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Research article

Solvability of a fractional Cauchy problem based on modified fixed point results of non-compactness measures

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Abstract: We study the solvability of a fractional Cauchy problem based on new development of fixed point theorem, where the operator is suggested to be non-compact on its domain. Moreover, we shall prove that the solution is bounded by a fractional entropy (entropy solution). For this purpose, we establish a collection of basic fixed point results, which generalizes and modifies some well known results. Our attention is toward the concept of a measure of non-compactness to generalize μ-set contractive condition, using three control functions.

Keywords: fractional calculus; fractional differential operator; fractional differential equation; fixed point theorem; measure of non-compactness; entropy solution
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1. Introduction

Fractional Cauchy problems switch the typical first-order time derivative by a fractional derivative. Newly fractional differential equations appeared as an original branch of applied mathematics which has been used for many mathematical simulations in science and engineering. In fact fractional differential equations are considered as an alternative model to nonlinear differential equations. In [13, 18] the researchers have showed the existence of solutions of abstract differential equations by using semigroup theory and fixed point theorem. Many authors have used Banach fixed point theorem. The non-local Cauchy problem for theoretical evolution differential equation was first planned by Byszewski [11]. Later several investigators have studied the problem for diverse kinds of nonlinear differential equations and integro-differential equations including functional differential equations in Banach spaces [14]. Motivated by those efforts we consider the existence of solutions of fractional differential equation, with a common factor function in Banach spaces by using Riemann-Liouville fractional calculus and generalized fixed point theorems of non-compactness measures.

2. Background

Darbo [12] used the notion of Kuratowski measure of noncompactness (MNC, for short) and define the classes of operators using this number. Thereafter various class of MNC have been explored. To discuss further we recall some scientific records to prove all the outcomes of this work.
Indicate by $\mathbb{R}$ the set of all real numbers and $\mathbb{R}^+ = [0, +\infty)$. Let $(E, \|.\|)$ be a real Banach space containing the zero element 0. Let $\mathcal{B}(x, r)$ indicate the closed ball centered at x with radius r. The notion $\mathcal{B}_r$ runs for the ball $\mathcal{B}(0, r)$. For $\bar{X} \neq \emptyset \subset E$, we symbolize by $\mathcal{X}$ and $Conv\mathcal{X}$ the closure and the convex closure of $\mathcal{X}$, respectively. Furthermore, formalize by $\mathfrak{M}_E$ the family of non-empty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily involving all relatively compact sets. We utilize the following definition of the MNC offered in [10].

Definition 2.1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}^+$ is said to be a MNC in E if it achieves the next assumptions:

(1^0) The set $ker\mu = \{\mathcal{X} \in \mathfrak{M}_E : \mu(\mathcal{X}) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$,
(2^0) $\mathcal{X} \subset \mathcal{Y} \Rightarrow \mu(\mathcal{X}) \leq \mu(\mathcal{Y})$,
(3^0) $\mu(\overline{\mathcal{X}}) = \mu(\mathcal{X})$,
(4^0) $\mu(Conv\mathcal{X}) = \mu(\mathcal{X})$,
(5^0) $\mu(\lambda\mathcal{X} + (1-\lambda)\mathcal{Y}) \leq \lambda\mu(\mathcal{X}) + (1-\lambda)(\mathcal{Y})$ for $\lambda \in [0, 1]$,
(6^0) If $(\mathcal{X}_n)$ is a sequence of closed sets from $\mathfrak{M}_E$ with $\mathcal{X}_{n+1} \subset \mathcal{X}_n$ ($n = 1, 2, \ldots$) and if $\lim_{n\to\infty} \mu(\mathcal{X}_n) = 0$, then the set $\mathcal{X}_\infty = \bigcap_{n=1}^{\infty} \mathcal{X}_n$ is nonempty.

The set $ker\mu$ defined in axiom (1^0) is referred to the kernel of the MNC μ. Obviously, the measure of non-compactness satisfies $\mathcal{X}_\infty \in ker\mu$. Moreover, by the inequality $\mu(\mathcal{X}_\infty) \leq \mu(\mathcal{X}_n)$ for $n \in \mathbb{N}$, we deduce that $\mu(\mathcal{X}_\infty) = 0$. The Kuratowski MNC is the map $\mu : \mathfrak{M}_E \to \mathbb{R}^+$ with

$$\mu(Q) = \inf\left\{\epsilon > 0 : Q \subset \bigcup_{k=1}^{n} \mathcal{S}_k, \mathcal{S}_k \subset E, diam(\mathcal{S}_k) < \epsilon\, (k \in \mathbb{N})\right\}.$$

We indicate the set $\Lambda = \{C : C \neq \emptyset$, closed, bounded and convex subset of a Banach space $E\}$. And the set of fixed point (coupled fixed point) of $\mathcal{T}$ by $Fix(\mathcal{T})$ ($CFix(\mathcal{T})$).

Theorem 2.2. *(SFPT [2]) Let C be a nonempty, bounded, closed and convex subset of a Banach space E and $\mathcal{T} : C \to C$ be a compact, continuous operator. Then $Fix(\mathcal{T}) \neq \emptyset$ in C.*

Lemma 2.3. *(DFPT [10]) Let $C \in \Lambda$, and $\mathcal{T} : C \to C$ be a continuous and μ-set contraction operator, that is, $\exists$ a constant $k \in [0, 1)$ with*

$$\mu(\mathcal{T}(\mathcal{X})) \leq k\mu(\mathcal{X})$$

$\forall\ \emptyset \neq \mathcal{X} \subset C$, where μ is the Kuratowski MNC on E. Then $Fix(\mathcal{T}) \neq \emptyset$ in C.

Thereafter some extension of DFPT and its coupled version by using different types of $\mu-$set contractive condition(for example, see [1–21]). Recently Aghajani et al. [3] proved the generalized version of Theorem 2.3 that stated as below:

Theorem 2.4. *Let $C \in \Lambda$, and $\mathcal{T} : C \to C$ be a continuous operator such that*

$$\psi(\mu(\mathcal{T}\mathcal{X})) \leq \psi(\mu(\mathcal{X})) - \phi(\mu(\mathcal{X})) \tag{2.1}$$

$\forall\ \emptyset \neq \mathcal{X} \subset C$, where μ is an arbitrary MNC and $\psi, \phi : \mathbb{R}_+ \to \mathbb{R}_+$ are given function such that ψ is continuous and ϕ is lower semicontinuous on $\mathbb{R}_+$. Moreover $\phi(0) = 0$ and $\psi(t) > 0$ for $t > 0$. Then $Fix(\mathcal{T}) \neq \emptyset$ in C.

In this work, we proposed a generalized $\mu-$set contractive condition in comparison with (2.1) using three control functions and prove some fixed point and coupled fixed point results that generalize and include the Theorem 2.4 and the work discussed in [10] and [8]. Our applications are based on fractional calculus by studying the solvability of fractional Cauchy problem.

3. Fixed point results

Here, we deal with the following set of functions:
$\Psi_1 = \{\psi_1 : [0, \infty) \to [0, \infty) \mid \psi_1$ is lower semicontinuous and nondecreasing$\}$,
$\Psi_2 = \{\psi_2 : [0, \infty) \to [0, \infty) \mid \psi_2$ is upper semicontinuous $\}$,
$\Psi_3 = \{\psi_3 : [0, \infty) \to [0, \infty) \mid \psi_3$ is lower semicontinuous $\}$.

Our first main result is the following :

Theorem 3.1. *Let $C \in \Lambda$ and $\mathcal{T} : C \to C$ be a continuous operator. Assume that $\exists\ \psi_1 \in \Psi_1$, $\psi_2 \in \Psi_2$ and $\psi_3 \in \Psi_3$ such that $\forall\ r > 0$,*

$$\psi_1(r) - \psi_2(r) + \psi_3(r) > 0, \tag{3.1}$$

and $\forall\ \mathcal{X} \neq \emptyset \subseteq C$,

$$\psi_1(\mu(\mathcal{T}(\mathcal{X})) + \varphi(\mu(\mathcal{T}(\mathcal{X})))) \leq \psi_2(\mu(\mathcal{X}) + \varphi(\mu(\mathcal{X}))) - \psi_3(\mu(\mathcal{X}) + \varphi(\mu(\mathcal{X}))), \tag{3.2}$$

where μ is an arbitrary MNC, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous mapping. Then $Fix(\mathcal{T}) \neq \emptyset$ in C.

Proof. Starting with $C_0 = C$ and construct a sequence $\{C_n\}$ as $C_{n+1} = Conv(\mathcal{T}C_n)$, for $n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$. If $\exists$ a natural number $n_0 \in \mathbb{N}^*$ such that $\mu(C_{n_0}) + \varphi(\mu(C_{n_0})) = 0$, i.e., $\mu(C_{n_0}) = 0$, then C_{n_0} is compact $\mathcal{T}(C_{n_0}) \subseteq Conv(\mathcal{T}C_{n_0}) = C_{n_0+1} \subseteq C_{n_0}$. Therefore, Theorem 2.2 indicated that $\mathcal{T}$ admits a fixed point. Consequently, we have

$$0 < \mu(C_n) + \varphi(\mu(C_n)), \forall\ n \geq 1.$$

In virtue of (3.2), we obtain

$$\begin{aligned} & \psi_1(\mu(C_{n+1}) + \varphi(\mu(C_{n+1}))) \\ = \ & \psi_1(\mu(Conv(\mathcal{T}C_n)) + \varphi(\mu(Conv(\mathcal{T}C_n)))) \\ = \ & \psi_1(\mu(\mathcal{T}C_n) + \varphi(\mu(\mathcal{T}C_n))) \\ \leq \ & \psi_2(\mu(C_n) + \varphi(\mu(C_n))) - \psi_3(\mu(C_n) + \varphi(\mu(C_n))). \end{aligned} \tag{3.3}$$

Now, since $C_{n+1} \subset C_n$, on the basis of axiom 2^o of Definition 2.1, the sequence $\{\mu(C_n) + \varphi(\mu(C_n))\}$ is nonincreasing and nonnegative. From this we conclude that $\mu(C_n) + \varphi(\mu(C_n)) \to \varrho$ when $n \to \infty$, where

$\varrho \leq 0$ is a non-negative real number. Using the properties of functions ψ_i ($i = 1, 2, 3$), and (3.3) we get that

$$\begin{aligned}\psi_1(\varrho) &\leq \liminf[\psi_1(\mu(C_{n+1}) + \varphi(\mu(C_{n+1})))]\\ &\leq \limsup[\psi_1(\mu(C_{n+1}) + \varphi(\mu(C_{n+1})))]\\ &\leq \limsup[\psi_2(\mu(C_n) + \varphi(\mu(C_n))) - \psi_3(\mu(C_n) + \varphi(\mu(C_n)))]\\ &\leq \limsup \psi_2(\mu(C_n) + \varphi(\mu(C_n))) - \liminf \psi_3(\mu(C_n) + \varphi(\mu(C_n))))\\ &\leq \psi_2(\varrho) - \psi_3(\varrho).\end{aligned}$$

Thus, we obtain $\psi_1(\varrho) \leq \psi_2(\varrho) - \psi_3(\varrho)$, which is not possible by the condition (3.1) if $\varrho > 0$. Hence, $\varrho = \lim_{n\to\infty}[\mu(C_n) + \varphi(\mu(C_n))] = 0$, that $\lim_{n\to\infty} \mu(C_n) = 0$.

Since $C_n \supseteq C_{n+1}$ and $\mathcal{T}C_n \subseteq C_n$ for all $n = 1, 2, \ldots$, then by (6^0) of Definition 2.1, $\mathcal{X}_\infty = \bigcap_{n=1}^{\infty} \mathcal{X}_n$ is nonempty convex closed set, invariant under $\mathcal{T}$ and belongs to $ker\mu$. Consequently, Theorem 2.2 implies that result in C_∞ but as $C_\infty \subset C$, the result is true in C.

If $\varphi(t) = 0$ in Theorem 3.1, then have following consequence:

Corollary 3.2. *Let $C \in \Lambda$ and $\mathcal{T} : C \to C$ be a continuous operator. Assume that $\exists\ \psi_1 \in \Psi_1$, $\psi_2 \in \Psi_2$ and $\psi_3 \in \Psi_3$ such that for all $r > 0$,*

$$\psi_1(r) - \psi_2(r) + \psi_3(r) > 0,$$

and $\forall\ \mathcal{X} \neq \emptyset \subseteq C$,

$$\psi_1(\mu(\mathcal{T}(\mathcal{X}))) \leq \psi_2(\mu(\mathcal{X})) - \psi_3(\mu(\mathcal{X})),$$

where μ is an arbitrary MNC. Then $Fix(\mathcal{T}) \neq \emptyset$ in C.

Proposition 3.3. *Let $C \in \Lambda$ and $\mathcal{T} : C \to C$ be a continuous operator. Assume that $\exists\ \psi_1 \in \Psi_1$, $\psi_2 \in \Psi_2$ and $\psi_3 \in \Psi_3$ such that for all $r > 0$,*

$$\psi_1(r) - \psi_2(r) + \psi_3(r) > 0, \tag{3.4}$$

and $\forall\ \mathcal{X} \neq \emptyset \subseteq C$,

$$\begin{aligned}&\psi_1(diam(\mathcal{T}(\mathcal{X})) + \varphi(diam(\mathcal{T}(\mathcal{X})))) \\ \leq\ &\psi_2(diam(\mathcal{X}) + \varphi(diam(\mathcal{X}))) - \psi_3(diam(\mathcal{X}) + \varphi(diam(\mathcal{X}))),\end{aligned} \tag{3.5}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous mapping. Then $Fix(\mathcal{T}) \neq \emptyset$ in C.

Proof. In view of (6^0), it is well know that $diam(\cdot)$ is a MNC and thus from Theorem 3.1, we get the existence of a $\mathcal{T}$-invariant nonempty closed convex subset $\mathcal{X}_\infty$ with $diam(\mathcal{X}_\infty) = 0$, consequently $\mathcal{X}_\infty$ is a singleton and therefore, $\mathcal{T}$ has a fixed point in C.
To attain the uniqueness, we assume that $\exists$ two distinct fixed points ζ, ξ in C, then we may define the set $\mathcal{Y} := \{\zeta, \xi\}$. In this case $diam(\mathcal{Y}) = diam(\mathcal{T}(\mathcal{Y})) = \|\xi - \zeta\| > 0$. Using (3.5), we obtain

$$\begin{aligned}&\psi_1(diam(\mathcal{T}(\mathcal{Y})) + \varphi(diam(\mathcal{T}(\mathcal{Y})))) \\ \leq\ &\psi_2(diam(\mathcal{Y}) + \varphi(diam(\mathcal{Y}))) - \psi_3(diam(\mathcal{Y}) + \varphi(diam(\mathcal{Y})))\end{aligned}$$

a contradiction to the condition (3.4) and hence the result.

Now we are in position to derive some classical fixed point result from Proposition 3.3 and Theorem 3.1.

Corollary 3.4. *Let $C \in \Lambda$ and $\mathcal{T} : C \to C$ be a continuous operator. Assume that $\exists\, \psi_1 \in \Psi_1$, $\psi_2 \in \Psi_2$ and $\psi_3 \in \Psi_3$ such that for all $r > 0$,*

$$\psi_1(r) - \psi_2(r) + \psi_3(r) > 0, \tag{3.6}$$

and $\forall\, u, v \subseteq C$,

$$\begin{aligned} & \psi_1(\|\mathcal{T}u - \mathcal{T}v\| + \varphi(\|\mathcal{T}u - \mathcal{T}v\|)) \\ \leq\ & \psi_2(\|u - v\| + \varphi(\|u - v\|)) - \psi_3(\|u - v\| + \varphi(\|u - v\|)), \end{aligned} \tag{3.7}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous mapping. Then $Fix(\mathcal{T}) \neq \emptyset$ in C.

Proof. Let $\mu : \mathfrak{M}_E \to \mathbb{R}^+$ be a set quantity defined by the formula $\mu(C) = diam(C)$, where $diam(C) = \sup\{\|u - v\| : u, v \in C\}$ stands for the diameter of C. It is easily seen that μ is a MNC in a space E in the sense of Definition 2.1. Therefore from (3.7) we have

$$\begin{aligned} & \sup_{u,v \in C} \psi_1(\|\mathcal{T}u - \mathcal{T}v)\| + \varphi(\|\mathcal{T}u - \mathcal{T}v\|)) \\ \leq\ & \psi_1(\sup_{u,v \in C} \|\mathcal{T}u - \mathcal{T}v)\| + \sup_{u,v \in C} \varphi(\|\mathcal{T}u - \mathcal{T}v\|)) \\ \leq\ & \sup_{u,v \in C} \psi_2(\|u - v\| + \varphi(\|u - v\|)) - \sup_{u,v \in C} \psi_3(\|u - v\| + \varphi(\|u - v\|)) \\ \leq\ & \psi_2(\sup_{u,v \in C} \|u - v\| + \varphi(\sup_{u,v \in C} \|u - v\|)) - \psi_3(\sup_{u,v \in C} \|u - v\| + \varphi(\sup_{u,v \in C} \|u - v\|)) \end{aligned}$$

which implies that

$$\begin{aligned} & \psi_1(diam(\mathcal{T}(C)) + \varphi(diam(\mathcal{T}(C)))) \\ \leq\ & \psi_2(diam(C) + \varphi(diam(C))) - \psi_3(diam(C) + \varphi(diam(C))). \end{aligned}$$

Thus following Proposition 3.3, $\mathcal{T}$ has an unique fixed point.

4. Coupled fixed point results

In this section, we prove coupled fixed point result of Theorem 3.1.

Definition 4.1. [15] An element $(u^*, v^*) \in \mathcal{E}^2$ is called a coupled fixed point (CFP) of a mapping $\mathcal{G} : \mathcal{E}^2 \to \mathcal{E}$ if $\mathcal{G}(u^*, v^*) = u^*$ and $\mathcal{G}(v^*, u^*) = v^*$.

Our first coupled fixed point result is the following:

Theorem 4.2. *Let $C \in \Lambda$ and $\mathcal{G} : C \times C \to C$ be continuous operator. Assume that $\exists\, \psi_1 \in \Psi_1$, $\psi_2 \in \Psi_2$ and $\psi_3 \in \Psi_3$ such that for all $r > 0$,*

$$\psi_1(r) - \psi_2(r) + \psi_3(r) > 0,$$

and for $i, j \in \{1,2\}$, $i \neq j$,

$$\begin{aligned} &\psi_1(\mu(\mathcal{G}(\mathcal{X}_i \times \mathcal{X}_j)) + \varphi(\mu(\mathcal{G}(\mathcal{X}_i \times \mathcal{X}_j)))) \\ \leq\ &\psi_2(\max\{\mu(\mathcal{X}_i), \mu(\mathcal{X}_j)\} + \varphi(\max\{\mu(\mathcal{X}_i), \mu(\mathcal{X}_j)\})) \\ &-\psi_3(\max\{\mu(\mathcal{X}_i), \mu(\mathcal{X}_j)\} + \varphi(\max\{\mu(\mathcal{X}_i), \mu(\mathcal{X}_j)\})) \end{aligned} \tag{4.1}$$

$\forall\ \mathcal{X}_1, \mathcal{X}_2 \in C$, *where* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous mapping. Then* $CFix(\mathcal{G}) \neq \emptyset$ *in* $C \times C$.

Proof. Consider the map $\widehat{\mathcal{G}} : C \times C \to C$ given by the formula

$$\widehat{\mathcal{G}}(u, v) = (\mathcal{G}(u, v), \mathcal{G}(v, u)).$$

Since $\mathcal{G}$ is continuous, $\widehat{\mathcal{G}}$ is also continuous. Following [3], we define a new MNC in the space C^2 as $\widehat{\mu}(\mathcal{X}) = \max\{\mu(\mathcal{X}_1), \mu(\mathcal{X}_2)\}$, where $\mathcal{X}_i$, $i = 1, 2$ denote the natural projections of C. Now let $\mathcal{X} \subset C^2$ be a nonempty subset. Now in view of (4.1) and the condition (2^0), we obtain

$$\begin{aligned} &\psi_1(\widehat{\mu}(\hat{\mathcal{G}}(\mathcal{X})) + \varphi(\widehat{\mu}(\hat{\mathcal{G}}(\mathcal{X})))) \\ \leq\ &\psi_1\begin{pmatrix} \widehat{\mu}(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2) \times \mathcal{G}(\mathcal{X}_2 \times \mathcal{X}_1)) \\ +\varphi(\widehat{\mu}(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2) \times \mathcal{G}(\mathcal{X}_2 \times \mathcal{X}_1))) \end{pmatrix} \\ =\ &\psi_1\begin{pmatrix} \max\{\mu(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2)), \mu(\mathcal{G}(\mathcal{X}_2 \times \mathcal{X}_1))\} \\ +\varphi(\max\{\mu(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2)), \mu(\mathcal{G}(\mathcal{X}_2 \times \mathcal{X}_1))\}) \end{pmatrix} \\ =\ &\max\begin{Bmatrix} \psi_1(\mu(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2)) + \varphi(\mu(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2)))), \\ \psi_1(\mu(\mathcal{G}(\mathcal{X}_2 \times \mathcal{X}_1)) + \varphi(\mu(\mathcal{G}(\mathcal{X}_2 \times \mathcal{X}_1)))) \end{Bmatrix} \\ \leq\ &\max\begin{Bmatrix} \psi_2(\max\{\mu(\mathcal{X}_1), \mu(\mathcal{X}_2)\} + \varphi(\max\{\mu(\mathcal{X}_1), \mu(\mathcal{X}_2)\})) \\ -\psi_3(\max\{\mu(\mathcal{X}_1), \mu(\mathcal{X}_2)\} + \varphi(\max\{\mu(\mathcal{X}_1), \mu(\mathcal{X}_2)\})), \\ \psi_2(\max\{\mu(\mathcal{X}_2), \mu(\mathcal{X}_1)\} + \varphi(\max\{\mu(\mathcal{X}_2), \mu(\mathcal{X}_1)\})) \\ -\psi_3(\max\{\mu(\mathcal{X}_2), \mu(\mathcal{X}_1)\} + \varphi(\max\{\mu(\mathcal{X}_2), \mu(\mathcal{X}_1)\})) \end{Bmatrix} \\ =\ &\psi_2(\max\{\mu(\mathcal{X}_1), \mu(\mathcal{X}_2)\} + \varphi(\max\{\mu(\mathcal{X}_1), \mu(\mathcal{X}_2)\})) \\ &-\psi_3(\max\{\mu(\mathcal{X}_1), \mu(\mathcal{X}_2)\} + \varphi(\max\{\mu(\mathcal{X}_1), \mu(\mathcal{X}_2)\})) \\ =\ &\psi_2(\widehat{\mu}(\mathcal{X}) + \varphi(\widehat{\mu}(\mathcal{X}))) - \psi_3(\widehat{\mu}(\mathcal{X}) + \varphi(\widehat{\mu}(\mathcal{X}))), \end{aligned}$$

that is,

$$\psi_2(\widehat{\mu}(\mathcal{X}) + \varphi(\widehat{\mu}(\mathcal{X}))) - \psi_3(\widehat{\mu}(\mathcal{X}) + \varphi(\widehat{\mu}(\mathcal{X}))).$$

Therefore, Theorem 3.1 implies that $\widehat{\mathcal{G}}$ has a fixed point, and consequently $\mathcal{G}$ has a CFP.

The second coupled fixed point result is the following:

Theorem 4.3. *Let* $C \in \Lambda$ *and* $\mathcal{G} : C \times C \to C$ *is continuous operator. Assume that* $\exists\ \psi_1 \in \Psi_1$, $\psi_2 \in \Psi_2$ *and* $\psi_3 \in \Psi_3$ *such that for all* $r > 0$,

$$\psi_1(r) - \psi_2(r) + \psi_3(r) > 0,$$

with ψ_1 *is sub-additive and for* $i, j \in \{1,2\}$, $i \neq j$,

$$\psi_1(\mu(\mathcal{G}(\mathcal{X}_i \times \mathcal{X}_j)) + \varphi(\mu(\mathcal{G}(\mathcal{X}_i \times \mathcal{X}_j))))$$

$$
\begin{aligned}
\leq\ & \frac{1}{2}\psi_2(\mu(\mathcal{X}_i)+\mu(\mathcal{X}_j)+\varphi(\mu(\mathcal{X}_i)+\mu(\mathcal{X}_j))) \\
& -\frac{1}{2}\psi_3(\mu(\mathcal{X}_i)+\mu(\mathcal{X}_j)+\varphi(\mu(\mathcal{X}_i)+\mu(\mathcal{X}_j)))
\end{aligned} \tag{4.2}
$$

$\forall\ \mathcal{X}_1, \mathcal{X}_2 \in C$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and sub-additive mapping. Then $CFix(\mathcal{G}) \neq \emptyset$ in $C \times C$.

Proof. Consider the map $\widehat{\mathcal{G}} : C \times C \to C$ defined by the formula

$$\widehat{\mathcal{G}}(u, v) = (\mathcal{G}(u, v), \mathcal{G}(v, u)).$$

Since $\mathcal{G}$ is continuous, $\widehat{\mathcal{G}}$ is also continuous. Following [3], we define a new MNC in the space C^2 as

$$\widehat{\mu}(\mathcal{X}) = \frac{\mu(\mathcal{X}_1) + \mu(\mathcal{X}_2)}{2}$$

where $\mathcal{X}_i$, $i = 1, 2$ denote the natural projections of C. Now let $\emptyset \neq \mathcal{X}$ and thus to (4.2) and the condition (2^0) of Definition 2.1 we conclude that

$$
\begin{aligned}
& \psi_1(\widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{X})) + \varphi(\widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{X})))) \\
\leq\ & \psi_1(\widehat{\mu}(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2) \times \mathcal{G}(\mathcal{X}_2 \times \mathcal{X}_1)) + \varphi(\widehat{\mu}(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2) \times \mathcal{G}(\mathcal{X}_2 \times \mathcal{X}_1)))) \\
=\ & \psi_1\Big(\frac{\mu(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2))}{2} + \varphi\Big(\frac{\mu(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2))}{2}\Big)\Big) \\
& +\psi_1\Big(\frac{\mu(\mathcal{G}(\mathcal{X}_2 \times \mathcal{X}_1))}{2} + \varphi\Big(\frac{\mu(\mathcal{G}(\mathcal{X}_2 \times \mathcal{X}_1))}{2}\Big)\Big) \\
\leq\ & \psi_2\Big(\frac{\mu(\mathcal{X}_1) + \mu(\mathcal{X}_2)}{2} + \varphi\Big(\frac{\mu(\mathcal{X}_1) + \mu(\mathcal{X}_2)}{2}\Big)\Big) \\
& -\psi_3\Big(\frac{\mu(\mathcal{X}_1) + \mu(\mathcal{X}_2)}{2} + \varphi\Big(\frac{\mu(\mathcal{X}_1) + \mu(\mathcal{X}_2)}{2}\Big)\Big) \\
=\ & \psi_2(\widehat{\mu}(\mathcal{X}) + \varphi(\widehat{\mu}(\mathcal{X}))) - \psi_3(\widehat{\mu}(\mathcal{X}) + \varphi(\widehat{\mu}(\mathcal{X}))),
\end{aligned}
$$

that is,

$$\psi_1(\widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{X})) + \varphi(\widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{X})))) \leq \psi_2(\widehat{\mu}(\mathcal{X}) + \varphi(\widehat{\mu}(\mathcal{X}))) - \psi_3(\widehat{\mu}(\mathcal{X}) + \varphi(\widehat{\mu}(\mathcal{X}))).$$

Thus Theorem 3.1 implies that $\widehat{\mathcal{G}}$ has a fixed point, and hence $\mathcal{G}$ has a CFP.

Remark 4.4. If $\varphi(t) = 0$ in Theorem 4.2 and 4.3, then we get CFP of Corollary 3.2.

5. Applications

This section deals with some practicing of the previous section. Our aim is to illustrate sufficient conditions for the existence of the fractional Cauchy problem taking the type

$$D_t^{\alpha} v(t) = F(t, v(t)), \quad v(0) = v_0; \tag{5.1}$$

where D_t^{α} is the Riemann-Liouville fractional calculus, ${}_0D_t^{\alpha}v(t) = \frac{d}{dt} I^{1-\alpha}v(t)$ corresponding to the fractional integral operator

$$I^{\alpha}(v)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\, ds.$$

In our discussion, we let the function $F(t, v(t))$ has a common factor function $\gamma^{\wp} : [0, T] \to (0, \infty)$; thus it becomes of the type

$$F(t, v(t)) := \gamma^{\wp}(t) f(t, v(t)).$$

Moreover, we assume that F is continuous and Lipschitz with the Lipschitz constant $\ell > 0$. We get the following boundedness result:

Theorem 5.1. *Suppose that* $f \in C([0, T] \times \mathbb{R}^2, \mathbb{R})$ *achieving*

$$\|f\| \le \frac{\Gamma(\alpha)}{(1-\wp)\, T^{\alpha-1}}, \quad \wp \in (0,1),\ t \in [0, T],\ \alpha \in (0, 1].$$

If $v_0 > \frac{1}{\wp}$ *then every solution of* (5.1) *is bounded.*

Proof. Let v be a solution of the form (see [19])

$$v(t) = v_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s, v(s)) ds.$$

Then by definition of the function F yields

$$\begin{aligned} v(t) &= v_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s, v(s)) ds \\ &= v_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \gamma^{\wp}(s) f(s, v(s)) ds \\ &\le v_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \gamma^{\wp}(s) \|f\| ds \\ &\le v_0 + \frac{T^{\alpha-1}\|f\|}{\Gamma(\alpha)} \int_0^t \gamma^{\wp}(s) ds \\ &\le v_0 + \frac{\int_0^t \gamma^{\wp}(s)\, ds}{1-\wp} \\ &= \frac{v_0 - v_0\wp}{1-\wp} + \frac{\int_0^t \gamma^{\wp}(s)\, ds}{1-\wp}. \end{aligned}$$

Since $v_0 > \frac{1}{\wp}$ then we have

$$\|v\| \le \frac{|v_0| + \left(\int_0^T \gamma^{\wp}(s)\, ds - 1 \right)}{1-\wp} := r.$$

This completes the proof.

Note that the term $\frac{\int_0^T \gamma^{\wp}(s)\,ds-1}{1-\wp}$ is called Tsallis fractional Entropy [22]. This entropy has been utilized beside with the Principle of maximum entropy to develop the Tsallis distribution. This entropy has been employed in many fields such as thermodynamics, chaos, statistical mechanics and information theory. Moreover, the value r is called the diameter of the entropy region (B_r). Therefore, any solution in this region is called the entropy solution (see [16, 17]). We proceed to prove the existence of (5.1) as follows:

Theorem 5.2. *Define the operator* $Q : B_r \to B_r$ *as follows:*

$$(Q\nu)(t) = \nu_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s,\nu(s))ds + \int_0^{t-\varepsilon} \frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)} F'(s,\nu(s))ds. \tag{5.2}$$

If

$$\|F\| \le \frac{\Gamma(\alpha+1)[\bar{\gamma} - 1 + \wp|\nu_0|]}{4T^{\alpha}(1-\wp)}, \quad \bar{\gamma} := \int_0^T \gamma^{\wp}(s)\,ds \ge 1,\ \wp \in (0,1),$$

then the equation 5.1 admits at least one solution.

Proof. Our aim is to achieve Corollary 3.4.

Boundedness. By the definition of the operator Q, a computation implies

$$\begin{aligned}(Q\nu)(t) =& \nu_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s,\nu(s))ds + \int_0^{t-\varepsilon} \frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)} F'(s,\nu(s))ds \\ =& \nu_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} F(s,\nu(s))ds + \alpha \int_0^{t-\varepsilon} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha+1)} F(s,\nu(s))ds \\ &+ \frac{\varepsilon^{\alpha}}{\Gamma(\alpha+1)} F(t-\varepsilon) - \frac{t^{\alpha}}{\Gamma(\alpha+1)} F(0,\nu(0)) \\ \le& |\nu_0| + \|F\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds + \alpha\|F\| \int_0^{t-\varepsilon} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha+1)} ds \\ &+ \frac{\varepsilon^{\alpha}}{\Gamma(\alpha+1)} F(t-\varepsilon) - \frac{t^{\alpha}}{\Gamma(\alpha+1)} F(0,\nu(0)) \\ \le& |\nu_0| + \|F\| \Big(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(t-\varepsilon)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\varepsilon^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} \Big).\end{aligned}$$

Taking the maximum norm on $t \in [0,T]$ we have

$$\|Q\nu\| \le |\nu_0| + \frac{4\|F\|T^{\alpha}}{\Gamma(\alpha+1)} \le r.$$

Hence Q is bounded in B_r.

Continuity. Let $\delta > 0$ and $\nu, \upsilon \in B_r$ such that $\|\nu - \upsilon\| \leq \delta$. Then a computation implies

$$\begin{aligned}
|(Q\nu)(t) - (Q\upsilon)(t)| \leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|F(s,\nu(s)) - F(s,\upsilon(s))\| ds \\
& + \int_0^{t-\varepsilon} \frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)} \|F'(s,\nu(s) - F'(s,\nu(s))\| ds \\
& \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|F(s,\nu(s)) - F(s,\upsilon(s))\| ds \\
& + \int_0^{t-\varepsilon} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha+1)} \|F(s,\nu(s) - F(s,\nu(s))\| ds \\
& + \frac{\varepsilon^{\alpha}}{\Gamma(\alpha+1)} \|F(s,\nu(s) - F(s,\nu(s))\|, \quad \nu_0 = \upsilon_0 \\
& \leq \|\nu - \upsilon\| \ell \Big(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(t-\varepsilon)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\varepsilon^{\alpha}}{\Gamma(\alpha+1)} \Big) \\
& \leq \frac{3\delta \ell T^{\alpha}}{\Gamma(\alpha+1)} = \epsilon,
\end{aligned}$$

where

$$\delta := \frac{\Gamma(\alpha+1)\epsilon}{3T^{\alpha}\ell}.$$

Measure of noncompactness. Here, we aim to prove

$$\mu(Q)(B_r) \leq \mu(B_r).$$

For ν and $\upsilon \in B_r$, we have

$$\begin{aligned}
|(Q\nu)(t) - (Q\upsilon)(t)| \leq & \|\nu - \upsilon\| \ell \Big(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{(t-\varepsilon)^{\alpha}}{\Gamma(\alpha+1)} + \frac{\varepsilon^{\alpha}}{\Gamma(\alpha+1)} \Big) \\
& \leq \|\nu - \upsilon\| \frac{3\ell T^{\alpha}}{\Gamma(\alpha+1)}
\end{aligned}$$

then we conclude that

$$diam(Q(B_r)) \leq K_{\alpha}\, diam(B_r),$$

where for sufficient value of $\ell > 0$ we have

$$K_{\alpha} := \frac{3\ell T^{\alpha}}{\Gamma(\alpha+1)} < \frac{1}{2}.$$

Consequently, $diam(Q(B_r)) \leq diam(B_r)$. Now we consider the functions $\psi_1, \psi_2, \psi_3 : (0,\infty) \to (0,\infty)$ as follows:

$$\psi_1(\varsigma) = \varsigma + \frac{1}{2}, \;\; \psi_2(\varsigma) = \varsigma + 1, \;\; \psi_3(\varsigma) = \frac{\varsigma + 1}{2}.$$

It is clear that

$$\psi_1(\varsigma) - \psi_2(\varsigma) + \psi_3(\varsigma) = \frac{\varsigma}{2} > 0$$

and

$$\psi_1(\mu(Q(B_r))) \le \psi_1(K_\alpha \mu(B_r)) = K_\alpha \mu(B_r) + \frac{1}{2}$$
$$< \frac{\mu(B_r)}{2} + \frac{1}{2}$$
$$= \psi_2(\mu(B_r) - \psi_3(\mu(B_r)).$$

Hence, Q admits a fixed point analogous to the solution of (5.1).

Theorem 5.3. *Consider the fractional Cauchy problem* (5.1). *If* $\ell < \frac{\Gamma(\alpha+1)}{6T^\alpha}$, *then the problem has a unique entropy solution* $\nu(t) \in B_r$ *and* $(\nu, \nu) \in (B_r, B_r)$ *is its unique coupled entropy solution.*

Proof. Define the operator $Q : B_r \to B_r$. Then for $\nu, \upsilon \in B_r$, we have

$$\begin{aligned}
|(Q\nu)(t) - (Q\upsilon)(t)| \le & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|F(s, \nu(s)) - F(s, \upsilon(s))\| ds \\
& + \int_0^{t-\varepsilon} \frac{(t-s)^{\alpha}}{\Gamma(\alpha+1)} \|F'(s, \nu(s) - F'(s, \nu(s))\| ds \\
& \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \|F(s, \nu(s)) - F(s, \upsilon(s))\| ds \\
& + \int_0^{t-\varepsilon} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha+1)} \|F(s, \nu(s) - F(s, \nu(s))\| ds \\
& + \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)} \|F(s, \nu(s) - F(s, \nu(s))\|, \quad \nu_0 = \upsilon_0 \\
& \le \|\nu - \upsilon\| \ell \Big(\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(t-\varepsilon)^\alpha}{\Gamma(\alpha+1)} + \frac{\varepsilon^\alpha}{\Gamma(\alpha+1)} \Big) \\
& \le \frac{3\ell T^\alpha}{\Gamma(\alpha+1)} \|\nu - \upsilon\|.
\end{aligned}$$

Hence, in virtue of the Banach fixed point theorem, Q admits a unique fixed point in B_r analogous to the solution of (5.1).
Now, define the operator $\mathfrak{Q} : B_r \times B_r \to B_r$ then we obtain

$$\|\mathfrak{Q}(u, \nu) - \mathfrak{Q}(u', \nu')\| \le 2\ell \max(\|u - u'\|, \|\nu - \nu'\|) \le \frac{6\ell T^\alpha}{\Gamma(\alpha+1)} < \ell.$$

Thus, Eq. (5.1) has a unique couple solution.

5.1. Numerical example

Consider the equation

$$D_t^\alpha \nu(t) = \gamma^{\wp}(t)\nu(t), \quad \nu(0) = \nu_0; \tag{5.3}$$

$$\Big(\wp = 0.5 \in (0, 1),\ \alpha = 0.5 \in (0, 1],\ t \in J = [0, 1],\ \nu \in C([J, 4J])\Big).$$

Thus, $\|v\| \leq \frac{\Gamma(\alpha)}{(1-\wp)T^{\alpha-1}} = 3.54 < 4$. This implies that the solution v is bounded by Tsallis entropy (Theorem 5.1). Moreover, let $\gamma^{\wp}(t) = t^{\wp}$, $\wp = 0.5, v_0 > 1/\wp$, we conclude that the assumption of Theorem 5.2 is achieved. This indicates that (5.3) admits a solution in B_r. Moreover, if we assume that $|\sqrt{\gamma(t)}| \leq \dfrac{0.88}{6} = 0.14$ then in view of Theorem 5.3, Eq. (5.3) has a unique couple solution in (B_r, B_r) for some r.

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Conflict of interest

The authors declare that there is no conflict of interest.

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