma_i||w_i(t)|)|\bar{d}_{ij}(t)||\widetilde{B}_j(w_j(t-q_{ij}(t)))|\}\\
&+\sum_{i=1}^{n}(\alpha_i^2|w_i'(t)|+|\alpha_i\gamma_i||w_i(t)|)\sum_{j=1}^{n}\sum_{l=1}^{n}|\theta_{ijl}(t)|\\
&\times(M_j^Q L_l^Q|w_l(t-\xi_{ijl}(t))|+L_j^Q|w_j(t-\eta_{ijl}(t))|M_l^Q)\},\quad \forall t\in[0,+\infty).
\end{aligned}\tag{2.5}
$$

It follows from (F_1) and $PQ \le \frac{1}{2}(P^2+Q^2)(P,Q\in\mathbb{R})$ that

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i^2|w_i'(t)|+|\alpha_i\gamma_i||w_i(t)|)|\bar{c}_{ij}(t)||\widetilde{A}_j(w_j(t))|\\
\le\ &\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i^2|\bar{c}_{ij}(t)|L_j^A((w_i'(t))^2+w_j^2(t))\\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha_i\gamma_i||\bar{c}_{ij}(t)|L_j^A(w_i^2(t)+w_j^2(t))\\
=\ &\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i^2|\bar{c}_{ij}(t)|L_j^A(w_i'(t))^2\\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(|\alpha_i\gamma_i||\bar{c}_{ij}(t)|L_j^A+\alpha_j^2|\bar{c}_{ji}(t)|L_i^A+|\alpha_j\gamma_j||\bar{c}_{ji}(t)|L_i^A)w_i^2(t),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i^2|w_i'(t)|+|\alpha_i\gamma_i||w_i(t)|)|\bar{d}_{ij}(t)||\widetilde{B}_j(w_j(t-q_{ij}(t)))|\\
\le\ &\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i^2|\bar{d}_{ij}(t)|L_j^B((w_i'(t))^2+w_j^2(t-q_{ij}(t)))\\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha_i\gamma_i||\bar{d}_{ij}(t)|L_j^B(w_i^2(t)+w_j^2(t-q_{ij}(t)))\\
=\ &\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i^2|\bar{d}_{ij}(t)|L_j^B(w_i'(t))^2+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|\alpha_i\gamma_i||\bar{d}_{ij}(t)|L_j^B w_i^2(t)\\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i^2|\bar{d}_{ij}(t)|L_j^B+|\alpha_i\gamma_i||\bar{d}_{ij}(t)|L_j^B)w_j^2(t-q_{ij}(t)),
\end{aligned}
$$

and

$$
\sum_{i=1}^{n}(\alpha_i^2|w_i'(t)|+|\alpha_i\gamma_i||w_i(t)|)
$$

$$
\begin{aligned}
& \times \sum_{j=1}^{n} \sum_{l=1}^{n} |\theta_{ijl}(t)|(M_j^Q L_l^Q |w_l(t-\xi_{ijl}(t))| + L_j^Q |w_j(t-\eta_{ijl}(t))| M_l^Q) \\
\leq\ & \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_i^2 |\theta_{ijl}(t)| M_j^Q L_l^Q ((w_i'(t))^2 + w_l^2(t-\xi_{ijl}(t))) \\
& + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} |\alpha_i\gamma_i||\theta_{ijl}(t)| M_j^Q L_l^Q ((w_i(t))^2 + w_l^2(t-\xi_{ijl}(t))) \\
& + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_i^2 |\theta_{ijl}(t)| L_j^Q M_l^Q ((w_i'(t))^2 + w_j^2(t-\eta_{ijl}(t))) \\
& + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} |\alpha_i\gamma_i||\theta_{ijl}(t)| L_j^Q M_l^Q ((w_i(t))^2 + w_j^2(t-\eta_{ijl}(t))) \\
=\ & \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_i^2 |\theta_{ijl}(t)| (M_j^Q L_l^Q + L_j^Q M_l^Q)(w_i'(t))^2 \\
& + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} |\alpha_i\gamma_i||\theta_{ijl}(t)| (M_j^Q L_l^Q + L_j^Q M_l^Q)(w_i(t))^2 \\
& + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} (\alpha_i^2 + |\alpha_i\gamma_i|)|\theta_{ijl}(t)| M_j^Q L_l^Q w_l^2(t-\xi_{ijl}(t)) \\
& + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} (\alpha_i^2 + |\alpha_i\gamma_i|)|\theta_{ijl}(t)| L_j^Q M_l^Q w_j^2(t-\eta_{ijl}(t)) \\
=\ & \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_i^2 |\theta_{ijl}(t)| (M_j^Q L_l^Q + L_j^Q M_l^Q)(w_i'(t))^2 \\
& + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} |\alpha_i\gamma_i||\theta_{ijl}(t)| (M_j^Q L_l^Q + L_j^Q M_l^Q)(w_i(t))^2 \\
& + \frac{1}{2} \sum_{l=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} (\alpha_l^2 + |\alpha_l\gamma_l|)|\theta_{lji}(t)| M_j^Q L_i^Q w_i^2(t-\xi_{lji}(t)) \\
& + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{l=1}^{n} (\alpha_j^2 + |\alpha_j\gamma_j|)|\theta_{jil}(t)| L_i^Q M_l^Q w_i^2(t-\eta_{jil}(t)),
\end{aligned}
$$

which, together with (2.4) and (2.5), entails that

$$
\begin{aligned}
K'(t) \ \leq\ & e^{2\lambda t}\{\sum_{i=1}^{n} (\beta_i + \gamma_i^2 + 2\lambda\alpha_i\gamma_i - \bar{a}_i(t)\alpha_i\gamma_i - \bar{b}_i(t)\alpha_i^2) w_i(t) w_i'(t) \\
& + \sum_{i=1}^{n} [\lambda\alpha_i^2 + \alpha_i\gamma_i - \bar{a}_i(t)\alpha_i^2 + \frac{1}{2}\alpha_i^2 \sum_{j=1}^{n} (|\bar{c}_{ij}(t)| L_j^A + |\bar{d}_{ij}(t)| L_j^B) \\
& + \frac{1}{2}\alpha_i^2 \sum_{j=1}^{n} \sum_{l=1}^{n} |\theta_{ijl}(t)| (M_j^Q L_l^Q + L_j^Q M_l^Q)](w_i'(t))^2
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{i=1}^{n}[-\bar{b}_i(t)\alpha_i\gamma_i+\lambda\beta_i+\lambda\gamma_i^2+\frac{1}{2}\sum_{j=1}^{n}(|\bar{c}_{ij}(t)|L_j^A+|\bar{d}_{ij}(t)|L_j^B)|\alpha_i\gamma_i| \\
& +\frac{1}{2}\sum_{j=1}^{n}\alpha_j^2(|\bar{c}_{ji}(t)|L_i^A+\bar{d}_{ji}^+L_i^B\frac{1}{1-\dot{q}_{ji}^+}e^{2\lambda q_{ji}^+}) \\
& +\frac{1}{2}\sum_{j=1}^{n}(|\bar{c}_{ji}(t)|L_i^A+\bar{d}_{ji}^+L_i^B\frac{1}{1-\dot{q}_{ji}^+}e^{2\lambda q_{ji}^+})|\alpha_j\gamma_j| \\
& +\frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}|\alpha_i\gamma_i||\theta_{ijl}(t)|(M_j^QL_l^Q+L_j^QM_l^Q) \\
& +\frac{1}{2}\sum_{l=1}^{n}\sum_{j=1}^{n}(\alpha_l^2+|\alpha_l\gamma_l|)\theta_{lji}^+M_j^QL_i^Qe^{2\lambda\xi_{lji}^+}\frac{1}{1-\dot{\xi}_{lji}^+} \\
& +\frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}(\alpha_j^2+|\alpha_j\gamma_j|)\theta_{jil}^+L_i^QM_l^Qe^{2\lambda\eta_{jil}^+}\frac{1}{1-\dot{\eta}_{jil}^+})]w_i^2(t)\} \\
= \quad & e^{2\lambda t}\{\sum_{i=1}^{n}(E_i^\lambda(t)(w_i'(t))^2+G_i^\lambda(t)w_i^2(t)+H_i^\lambda(t)w_i(t)w_i'(t))\} \\
= \quad & e^{2\lambda t}\{\sum_{i=1}^{n}E_i^\lambda(t)(w_i'(t)+\frac{H_i^\lambda(t)}{2E_i^\lambda(t)}w_i(t))^2+\sum_{i=1}^{n}(G_i^\lambda(t)-\frac{(H_i^\lambda(t))^2}{4E_i^\lambda(t)})w_i^2(t)\} \\
\leq \quad & 0, \quad \forall t\in[0,+\infty).
\end{aligned}
$$

This indicates that $K(t)\leq K(0)$ for all $t\in[0,+\infty)$, and

$$
\frac{1}{2}\sum_{i=1}^{n}\beta_iw_i^2(t)e^{2\lambda t}+\frac{1}{2}\sum_{i=1}^{n}(\alpha_iw_i'(t)+\gamma_iw_i(t))^2e^{2\lambda t}\leq K(0),\ t\in[0,+\infty).
$$

Note that

$$
(\alpha_iw_i'(t)e^{\lambda t}+\gamma_iw_i(t)e^{\lambda t})^2=(\alpha_iw_i'(t)+\gamma_iw_i(t))^2e^{2\lambda t}
$$

and

$$
\alpha_i|w_i'(t)|e^{\lambda t}\leq|\alpha_iw_i'(t)e^{\lambda t}+\gamma_iw_i(t)e^{\lambda t}|+|\gamma_iw_i(t)e^{\lambda t}|,
$$

one can find a constant $M>0$ such that

$$
|w_i'(t)|\leq Me^{-\lambda t},\quad |w_i(t)|\leq Me^{-\lambda t},\ \ t\geq 0,\ i\in D,
$$

which proves Lemma 2.1.
Remark 2.2. Under the assumptions adopted in Lemma 2.1, if $y(t)$ is an equilibrium point or a periodic solution of (1.1), one can see $y(t)$ is globally exponentially stable. Moreover, the definition of global exponential stability can be also seen in [13, 16].

3. Anti-periodicity of system (1.1)

Now, we set out the main result of this paper as follows.

Theorem 3.1. Under assumptions (F_1)–(F_3), system (1.1) possesses a global exponential stable T-anti-periodic solution.

Proof. Denote $\kappa(t) = (\kappa_1(t), \kappa_2(t), \cdots, \kappa_n(t))$ be a solution of system (1.1) satisfying:

$$\kappa_i(s) = \varphi_i^\kappa(s),\ \ \kappa_i'(s) = \psi_i^\kappa(s),\ -\tau_i \le s \le 0,\ \varphi_i^\kappa, \psi_i^\kappa \in C([-\tau_i, 0], \mathbb{R}),\ i \in D. \tag{3.1}$$

With the aid of (F_1), one can see that

$$\bar{a}_i(t+T) = \bar{a}_i(t),\ \ \bar{b}_i(t+T) = \bar{b}_i(t),\ \ q_{ij}(t+T) = q_{ij}(t),$$

$$\eta_{ijl}(t+T) = \eta_{ijl}(t),\ \ \xi_{ijl}(t+T) = \xi_{ijl}(t),\ \ J_i(t+T) = -J_i(t),$$
$$(-1)^{m+1}\bar{c}_{ij}(t+(m+1)T)A_j(\kappa_j(t+(m+1)T)) = \bar{c}_{ij}(t)A_j((-1)^{m+1}\kappa_j(t+(m+1)T)),$$
$$(-1)^{m+1}\bar{d}_{ij}(t+(m+1)T)B_j(\kappa_j(t+(m+1)T-q_{ij}(t))) = \bar{d}_{ij}(t)B_j((-1)^{m+1}\kappa_j(t+(m+1)T-q_{ij}(t))),$$

and

$$\begin{aligned}&(-1)^{m+1}\theta_{ijl}(t+(m+1)T)Q_j(\kappa_j(t+(m+1)T-\eta_{ijl}(t)))Q_l(\kappa_l(t+(m+1)T-\xi_{ijl}(t)))\\ &= \theta_{ijl}(t)Q_j((-1)^{m+1}\kappa_j(t+(m+1)T-\eta_{ijl}(t)))Q_l((-1)^{m+1}\kappa_l(t+(m+1)T-\xi_{ijl}(t))),\end{aligned}$$

where $t \in \mathbb{R}$ and $i, j, l \in D$.

Consequently, for any nonnegative integer m,

$$\begin{aligned}&((-1)^{m+1}\kappa_i(t+(m+1)T))''\\ =\ & -\bar{a}_i(t)((-1)^{m+1}\kappa_i(t+(m+1)T))' - \bar{b}_i(t)((-1)^{m+1}\kappa_i(t+(m+1)T))\\ &+\sum_{j=1}^n \bar{c}_{ij}(t)A_j((-1)^{m+1}\kappa_j(t+(m+1)T))\\ &+\sum_{j=1}^n \bar{d}_{ij}(t)B_j((-1)^{m+1}\kappa_j(t+(m+1)T-q_{ij}(t))) + \sum_{j=1}^n\sum_{l=1}^n \theta_{ijl}(t)\\ &\times Q_j((-1)^{m+1}\kappa_j(t+(m+1)T-\eta_{ijl}(t)))\\ &\times Q_l((-1)^{m+1}\kappa_l(t+(m+1)T-\xi_{ijl}(t)))\\ &+J_i(t),\ \text{for all } i \in D, t+(m+1)T \ge 0.\end{aligned} \tag{3.2}$$

Clearly, $(-1)^{m+1}\kappa(t+(m+1)T)$ $(t+(m+1)T \ge 0)$ satisfies (1.1), and $\nu(t) = -\kappa(t+T)$ is a solution of system (1.1) involving initial values:

$$\varphi_i^\nu(s) = -\kappa_i(s+T), \psi_i^\nu(s) = -\kappa_i'(s+T),\ \text{for all } s \in [-\tau_i, 0], i \in D.$$

Thus, with the aid of Lemma 2.1, we can pick a constant $M = M(\varphi^\kappa, \psi^\kappa, \varphi^\nu, \psi^\nu)$ satisfying

$$|\kappa_i(t) - \nu_i(t)| \le Me^{-\lambda t},\ \ |\kappa_i'(t) - \nu_i'(t)| \le Me^{-\lambda t},\ \ \text{for all } t \ge 0, i \in D.$$

Hence,

$$\left.\begin{aligned}&|(-1)^p\kappa_i(t+pT) - (-1)^{p+1}\kappa_i(t+(p+1)T)|\\ &= |\kappa_i(t+pT) - \nu_i(t+pT)| \le Me^{-\lambda(t+pT)},\\ &|((-1)^p\kappa_i(t+pT))' - ((-1)^{p+1}\kappa_i(t+(p+1)T))'|\\ &= |\kappa_i'(t+pT) - \nu_i'(t+pT)| \le Me^{-\lambda(t+pT)},\end{aligned}\right\}\ \forall i \in D,\ t+pT \ge 0. \tag{3.3}$$

Consequently,

$$(-1)^{m+1}\kappa_i(t+(m+1)T) = \kappa_i(t) + \sum_{p=0}^{m}[(-1)^{p+1}\kappa_i(t+(p+1)T) - (-1)^p\kappa_i(t+pT)] \ (i \in D)$$

and

$$\begin{aligned} &((-1)^{m+1}\kappa_i(t+(m+1)T))' \\ = \ & \kappa_i'(t) + \sum_{p=0}^{m}[((-1)^{p+1}\kappa_i(t+(p+1)T))' - ((-1)^p\kappa_i(t+pT))'] \ (i \in D). \end{aligned}$$

Therefore, (3.3) suggests that there exists a continuous differentiable function $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))$ such that $\{(-1)^m\kappa(t+mT)\}_{m\geq 1}$ and $\{((-1)^m\kappa(t+mT))'\}_{m\geq 1}$ are uniformly convergent to $y(t)$ and $y'(t)$ on any compact set of $\mathbb{R}$, respectively.

Moreover,

$$y(t+T) = \lim_{m\to+\infty}(-1)^m\kappa(t+T+mT) = -\lim_{(m+1)\to+\infty}(-1)^{m+1}\kappa(t+(m+1)T) = -y(t)$$

involves that $y(t)$ is $T-$anti-periodic on $\mathbb{R}$. It follows from (F_1)-(F_3) and the continuity on (3.2) that $\{(\kappa''(t+(m+1)T)\}_{m\geq 1}$ uniformly converges to a continuous function on any compact set of $\mathbb{R}$. Furthermore, for any compact set of $\mathbb{R}$, setting $m \longrightarrow +\infty$, we obtain

$$\begin{aligned} y_i''(t) \ = \ & -\bar{a}_i(t)y_i'(t) - \bar{b}_i(t)y_i(t) + \sum_{j=1}^{n}\bar{c}_{ij}(t)A_j(y_j(t)) + \sum_{j=1}^{n}\bar{d}_{ij}(t)B_j(y_j(t-q_{ij}(t))) \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n}\theta_{ijl}(t)Q_j(y_j(t-\eta_{ijl}(t)))Q_l(y_l(t-\xi_{ijl}(t))) + J_i(t),, i \in D, \end{aligned}$$

which involves that $y(t)$ is a $T-$anti-periodic solution of (1.1). Again from Lemma 2.1, we gain that $y(t)$ is globally exponentially stable. This finishes the proof of Theorem 3.1.

Remark 3.1. For inertial neural networks without high-order terms respectively, suppose

$$|\bar{a}_i - \bar{b}_i| < 2, \ A_i \ \text{and} \ B_i \ \text{are bounded}, \ i \in D, \tag{3.4}$$

and

$$|\bar{a}_i - \bar{b}_i + 1| < 1, i \in D, \tag{3.5}$$

the authors gained the existence and stability on periodic solutions in [10, 11] and anti-periodic solutions in [12]. Moreover, the reduced-order method was crucial in [10–12] when anti-periodicity and periodicity of second-order inertial neural networks were considered. However, (3.4) and (3.5) have been abandoned in Theorem 3.1 and the reduced-order method has been substituted in this paper. Therefore, our results on anti-periodicity of high-order inertial Hopfield neural networks are new and supplemental in nature.

4. Examples and numerical simulations

Example 4.1. Let $n = 2$, and consider a class of high-order inertial Hopfield neural networks in the form of

$$
\begin{cases}
x_1''(t) \\
= -14.92x_1'(t) - 27.89x_1(t) + 2.28(\sin t)A_1(x_1(t)) \\
\quad +2.19(\cos t)A_2(x_2(t)) \\
\quad -0.84(\cos 2t)B_1(x_1(t-0.2\sin^2 t)) + 2.41(\cos 2t)B_2(x_2(t-0.3\sin^2 t)) \\
\quad +4(\sin 2t)Q_1(x_1(t-0.4\sin^2 t))Q_2(x_2(t-0.5\sin^2 t)) + 55\sin t, \\
x_2''(t) \\
= -15.11x_2'(t) - 31.05x_2(t) - 1.88(\sin t)A_1(x_1(t)) \\
\quad -2.33(\cos t)A_2(x_2(t)) \\
\quad -2.18(\sin 2t)B_1(x_1(t-0.2\cos^2 t)) + 3.18(\cos 2t)B_2(x_2(t-0.3\cos^2 t)) \\
\quad +3.8(\sin 2t)Q_1(x_1(t-0.4\cos^2 t))Q_2(x_2(t-0.5\cos^2 t)) + 48\sin t,
\end{cases} \tag{4.1}
$$

where $t \geq 0,\ A_1(u) = A_2(u) = \frac{1}{35}|u|,\ B_1(u) = B_2(u) = \frac{1}{48}u,\ Q_1(u) = Q_2(u) = \frac{1}{55}\arctan u.$

Using a direct calculation, one can check that (4.1) satisfies (2.4) and $(F_1) - (F_3)$. Applying Theorem 3.1, it is obvious that system (4.1) has a globally exponentially stable π-anti-periodic solution. Simulations reflect that the theoretical anti-periodicity is in sympathy with the numerically observed behavior (Figures 1 and 2).

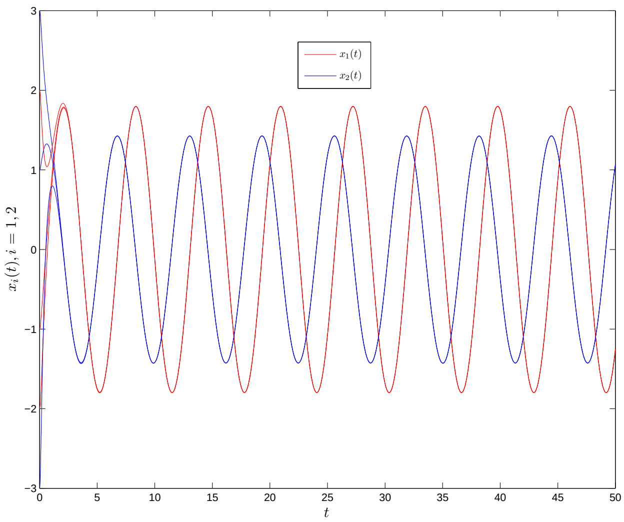

Figure 1. Numerical solutions $x(t)$ to system (4.1) with initial values: $(\varphi_1(s), \varphi_2(s), \psi_1(s), \psi_2(s)) \equiv (-1, 3, 0, 0), (-2, 1, 0, 0), (2, -3, 0, 0),\ s \in [-5, 0]$.

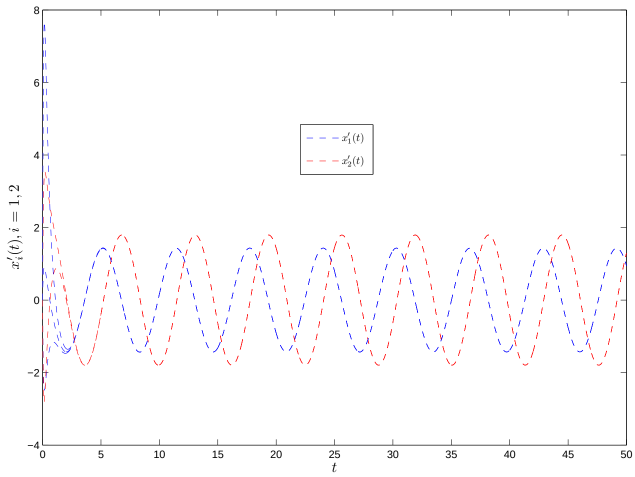

Figure 2. Numerical solutions $x'(t)$ to system (4.1) with initial value $(\varphi_1(s), \varphi_2(s), \psi_1(s), \psi_2(s)) \equiv (-1, 3, 0, 0), (-2, 1, 0, 0), (2, -3, 0, 0)$, $s \in [-5, 0]$.

Example 4.2. Regard the following high-order inertial Hopfield neural networks involving time-varying delays and coefficients:

$$\begin{cases} x_1''(t) \\ = -(14 + 0.9|\sin t|)x_1'(t) - (27 + 0.8|\cos t|)x_1(t) + 2.28(\sin t)A_1(x_1(t)) \\ \quad +2.19(\cos t)A_2(x_2(t)) \\ \quad -0.84(\cos 2t)B_1(x_1(t - 0.2\sin^2 t)) + 2.41(\cos 2t)B_2(x_2(t - 0.3\sin^2 t)) \\ \quad +4(\sin t)Q_1(x_1(t - 0.4\sin^2 t))Q_2(x_2(t - 0.5\sin^2 t)) + 100\sin t, \\ x_2''(t) \\ = -(15 + 0.1|\cos t|)x_2'(t) - (31 + 0.1|\sin t|)x_2(t) - 1.88(\sin t)A_1(x_1(t)) \\ \quad -2.33(\cos t)A_2(x_2(t)) \\ \quad -2.18(\sin 2t)B_1(x_1(t - 0.2\cos^2 t)) + 3.18(\cos 2t)B_2(x_2(t - 0.3\cos^2 t)) \\ \quad +3.8(\sin t)Q_1(x_1(t - 0.4\cos^2 t))Q_2(x_2(t - 0.5\cos^2 t)) + 200\sin t, \end{cases} \tag{4.2}$$

where $t \geq 0$, $A_1(u) = A_2(u) = \frac{1}{35}|u|$, $B_1(u) = B_2(u) = \frac{1}{48}u$, $Q_1(u) = Q_2(u) = \frac{1}{110}(|x + 1| - |x - 1|)$. Then, by Theorem 3.1, one can find that all solutions of networks (4.2) are convergent to a π-anti-periodic solution (See Figures 3 and 4).

Remark 4.1. From the figures 1–4, one can see that the solution is similar to sinusoidal oscillation, and there exists a π-anti-periodic solution satisfying $x(t + \pi) = -x(t)$. To the author's knowledge, the anti-periodicity on high-order inertial Hopfield neural networks involving time-varying delays has never been touched by using the non-reduced order method. Manifestly, the assumptions (3.4) and (3.5) adopted in [10, 11] are invalid in systems (4.1) and (4.2). In addition, the most recently papers [10, 11] only considered the polynomial power stability of some proportional time-delay systems, but not involved the exponential power stability of the addressed systems. And the results in [35–82] have not touched on the anti-periodicity of inertial neural networks. This entails that the corresponding conclusions in [10–82] and the references cited therein can not be applied to show the anti-periodic convergence for systems (4.1) and (4.2).

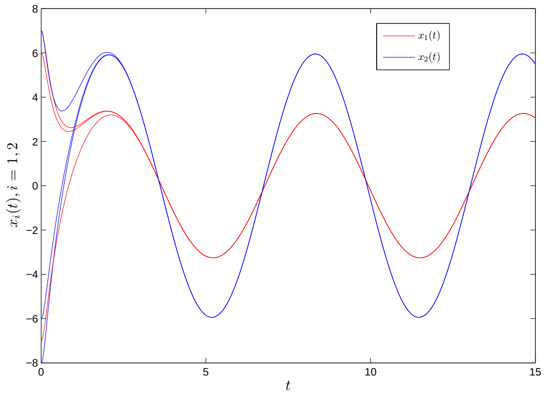

Figure 3. Numerical solutions $x(t)$ to system (4.2) with initial values: $(\varphi_1(s), \varphi_2(s), \psi_1(s), \psi_2(s)) \equiv (6, -8, 0, 0), (7, -6, 0, 0), (-7, 7, 0, 0)$, $s \in [-0.5, 0]$.

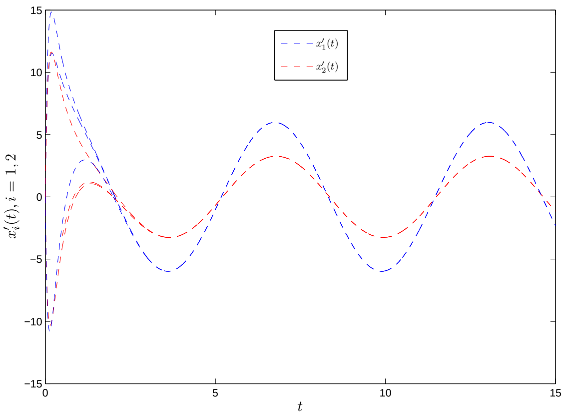

Figure 4. Numerical solutions $x'(t)$ to system (4.2) with initial value $(\varphi_1(s), \varphi_2(s), \psi_1(s), \psi_2(s)) \equiv (6, -8, 0, 0), (7, -6, 0, 0), (-7, 7, 0, 0)$, $s \in [-0.5, 0]$.

5. Conclusion

In this paper, abandoning the reduced order method, we apply inequality techniques and Lyapunov function method to establish the existence and global exponential stability of anti-periodic solutions for a class of high-order inertial Hopfield neural networks involving time-varying delays and anti-periodic environments. The obtained results are essentially new and complement some recently published results. The method proposed in this article furnishes a possible approach for studying anti-periodic on other types high-order inertial neural networks such as shunting inhibitory cellular neural networks, BAM neural networks, Cohen-Grossberg neural networks and so on.

Acknowledgments

The authors would like to express the sincere appreciation to the editor and reviewers for their helpful comments in improving the presentation and quality of the paper.

Conflict of interest

The authors confirm that they have no conflict of interest.

References

1. K. Babcock, R. Westervelt, *Stability and dynamics of simple electronic neural networks with added inertia*, Physica D, **23** (1986), 464–469.
2. K. Babcock, R. Westervelt, *Dynamics of simple electronic neural networks*, Physica D, **28** (1987), 305–316.
3. L. Duan, L. Huang, Z. Guo, et al. *Periodic attractor for reaction-diffusion high-order hopfield neural networks with time-varying delays*, Comput. Math. Appl., **73** (2017), 233–245.
4. J. Wang, X. Chen, L. Huang, *The number and stability of limit cycles for planar piecewise linear systems of node-saddle type*, J. Math. Anal. Appl., **469** (2019), 405–427.
5. J. Wang, C. Huang, L. Huang, *Discontinuity-induced limit cycles in a general planar piecewise linear system of saddle-focus type*, Nonlinear Anal. Hybrid Syst., **33** (2019), 162–178.
6. Z. Cai, J. Huang, L. Huang, *Periodic orbit analysis for the delayed Filippov system*, P. Am. Math. Soc., **146** (2018), 4667–4682.
7. T. Chen, L. Huang, P. Yu, et al. *Bifurcation of limit cycles at infinity in piecewise polynomial systems*, Nonlinear Anal. Real., **41** (2018), 82–106.
8. C. Huang, Y. Qiao, L. Huang, et al. *Dynamical behaviors of a food-chain model with stage structure and time delays*, Adv. Differ. Equ., **2018** (2018), 1–13.
9. C. Huang, S. Wen, L. Huang, *Dynamics of anti-periodic solutions on shunting inhibitory cellular neural networks with multi-proportional delays*, Neurocomputing, **357** (2019), 47–52.
10. Y. Ke, C. Miao, *Stability and existence of periodic solutions in inertial BAM neural networks with time delay*, Neural Comput. Appl., **23** (2013), 1089–1099.
11. Y. Ke, C. Miao, *Anti-periodic solutions of inertial neural networks with time delays*, Neural Process. Lett., **45** (2017), 523–538.
12. C. Xu, Q. Zhang, *Existence and global exponential stability of anti-periodic solutions for BAM neural networks with inertial term and delay*, Neurocomputing, **153** (2015), 108–116.
13. C. Huang, B. Liu, *New studies on dynamic analysis of inertial neural networks involving non-reduced order method*, Neurocomputing, **325** (2019), 283–287.
14. X. Li, X. Li, C. Hu, *Some new results on stability and synchronization for delayed inertial neural networks based on non-reduced order method*, Neural Networks, **96** (2017), 91–100.

15. C. Huang, H. Zhang, *Periodicity of non-autonomous inertial neural networks involving proportional delays and non-reduced order method*, Int. J. Biomath., **12** (2019), 1–13.

16. C. Huang, L. Yang, B. Liu, *New results on periodicity of non-autonomous inertial neural networks involving non-reduced order method*, Neural Process. Lett., **50** (2019), 595–606.

17. B. Liu, *Anti-periodic solutions for forced Rayleigh-type equations*, Nonlinear Anal. Real., **10** (2009), 2850–2856.

18. J. M. Belley, E. Bondo, *Anti-periodic solutions of Liénard equations with state dependent impulses*, J. Differ. Equ., **261** (2016), 4164–4187.

19. Z. Long, *New results on anti-periodic solutions for SICNNs with oscillating coefficients in leakage terms*, Neurocomputing, **171** (2016), 503–509.

20. C. Huang, *Exponential stability of inertial neural networks involving proportional delays and non-reduced order method*, J. Exp. Theor. Artif. Intell., **32** (2020), 133–146.

21. M. Zhang, D. Wang, *Robust dissipativity analysis for delayed memristor-based inertial neural network*, Neurocomputing, **366** (2019), 340–351.

22. M. Iswarya, R. Raja, G. Rajchakit, et al. *Existence, uniqueness and exponential stability of periodic solution for discrete-time delayed BAM neural networks based on coincidence degree theory and graph theoretic method*, Mathematics, **7** (2019), 1–18.

23. H. Zhang, *Global Large Smooth Solutions for 3-D Hall-magnetohydrodynamics*, Discrete Contin. Dyn. Syst., **39** (2019), 6669–6682.

24. X. Li, Z. Liu, J. Li, *Existence and controllability for nonlinear fractional control systems with damping in Hilbert spaces*, Acta Mech. Sin. Engl. Ser., **39** (2019), 229–242.

25. K. Zhu, Y. Xie, F. Zhou, *Pullback attractors for a damped semilinear wave equation with delays*, Acta Mech. Sin. Engl. Ser., **34** (2018), 1131–1150.

26. J. Zhao, J. Liu, L. Fang, *Anti-periodic boundary value problems of second-order functional differential equations*, Malays. Math. Sci. Soc., **37** (2014), 311–320.

27. X. Yang, S. Wen, Z. Liu, et al. *Dynamic properties of foreign exchange complex network*, Mathematics, **7** (2019), 1–19.

28. N. Huo, B. Li, Y. Li, *Existence and exponential stability of anti-periodic solutions for inertial quaternion-valued high-order Hopfield neural networks with state-dependent delays*, IEEE Access, **7** (2019), 60010–60019.

29. Z. X. Zheng, *Theory of Functional Differential Equations*, Heifei: Anhui Education Press, 1994.

30. J. Li, J. Ying, D. Xie, *On the analysis and application of an ion size-modified Poisson-Boltzmann equation*, Nonlinear Anal. Real., **47** (2019), 188–203.

31. C. Huang, X. Long, J. Cao, *Stability of anti-periodic recurrent neural networks with multi-proportional delays*, Math. Meth. Appl. Sci., **43** (2020), 6093–6102.

32. J. Zhang, C. Huang, *Dynamics analysis on a class of delayed neural networks involving inertial terms*, Adv. Differ. Equ., **120** (2020), 1–12.

33. C. Huang, H. Yang, J. Cao, *Weighted pseudo almost periodicity of multi-proportional delayed shunting inhibitory cellular neural networks with D operator*, Discrete Contin. Dyn. Syst. Ser. S, (2020), DOI:10.3934/dcdss.2020372.

34. L. Yao, *Global exponential stability on anti-periodic solutions in proportional delayed HIHNNs*, J. Exp. Theor. Artif. Intell., (2020), 1–15.

35. Y. Xu, Q. Cao, X. Guo, *Stability on a patch structure Nicholson's blowflies system involving distinctive delays*, Appl. Math. Lett., **105** (2020), 106340.

36. W. Li, L. Huang, J. Ji, *Periodic solution and its stability of a delayed Beddington–DeAngelis type predator-prey system with discontinuous control strategy*, Math. Meth. Appl. Sci., **42** (2019), 4498–4515.

37. X. Gong, F. Wen, Z. He, et al. *Extreme return, extreme volatility and investor sentiment*, Filomat, **30** (2016), 3949–3961.

38. C. Huang, L. Yang, J. Cao, *Asymptotic behavior for a class of population dynamics*, AIMS Mathematics, **5** (2020), 3378–3390.

39. X. Long, S. Gong, *New results on stability of Nicholson's blowflies equation with multiple pairs of time-varying delays*, Appl. Math. Lett., **100** (2020), 106027.

40. C. Huang, H. Zhang, L. Huang, *Almost periodicity analysis for a delayed Nicholson's blowflies model with nonlinear density-dependent mortality term*, Commun. Pure Appl. Anal., **18** (2019), 3337–3349.

41. C. Huang, C. Peng, X. Chen, et al. *Dynamics analysis of a class of delayed economic model*, Abstr. Appl. Anal., **2013** (2013), 1–12.

42. C. Huang, H. Zhang, J. Cao, et al. *Stability and Hopf bifurcation of a delayed prey-predator model with disease in the predator*, Int. J. Bifurcat. Chaos, **29** (2019), 1950091.

43. C. Huang, X. Yang, J. Cao, *Stability analysis of Nicholson's blowflies equation with two different delays*, Math. Comput. Simulation, **171** (2020), 201–206.

44. C. Huang, H. Kuang, X. Chen, et al. *An LMI approach for dynamics of switched cellular neural networks with mixed delays*, Abstr. Appl. Anal., **2013** (2013), 1–8.

45. Y. Zhou, X. Wan, C. Huang, et al. *Finite-time stochastic synchronization of dynamic networks with nonlinear coupling strength via quantized intermittent control*, Appl. Math. Comput., **376** (2020), 125157.

46. X. Zhang, H. Hu, *Convergence in a system of critical neutral functional differential equations*, Appl. Math. Lett., **107** (2020), 106385.

47. Y. Zhang, *Some observations on the diophantine equation f(x)f(y) - f(z)(2)*, Colloq. Math., **142** (2016), 275–283.

48. C. Huang, R. Su, J. Cao, et al. *Asymptotically stable high-order neutral cellular neural networks with proportional delays and D operators*, Math. Comput. Simul., **171** (2020), 127–135.

49. C. Qian, *New periodic stability for a class of Nicholson's blowflies models with multiple different delays*, Int. J. Control, (2020), 1–13.

50. L. Huang, H. Su, G. Tang, et al. *Bilinear forms graphs over residue class rings*, Linear Algebra Appl., **523** (2017), 13–32.

51. Q. Cao, G. Wang, C. Qian, *New results on global exponential stability for a periodic Nicholson's blowflies model involving time-varying delays, delays*, Adv. Differ. Equ., **2020** (2020), 1–12.

52. C. Huang, X. Long, L. Huang, et al. *Stability of almost periodic Nicholson's blowflies model involving patch structure and mortality terms*, Canad. Math. Bull., **63** (2020), 405–422.

53. J. Peng, Y. Zhang, *Heron triangles with figurate number sides*, Acta Math. Hungar., **157** (2019), 478–488.

54. F. Wang, Z. Yao, *Approximate controllability of fractional neutral differential systems with bounded delay*, Fixed Point Theor., **17** (2016), 495–507.

55. W. Liu, *An incremental approach to obtaining attribute reduction for dynamic decision systems*, Open Math., **14** (2016), 875–888.

56. L. Huang, B. Lv, *Cores and independence numbers of Grassmann graphs*, Graphs Combin., **33** (2017), 1607–1620.

57. L. Huang, J. Huang, K. Zhao, *On endomorphisms of alternating forms graph*, Discrete Math., **338** (2015), 110–121.

58. Y. Xu, Q. Cao, X. Guo, *Stability on a patch structure Nicholson's blowflies system involving distinctive delays*, Appl. Math. Lett., **105** (2020), 106340.

59. H. Hu, X. Yuan, L. Huang, et al. *Global dynamics of an SIRS model with demographics and transfer from infectious to susceptible on heterogeneous networks*, Math. Biosci. Eng., **16** (2019), 5729–5749.

60. L. Huang, B. Lv, K. Wang, *The endomorphisms of Grassmann graphs*, Ars Math. Contemp., **10** (2016), 383–392.

61. Y. Zhang, *Right triangle and parallelogram pairs with a common area and a common perimeter*, J. Number Theory, **164** (2016), 179–190.

62. H. Hu, L. Liu, *Weighted inequalities for a general commutator associated to a singular integral operator satisfying a variant of Hormander's condition*, Math. Notes, **101** (2017), 830–840.

63. L. Huang, B. Lv, K. Wang, *Erdos-Ko-Rado theorem, Grassmann graphs and p(s)-Kneser graphs for vector spaces over a residue class ring*, J. Combin. Theory Ser. A, **164** (2019), 125–158.

64. Y. Li, M. Vuorinen, Q. Zhou, *Characterizations of John spaces*, Monatsh. Math, **188** (2019), 547–559.

65. L. Li, Q. Jin, B. Yao, *Regularity of fuzzy convergence spaces*, Open Math., **16** (2018), 1455–1465.

66. C. Huang, L. Liu, *Boundedness of multilinear singular integral operator with non-smooth kernels and mean oscillation*, Quaest. Math., **40** (2017), 295–312.

67. C. Huang, J. Cao, F. Wen, et al. *Stability analysis of SIR model with distributed delay on complex networks*, PLoS One, **11** (2016), e0158813.

68. X. Li, Y. Liu, J. Wu, *Flocking and pattern motion in a modified cucker-smale model*, Bull. Korean Math. Soc., **53** (2016), 1327–1339.

69. Y. Xie, Q. Li, K. Zhu, *Attractors for nonclassical diffusion equations with arbitrary polynomial growth nonlinearity*, Nonlinear Anal. Real., **31** (2016), 23–37.
70. Y. Xie, Y. Li, Y. Zeng, *Uniform attractors for nonclassical diffusion equations with memory*, J. Funct. Space., **2016** (2016), 1–11.
71. F. Wang, P. Wang, Z. Yao, *Approximate controllability of fractional partial differential equation*, Adv. Differ. Equ., **2015** (2015), 1–10.
72. Y. Liu, J. Wu, *Multiple solutions of ordinary differential systems with min-max terms and applications to the fuzzy differential equations*, Adv. Differ. Equ., **2015** (2015), 1–13, https://doi.org/10.1186/s13662-015-0708-z.
73. L. Yan, J. Liu, Z. Luo, *Existence and multiplicity of solutions for second-order impulsive differential equations on the half-line*, Adv. Differ. Equ., **2013** (2013), 1–12.
74. Y. Liu, J. Wu, *Fixed point theorems in piecewise continuous function spaces and applications to some nonlinear problems*, Math. Meth. Appl. Sci., **37** (2014), 508–517.
75. D. Tong, W. Wang, *Conditional regularity for the 3D MHD equations in the critical Besov space*, Appl. Math. Lett., **102** (2020), 106119.
76. Y. Cai, K. Wang, W. Wang, *Global transmission dynamics of a Zika virus model*, Appl. Math. Lett., **92** (2019), 190–195.
77. C. Huang, J. Wang, L. Huang, *Asymptotically almost periodicity of delayed Nicholson-type system involving patch structure*, Electron. J. Differ. Equ., **2020** (2020), 1–17.
78. H. Zhang, Q. Cao, H. Yang, *Asymptotically almost periodicity of delayed Nicholson-type system involving patch structure*, J. Inequal. Appl., **2020** (2020), 1–27.
79. C. Qian, Y. Hu, *Novel stability criteria on nonlinear density-dependent mortality Nicholson's blowflies systems in asymptotically almost periodic environments*, J. Inequal. Appl., **2020** (2020), 1–18.
80. S. Zhou, Y. Jiang, *Finite volume methods for N-dimensional time fractional Fokker-Planck equations*, Bull. Malays. Math. Sci. Soc., **42** (2019), 3167–3186.
81. C. Huang, S. Wen, M. Li, et al. *An empirical evaluation of the influential nodes for stock market network: Chinese A shares case*, Financ. Res. Lett., (2020), 101517.
82. L. Huang, *Endomorphisms and cores of quadratic forms graphs in odd characteristic*, Finite Fields Appl., **55** (2019), 284–304.