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Research article

Non-extensive minimal entropy martingale measures and semi-Markov regime switching interest rate modeling

Muhammad Sheraz[1,*], Vasile Preda[2,3,4] and Silvia Dedu[5]

[1] Department of Mathematical Sciences, Institute of Business Administration Karachi, Pakistan
[2] University of Bucharest, Romania
[3] "Gh. Mihoc-C. Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy
[4] National Institute for Economic Research "Costin C. Kiritescu", Bucharest, Romania
[5] Bucharest University of Economic Studies, Bucharest, Romania

* **Correspondence:** Email: msheraz@iba.edu.pk; Tel: +92021381047002683.

Abstract: A minimal entropy martingale measure problem is studied to investigate risk-neutral densities and interest rate modelling. Hunt & Devolder focused on the method of Shannon minimal entropy martingale measure to select the best measure among all the equivalent martingale measures and, proposed a generalization of the Ho & Lee model in the semi-Markov regime-switching framework [1]. We formulate and solve the optimization problem of Hunt & Devolder for deriving risk-neutral densities using a new non-extensive entropy measure [2]. We use the Lambert function and a new type of approach to obtain results without depending on stochastic calculus techniques.

Keywords: entropy; minimal entropy martingale; interest rate models; semi-Markov processes; risk neutral density
Mathematics Subject Classification: 60Gxx, 60Jxx, 90Bxx

1. Introduction

The application of entropy measures in finance can be regarded as the extension of both information and probability theory. Recently the use of information measures has become an important tool for solving problems arising in finance, such as portfolio selection or asset pricing. Shannon introduced the concept of information entropy [3]. The entropy of a random variable measures uncertainty in probability theory. The Shannon entropy can be used in particular manners to evaluate the entropy of probability density distribution around some points but, in the case of special events, for example, deviation from mean and any sudden news for stock returns going up (down), additional information is needed. Therefore, the concept of Shannon entropy can be generalized. The Shannon entropy assumes certain implicit exchange that occurs as a compromise between contributions from the tail and main mass of the distribution. Therefore, it misleads in comparison to the entropy of two distributions by considering the preceding two events. The use of entropy measures that depend on the powers of probability, for example, Shafee, Kaniadakis, Rényi, Tsallis, and Ubriaco, provide such control [2,4–7]. The measure is more sensitive to events that occur many times in a short interval if the power of the probability is positive large. On the other hand, for a negative value, it is more sensitive to the events that occur seldom.

In quantitative finance and life insurance, various methodologies have been developed to model the interest rate structures. In a discrete setting, a binomial like the model of Ho & Lee is well known [8]. Ho & Lee developed a revolutionary approach for modeling the yield curve movements using a binomial tree. Similarly, continuous-time models of CIR or Hull & White and Vasicek are famous [9–11]. On the other hand, regime-switching models have been used in financial derivatives, interest rates, and portfolio optimization. Hunt & Devolder used a minimal-entropy martingale measure for the semi-Markov regime-switching interest rate model in a discrete framework [1].

The process which has mean value at any future time, conditional on the present, equal to its present value, is called martingale. Martingales represent the most crucial tool in modern probability theory and applied finance. In an incomplete market, an infinite number of martingale measures exist. Therefore, measure selection has always been a crucial step in making a model useful. In the theory of derivative pricing, one starts with the model of the asset price process in an economy with a probability measure $\mathbb{P}$, and then, by choosing a numeraire $\mathbb{N}$, changes the probability measure $\mathbb{P}$ to equivalent martingale measure $\mathbb{Q}$ (see [12] and their references). In recent studies, the application of entropy in finance and economics has received significant attraction from researchers. It can be an important tool in portfolio selection and asset pricing. Philippatos & Wilson were the first two researchers who applied the concept of entropy to portfolio selection [13]. Similarly, entropy has used in the field of option pricing. A typical example is the Entropy Pricing Theory (EPT) introduced by Gulko [14]. The method of minimal entropy martingale measure has been used recently to study the generalization of the Ho & Lee model in a semi-Markov regime-switching framework by Hunt & Devolder [1]. Trivellato studied the Kaniadakis minimal entropy martingale measure, as well as its connections with the Tsallis and the well-known entropy martingale measure, in a general semi-martingale pricing model [15–16]. Preda et al. introduced a new measure selection for the Hunt-Devolder semi-Markov regime-switching interest rate model using Tsallis and Kaniadakis entropy measures [17]. Preda et al. introduced the new classes of Lorenz curves by maximizing Tsallis entropy under mean and Gini's equality and inequality constraints [18]. Shafee proposed a new way of defining the entropy of a system and proposed a new entropy measure that is non-extensive like Tsallis entropy, but it is linearly dependent on component entropies, like Rényi entropy, which is an extensive measure [2]. Preda & Sheraz have recently used the Shafee entropy measure for the case of risk-neutral densities using a framework of entropy pricing theory [19]. Sheraz et al. have used some general frameworks to obtain risk-neutral densities. The results are based on Tsallis-weighted-Tsallis-Kaniadakis and weighted Kanidakis entropy measures [20].

In this paper, we consider the Shafee entropy measure to obtain the results for risk-neutral probabilities, using the framework of the Hunt-Devolder model. In Section 2 the framework of the regime-switching interest rate model is introduced, as it provides tools for the development of the risk-neutral probabilities. In Section 3 we present our new results for risk-neutral probabilities using Shafee entropy measure and another new approach. Section 4 concludes our results.

2. Preliminaries and framework

We will use some concepts developed in Hunt & Devolder semi-Markov regime-switching model. In the unstable economic environment, regime-switching processes have been used to model regular changes at some non-predictable stopping time. A significant part of the financial literature focuses on Markov switching models. The major disadvantage of these models is the memoryless property of the Markov process. semi-Markov processes depend on backward recurrence time, i.e., the time elapsed since the last jump of the process. Therefore, the use of these models is advantageous on simple homogeneous Markov models. Moreover, since the Markov process are a subclass of semi-Markov processes, semi-Markov switching models should always perform at least as well as Markov switching models.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a probability space and $(\mathcal{F}_t)$ a filtration, where $t \in [0, 1, 2, \dots T]$ and T is a fixed time horizon. Consider an economy $\mathcal{E}$ with assets $A_t^i(\omega_j)$, i.e., the price of asset i at time t if the economy is in state ω_j. The arbitrage-free model is the most critical assumption in the pricing of any financial model. See details in [12,21].

Definition 2.1. ([12]) A one period economy $\mathcal{E}$ admits arbitrage if there exists a portfolio Φ such that one of the following conditions hold.

i. $\Phi. A_0 < 0$ and $\Phi. A_1(\omega_j) \geq 0$, for all j.

ii. $\Phi. A_0 \leq 0$ and $\Phi. A_1(\omega_j) > 0$, for all j.

If there is no such Φ, then the economy $\mathcal{E}$ is arbitrage free. In any financial modeling process it is crucial to find an arbitrage free economy $\mathcal{E}$. The above definition refers to a one period model.

The binomial option pricing model provides a numerical method for the valuation of options [22]. In the one period binomial model if B_t denotes price of a bond, then we can write B_0 = 1 at time $t = 0$ and $B_1 = 1$, at future time $t = 1$. Similarly, for the stock price process $\{S_t\}_{t \in T}$, the initial value of the stock at time $t = 0$ is $S_0 = s$ and the value of the stock corresponding to the future time $t = 1$ is $S_1 = us$, with probability pu (up movement) or $S_1 = ds$, with probability qd (down movement), with the condition $d < u$ and of course $pu + qd = 1$.

Proposition 2.1. ([21]) The binomial model is free of arbitrage if and only if $d < 1 + r > u$ holds.

A market model is arbitrage-free if there exists a martingale measure such that the return on the stock is not allowed to dominate the returns on the bond and vice versa. In Hunt & Devolder's discrete-time regime-switching binomial-like model of the term structure where the regime switches are governed by the semi-Markov process, the minimal Shannon entropy martingale measure characterized one period and *n*-period binomial model. Now we introduce the framework of the Hunt & Devolder model. We consider a set $E = \{1,2, . . \ , m\}$ for m finite, Σ a sigma-algebra on E and $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ a probability space which carries a pair of processes (X_n, T_n) taking values in $E \times \mathbb{N}$. See details in [1].

Definition 2.2. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a probability space and $(\mathrm{S}, \mathcal{S})$ be a measureable space. A stochastic process $\{X_t\}_{t\in T}$ adapted to the filtration $(\mathcal{F}_t)$ is called Markov process w.r.t. $(\mathcal{F}_t)$, if for each $A \in \mathcal{S}$ and each $s, t \in T$ with $s < t$

$$\mathbb{P}(X_t \in A \mid \mathcal{F}_s) = \mathbb{P}(X_t \in A \mid X_s). \tag{2.1}$$

Definition 2.3. ([1]) A homogeneous Markov-renewal process of semi-Markov kernel Q can be defined as:

$$\begin{aligned}\mathbb{P}(X_{n+1} = j;\ T_{n+1} - T_n \le t | X_0, T_0,\ \ . \ . \ . , X_n, T_n)\\ = \mathbb{P}(X_{n+1} = j\ ;\ T_{n+1} - T_n \le t | \ X_n)\\ \coloneqq Q_{x_{nj}}(t)\end{aligned} \tag{2.2}$$

In the above definition, (X_n, T_n) is a homogeneous Markov-renewal process of semi-Markov kernel Q. Then the semi-Markov process Y of kernel Q is defined by $Y_t = X_{v_t}$, where, $v_t = \max\{n | T_n \le t\}$ and Y_0 is supposed to be known and non-random. The process Y controls the "regime" or "state" of the economy. Similarly, the process K_t which represents the time elapsed since the last jump, is given by $K_t = t - T_{v_t}$, where K_0 is supposed to be known and non-random and K_t is known as backword recurrence process. The pair (Y_t, K_t) satisfies the Markov property. Assume the existence of two real values $u_i(t)$ and $d_i(t)$ up and down movements respectively for every $\tau = T - t$, $u_i(t) > d_i(t)$ and strictly positive. The vector stochastic process $\{\zeta_t\}_{t\in T}$ of the term structure from time t to $t + 1$ is given by

$$\mathbb{P}\big(\zeta_{t+1} = u_{Y_t} | \mathcal{F}_t\big) = 1 - \mathbb{P}\big(\zeta_{t+1} = d_{Y_t} | \mathcal{F}_t\big) \tag{2.3}$$

where ζ_{t+1} and Y_{t+1} are conditionally independent given $(\mathcal{F}_t)$ for all $t \ge 1$. Therefore we can write:

$$\mathbb{P}\big(\zeta_{t+1} = u_{Y_t}, Y_{t+1} = j | \mathcal{F}_t\big) = \mathbb{P}\big(\zeta_{t+1} = u_{Y_t} | \mathcal{F}_t\big)\mathbb{P}(Y_{t+1} = j | \ \mathcal{F}_t) \tag{2.4}$$

We denote by:

$$v_t^j = \mathbb{P}(Y_{t+1} = j | \mathcal{F}_t) \tag{2.5}$$

$$z_t^j = \mathbb{P}\big(\zeta_{t+1} = u_{Y_t} | \mathcal{F}_t\big) \tag{2.6}$$

$$\pi_t^j = \mathbb{P}\big(\zeta_{t+1} = u_{Y_t}, Y_{t+1} = j | \mathcal{F}_t\big) = z_t^j v_t^j \tag{2.7}$$

$$k_t^j = \mathbb{P}\big(\zeta_{t+1} = d_{Y_t}, Y_{t+1} = j | \mathcal{F}_t\big) = (1 - z_t^j) v_t^j \tag{2.8}$$

Since for every t, it follows that

$$\textstyle\sum_{j=1}^m \big(\pi_t^j + k_t^j\big) = 1 \tag{2.9}$$

The system composed of process ζ, Y and K can take $2m$ different values and these values are all determined by the following state of events.

$$A_{t+1}^{j,u} \coloneqq \{\omega \in \Omega | Y_{t+1} = j\,, \zeta_{t+1} = u_{Y_t}\} \tag{2.10}$$

$$A_{t+1}^{j,d} \coloneqq \{\omega \in \Omega | Y_{t+1} = j\,, \zeta_{t+1} = d_{Y_t}\} \tag{2.11}$$

In order to maintain the no arbitrage condition in a market one needs an equivalent martingale measure.

Lemma 2.1. ([1]) Let $D_t = \prod_{s=0}^{t-1} \sum_{j=1}^{m} \left(\frac{p_s^j}{\pi_s^j} 1_{A_{s+1}^{j,u}} + \frac{q_s^j}{k_s^j} 1_{A_{s+1}^{j,d}}\right)$. Then $D_t > 0$ for all t and $\mathbb{E}(D_t) = 1$, $\mathbb{E}(D_{t+1} | \mathcal{F}_t) = D_t$.

Hunt & Devolder used Lemma 2.1 to define an equivalent martingale measure and studied the minimal entropy martingale measure for the binomial model [1].

Definition 2.4. Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures. The relative Shannon entropy is given by

$$\mathrm{I}(\mathbb{P}, \mathbb{Q}) = \begin{cases} \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] & \mathbb{Q} \ll \mathbb{P} \\ +\infty & \text{otherwise} \end{cases} \tag{2.12}$$

A measure $\mathbb{Q}$ is called a minimal (Shannon) entropy martingale measure if it minimizes the relative entropy over the set of all equivalent martingale measures. The minimal entropy martingale is an efficient method because the minimization of the relative entropy finds a probability distribution that is closest to the prior. Also, it contains all the information we know about the random variable of interest. The martingale probabilities for the one-period binomial model, with minimal Shannon entropy martingale measure, are given by

$$p_0^j = v_0^j \left(\frac{1 - d_{Y_0}}{u_{Y_0} - d_{Y_0}}\right) \tag{2.13}$$

$$q_0^j = v_0^j \left(\frac{u_{Y_0} - 1}{u_{Y_0} - d_{Y_0}}\right) \tag{2.14}$$

where Y_t is a semi-Markov process.

We select Shafee entropy measure to compute risk-neutral probabilities for the semi-Markov regime-switching interest rate model of the term structure. Optimization of this entropy measure leads to a probability distribution function involving the Lambert function.

A Lambert function W is a multivalued complex function which is defined as the solution of the equation $W(\mathbf{z})e^{W(\mathbf{z})} = \mathbf{z}$, where $\mathbf{z}$ is a complex number. If $\mathbf{z}$ is a real number such that $\mathbf{z} \geq e^{-1}$, then $W(\mathbf{z})$ becomes a real function with two possible values in $(-\infty, 1]$ and $[1, \infty)$[23].

3. Measure selection: Shafee entropy

In 2007 a non-additive entropy measure was proposed by Shafee [2]. It is non-extensive like Tsallis entropy, but it is linearly dependent on component entropies, like Rényi entropy, which is an extensive measure. Formally the Shafee entropy is given by:

$$H(a) = -\sum_i p_i^a \log p_i \tag{3.1}$$

The continuous cases as an analogue of the discrete case is given by:

$$H(a) = -E^{p}[p^{a-1}\ln p],\ a > 0,\ a \neq 1. \tag{3.2}$$

We note that for $a \to 1$ we obtain the corresponding Shannon entropy.

Definition 3.1. Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures. The relative Shafee entropy for the one period minimal martingale is given by

$$\mathrm{S}(\mathbb{P},\mathbb{Q}) = \begin{cases} \mathbb{E}^{\mathbb{P}}\left[(\frac{d\mathbb{Q}}{d\mathbb{P}})^{a}\ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] & \mathbb{Q} \ll \mathbb{P}, a > 0 \\ +\infty & \text{otherwise} \end{cases} \tag{3.3}$$

We note that for $a \to 1$ we obtain the corresponding relative Shannon entropy.

Definition 3.2. A measure $\mathbb{Q}$ is called a minimal Shafee entropy martingale measure if it minimizes the relative entropy over the set of all equivalent martingale measures.

This idea leads to closest to real-world martingale measure, in terms of Shafee entropy measure subject to the following constraints that ensure that the measure will be an equivalent martingale measure.

$$\sum_{j=1}^{m}\left(p_0^j + q_0^j\right) = 1 \tag{3.4}$$

$$u_{Y_0}\sum_{j=1}^{m} p_0^j + d_{Y_0}\sum_{j=1}^{m} q_0^j = 1 \tag{3.5}$$

Constructing the minimal entropy martingale measure is equivalent to giving the parameters p_t^j in the density process associated to the minimal entropy martingale measure. We study the one-period case. We present the computation of risk-neutral probabilities for one period binomial model of the term structure. We assume that Lambert's function is well defined for the quantities which have been considered.

Theorem 3.1. Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures. The relative Shafee entropy for the one period minimal martingale is given by Eq. (3.3) and constraints (3.4) and (3.5) respectively. We obtain risk neutral probabilities p_0^j and q_0^j, given by:

$$p_0^j = \pi_0^j\left(\frac{aW\left((\lambda_0+\lambda_1 u_{Y_0})\left(\frac{1-a}{a}\right)\left(\pi_0^j\right)^{1-a}.\, e^{\frac{a-1}{a}}\right)}{(\lambda_0+\lambda_1 u_{Y_0})(1-a)\left(\pi_0^j\right)^{1-a}}\right)^{\frac{1}{1-a}} \tag{3.6}$$

$$q_0^j = k_0^j\left(\frac{aW\left((\lambda_0+\lambda_1 d_{Y_0})\left(\frac{1-a}{a}\right)\left(k_0^j\right)^{1-a}.\, e^{\frac{a-1}{a}}\right)}{(\lambda_0+\lambda_1 d_{Y_0})(1-a)\left(k_0^j\right)^{1-a}}\right)^{\frac{1}{1-a}} \tag{3.7}$$

where λ_0 and λ_1 are Lagrange multipliers and λ_0 and λ_1 are chosen so that the risk neutral densities satisfy given constraints.

Proof. We can write the Lagrangian, and using calculus of variation for optimization of functional (see for example [24–25]).

$$\mathrm{L} = \sum_{j=1}^{m}\left(\left(p_0^j\right)^a \ln\left(\frac{p_0^j}{\pi_0^j}\right) + \left(q_0^j\right)^a \ln\left(\frac{q_0^j}{k_0^j}\right)\right) + \lambda_0\left(\sum_{j=1}^{m}\left(p_0^j + q_0^j\right) - 1\right) +$$

$$+\lambda_1\left(u_{Y_0}\sum_{j=1}^{m} p_0^j + d_{Y_0}\sum_{j=1}^{m} q_0^j - 1\right) \tag{3.8}$$

Applying derivative on the above equation of Lagrangian L, w. r. t p_0^j and q_0^j and equating with zero we get:

$$a\left(p_0^j\right)^{a-1}\ln\left(\frac{p_0^j}{\pi_0^j}\right) + \left(p_0^j\right)^{a-1} = -\left(\lambda_0 + \lambda_1 u_{Y_0}\right)$$

$$a\left(q_0^j\right)^{a-1}\ln\left(\frac{q_0^j}{k_0^j}\right) + \left(q_0^j\right)^{a-1} = -\left(\lambda_0 + \lambda_1 d_{Y_0}\right)$$

which is equivalent to

$$\left(p_0^j\right)^{a-1}\left(a\ln\left(\frac{p_0^j}{\pi_0^j}\right) + 1\right) = -\left(\lambda_0 + \lambda_1 u_{Y_0}\right)$$

$$\left(q_0^j\right)^{a-1}\left(a\ln\left(\frac{q_0^j}{k_0^j}\right) + 1\right) = -\left(\lambda_0 + \lambda_1 d_{Y_0}\right)$$

We can rewrite the above two equations:

$$\left(\frac{p_0^j}{\pi_0^j}\right)^{a-1}\left(\pi_0^j\right)^{a-1}\left(a\ln\left(\frac{p_0^j}{\pi_0^j}\right) + 1\right) = -\left(\lambda_0 + \lambda_1 u_{Y_0}\right)$$

$$\left(\frac{q_0^j}{k_0^j}\right)^{a-1}\left(k_0^j\right)^{a-1}\left(a\ln\left(\frac{q_0^j}{k_0^j}\right) + 1\right) = -\left(\lambda_0 + \lambda_1 d_{Y_0}\right)$$

Equivalently,

$$a\ln\left(\frac{p_0^j}{\pi_0^j}\right) + 1 = \left(-\lambda_0 - \lambda_1 u_{Y_0}\right)\left(\frac{p_0^j}{\pi_0^j}\right)^{1-a}\left(\pi_0^j\right)^{1-a} \tag{3.9}$$

$$a\ln\left(\frac{q_0^j}{k_0^j}\right) + 1 = \left(-\lambda_0 - \lambda_1 d_{Y_0}\right)\left(\frac{q_0^j}{k_0^j}\right)^{1-a}\left(k_0^j\right)^{1-a} \tag{3.10}$$

Let us suppose $\left(\frac{p_0^j}{\pi_0^j}\right)^{1-a} = y_1 \Rightarrow (1-a)\ln\left(\frac{p_0^j}{\pi_0^j}\right) = \ln y_1$, and we get $\frac{1}{1-a}\ln y_1 = \ln\left(\frac{p_0^j}{\pi_0^j}\right)$. Similarly, we can put $\left(\frac{q_0^j}{k_0^j}\right)^{1-a} = y_2$ and we get $\frac{1}{1-a}\ln y_2 = \ln\left(\frac{q_0^j}{k_0^j}\right)$. Now using Eqs. (3.9) and (3.10) to compute p_0^j and q_0^j respectively. Therefore, we can write:

$$\frac{a}{1-a}\ln y_1 + 1 = \left(-\lambda_0 - \lambda_1 u_{Y_0}\right) y_1\left(\pi_0^j\right)^{1-a}$$

Equivalently,

$$\ln y_1 = -\left(\lambda_0 + \lambda_1 u_{Y_0}\right)\left(\frac{1-a}{a}\right) y_1 \left(\pi_0^j\right)^{1-a} + \frac{a-1}{a}$$

$$\ln y_1 + \left(\frac{1-a}{a}\right)\left(\pi_0^j\right)^{1-a}\left(\lambda_0 + \lambda_1 u_{Y_0}\right) y_1 = \frac{a-1}{a}$$

$$\ln\left(\left(y_1 . e^{\left(\frac{1-a}{a}\right)\left(\pi_0^j\right)^{1-a}\left(\lambda_0+\lambda_1 u_{Y_0}\right) y_1}\right)\right) = \frac{a-1}{a}$$

Let us use definition of the Lambert function and suppose $\mathbf{z} = \left(\frac{1-a}{a}\right)\left(\pi_0^j\right)^{1-a}\left(\lambda_0 + \lambda_1 u_{Y_0}\right) y_1$ i.e., $y_1 = \frac{\mathbf{z}}{\lambda_0+\lambda_1 u_{Y_0}}\left(\frac{a}{1-a}\right)\left(\pi_0^j\right)^{a-1}$. Therefore, we can write:

$$\frac{\mathbf{z}}{\lambda_0+\lambda_1 u_{Y_0}}\left(\frac{a}{1-a}\right)\left(\pi_0^j\right)^{a-1} . e^{\mathbf{z}} = e^{\frac{a-1}{a}}$$

$$\mathbf{z}. e^{\mathbf{z}} = \left(\lambda_0 + \lambda_1 u_{Y_0}\right)\left(\frac{1-a}{a}\right)\left(\pi_0^j\right)^{1-a} . e^{\frac{a-1}{a}}$$

$$\mathbf{z} = W\left(\left(\lambda_0 + \lambda_1 u_{Y_0}\right)\left(\frac{1-a}{a}\right)\left(\pi_0^j\right)^{1-a} . e^{\frac{a-1}{a}}\right)$$

$$\left(\frac{1-a}{a}\right)\left(\pi_0^j\right)^{1-a}\left(\lambda_0 + \lambda_1 u_{Y_0}\right)\left(\frac{p_0^j}{\pi_0^j}\right)^{1-a} = W\left(\left(\lambda_0 + \lambda_1 u_{Y_0}\right)\left(\frac{1-a}{a}\right)\left(\pi_0^j\right)^{1-a} . e^{\frac{a-1}{a}}\right)$$

Thus we obtain

$$p_0^j = \pi_0^j\left(\frac{aW\left(\left(\lambda_0+\lambda_1 u_{Y_0}\right)\left(\frac{1-a}{a}\right)\left(\pi_0^j\right)^{1-a} . e^{\frac{a-1}{a}}\right)}{\left(\lambda_0+\lambda_1 u_{Y_0}\right)(1-a)\left(\pi_0^j\right)^{1-a}}\right)^{\frac{1}{1-a}}$$

Similarly, using Eq. (3.10) q_0^j is given by

$$q_0^j = k_0^j\left(\frac{aW\left(\left(\lambda_0+\lambda_1 d_{Y_0}\right)\left(\frac{1-a}{a}\right)\left(k_0^j\right)^{1-a} . e^{\frac{a-1}{a}}\right)}{\left(\lambda_0+\lambda_1 d_{Y_0}\right)(1-a)\left(k_0^j\right)^{1-a}}\right)^{\frac{1}{1-a}}$$

where λ_0 and λ_1 are Lagrange multipliers and λ_0 and λ_1 are chosen so that risk neutral densities satisfy given constraints.

We present another approach to compute p_0^j and q_0^j.

Lemma 3.1. If $\psi = x^b \ln x$ and $\psi: (0,\infty) \to (0,\infty)$ where $b > 0$ and $b \neq 1$, then $\psi^{/}$, the first derivative of ψ admits an inverse. Therefore,

a. $\psi^{/}: \left(0\,, e^{\frac{1-2b}{b(b-1)}}\right) \to \left(0, \frac{-be^{\frac{1-2b}{b}}}{b-1}\right)$

b. $\psi^{/}: \left(0\,, e^{\frac{1-2b}{b(b-1)}}\right) \to \left(\frac{-be^{\frac{1-2b}{b}}}{b-1}, \infty\right)$

c. If $b \in (0,1)$ then ψ is convex on $\left(0, e^{\frac{1-2b}{b(b-1)}}\right)$ and concave on $\left(\frac{-be^{\frac{1-2b}{b}}}{b-1}, \infty\right)$

Theorem 3.2. Let $\mathbb{P}$ and $\mathbb{Q}$ be two probability measures. The relative Shafee entropy for the one period minimal martingale is given by

$$S(\mathbb{P},\mathbb{Q}) = \begin{cases} \mathbb{E}^{\mathbb{P}}\left[(\frac{d\mathbb{Q}}{d\mathbb{P}})^a \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] & \mathbb{Q} \ll \mathbb{P}, a > 0 \\ +\infty & \text{otherwise} \end{cases}$$

subject to constraints (3.4) and (3.5). Consider $\psi_1(x) = x^a \ln\left(\frac{x}{\pi_0^j}\right)$ and $\psi_2(x) = x^a \ln\left(\frac{x}{k_0^j}\right)$. Then risk neutral probabilities p_0^j and q_0^j are given by:

$$p_0^j = \left(\psi_1{}^{/}\right)^{-1}\left(-\lambda_0 - \lambda_1 u_{Y_0}\right) \tag{3.11}$$

$$q_0^j = \left(\psi_2{}^{/}\right)^{-1}\left(-\lambda_0 - \lambda_1 d_{Y_0}\right) \tag{3.12}$$

where λ_0 and λ_1 are chosen so that risk neutral density satisfies given constraints.

Proof. Using Eq. (3.8) of Lagrangian L and Eqs. (3.9) and (3.10) respectively. We put $\psi_1\left(p_0^j\right) = \left(p_0^j\right)^a \ln\left(\frac{p_0^j}{\pi_0^j}\right)$ and $\psi_2\left(q_0^j\right) = \left(q_0^j\right)^a \ln\left(\frac{q_0^j}{k_0^j}\right)$. Then, p_0^j and q_0^j are given by (3.11) and (3.12).

Example. Let $\left(v_0^j\right)_{j=\overline{1,4}} = \left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}\right)$, $\left(z_0^j\right)_{j=\overline{1,4}} = \left(\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{2}{8}\right)$, $d_{Y_0} = 0.9$ and $u_{Y_0} = 1.2$, therefore $\left(\pi_0^j\right)_{j=\overline{1,4}} = \left(\frac{1}{40}, \frac{4}{40}, \frac{3}{40}, \frac{2}{40}\right)$ and $\left(k_0^j\right)_{j=\overline{1,4}} = \left(\frac{7}{40}, \frac{12}{40}, \frac{5}{40}, \frac{6}{40}\right)$.

We use Theorem 3.2 to derive the minimal second order Shafee entropy martingale measure. By using (3.8), (3.11), (3.12) it results that the martingale probabilities for the one-period binomial model with minimal second order Shafee entropy martingale measure are given by

$$\left(p_0^j\right)_{j=\overline{1,4}} = (0.08314; 0.10435; 0.2146; 0.12438)$$

and

$$\left(q_0^j\right)_{j=\overline{1,4}} = (0.08957; 0.09485; 0.1835; 0.29879).$$

4. Conclusion

We have presented two different approaches to obtain risk-neutral probabilities. The problem of extracting risk-neutral probabilities is crucial in mathematical finance. This approach is an alternative structure to solve such problems without depending on the stochastic calculus techniques. We have considered a non-extensive entropy, which is mathematically simple, and it leads to solutions using the Lambert function.

Conflict of interest

The authors declare no conflict of interest.

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