Mathematics

## Research article

# Deferred statistical convergence of order $\alpha$ in metric spaces 

Mikail Et ${ }^{1, *}$, Muhammed Cinar ${ }^{2}$ and Hacer Sengul Kandemir ${ }^{3}$<br>${ }^{1}$ Department of Mathematics, Firat University, 23119 Elazıg, TURKEY<br>${ }^{2}$ Faculty of Education, Department of Mathematics Education, Mus Alparslan University, 49100 Mus, TURKEY<br>${ }^{3}$ Faculty of Education, Harran University, Osmanbey Campus 63190, Sanlıurfa, TURKEY<br>* Correspondence: Email: mikailet68@gmail.com.


#### Abstract

In this paper, the concepts of deferred statistical convergence of order $\alpha$ and deferred strong Cesàro summability are generalized to general metric spaces and some relations between deferred strong Cesàro summability of order $\alpha$ and deferred statistical convergence of order $\alpha$ are given in general metric spaces.


Keywords: metric space; statistical convergence; deferred statistical convergence
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## 1. Introduction

The idea of statistical convergence was given by Zygmund [1] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [2] and Fast [3] and then reintroduced independently by Schoenberg [4]. Over the years and under different names, statistical convergence has been discussed in the Theory of Fourier Analysis, Ergodic Theory, Number Theory, Measure Theory, Trigonometric Series, Turnpike Theory and Banach Spaces. Later on it was further investigated from the sequence spaces point of view and linked with summability theory by Bilalov and Nazarova [5], Braha et al. [6], Cinar et al. [7], Colak [8], Connor [9], Et et al. ( [10-14]), Fridy [15], Isik et al. ( [16-18]), Kayan et al. [19], Kucukaslan et al. ( [20,21]), Mohiuddine et al. [22], Nuray [23], Nuray and Aydın [24], Salat [25], Sengul et al. ( [26-29]), Srivastava et al. ( $[30,31]$ ) and many others.

The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$ of natural
numbers. The density of a subset $\mathbb{E}$ of $\mathbb{N}$ is defined by

$$
\delta(\mathbb{E})=\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{\mathbb{E}}(k),
$$

provided that the limit exists, where $\chi_{\mathbb{E}}$ is the characteristic function of the set $\mathbb{E}$. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and that

$$
\delta\left(\mathbb{E}^{c}\right)=1-\delta(\mathbb{E}) .
$$

A sequence $x=\left(x_{k}\right)_{k \in \mathbb{N}}$ is said to be statistically convergent to $L$ if, for every $\varepsilon>0$, we have

$$
\delta\left(\left\{k \in \mathbb{N}:\left|x_{k}-L\right| \geq \varepsilon\right\}\right)=0 .
$$

In this case, we write

$$
x_{k} \xrightarrow{\text { stat }} L \quad \text { as } \quad k \rightarrow \infty \quad \text { or } \quad S-\lim _{k \rightarrow \infty} x_{k}=L .
$$

In 1932, Agnew [32] introduced the concept of deferred Cesaro mean of real (or complex) valued sequences $x=\left(x_{k}\right)$ defined by

$$
\left(D_{p, q} x\right)_{n}=\frac{1}{q_{n}-p_{n}} \sum_{k=p_{n}+1}^{q_{n}} x_{k}, n=1,2,3, \ldots
$$

where $p=\left(p_{n}\right)$ and $q=\left(q_{n}\right)$ are the sequences of non-negative integers satisfying

$$
\begin{equation*}
p_{n}<q_{n} \text { and } \lim _{n \rightarrow \infty} q_{n}=\infty . \tag{1}
\end{equation*}
$$

Let $K$ be a subset of $\mathbb{N}$ and denote the set $\left\{k: k \in\left(p_{n}, q_{n}\right], k \in K\right\}$ by $K_{p, q}(n)$.
Deferred density of $K$ is defined by

$$
\delta_{p, q}(K)=\lim _{n \rightarrow \infty} \frac{1}{\left(q_{n}-p_{n}\right)}\left|K_{p, q}(n)\right|, \text { provided the limit exists }
$$

where, vertical bars indicate the cardinality of the enclosed set $K_{p, q}(n)$. If $q_{n}=n, p_{n}=0$, then the deferred density coincides with natural density of $K$.

A real valued sequence $x=\left(x_{k}\right)$ is said to be deferred statistically convergent to $L$, if for each $\varepsilon>0$

$$
\lim _{n \rightarrow \infty} \frac{1}{\left(q_{n}-p_{n}\right)}\left|\left\{k \in\left(p_{n}, q_{n}\right]:\left|x_{k}-L\right| \geq \varepsilon\right\}\right|=0 .
$$

In this case we write $S_{p, q}-\lim x_{k}=L$. If $q_{n}=n$, $p_{n}=0$, for all $n \in \mathbb{N}$, then deferred statistical convergence coincides with usual statistical convergence [20].

## 2. Main results

In this section, we give some inclusion relations between statistical convergence of order $\alpha$, deferred strong Cesàro summability of order $\alpha$ and deferred statistical convergence of order $\alpha$ in general metric spaces.
Definition 1. Let $(X, d)$ be a metric space, $\left(p_{n}\right)$ and $\left(q_{n}\right)$ be two sequences as above and $0<\alpha \leq 1$. A metric valued sequence $x=\left(x_{k}\right)$ is said to be $S_{p, q}^{d, \alpha}$-convergent (or deferred $d$-statistically convergent of order $\alpha$ ) to $x_{0}$ if there is $x_{0} \in X$ such that

$$
\lim _{n \rightarrow \infty} \frac{1}{\left(q_{n}-p_{n}\right)^{\alpha}}\left|\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right|=0,
$$

where $B_{\varepsilon}\left(x_{0}\right)=\left\{x \in X: d\left(x, x_{0}\right)<\varepsilon\right\}$ is the open ball of radius $\varepsilon$ and center $x_{0}$. In this case we write $S_{p, q}^{d, \alpha}-\lim x_{k}=x_{0}$ or $x_{k} \rightarrow x_{0}\left(S_{p, q}^{d, \alpha}\right)$. The set of all $S_{p, q}^{d, \alpha}$-statistically convergent sequences will be denoted by $S_{p, q}^{d, \alpha}$. If $q_{n}=n$ and $p_{n}=0$, then deferred $d$-statistical convergence of order $\alpha$ coincides $d$-statistical convergence of order $\alpha$ denoted by $S^{d, \alpha}$. In the special cases $q_{n}=n, p_{n}=0$ and $\alpha=1$ then deferred $d$-statistical convergence of order $\alpha$ coincides $d$-statistical convergence denoted by $S^{d}$.
Definition 2. Let $(X, d)$ be a metric space, $\left(p_{n}\right)$ and $\left(q_{n}\right)$ be two sequences as above and $0<\alpha \leq 1$. A metric valued sequence $x=\left(x_{k}\right)$ is said to be strongly $w_{p, q}^{d, \alpha}$-summable (or deferred strongly $d$-Cesàro summable of order $\alpha$ ) to $x_{0}$ if there is $x_{0} \in X$ such that

$$
\lim _{n \rightarrow \infty} \frac{1}{\left(q_{n}-p_{n}\right)^{\alpha}} \sum_{k=p_{n}+1}^{q_{n}} d\left(x_{k}, x_{0}\right)=0 .
$$

In this case we write $w_{p, q}^{d, \alpha}-\lim x_{k}=x_{0}$ or $x_{k} \rightarrow x_{0}\left(w_{p, q}^{d, \alpha}\right)$. The set of all strongly $w_{p, q}^{d, \alpha}$-summable sequences will be denoted by $w_{p, q}^{d, \alpha}$. If $q_{n}=n$ and $p_{n}=0$, for all $n \in \mathbb{N}$, then deferred strong $d$-Cesàro summability of order $\alpha$ coincides strong $d$-Cesàro summability of order $\alpha$ denoted by $w^{d, \alpha}$. In the special cases $q_{n}=n, p_{n}=0$ and $\alpha=1$ then deferred strong $d$-Cesàro summability of order $\alpha$ coincides strong $d$-Cesàro summability denoted by $w^{d}$.
Theorem 1. Let $\left(p_{n}\right)$ and $\left(q_{n}\right)$ be sequences of non-negative integers satisfying the condition (1), (X, $d$ ) be a linear metric space and $x=\left(x_{k}\right), y=\left(y_{k}\right)$ be metric valued sequences, then

(ii)If $S_{p, q}^{d, \alpha}-\lim x_{k}=x_{0}$ and $c \in \mathbb{C}$, then $S_{p, q}^{d, \alpha}-\lim \left(c x_{k}\right)=c x_{0}$,
(iii) If $S_{p, q}^{d, \alpha}-\lim x_{k}=x_{0}, S_{p, q}^{d, \alpha}-\lim y_{k}=y_{0}$ and $x, y \in \ell_{\infty}(X)$, then $S_{p, q}^{d, \alpha}-\lim \left(x_{k} y_{k}\right)=x_{0} y_{0}$.

## Proof. Omitted.

Theorem 2. Let $\left(p_{n}\right)$ and $\left(q_{n}\right)$ be sequences of non-negative integers satisfying the condition (1) and $\alpha$ and $\beta$ be two real numbers such that $0<\alpha \leq \beta \leq 1$. If a sequence $x=\left(x_{k}\right)$ is deferred strongly $d$-Cesàro summable of order $\alpha$ to $x_{0}$, then it is deferred $d$-statistically convergent of order $\beta$ to $x_{0}$, but the converse is not true.
Proof. First part of the proof is easy, so omitted. For the converse, take $X=\mathbb{R}$ and choose $q_{n}=n, p_{n}=0$ (for all $n \in \mathbb{N}$ ), $d(x, y)=|x-y|$ and define a sequence $x=\left(x_{k}\right)$ by

$$
x_{k}=\left\{\begin{array}{cc}
\sqrt[3]{n}, & k=n^{2} \\
0, & k \neq n^{2}
\end{array}\right.
$$

Then for every $\varepsilon>0$, we have

$$
\frac{1}{\left(q_{n}-p_{n}\right)^{\alpha}}\left|\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}(0)\right\}\right| \leq \frac{[\sqrt{n}]}{n^{\alpha}} \rightarrow 0, \text { as } n \rightarrow \infty,
$$

where $\frac{1}{2}<\alpha \leq 1$, that is $x_{k} \rightarrow 0\left(S_{p, q}^{d, \alpha}\right)$. At the same time, we get

$$
\frac{1}{\left(q_{n}-p_{n}\right)^{\alpha}} \sum_{k=p_{n}+1}^{q_{n}} d\left(x_{k}, 0\right) \leq \frac{[\sqrt{n}][\sqrt[3]{n}]}{n^{\alpha}} \rightarrow 1
$$

for $\alpha=\frac{1}{6}$ and

$$
\frac{1}{\left(q_{n}-p_{n}\right)^{\alpha}} \sum_{k=p_{n}+1}^{q_{n}} d\left(x_{k}, 0\right) \leq \frac{[\sqrt{n}][\sqrt[3]{n}]}{n^{\alpha}} \rightarrow \infty
$$

for $0<\alpha<\frac{1}{6}$, i.e., $x_{k} \rightarrow 0\left(w_{p, q}^{d, \alpha}\right)$ for $0<\alpha \leq \frac{1}{6}$.
From Theorem 2 we have the following results.
Corollary 1. i) Let $\left(p_{n}\right)$ and ( $q_{n}$ ) be sequences of non-negative integers satisfying the condition (1) and $\alpha$ be a real number such that $0<\alpha \leq 1$. If a sequence $x=\left(x_{k}\right)$ is deferred strongly $d$-Cesàro summable of order $\alpha$ to $x_{0}$, then it is deferred $d$-statistically convergent of order $\alpha$ to $x_{0}$, but the converse is not true.
ii) Let $\left(p_{n}\right)$ and $\left(q_{n}\right)$ be sequences of non-negative integers satisfying the condition (1) and $\alpha$ be a real number such that $0<\alpha \leq 1$. If a sequence $x=\left(x_{k}\right)$ is deferred strongly $d$-Cesàro summable of order $\alpha$ to $x_{0}$, then it is deferred $d$-statistically convergent to $x_{0}$, but the converse is not true.
iii) Let $\left(p_{n}\right)$ and $\left(q_{n}\right)$ be sequences of non-negative integers satisfying the condition (1). If a sequence $x=\left(x_{k}\right)$ is deferred strongly $d$-Cesàro summable to $x_{0}$, then it is deferred $d$-statistically convergent to $x_{0}$, but the converse is not true.
Remark Even if $x=\left(x_{k}\right)$ is a bounded sequence in a metric space, the converse of Theorem 2 ( So Corollary 1 i) and ii) ) does not hold, in general. To show this we give the following example.

Example 1. Take $X=\mathbb{R}$ and choose $q_{n}=n, p_{n}=0$ (for all $n \in \mathbb{N}$ ), $d(x, y)=|x-y|$ and define a sequence $x=\left(x_{k}\right)$ by

$$
x_{k}=\left\{\begin{array}{cl}
\frac{1}{\sqrt{k}}, & k \neq n^{3} \\
0, & k=n^{3}
\end{array} \quad n=1,2, \ldots\right.
$$

It is clear that $x \in \ell_{\infty}$ and it can be shown that $x \in S^{d, \alpha}-w^{d, \alpha}$ for $\frac{1}{3}<\alpha<\frac{1}{2}$.
In the special case $\alpha=1$, we can give the followig result.
Theorem 3. Let $\left(p_{n}\right)$ and $\left(q_{n}\right)$ be sequences of non-negative integers satisfying the condition (1) and $x=\left(x_{k}\right)$ is a bounded sequence in a metric space. If a sequence $x=\left(x_{k}\right)$ is deferred $d$-statistically convergent to $x_{0}$, then it is deferred strongly $d$-Cesàro summable to $x_{0}$.

Proof. Let $x=\left(x_{k}\right)$ be deferred $d$-statistically convergent to $x_{0}$ and $\varepsilon>0$ be given. Then there exists $x_{0} \in X$ such that

$$
\lim _{n \rightarrow \infty} \frac{1}{\left(q_{n}-p_{n}\right)}\left|\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right|=0,
$$

Since $x=\left(x_{k}\right)$ is a bounded sequence in a metric space $X$, there exists $x_{0} \in X$ and a positive real number $M$ such that $d\left(x_{k}, x_{0}\right)<M$ for all $k \in \mathbb{N}$. So we have

$$
\begin{aligned}
\frac{1}{\left(q_{n}-p_{n}\right)} \sum_{k=p_{n}+1}^{q_{n}} d\left(x_{k}, x_{0}\right)= & \frac{1}{\left(q_{n}-p_{n}\right)} \sum_{\substack{k=p_{n}+1 \\
d\left(x_{k}, x_{0}\right) \geq \varepsilon}}^{q_{n}} d\left(x_{k}, x_{0}\right) \\
& +\frac{1}{\left(q_{n}-p_{n}\right)} \sum_{\substack{k=p_{n}+1 \\
q_{n}}}^{d\left(x_{k}, x_{0}\right)<\varepsilon} \\
\leq & \left.\frac{M}{\left(q_{n}-p_{n}\right)} \right\rvert\,\left\{k \in\left(x_{k}, x_{0}\right)\right. \\
& \left.\left.+\varepsilon \in q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\} \mid
\end{aligned}
$$

Takin limit $n \rightarrow \infty$, we get $w_{p, q}^{d}-\lim x_{k}=x_{0}$.
Theorem 4. Let $\left(p_{n}\right)$ and ( $q_{n}$ ) be sequences of non-negative integers satisfying the condition (1) and $\alpha$ be a real number such that $0<\alpha \leq 1$. If $\liminf _{n} \frac{q_{n}}{p_{n}}>1$, then $S^{d, \alpha} \subseteq S_{p, q}^{d, \alpha}$.
Proof. Suppose that $\lim \inf _{n} \frac{q_{n}}{p_{n}}>1$; then there exists a $v>0$ such that $\frac{q_{n}}{p_{n}} \geq 1+v$ for sufficiently large $n$, which implies that

$$
\left(\frac{q_{n}-p_{n}}{q_{n}}\right)^{\alpha} \geq\left(\frac{v}{1+v}\right)^{\alpha} \Longrightarrow \frac{1}{q_{n}^{\alpha}} \geq \frac{v^{\alpha}}{(1+v)^{\alpha}} \frac{1}{\left(q_{n}-p_{n}\right)^{\alpha}} .
$$

If $x_{k} \rightarrow x_{0}\left(S^{d, \alpha}\right)$, then for every $\varepsilon>0$ and for sufficiently large $n$, we have

$$
\begin{aligned}
\frac{1}{q_{n}^{\alpha}}\left|\left\{k \leq q_{n}: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right| & \geq \frac{1}{q_{n}^{\alpha}}\left|\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right| \\
& \geq \frac{v^{\alpha}}{(1+v)^{\alpha}} \frac{1}{\left(q_{n}-p_{n}\right)^{\alpha}}\left|\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right| .
\end{aligned}
$$

This proves the proof.
Theorem 5. Let ( $p_{n}$ ) and ( $q_{n}$ ) be sequences of non-negative integers satisfying the condition (1) and $\alpha$ and $\beta$ be two real numbers such that $0<\alpha \leq \beta \leq 1$. If $\lim _{n \rightarrow \infty} \frac{\left(q_{n}-p_{n}\right)^{\alpha}}{q_{n}^{\beta}}=s>0$, then $S^{d, \alpha} \subseteq S_{p, q}^{d, \beta}$. Proof. Let $\lim _{n \rightarrow \infty} \frac{\left(q_{n}-p_{n}\right)^{\alpha}}{q_{n}^{\beta}}=s>0$. Notice that for each $\varepsilon>0$ the inclusion

$$
\left\{k \leq q_{n}: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\} \supset\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}
$$

is satisfied and so we have the following inequality

$$
\begin{aligned}
\frac{1}{q_{n}^{\alpha}}\left|\left\{k \leq q_{n}: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right| & \geq \frac{1}{q_{n}^{\alpha}}\left|\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right| \\
& \geq \frac{1}{q_{n}^{\beta}}\left|\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right| \\
& =\frac{\left(q_{n}-p_{n}\right)^{\alpha}}{q_{n}^{\beta}} \frac{1}{\left(q_{n}-p_{n}\right)^{\alpha}}\left|\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right| \\
& \geq \frac{\left(q_{n}-p_{n}\right)^{\alpha}}{q_{n}^{\beta}} \frac{1}{\left(q_{n}-p_{n}\right)^{\beta}}\left|\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right| .
\end{aligned}
$$

Therefore $S^{d, \alpha} \subseteq S_{p, q}^{d, \beta}$.
Theorem 6. Let $\left(p_{n}\right),\left(q_{n}\right),\left(p_{n}^{\prime}\right)$ and $\left(q_{n}^{\prime}\right)$ be four sequences of non-negative real numbers such that

$$
\begin{equation*}
p_{n}^{\prime}<p_{n}<q_{n}<q_{n}^{\prime} \text { for all } n \in \mathbb{N} \tag{2}
\end{equation*}
$$

and $\alpha, \beta$ be fixed real numbers such that $0<\alpha \leq \beta \leq 1$, then
(i) If

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\left(q_{n}-p_{n}\right)^{\alpha}}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}}=a>0 \tag{3}
\end{equation*}
$$

then $S_{p^{\prime}, q^{\prime}}^{d, \beta} \subseteq S_{p, q}^{d, \alpha}$,
(ii) If

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{q_{n}^{\prime}-p_{n}^{\prime}}{\left(q_{n}-p_{n}\right)^{\beta}}=1 \tag{4}
\end{equation*}
$$

then $S_{p, q}^{d, \alpha} \subseteq S_{p^{\prime}, q^{\prime}}^{d,}$.
Proof. (i) Let (3) be satisfied. For given $\varepsilon>0$ we have

$$
\left\{k \in\left(p_{n}^{\prime}, q_{n}^{\prime}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\} \supseteq\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\},
$$

and so

$$
\begin{aligned}
& \frac{1}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}}\left|\left\{k \in\left(p_{n}^{\prime}, q_{n}^{\prime}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right| \\
\geq & \frac{\left(q_{n}-p_{n}\right)^{\alpha}}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}} \frac{1}{\left(q_{n}-p_{n}\right)^{\alpha}}\left|\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right| .
\end{aligned}
$$

Therefore $S_{p^{\prime}, q^{\prime}}^{d, \beta} \subseteq S_{p, q}^{d, \alpha}$.
(ii) Let (4) be satisfied and $x=\left(x_{k}\right)$ be a deferred $d$-statistically convergent sequence of order $\alpha$ to $x_{0}$. Then for given $\varepsilon>0$, we have

$$
\frac{1}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}}\left|\left\{k \in\left(p_{n}^{\prime}, q_{n}^{\prime}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right|
$$

$$
\begin{aligned}
\leq & \frac{1}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}}\left|\left\{k \in\left(p_{n}^{\prime}, p_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right| \\
& +\frac{1}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}}\left|\left\{k \in\left(q_{n}, q_{n}^{\prime}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right|+\frac{1}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}}\left|\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right| \\
\leq & \frac{p_{n}-p_{n}^{\prime}+q_{n}^{\prime}-q_{n}}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}}+\frac{1}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}}\left|\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right| \\
= & \frac{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)-\left(q_{n}-p_{n}\right)}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}}+\frac{1}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}}\left|\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right| \\
\leq & \frac{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)-\left(q_{n}-p_{n}\right)^{\beta}}{\left(q_{n}-p_{n}\right)^{\beta}}+\frac{1}{\left(q_{n}-p_{n}\right)^{\beta}}\left|\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right| \\
\leq & \left(\frac{q_{n}^{\prime}-p_{n}^{\prime}}{\left(q_{n}-p_{n}\right)^{\beta}}-1\right)+\frac{1}{\left(q_{n}-p_{n}\right)^{)^{2}}}\left|\left\{k \in\left(p_{n}, q_{n}\right]: x_{k} \notin B_{\varepsilon}\left(x_{0}\right)\right\}\right|
\end{aligned}
$$

Therefore $S_{p, q}^{d, \alpha} \subseteq S_{p^{\prime}, q^{\prime}}^{d, \beta}$.
Theorem 7. Let $\left(p_{n}\right),\left(q_{n}\right),\left(p_{n}^{\prime}\right)$ and $\left(q_{n}^{\prime}\right)$ be four sequences of non-negative integers defined as in (2) and $\alpha, \beta$ be fixed real numbers such that $0<\alpha \leq \beta \leq 1$.
(i) If (3) holds then $w_{p^{\prime}, q^{\prime}}^{d, \beta} \subset w_{p, q}^{d, \alpha}$,
(ii) If (4) holds and $x=\left(x_{k}\right)$ be a bounded sequence, then $w_{p, q}^{d, \alpha} \subset w_{p^{\prime}, q^{\prime}}^{d,}$

## Proof.

i) Omitted.
ii) Suppose that $w_{p, q}^{d, \alpha}-\lim x_{k}=x_{0}$ and $\left(x_{k}\right) \in \ell_{\infty}(X)$. Then there exists some $M>0$ such that $d\left(x_{k}, x_{0}\right)<M$ for all $k$, then

$$
\begin{aligned}
\frac{1}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}} \sum_{k=p_{n}^{\prime}+1}^{q_{n}^{\prime}} d\left(x_{k}, x_{0}\right) & =\frac{1}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}}\left[\sum_{k=p_{n}^{\prime}+1}^{p_{n}}+\sum_{k=p_{n}+1}^{q_{n}}+\sum_{k=q_{n}+1}^{q_{n}^{\prime}}\right] d\left(x_{k}, x_{0}\right) \\
& \leq \frac{p_{n}-p_{n}^{\prime}+q_{n}^{\prime}-q_{n}}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}} M+\frac{1}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}} \sum_{k=p_{n}+1}^{q_{n}} d\left(x_{k}, x_{0}\right) \\
& \leq \frac{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)-\left(q_{n}-p_{n}\right)^{\beta}}{\left(q_{n}-p_{n}\right)^{\beta}} M+\frac{1}{\left(q_{n}-p_{n}\right)^{\alpha}} \sum_{k=p_{n}+1}^{q_{n}} d\left(x_{k}, x_{0}\right) \\
& =\left(\frac{q_{n}^{\prime}-p_{n}^{\prime}}{\left(q_{n}-p_{n}\right)^{\beta}}-1\right) M+\frac{1}{\left(q_{n}-p_{n}\right)^{\alpha}} \sum_{k=p_{n}+1}^{q_{n}} d\left(x_{k}, x_{0}\right)
\end{aligned}
$$

Theorem 8. Let $\left(p_{n}\right),\left(q_{n}\right),\left(p_{n}^{\prime}\right)$ and $\left(q_{n}^{\prime}\right)$ be four sequences of non-negative integers defined as in (2) and $\alpha, \beta$ be fixed real numbers such that $0<\alpha \leq \beta \leq 1$. Then
(i) Let (3) holds, if a sequence is strongly $w_{p^{\prime}, q^{\prime}}^{d, \beta}$-summable to $x_{0}$, then it is $S_{p, q^{-}}^{d, \alpha}$-convergent to $x_{0}$,
(ii) Let (4) holds and $x=\left(x_{k}\right)$ be a bounded sequence in $(X, d)$, if a sequence is $S_{p, q}^{d, \alpha}$-convergent to $x_{0}$ then it is strongly $w_{p^{\prime}, q^{\prime}}^{d,}$-summable to $x_{0}$.

Proof. (i) Omitted.
(ii) Suppose that $S_{p, q}^{d, \alpha} \lim x_{k}=x_{0}$ and $\left(x_{k}\right) \in \ell_{\infty}$. Then there exists some $M>0$ such that $d\left(x_{k}, x_{0}\right)<$ $M$ for all $k$, then for every $\varepsilon>0$ we may write

$$
\begin{aligned}
& \frac{1}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}} \sum_{k=p_{n}^{\prime}+1}^{q_{n}^{\prime}} d\left(x_{k}, x_{0}\right) \\
&= \frac{1}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}} \sum_{k=q_{n}-p_{n}+1}^{q_{n}-p_{n}^{\prime}} d\left(x_{k}, x_{0}\right)+\frac{1}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}} \sum_{k=p_{n}+1}^{q_{n}} d\left(x_{k}, x_{0}\right) \\
& \leq \frac{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)-\left(q_{n}-p_{n}\right)}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}} M+\frac{1}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}} \sum_{k=p_{n}+1}^{q_{n}} d\left(x_{k}, x_{0}\right) \\
& \leq \frac{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)-\left(q_{n}-p_{n}\right)^{\beta}}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}} M+\frac{1}{\left(q_{n}^{\prime}-p_{n}^{\prime}\right)^{\beta}} \sum_{k=p_{n}+1}^{q_{n}} d\left(x_{k}, x_{0}\right) \\
& \leq\left(\frac{q_{n}^{\prime}-p_{n}^{\prime}}{\left(q_{n}-p_{n}\right)^{\beta}}-1\right) M+\frac{1}{\left(q_{n}-p_{n}\right)^{\beta}} \sum_{k=p_{n}+1}^{q_{n}} d\left(x_{k}, x_{0}\right) \\
&+\frac{1}{\left(q_{n}-p_{n}\right)^{\beta}} \sum_{k=p_{n}+1}^{q_{n}} d\left(x_{k}, x_{0}\right) \\
& d\left(x_{k}, x_{0}\right)<\varepsilon \\
& \leq\left(\frac{q_{n}^{\prime}-p_{n}^{\prime}}{\left(q_{n}-p_{n}\right)^{\beta}}-1\right) M+\frac{M}{\left(q_{n}-p_{n}\right)^{\alpha}}\left|\left\{k \in\left(p_{n}, q_{n}\right]: d\left(x_{k}, x_{0}\right) \geq \varepsilon\right\}\right| \\
&+\frac{q_{n}^{\prime}-p_{n}^{\prime}}{\left(q_{n}-p_{n}\right)^{\beta}} \varepsilon .
\end{aligned}
$$

This completes the proof.

## Conflict of interest

The authors declare that they have no conflict of interests.

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