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Research article

Sharp bounds for Heinz mean by Heron mean and other means

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Abstract: In this paper, some sharp bounds for Heinz mean by Heron mean and other means are presented. Further, we point out a mistake in [1] and correct it. Finally, we extend the results to the corresponding operator means.

Keywords: bounds; Heinz operator mean; Heron operator mean; other operator means
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1. Introduction

The ν-weighted arithmetic mean−geometric mean inequality or ν-weighted $AM-GM$ inequality is the following statement: If $a, b \geq 0$ and $0 \leq \nu \leq 1$, then

$$a^{\nu}b^{1-\nu} \leq \nu a + (1-\nu)b \tag{1.1}$$

holds with equality if and only if $a = b$. The inequality (1.1) for $\nu = 1/2$ reduces to the arithmetic mean-geometric mean inequality or $AM-GM$ inequality

$$\sqrt{ab} \leq \frac{a+b}{2}. \tag{1.2}$$

The Heinz mean, introduced in Bhatia and Davis [2], is defined by

$$H_{\nu}(a,b) = \frac{a^{\nu}b^{1-\nu} + a^{1-\nu}b^{\nu}}{2} \tag{1.3}$$

for $0 \leq \nu \leq 1$ and $a, b \geq 0$. We know that

$$\sqrt{ab} \leq H_{\nu}(a,b) \leq \frac{a+b}{2}. \tag{1.4}$$

In [3], Bhatia defined the weighted mean of arithmetic mean and geometric mean as Heron means, wrote it as

$$F_\alpha(a,b) = (1-\alpha)G(a,b) + \alpha A(a,b) = (1-\alpha)\sqrt{ab} + \alpha\frac{a+b}{2}, \tag{1.5}$$

and got the following result about Heinz and the Heron means.

Proposition 1.1. *Let $a, b \geq 0$, $0 \leq \nu \leq 1$, and $\alpha(\nu) = 1 - 4\left(\nu - \nu^2\right)$. Then*

$$H_\nu(a,b) \leq F_{\alpha(\nu)}(a,b). \tag{1.6}$$

Kittaneh, Moslehian and Sababheh come to the above conclusion from the main one (see [4, Theorem 2.1]), which is handled skillfully and is also used by Sababheh [5] on other means. The first objective of this paper is to sharpen the above inequality and obtain the result.

Theorem 1.1. *Let $a, b > 0$, $a \neq b$, $0 < \nu < 1$, and $\theta = 1 - 2\nu$. Then*

$$H_\nu(a,b) < F_{\theta^2}(a,b) \tag{1.7}$$

holds, where θ^2 can not be replaced by any smaller number.

We can find that (1.7) contains (1.6) and indicates that θ^2 is the best constant in (1.7).

In addition, recently, in [1] Shi gave a lower bound for Heinz mean.

Proposition 1.2. *Let $a, b > 0$, $a \neq b$, $0 \leq \nu \leq 1$, $\alpha = 1 - 2t$, $\beta = 1 - 2s$, and $\beta^2 > \alpha^2/3$. Then*

$$H_s(a,b) \geq \frac{a^{1-t}b^t - a^t b^{1-t}}{(1-2t)(\ln a - \ln b)}\frac{1}{\sqrt{ab}} \tag{1.8}$$

holds.

We find that there are two issues in above result. First, there should be no $\sqrt{ab}$ on the right hand side of (1.8). This judgement is verified by calculation. Second, under the condition $\beta^2 > \alpha^2/3$, [1] draw the following conclusion:

$$\cosh\beta x = 1 + \frac{\beta^2 x^2}{2!} + \frac{\beta^4 x^4}{4!} + \cdots \geq 1 + \frac{\alpha^2 x^2}{3!} + \frac{\alpha^4 x^4}{5!} \cdots = \frac{\sinh\alpha x}{\alpha x}.$$

In fact, the coefficients of the same power in the left-hand side series are not greater than or equal to the right one, for example, the relationship $\beta^4/4! \geq \alpha^4/5!$ is not true.

The second objective of this paper is to reconstruct relevant results about (1.8) as follows.

Theorem 1.2. *Let $a, b > 0$, $a \neq b$, $\alpha = 1 - 2t$, and $\beta = 1 - 2s$. Then*

(i) when $\beta^2 \geq \alpha^2$, we have

$$\frac{a^{1-t}b^t - a^t b^{1-t}}{(1-2t)(\ln a - \ln b)} < H_s(a,b); \tag{1.9}$$

(ii) when $\beta^2 \leq \alpha^2/3$, we have

$$\frac{a^{1-t}b^t - a^t b^{1-t}}{(1-2t)(\ln a - \ln b)} > H_s(a,b). \tag{1.10}$$

2. Lemmas

Lemma 2.1 ([6]). *Let a_n and b_n $(n = 0, 1, 2, \cdots)$ be real numbers, and let the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent for $|x| < R$ $(R \le +\infty)$. If $b_n > 0$ for $n = 0, 1, 2, \cdots$, and if $c_n = a_n/b_n$ is strictly increasing (or decreasing) for $n = 0, 1, 2, \cdots$, then the function $A(x)/B(x)$ is strictly increasing (or decreasing) on $(0, R)$ $(R \le +\infty)$.*

Lemma 2.2. *Let $x \neq 0$, and $|\theta| \neq 1$. Then*

(i) when $|\theta| < 1$ the double inequality

$$0 < \frac{\cosh \theta x - 1}{\cosh x - 1} < \theta^2 \tag{2.1}$$

holds with the best constants 0 and θ^2;

(ii) when $|\theta| > 1$ the inequality

$$\frac{\cosh \theta x - 1}{\cosh x - 1} > \theta^2 \tag{2.2}$$

holds with the best constant θ^2.

Proof Because the functions involved in this lemma are all even functions, we can assume that $x \in (0, \infty)$. Let

$$\begin{aligned} A(x) &= \cosh \theta x - 1 = \sum_{n=1}^{\infty} \frac{\theta^{2n}}{(2n)!} x^{2n} := \sum_{n=1}^{\infty} a_n x^{2n}, \\ B(x) &= \cosh x - 1 = \sum_{n=1}^{\infty} \frac{1}{(2n)!} x^{2n} := \sum_{n=1}^{\infty} b_n x^{2n}, \end{aligned}$$

where

$$\begin{aligned} a_n &= \frac{\theta^{2n}}{(2n)!}, \\ b_n &= \frac{1}{(2n)!} > 0. \end{aligned}$$

Then

$$c_n := \frac{a_n}{b_n} = \theta^{2n}.$$

Since $\{c_n\}_{n\ge1}$ is decreasing for $|\theta| < 1$ and increasing for $|\theta| > 1$, by Lemma 2.1 we have that the function $(\cosh \theta x - 1) / (\cosh x - 1)$ is decreasing on $(0, \infty)$ for $|\theta| < 1$ and increasing on $(0, \infty)$ for $|\theta| > 1$. In view of

$$\begin{aligned} \lim_{x\to0^+} \frac{\cosh \theta x - 1}{\cosh x - 1} &= c_1 = \theta^2, \\ \lim_{x\to\infty} \frac{\cosh \theta x - 1}{\cosh x - 1} &= \lim_{n\to\infty} c_n = \lim_{n\to\infty} \theta^{2n} = \begin{cases} 0, & |\theta| < 1 \\ \infty, & |\theta| > 1 \end{cases}, \end{aligned}$$

the proof of Lemma 2.2 is complete.

Lemma 2.3. *Let $t \neq 0$. Then the double inequality*

$$\cosh \frac{1}{\sqrt{3}} t < \frac{\sinh t}{t} < \cosh t \tag{2.3}$$

holds with the best constants $1/\sqrt{3}$ *and* 1.

Proof Using the power series expansions

$$\sinh kt = \sum_{n=0}^{\infty} \frac{k^{2n+1}}{(2n+1)!} t^{2n+1}, \quad \cosh kt = \sum_{n=0}^{\infty} \frac{k^{2n}}{(2n)!} t^{2n}$$

we have

$$\begin{aligned} \cosh t - \frac{\sinh t}{t} &= \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} t^{2n} > 0, \\ \frac{\sinh t}{t} - \cosh \frac{1}{\sqrt{3}} t &= \sum_{n=2}^{\infty} \frac{3^n - (2n+1)}{3^n (2n+1)!} t^{2n} > 0. \end{aligned}$$

For the reason

$$\cosh \frac{1}{\sqrt{3}} t < \frac{\sinh t}{t} < \cosh t \Longleftrightarrow \frac{1}{\sqrt{3}} < \frac{\operatorname{arccosh}\left(\frac{\sinh t}{t}\right)}{t} < 1,$$

we complete the proof of Lemma 2.3 when proving

$$\begin{aligned} \lim_{t\to 0^+} \frac{\operatorname{arccosh}\left(\frac{\sinh t}{t}\right)}{t} &= \frac{1}{\sqrt{3}}, \\ \lim_{t\to \infty} \frac{\operatorname{arccosh}\left(\frac{\sinh t}{t}\right)}{t} &= 1. \end{aligned}$$

In fact, since

$$\begin{aligned} \operatorname{arccosh} x &= \ln\left(x + \sqrt{x^2 - 1}\right), \\ \left[\operatorname{arccosh}\left(\frac{\sinh t}{t}\right)\right]_t' &= \frac{t\cosh t - \sinh t}{t\sqrt{\sinh^2 t - t^2}}, \end{aligned}$$

by l'Hospital's rule we have

$$\begin{aligned} \lim_{t\to 0^+} \frac{\operatorname{arccosh}\left(\frac{\sinh t}{t}\right)}{t} &= \lim_{t\to 0^+} \frac{\left[\operatorname{arccosh}\left(\frac{\sinh t}{t}\right)\right]'}{t'} \\ &= \lim_{t\to 0^+} \frac{t\cosh t - \sinh t}{t\sqrt{\sinh^2 t - t^2}} = \frac{1}{\sqrt{3}}, \\ \lim_{t\to \infty} \frac{\operatorname{arccosh}\left(\frac{\sinh t}{t}\right)}{t} &= \lim_{t\to \infty} \frac{\left[\operatorname{arccosh}\left(\frac{\sinh t}{t}\right)\right]'}{t'} \\ &= \lim_{t\to \infty} \frac{t\cosh t - \sinh t}{t\sqrt{\sinh^2 t - t^2}} = 1. \end{aligned}$$

3. Proofs of main results

3.1. The proof of Theorem 1.1

Since

$$H_\nu(a,b) < F_{\theta^2}(a,b) \Longleftrightarrow \frac{1}{2}\left(\left(\frac{b}{a}\right)^\nu + \left(\frac{b}{a}\right)^{1-\nu}\right) < \left(1-\theta^2\right)\left(\frac{b}{a}\right)^{1/2} + \frac{\theta^2}{2}\left(1+\frac{b}{a}\right),$$

when letting $\sqrt{b/a} = e^t$ the last inequality is equivalent to

$$\begin{aligned}
\frac{e^{2(1-\nu)t} + e^{2\nu t}}{2} &< \left(1-\theta^2\right)e^t + \theta^2\frac{1+e^{2t}}{2} \Longleftrightarrow \\
\frac{e^{(1-2\nu)t} + e^{(2\nu-1)t}}{2} &< \left(1-\theta^2\right) + \theta^2\frac{e^t + e^{-t}}{2} \Longleftrightarrow \\
\cosh(1-2\nu)t &< \left(1-\theta^2\right) + \theta^2\cosh t \Longleftrightarrow \\
\frac{\cosh(1-2\nu)t - 1}{\cosh t - 1} &< \theta^2.
\end{aligned}$$

Letting $\theta = 1 - 2\nu$ in Lemma 2.2 we can obtain the above inequality. This completes the proof of Theorem 1.1.

3.2. The proof of Theorem 1.2

With transformations

$$\begin{aligned}
\alpha &= 1-2t,\ \beta = 1-2s, \\
x &= \frac{1}{2}\ln\frac{a}{b},
\end{aligned}$$

we may come to the conclusions

$$\begin{aligned}
\frac{a^{1-s}b^s + a^s b^{1-s}}{2\sqrt{ab}} &= \frac{\left(\frac{a}{b}\right)^{1-s} + \left(\frac{a}{b}\right)^s}{2\sqrt{\frac{a}{b}}} = \frac{e^{2x(1-s)+}e^{2xs}}{2e^x} = \frac{e^{(1-2s)x} + e^{-(1-2s)x}}{2} \\
&= \frac{e^{\beta x} + e^{-\beta x}}{2} = \cosh\beta x,
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
\frac{a^{1-t}b^t - a^t b^{1-t}}{(1-2t)\ln\frac{a}{b}}\frac{1}{\sqrt{ab}} &= \frac{\left(\frac{a}{b}\right)^{1-t} - \left(\frac{a}{b}\right)^t}{(1-2t)\ln\frac{a}{b}}\frac{1}{\sqrt{\frac{a}{b}}} = \frac{e^{2x(1-t)-}e^{2xt}}{(1-2t)\ln\frac{a}{b}}\frac{1}{e^x} \\
&= \frac{\left[e^{(1-2t)x} - e^{-(1-2t)x}\right]/2}{(1-2t)\left(\frac{1}{2}\ln\frac{a}{b}\right)} = \frac{\left[e^{\alpha x} - e^{-\alpha x}\right]/2}{\alpha x} = \frac{\sinh\alpha x}{\alpha x}.
\end{aligned} \tag{3.2}$$

Via (3.1) and (3.2), letting $t = \alpha x$ in Lemma 2.3 we can complete the proof of Theorem 1.2.

4. Corollary, comparation, and inequalities related to the Heinz operator mean

4.1. Corollary

At the beginning of this section, we want to draw some useful inferences of Theorem 1.2.

(a) Let $\beta = \alpha$ in (1.9), that is $s = t := \nu$. Then we have

$$\frac{a^{1-\nu}b^{\nu} - a^{\nu}b^{1-\nu}}{(1-2\nu)(\ln a - \ln b)} < H_\nu(a,b).$$

(b) Let $\beta = -\alpha$ in (1.9), that is $1 - t = s := \nu$. Then we obtain

$$\frac{a^{1-\nu}b^{\nu} - a^{\nu}b^{1-\nu}}{(1-2\nu)(\ln a - \ln b)} = \frac{a^{\nu}b^{1-\nu} - a^{1-\nu}b^{\nu}}{(2\nu-1)(\ln a - \ln b)} < H_\nu(a,b).$$

(c) Let $\sqrt{3}\beta = \alpha$ or $\sqrt{3}(1-2s) = 1-2t$ in (1.10), and

$$\frac{\sqrt{3}-1}{2} = \varkappa_1, \quad \frac{\sqrt{3}+1}{2} = \varkappa_2,$$

that is

$$t = \sqrt{3}s - \frac{\sqrt{3}-1}{2} = \sqrt{3}s - \varkappa_1 := \sqrt{3}\nu - \varkappa_1.$$

Then we obtain

$$\frac{a^{\varkappa_2 - \sqrt{3}\nu} b^{\sqrt{3}\nu - \varkappa_1} - a^{\sqrt{3}\nu - \varkappa_1} b^{\varkappa_2 - \sqrt{3}\nu}}{\sqrt{3}(1-2\nu)(\ln a - \ln b)} > H_\nu(a,b).$$

Similarly, letting $\sqrt{3}\beta = -\alpha$ in (1.10) gives the above inequality too.

In this way, from Theorem 1.2, we can get the following corollary.

Corollary 4.1. *Let $a, b > 0$, $a \neq b$, $0 \leq \nu \leq 1$, $\nu \neq 1/2$, and*

$$\frac{\sqrt{3}-1}{2} = \varkappa_1, \quad \frac{\sqrt{3}+1}{2} = \varkappa_2. \tag{4.1}$$

Then

$$\frac{a^{1-\nu}b^{\nu} - a^{\nu}b^{1-\nu}}{(1-2\nu)(\ln a - \ln b)} < H_\nu(a,b), \tag{4.2}$$

and

$$H_\nu(a,b) < \frac{a^{\varkappa_2 - \sqrt{3}\nu} b^{\sqrt{3}\nu - \varkappa_1} - a^{\sqrt{3}\nu - \varkappa_1} b^{\varkappa_2 - \sqrt{3}\nu}}{\sqrt{3}(1-2\nu)(\ln a - \ln b)} \tag{4.3}$$

hold.

4.2. Comparation

In fact, we see that (4.2) gives a lower bound for $H_\nu(a,b)$, while (1.6) or (1.7) and (4.3) show different upper bounds for $H_\nu(a,b)$. Here, we get the following result by transforming $t = a/b$: (1.6) or (1.7) and (4.3) are equivalent to

$$\frac{t^\nu + t^{1-\nu}}{2} < \frac{1+t}{2} - 2\nu(1-\nu)\left(1-\sqrt{t}\right)^2 := r_1(\nu,t), \tag{4.4}$$

and

$$\frac{t^{1-\nu} + t^\nu}{2} < \frac{t^{\varkappa_2 - \sqrt{3}\nu} - t^{\sqrt{3}\nu - \varkappa_1}}{\sqrt{3}(1-2\nu)(\ln t)} = r_2(\nu,t), \tag{4.5}$$

respectively, where $\varkappa_1$ and $\varkappa_2$ are defined as (4.1). Since $r_i(\nu,t) = r_i(1-\nu,t)$ for $i = 1,2$, we want to compare the advantages and disadvantages of these two inequalities above as long as we discuss them in this case $\nu \in (0,1/2)$. Numerical experiments show that

$$r_2(\nu,t) < r_1(\nu,t) \tag{4.6}$$

holds for all $t > 0$ and $0 < \nu < 1/2$. So the inequality (4.6) holds for all $t > 0$ and $0 < \nu < 1$. Then we have the following note.

Remark 4.1. *The inequality (4.3) is better than the one (1.6) or (1.7).*

4.3. Inequalities related to the Heinz operator mean

Now that we have the fact above, we can apply our conclusions to Heinz operator mean on Hilbert spaces.

Let $\mathcal{B}^+$ denotes the set of all positive invertible operators on a Hilbert space $\mathcal{H}$. For $A, B \in \mathcal{B}^+$ and $\nu \in [0,1]$, the weighted arithmetic operator mean $A\nabla_\nu B$, geometric mean $A\sharp_\nu B$, and the Heinz operator mean $H_\nu(A,B)$ are defined as

$$\begin{aligned} A\nabla_\nu B &= (1-\nu)A + \nu B, \\ A\sharp_\nu B &= A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\nu A^{1/2}, \\ H_\nu(A,B) &= \left(A\sharp_\nu B + A\sharp_{1-\nu}B\right)/2. \end{aligned}$$

Let $\nu \in [0,1]$ and define the function $K_\nu : \mathbb{R}_+ \longrightarrow \mathbb{R}$ by

$$K_\nu(x) = \begin{cases} \frac{x^\nu - x^{1-\nu}}{\ln x}, & x > 0 \text{ and } x \neq 1 \\ 2\nu - 1, & x = 1 \end{cases}.$$

The function above was first introduced by Kittaneh and Krnic in [7].

Kittaneh and Krnic [7], Zhao, Wu, Cao, and Liao [8] obtained the following result.

Proposition 4.1. *Let A and B be two positive and invertible operators. Then*

$$H_\nu(A,B) \le F_{\alpha(\nu)}(A,B)$$

for $\nu \in [0,1]$, *where* $\alpha(\nu) = 1 - 4(\nu - \nu^2)$, *and* $F_\alpha(A,B)$ *is defined by*

$$F_\alpha(A,B) = (1-\alpha)A\sharp_{1/2}B + \alpha A\nabla_{1/2}B.$$

From Theorem 1.1, we can obtain that

Theorem 4.1. *Let A and B be two positive and invertible operators, and $\theta = 1 - 2v$. Then*

$$H_v(A, B) \leq F_{\theta^2}(A, B) \tag{4.7}$$

holds, where θ^2 can not be replaced by any smaller number.

As for the application of Theorem 1.2 on the same topic, it is not difficult to get the following results.

Theorem 4.2. *Let A and B be two different positive and invertible operators, $\alpha = 1-2t$, and $\beta = 1-2s$. Then*

(i) when $\beta^2 \geq \alpha^2$, we have

$$\frac{1}{2t-1}A^{1/2}K_t\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} < H_s(A, B); \tag{4.8}$$

(ii) when $\beta^2 \leq \alpha^2/3$, we have

$$\frac{1}{2t-1}A^{1/2}K_t\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} > H_s(A, B). \tag{4.9}$$

Remark 4.2. *Let $s = t$. Then $\beta = \alpha$, and by (4.8) we have*

$$\frac{1}{2t-1}A^{1/2}K_t\left(A^{-1/2}BA^{-1/2}\right)A^{1/2} < H_t(A, B), \tag{4.10}$$

which is the result of Remark 2.6 in [1].

Remark 4.3. *(4.9) is to correct the conclusion of Theorem 2.5 in [1]. The conclusion of Theorem 2.5 comes from (1.8) which is wrong.*

Remark 4.4. *For further results of Heinz operator inequalities, interested readers can refer to [9–13].*

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Conflict of interest

The author declares no conflict of interest in this paper.

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