Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(1): 717–722.
DOI:10.3934/math.2020048
Received: 23 October 2019
Accepted: 16 December 2019
Published: 20 December 2019

Research article

Some new results for the Srivastava-Luo-Raina $\mathbb{M}$-transform pertaining to the incomplete H-functions

M. K. Bansal[1], D. Kumar[2], J. Singh[3], A. Tassaddiq[4] and K. S. Nisar[5,*]

[1] Department of Applied Sciences, Government Engineering College, Banswara-327001, Rajasthan, India

[2] Department of Mathematics, University of Rajasthan, Jaipur 302004, Rajasthan, India

[3] Department of Mathematics, JECRC University, Jaipur-303905, Rajasthan, India

[4] College of Computer and Information Sciences, Majmaah University, Al Majmaah 11952, Saudi Arabia

[5] Department of Mathematics, College of Arts and Sciences, Wadi Aldawaser, 11991, Prince Sattam bin Abdulaziz University, Saudi Arabia

* **Correspondence:** Email: n.sooppy@psau.edu.sa; Tel: +966563456976.

Abstract: The image formula of the incomplete H-functions is derived in this paper by using the Srivastava-Luo-Raina $\mathbb{M}$-Transform. Also, we obtained the image formulae of some useful and important cases of incomplete H-function. The results derived in this paper are in general form and several known and new results can be obtained by giving particular values to the parameters involved in the main results.

Keywords: gamma function; incomplete gamma functions; Srivastava-Luo-Raina $\mathbb{M}$-transform; incomplete H-functions; Mellin-Barnes type contour; incomplete Fox-Wright generalized hypergeometric functions
Mathematics Subject Classification: 26A33, 33B20, 33C60, 33E20, 44A40

1. Introduction

There are numerous integral transforms available in the literature such as the Laplace transform, Mellin transform, Z-transform, Fourier transform and many more [1, 2]. The integral transforms are very useful in solving differential equations of integer as well as fractional orders arising in scientific problems. In this sequence, recently, Srivastava et al. [3, p.1386, Eq.(1.1)] introduced the following

$\mathbb{M}$-Transform:

$$\mathbb{M}_{\lambda,\mu}[g(x)](p,q) = \int_0^\infty \frac{e^{-px} g(qx)}{(x^\mu + q^\mu)^\lambda} dx \tag{1.1}$$

provided that $\quad (\lambda \in \mathbb{C}; \quad \Re(\lambda) \geq 0; \mu \in \mathbb{Z}_+ = \{1, 2, 3, \cdots\})$

where both $p \in \mathbb{C}$ and $q \in \mathbb{R}_+$ are the transform variables. The classical Stieltjes transform [4] is a special case of the Srivastava-Luo-Raina transform (1.1) when $p = 0$.
Convergence conditions of the integral transform (1.1) is given in the theorem [3, p. 1388].

The incomplete H-functions (IHF) $\gamma_{p,q}^{m,n}(w)$ and $\Gamma_{p,q}^{m,n}(w)$ have introduced and investigated by Srivastava et al. [5, Eqs.(2.1)-(2.4)] in the following manner:

$$\begin{aligned} \gamma_{p,q}^{m,n}(w) =& \gamma_{p,q}^{m,n}\left[w \,\middle|\, \begin{matrix} (e_1, E_1, u), (e_j, E_j)_{2,p} \\ \\ (f_j, F_j)_{1,q} \end{matrix} \right] \\ =& \gamma_{p,q}^{m,n}\left[w \,\middle|\, \begin{matrix} (e_1, E_1, u), (e_2, E_2), \cdots, (e_p, E_p) \\ \\ (f_1, F_1), \cdots, (f_q, F_q) \end{matrix} \right] \\ =& \frac{1}{2\pi i} \int_{\mathfrak{L}} k(\xi, u) w^{-\xi} d\xi, \end{aligned} \tag{1.2}$$

where

$$k(\xi, u) = \frac{\gamma(1 - e_1 - E_1\xi, u) \prod_{j=1}^{m} \Gamma(f_j + F_j\xi) \prod_{j=2}^{n} \Gamma(1 - e_j - E_j\xi)}{\prod_{j=m+1}^{q} \Gamma(1 - f_j - F_j\xi) \prod_{j=n+1}^{p} \Gamma(e_j + E_j\xi)} \tag{1.3}$$

and

$$\begin{aligned} \Gamma_{p,q}^{m,n}(w) =& \Gamma_{p,q}^{m,n}\left[w \,\middle|\, \begin{matrix} (e_1, E_1, u), (e_j, E_j)_{2,p} \\ \\ (f_j, F_j)_{1,q} \end{matrix} \right] \\ =& \Gamma_{p,q}^{m,n}\left[w \,\middle|\, \begin{matrix} (e_1, E_1, u), (e_2, E_2), \cdots, (e_p, E_p) \\ \\ (f_1, F_1), \cdots, (f_q, F_q) \end{matrix} \right] \\ =& \frac{1}{2\pi i} \int_{\mathfrak{L}} K(\xi, u) w^{-\xi} d\xi, \end{aligned} \tag{1.4}$$

where

$$K(\xi, u) = \frac{\gamma(1 - e_1 - E_1\xi, u) \prod_{j=1}^{m} \Gamma(f_j + F_j\xi) \prod_{j=2}^{n} \Gamma(1 - e_j - E_j\xi)}{\prod_{j=m+1}^{q} \Gamma(1 - f_j - F_j\xi) \prod_{j=n+1}^{p} \Gamma(e_j + E_j\xi)}. \tag{1.5}$$

The IHF $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$ in (1.2) and (1.4) exist for all $x \geqq 0$ under the same contour and the same set of conditions as stated in [5] (see, also, for details, [6–9]). A complete description of (1.2) and (1.4) can be found in [5].
Some interesting and important special cases of incomplete $H-$ functions are given below (see also [10]):

(i) The case $u = 0$ of (1.4) reduces to the Fox's H-function

$$\Gamma_{p,q}^{m,n}\left[w \left| \begin{matrix} (e_1, E_1, 0), (e_2, E_2), \cdots, (e_p, E_p) \\ \\ (f_1, F_1), \cdots, (f_q, F_q) \end{matrix} \right.\right] = H_{p,q}^{m,n}\left[w \left| \begin{matrix} (e_1, E_1), \cdots, (e_p, E_p) \\ \\ (f_1, F_1), \cdots, (f_q, F_q) \end{matrix} \right.\right]. \tag{1.6}$$

(ii) Letting $m = 1, n = p, q$ being replaced by $q+1$ and taking suitable parameters, the functions (1.2) and (1.4) convert, respectively, to the incomplete Fox-Wright functions(IFWF) ${}_p\Psi_q^{(\gamma)}$ and ${}_p\Psi_q^{(\Gamma)}$ (see [5, Eqs. (6.3) and (6.4)]):

$$\gamma_{p,q+1}^{1,p}\left[-w \left| \begin{matrix} (1-e_1, E_1, u), (1-e_j, E_j)_{2,p} \\ \\ (0,1), (1-f_j, F_j)_{1,q} \end{matrix} \right.\right] = {}_p\Psi_q^{(\gamma)}\left[\begin{matrix} (e_1, E_1, u), (e_j, E_j)_{2,p}; \\ \\ (f_j, F_j)_{1,q}; \end{matrix} \; w \right] \tag{1.7}$$

and

$$\Gamma_{p,q+1}^{1,p}\left[-w \left| \begin{matrix} (1-e_1, E_1, u), (1-e_j, E_j)_{2,p} \\ \\ (0,1), (1-f_j, F_j)_{1,q} \end{matrix} \right.\right] = {}_p\Psi_q^{(\Gamma)}\left[\begin{matrix} (e_1, E_1, u), (e_j, E_j)_{2,p}; \\ \\ (f_j, F_j)_{1,q}; \end{matrix} \; w \right] \tag{1.8}$$

(iii) Taking $x = 0$, $m = n = p = 1$, q being replaced by $m+1$ and choosing proper parameters, the function (1.4) reduces to the generalized multi-index Mittag-Leffler function (GMIMLF) (see, details, [11])

$$\begin{aligned} &\Gamma_{1,m+1}^{1,1}\left[-z \left| \begin{matrix} (1-\lambda, \mu, 0) \\ \\ (0,1), (1-\beta_j, \alpha_j)_{1,m} \end{matrix} \right.\right] \\ &= \frac{1}{\Gamma(\lambda)}\, {}_1\Psi_m\left[\begin{matrix} (\lambda, \mu); \\ \\ (\beta_j, \alpha_j)_{1,m}; \end{matrix} \; z \right] = E_{(\alpha_j, \beta_j)_m}^{\lambda,\mu}[z]. \end{aligned} \tag{1.9}$$

In this paper, we aim to establish the images of incomplete H-functions and then, it is reduces to the some useful interesting special functions under this transform.

2. The Srivastava-Luo-Raina $\mathbb{M}$-transform of IHF

In this section, we establish the Srivastava-Luo-Raina $\mathbb{M}$-transform of the incomplete H-functions (1.2) and (1.3).

Theorem 1. *If* $\Re(\lambda) \geq 0$*,* $\mu \in \mathbb{Z}_{+}$*,* $p \in \mathbb{C}$ *and* $q \in \mathbb{R}_{+}$ *and with the conditions presented along with* (1.2)*, the following* $\mathbb{M}$*-transform result exits for the IHF* $\gamma_{P,Q}^{M,N}[z]$*:*

$$\mathbb{M}_{\lambda,\mu}\left\{\gamma_{P,Q}^{M,N}\left[zx\left|\begin{matrix}(a_1,A_1,t),(a_j,A_j)_{2,p}\\ \\ (b_j,B_j)_{1,q}\end{matrix}\right.\right]\right\}(p,q)=\frac{q^{-\mu\lambda}}{p\mu}\frac{1}{2\pi i}\int_{\mathfrak{L}}\mathbb{B}\left(\lambda-\frac{\xi}{\mu},\frac{\xi}{\mu}\right)(pq)^{\xi}$$
$$\gamma_{P+1,Q}^{M,N+1}\left[\frac{zq}{p}\left|\begin{matrix}(a_1,A_1,t),(\xi,1),(a_j,A_j)_{2,p}\\ \\ (b_j,B_j)_{1,q}\end{matrix}\right.\right]d\xi \quad (2.1)$$

In the above equation $\mathbb{B}(x,y)$ *indicates the classical Euler-Beta function.*

Proof. To prove the result (2.1), by applying the Srivastava-Luo-Raina $\mathbb{M}$-transform given in (1.1) of (1.2), then interchanging the order of integral and contour integral (which is allowable under the conditions presented) and with the help of [3, p. 1389, Eq. (2.7)], we easily achieve the right hand side of (2.1) after a small adjustment of terms. □

Theorem 2. *If* $\Re(\lambda) \geq 0$*,* $\mu \in \mathbb{Z}_{+}$*,* $p \in \mathbb{C}$ *and* $q \in \mathbb{R}_{+}$ *and with the conditions presented along with* (1.3)*, the following Srivastava-Luo-Raina* $\mathbb{M}$*-transform result exits for the IHF* $\Gamma_{P,Q}^{M,N}[z]$*:*

$$\mathbb{M}_{\lambda,\mu}\left\{\Gamma_{P,Q}^{M,N}\left[zx\left|\begin{matrix}(a_1,A_1,t),(a_j,A_j)_{2,p}\\ \\ (b_j,B_j)_{1,q}\end{matrix}\right.\right]\right\}(p,q)=\frac{q^{-\mu\lambda}}{p\mu}\frac{1}{2\pi i}\int_{\mathfrak{L}}\mathbb{B}\left(\lambda-\frac{\xi}{\mu},\frac{\xi}{\mu}\right)(pq)^{\xi}$$
$$\Gamma_{P+1,Q}^{M,N+1}\left[\frac{zq}{p}\left|\begin{matrix}(a_1,A_1,t),(\xi,1),(a_j,A_j)_{2,p}\\ \\ (b_j,B_j)_{1,q}\end{matrix}\right.\right]d\xi \quad (2.2)$$

In the above equation $\mathbb{B}(x,y)$ *is the famous Euler-Beta function.*

Proof. Similarly as in the proof of Theorem 1, one can easily derive the result. So, we omit the details. □

Among many particular cases of the results in Theorems 1 and 2, we give some of them in the subsequent corollaries.

Corollary 2.1. *If* $\Re(\lambda) \geq 0$*,* $\mu \in \mathbb{Z}_{+}$*,* $p \in \mathbb{C}$ *and* $q \in \mathbb{R}_{+}$*, the following* $\mathbb{M}$*-transform of the IFWF* ${}_{P}\Psi_{Q}^{(\gamma)}[z]$ *and* ${}_{P}\Psi_{Q}^{(\Gamma)}[z]$ *and MIMLF* $E_{(A_j,B_j)_M}^{\rho,\sigma}[z]$ *exist:*

$$\mathbb{M}_{\lambda,\mu}\left\{{}_{P}\Psi_{Q}^{(\gamma)}\left[zx\left|\begin{matrix}(a_1,A_1,t),(a_j,A_j)_{2,p}\\ \\ (b_j,B_j)_{1,q}\end{matrix}\right.\right]\right\}(p,q)=\frac{q^{-\mu\lambda}}{p\mu}\frac{1}{2\pi i}\int_{\mathfrak{L}}\mathbb{B}\left(\lambda-\frac{\xi}{\mu},\frac{\xi}{\mu}\right)(pq)^{\xi}$$
$${}_{P+1}\Psi_{Q}^{(\gamma)}\left[\frac{zq}{p}\left|\begin{matrix}(a_1,A_1,t),(\xi,1),(a_j,A_j)_{2,p}\\ \\ (b_j,B_j)_{1,q}\end{matrix}\right.\right]d\xi \quad (2.3)$$

and

$$\mathbb{M}_{\lambda,\mu}\left\{ {}_{P}\Psi_{Q}^{(\Gamma)}\left[zx \left| \begin{array}{c} (a_1, A_1, t), (a_j, A_j)_{2,p} \\ \\ (b_j, B_j)_{1,q} \end{array} \right. \right] \right\}(p,q) = \frac{q^{-\mu\lambda}}{p\mu}\frac{1}{2\pi i}\int_{\mathfrak{L}} \mathbb{B}\left(\lambda - \frac{\xi}{\mu}, \frac{\xi}{\mu}\right)(pq)^{\xi}$$

$${}_{P+1}\Psi_{Q}^{(\Gamma)}\left[\frac{zq}{p} \left| \begin{array}{c} (a_1, A_1, t), (\xi, 1), (a_j, A_j)_{2,p} \\ \\ (b_j, B_j)_{1,q} \end{array} \right. \right] d\xi \quad (2.4)$$

and

$$\mathbb{M}_{\lambda,\mu}\left\{E^{\rho,\sigma}_{(A_j,B_j)_M}[zx]\right\}(p,q) = \frac{q^{-\mu\lambda}}{p\mu\Gamma(\lambda)}$$

$$H^{0,1:1,1;1,1}_{1,0:1,M+1;1,1}\left[\begin{array}{c} -\frac{zq}{p} \\ \\ \frac{1}{pq} \end{array} \left| \begin{array}{ccc} (0;1,1) & : (1-\rho, \sigma) & ; (1-\lambda, \frac{1}{\mu}) \\ \\ - & : (0,1), (1-B_j, A_j)_{1,M} & ; (0, \frac{1}{\mu}) \end{array} \right. \right] \quad (2.5)$$

here $\mathbb{B}(x,y)$ is the standards Euler-Beta function and provided both sided members of above identities exist.

Proof. Considering (1.7), (1.8) and (1.9), we can obtain the results here from those in Theorems 1 and 2. □

3. Conclusions

In this work, we have established image of incomplete H-functions in the Srivastava-Luo-Raina $\mathbb{M}$-transform and obtained the images of some useful and important cases of incomplete H-function in it. The results presented in this article are new and and can be utilized to derivative various new and known results having applications in science and engineering.

Acknowledgments

The author K. S. Nisar thanks to Prince Sattam bin Abdulaziz University, Saudi Arabia for providing facilities and support.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this article.

References

1. L. Debnath, D. Bhatta, *Integral transforms and their applications*, Chapman & Hall/CRC, Boca Raton, London, New York, 2007.
2. I. N. Sneddon, *The use of integral transforms*, Tata McGraw-Hill, New York, 1974.

3. H. M. Srivastava, M. J. Luo, R. K. Raina, *A new integral transform and its applications*, Acta Math. Sci., **35** (2015), 1386–1400.

4. H. M. Srivastava, *Some remarks on a generalization of the Stieltjes transform*, Publ Math Debrecen, **23** (1976), 119–122.

5. H. M. Srivastava, R. K. Saxena, R. K. Parmar, *Some families of the incomplete H-functions and the incomplete $\overline{H}$-functions and associated integral transforms and operators of fractional calculus with applications*, Russian J. Math. Phys., **25** (2018), 116–138.

6. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematical Studies, Vol. 204, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.

7. A. M. Mathai, R. K. Saxena, H. J. Haubold, *The H-function theory and applications*, Springer-Verlag, New York, 2010.

8. A. M. Mathai, R. K. Saxena, *The H-function with applications in statistics and other disciplines*, Wiley Eastern Limited, New Delhi; John Wiley and Sons, New York, 1978.

9. H. M. Srivastava, K. C. Gupta, S. P. Goyal, *The H-functions of one and two variables with applications*, South Asian Publishers, New Delhi and Madras, 1982.

10. M. K. Bansal, J. Choi, *A note on pathway fractional integral formulas associated with the incomplete $H-$functions*, Int. J. Appl. Comput. Math., **5** (2019), 133.

11. H. M. Srivastava, M. K. Bansal, P. Harjule, *A study of fractional integral operators involving a certain generalized multi-index Mittag-Leffler function*, Math. Meth. Appl. Sci., **41** (2018), 6108–6121.