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Research article

Existence and uniqueness of solutions for nonlinear fractional differential equations depending on lower-order derivative with non-separated type integral boundary conditions

Djamila Chergui, Taki Eddine Oussaeif* and Merad Ahcene

Department of Mathematics, Oum El Bouaghi University, Algeria

* **Correspondence:** Email: taki_maths@live.fr; Tel: +0671420203.

Abstract: We study a new class of boundary value problems of nonlinear fractional differential equations whose nonlinear term depends on a lower-order derivative with fractional non-separated type integral boundary conditions. Some existence and uniqueness results are obtained by using standard fixed point theorems. Examples are given to illustrate the results.

Keywords: fractional differential equations; fractional non-separated boundary conditions; fixed point theorems; existence
Mathematics Subject Classification: 34A12, 34B15

1. Introduction

Boundary value problems for nonlinear fractional differential equations have recently been investigated by several researchers. The study of fractional equations ranges from the theoretical aspects of existence and uniqueness of solutions to the analytic and numerical methods for finding solutions. A strong motivation for studying fractional differential equations comes from the fact that they have been proved to be valuable tools in the modeling of many phenomena in engineering and sciences such as physics, mechanics, chemistry, economics and biology, etc. [4,7,10,14]

Ahmad and Ntouyas [6] investigated the existence of solutions for a fractional boundary value problem with fractional separated boundary conditions given by

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad t \in [0, T], \quad 1 \le q \le 2, \\ \alpha_1 x(0) + \beta_1 ({}^cD^p x(0)) = \gamma_1, \\ \alpha_2 x(1) + \beta_2 ({}^cD^p x(1)) = \gamma_2, \quad 0 < p < 1. \end{cases}$$

Where ${}^cD^q$ denotes the Caputo fractional derivative of order q, f is continuous function on $[0,T]\times\mathbb{R}$ and $\alpha_i,\beta_i,\gamma_i,(i=1,2)$ are real constants, with $\alpha_i\neq 0$.

Also Xiaoyou Liu and Zhenhai Liu [17] investigated the existence and uniqueness of solutions for the nonlinear fractional boundary value problem with fractional separated boundary conditions given by :

$$\begin{cases} {}^cD^\alpha x(t)=f(t,x(t),{}^cD^\beta x(t)), \quad t\in[0,T], \quad 1\leq q\leq 2, \quad 1<\beta\leq 1\\ a_1x(0)+b_1({}^cD^\gamma x(0))=c_1,\\ a_2x(T)+b_2({}^cD^\gamma x(T))=c_2, \quad 0<\gamma<1. \end{cases}$$

Where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α, f is continuous function on $[0,T]\times\mathbb{R}\times\mathbb{R}$ and $a_i,b_i,c_i,(i=1,2)$ are real constants, with $a_1\neq 0$ and $T>0$.

Bashir Ahmad, Juan J, Nieto and Ahmed Alsaedi [5] investigated the existence and uniqueness of the solutions for a new class of boundary value problems of nonlinear fractional differential equations with non-separated type integral boundary conditions. Precisely, they consider the following problem

$$\begin{cases} {}^cD^q x(t)=f(t,x(t)), \quad t\in[0,T], \quad T>0, \quad 1<q\leq 2,\\ x(0)-\lambda_1x(T)=\mu_1\int_0^T g(s,x(s))ds,\\ x'(0)-\lambda_2x'(T)=\mu_2\int_0^T h(s,x(s))ds. \end{cases}$$

Where ${}^cD^q$ denotes the Caputo fractional derivative of order q, and $f,g,h:[0,T]\times\mathbb{R}\to\mathbb{R}$ are given continuous functions and $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}$ with $\lambda_1\neq 1,\lambda_2\neq 1$.

In this paper, we discuss the existence and uniqueness of solutions for a new class of boundary value problems of nonlinear fractional differential equations depending with non-separated type integral boundary conditions. Precisely, we consider the following problem

$$\begin{cases} {}^cD^q x(t)=f(t,x(t),{}^cD^r x(t)), \quad t\in[0,T], \quad T>0, \quad 1<q\leq 2, \quad 0<r\leq 1\\ x(0)-\lambda_1x(T)=\mu_1\int_0^T g(s,x(s))ds,\\ x'(0)-\lambda_2x'(T)=\mu_2\int_0^T h(s,x(s))ds. \end{cases} \tag{1.1}$$

Where ${}^cD^q$ denotes the Caputo fractional derivative of order q, and $f\in C([0,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R}), g,h:[0,T]\times\mathbb{R}\to\mathbb{R}$ are given continuous functions and $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{R}$ with $\lambda_1\neq 1,\lambda_2\neq 1$.

The rest of the paper is arranged as follows. In Section 2, we establish a basic result that lays the foundation for defining a fixed point problem equivalent to the given problem (1.1). The main results, based on Banach's contraction mapping principal, Schauder fixed point theorem and nonlinear alternative of Leray-Schauder type, are obtained in Section 3. Illustrating examples are discussed in Section 4.

2. Preliminaries

For convenience of the reader, we present here some necessary definitions about fractional calculus theory, which can be found in [1,9,12,13].

Definition 2.1. The Riemann-Liouville fractional integral of order q for a continuous function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \quad q > 0,$$

provided the right-hand side is point-wise defined on $(0, \infty)$, where $\Gamma(.)$ is the gamma function, which is defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

Definition 2.2. For a at least n-times continuously differentiable function $f : (0, \infty) \longrightarrow \mathbb{R}$, the Caputo derivative of order $q > 0$ is defined as

$$^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Lemma 2.3. *Let $\alpha > 0$, then the differential equation*

$$^cD^\alpha h(t) = 0$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$ and

$$I^{\alpha c} D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

here $c_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n-1$ and $n = [\alpha] + 1$.

Theorem 2.4. ***(Schauder fixed Point theorem**)(see [2]) Let U be a closed, convex and nonempty subset of a Banach space X, let $P : U \longrightarrow U$ be a continuous mapping such that $P(U)$ is a relatively compact subset of X. Then P has at least one fixed point in U.*

Theorem 2.5. ***(Nonlineair alternative of Leray-Schauder type**)(see [2]) Let E be a Banach space, C a closed, convex subset of E, U an open subset of C, and $0 \in U$. Suppose that $F : \bar{U} \longrightarrow C$ is a continuous, compact (that is, $F(\bar{U})$ is a relatively compact subset of C) map. Then either*

(i) F has a fixed point in $\bar{U}$, or
(ii) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(x)$.

3. Existence and uniqueness results

Let $I = [0, T]$ and $C(I, \mathbb{R})$ be the space of all continuous real functions defined on I. Define the space $X = \{x(t) : x(t) \in C(I, \mathbb{R}) \quad and \quad {}^cD^r x \in C(I, \mathbb{R})\}, \quad (0 < r \leq 1)$ endowed with the norm $\|x\| = c_1 \max_{t\in I}|x(t)| + c_2 \max_{t\in I}|{}^cD^r x(t)|, \quad c_1, c_2 \in \mathbb{R}^+_*$, we know that $(X, \|.\|)$ is a Banach space.

Now we present the Green's function for boundary value problem of fractional differential equation.

Lemma 3.1. *For a given $y \in C([0, T], \mathbb{R})$, the unique solution of the fractional non separated boundary-value problem*

$$\begin{cases} {}^cD^q x(t) = y(t), \quad t \in [0, T], \quad T > 0, \quad 1 < q \leq 2, \\ x(0) - \lambda_1 x(T) = \mu_1 \displaystyle\int_0^T g(s, x(s))ds, \\ x'(0) - \lambda_2 x'(T) = \mu_2 \displaystyle\int_0^T h(s, x(s))ds, \end{cases} \tag{3.1}$$

is given by :

$$\begin{aligned} x(t) &= \int_0^T G(t, s) f(s, x(s), {}^cD^r x(s))ds + \frac{\mu_2[\lambda_1 T + (1 - \lambda_1)t]}{(\lambda_2 - 1)(\lambda_1 - 1)} \int_0^T h(s, x(s))ds \\ &\quad - \frac{\mu_1}{\lambda_1 - 1} \int_0^T g(s, x(s))ds, \end{aligned}$$

where $G(t, s)$ is the Green's function given by

$$G(t, s) = \begin{cases} \dfrac{(t - s)^{q-1}}{\Gamma(q)} - \dfrac{\lambda_1 (t - s)^{q-1}}{(\lambda_1 - 1)\Gamma(q)} + \dfrac{\lambda_2[\lambda_1 T + (1 - \lambda_1)t](t - s)^{q-1}}{(\lambda_2 - 1)(\lambda_1 - 1)\Gamma(q - 1)} & s \leq t, \\ \\ -\dfrac{\lambda_1 (t - s)^{q-1}}{(\lambda_1 - 1)\Gamma(q)} + \dfrac{\lambda_2[\lambda_1 T + (1 - \lambda_1)t](t - s)^{q-1}}{(\lambda_2 - 1)(\lambda_1 - 1)\Gamma(q - 1)} & t \leq s, \end{cases} \tag{3.2}$$

Proof. We omit the proof as it employs the standard arguments for instance, see [3]. □

In this section, we given some existence results for the problem (1.1) In view of Lemma 3.1 we define an operator $F : X \longrightarrow X$

$$\begin{aligned} (Fx)(t) &= \int_0^t \frac{(t - s)^{q-1}}{\Gamma(q)} f(s, x(s), {}^cD^r x(s))ds \\ &\quad - \xi_1 \lambda_1 \int_0^T \frac{(T - s)^{q-1}}{\Gamma(q)} f(s, x(s), {}^cD^r x(s))ds \\ &\quad + \xi_2 \lambda_2 [\lambda_1 (T - t) + t] \int_0^T \frac{(T - s)^{q-2}}{\Gamma(q - 1)} f(s, x(s), {}^cD^r x(s))ds \\ &\quad + \xi_2 \mu_2 [\lambda_1 (T - t) + t] \int_0^T h(s, x(s))ds - \mu_1 \xi_1 \int_0^T g(s, x(s))ds, \quad t \in [0, T], \end{aligned} \tag{3.3}$$

$$\xi_1 = \frac{1}{\lambda_1 - 1}, \qquad \xi_2 = \frac{1}{(\lambda_2 - 1)(\lambda_1 - 1)}.$$

It is clear that the problem (1.1) has solutions if and only if the operator equation $Fx = x$ has fixed points. For any $x \in X$, let

$$(Nx)(t) = f(t, x(t), {}^cD^r x(t)), \quad t \in [0, T].$$

Since the function f is continuous and

$$({}^cD^r Fx)(t) = (I^{q-r}Nx)(t) - \frac{kt^{1-r}}{\Gamma(2-r)}. \tag{3.4}$$

We know that the operator F maps X into X. Here k is constant given by

$$k = \frac{\lambda_2}{(\lambda_2 - 1)\Gamma(q-1)} \int_0^T (T-s)^{q-2} f(s, x(s), {}^cD^r x(s))ds + \frac{\mu_2}{(\lambda_2 - 1)} \int_0^T h(s, x(s))ds.$$

We put $Fx = F_1x + F_2x$, where

$$(F_1x)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s), {}^cD^r x(s))ds,$$

$$\begin{aligned}(F_2x)(t) \quad = \quad & -\xi_1\lambda_1 \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s, x(s), {}^cD^r x(s))ds \\ & +\xi_2\lambda_2[\lambda_1(T-t) + t] \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} f(s, x(s), {}^cD^r x(s))ds \\ & +\xi_2\mu_2[\lambda_1(T-t) + t] \int_0^T h(s, x(s))ds - \mu_1\xi_1 \int_0^T g(s, x(s))ds.\end{aligned}$$

Observe that problem (1.1) has solution if the operator Eq. (3.3) has fixed points, our first result is based on the Banach fixed point theorem (see [11]).

Theorem 3.2. *We suppose that*

(A_1) *The function* $g, h \in C([0, T] \times \mathbb{R}, \mathbb{R})$, *there exist* $L_1, L_2 > 0$ *and* $0 < L < 1$, *such that*

$$|g(t, x) - g(t, y)| \le L_1|x - y|, \quad |h(t, x) - h(t, y)| \le L_2|x - y|, \quad \textit{for} \quad t \in [0, T], x, y \in \mathbb{R},$$

(A_2) $f \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and there exist constants*

$$0 < c_2 < \frac{\Gamma(2-r)[Lc_1 - c_1|\xi_2\mu_2||1 + \lambda_1|T^2L_2 - c_1|\mu_1\xi_1|TL_1]}{T^{2-r}|\mu_2\xi_2||\lambda_1 - 1|L_2},$$

$$L > |\xi_2\mu_2||1 + \lambda_1|T^2L_2 + |\mu_1\xi_1|TL_1$$

and

$$\theta_1, \theta_2 \geq 0$$

with

$$\theta_1 \leq \frac{M_1'}{N_1'}, \qquad \theta_2 \leq \frac{M_2'}{N_2'},$$

such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq \theta_1|x_1 - x_2| + \theta_2|y_1 - y_2|, \quad \textit{for} \quad t \in [0, T], x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

where :

$$\begin{aligned} M_1' &= \Gamma(q+1)\Gamma(2-r)\Gamma(q-r+1)\{\Gamma(2-r)[Lc_1 - c_1|\xi_2\mu_2||1+\lambda_1|T^2L_2 \\ &\quad -c_1|\mu_1\xi_1|TL_1] - T^{2-r}|\mu_2\xi_2||\lambda_1 - 1|L_2c_2\}, \end{aligned}$$

$$\begin{aligned} N_1' &= \Gamma(2-r)^2\Gamma(q-r+1)c_1T^q + \Gamma(2-r)^2\Gamma(q-r+1)c_1|\xi_1\lambda_1|T^q \\ &\quad +\Gamma(2-r)^2\Gamma(q-r+1)c_1|\xi_2\lambda_2||1+\lambda_1|T^q q + \Gamma(2-r)^2\Gamma(q+1)T^{q-r}c_2 \\ &\quad +\Gamma(2-r)\Gamma(q-r+1)T^{q-r}|\lambda_2\xi_2||\lambda_1 - 1|c_2 q \end{aligned}$$

and

$$M_2' = Lc_2\Gamma(q+1)\Gamma(2-r)\Gamma(q-r+1),$$

$$\begin{aligned} N_2' &= \Gamma(2-r)\Gamma(q-r+1)T^qc_1 + \Gamma(2-r)\Gamma(q-r+1)|\xi_1\lambda_1|T^qc_1 \\ &\quad +\Gamma(2-r)\Gamma(q-r+1)|\xi_2\lambda_2||1+\lambda_1|T^qc_1q \\ &\quad +\Gamma(q+1)\Gamma(2-r)T^{q-r}c_2 + \Gamma(q-r+1)T^{q-r}|\lambda_2\xi_2||\lambda_1 - 1|c_2q. \end{aligned}$$

Then the boundary value problem (1.1) *has a unique solution.*

Proof. Let us set

$\sup_{t\in I}|f(t,0,0)| = M, \quad \sup_{t\in I}|g(t,0)| = M_1, \quad \sup_{t\in I}|h(t,0)| = M_2,$

$B_R = \{x \in X, \quad \|x\| \leq R\}$, where $R \geq \dfrac{\gamma}{1-L}$ with :

$$\begin{aligned} \gamma &= \frac{MT^qc_1}{\Gamma(q+1)} + \frac{|\xi_1\lambda_1|T^qMc_1}{\Gamma(q+1)} + \frac{|\xi_2\lambda_2||1+\lambda_1|T^qMc_1}{\Gamma(q)} + |\xi_2\mu_2||1+\lambda_1|T^2M_2c_1 \\ &\quad +|\mu_1\xi_1|TM_1c_1 + \frac{T^{q-r}Mc_2}{\Gamma(q-r+1)} + \frac{T^{q-r}|\lambda_2\xi_2||\lambda_1-1|Mc_2}{\Gamma(2-r)\Gamma(q)} \end{aligned}$$

$$+\frac{T^{2-r}|\mu_2\xi_2||\lambda_1-1|M_2c_2}{\Gamma(2-r)}.$$

Now we show that $FB_R \subset B_R$, where $F : X \longrightarrow X$ is defined by Eq. (3.3) for $x \in B_R$, we have:

$$\begin{aligned}
|(Fx)(t)| \leq & \Big(\frac{T^q\theta_1}{\Gamma(q+1)} + |\xi_1\lambda_1|\frac{T^q\theta_1}{\Gamma(q+1)} + |\xi_2\lambda_2||1+\lambda_1|\frac{T^q\theta_1}{\Gamma(q)} \\
& + |\xi_2\mu_2||1+\lambda_1|T^2L_2 + |\mu_1\xi_1|TL_1\Big)|x| \\
& + \leq \Big(\frac{T^q\theta_2}{\Gamma(q+1)} + |\xi_1\lambda_1|\frac{T^q\theta_2}{\Gamma(q+1)} + |\xi_2\lambda_2||1+\lambda_1|\frac{T^q\theta_2}{\Gamma(q)}\Big)|{}^cD^rx| \\
& + \frac{T^qM}{\Gamma(q+1)} + |\xi_1\lambda_1|\frac{T^qM}{\Gamma(q+1)} + |\xi_2\lambda_2||1+\lambda_1|\frac{T^qM}{\Gamma(q)} \\
& + |\xi_2\mu_2||1+\lambda_1|T^2M_2 + |\mu_1\xi_1|TM_1.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
|({}^cD^rfx)(t)| \leq & \frac{1}{\Gamma(q-r)}\int_0^t (t-s)^{q-r-1}(\theta_1|x| + \theta_2|{}^cD^rx| + M)ds \\
& + \frac{T^{1-r}}{\Gamma(2-r)}\frac{|\lambda_2|}{|\lambda_2-1|\Gamma(q-1)}\int_0^T (T-s)^{q-2}(\theta_1|x| + \theta_2|{}^cD^rx| + M)ds \\
& + \frac{T^{1-r}}{\Gamma(2-r)}\frac{|\mu_2|}{|\lambda_2-1|}\int_0^T (L_2|x| + M_2)ds \\
& \leq \Big(\frac{T^{q-r}\theta_1}{\Gamma(q-r+1)} + \frac{T^{q-r}|\lambda_2\xi_2||\lambda_1-1|\theta_1}{\Gamma(2-r)\Gamma(q)} + \frac{T^{2-r}|\mu_2\xi_2||\lambda_1-1|L_2}{\Gamma(2-r)}\Big)|x| \\
& + \Big(\frac{T^{q-r}\theta_2}{\Gamma(q-r+1)} + \frac{T^{q-r}|\lambda_2\xi_2||\lambda_1-1|\theta_2}{\Gamma(2-r)\Gamma(q)}\Big)|{}^cD^rx| \\
& + \frac{T^{q-r}M}{\Gamma(q-r+1)} + \frac{T^{q-r}|\lambda_2\xi_2||\lambda_1-1|M}{\Gamma(2-r)\Gamma(q)} + \frac{T^{2-r}|\mu_2\xi_2||\lambda_1-1|M_2}{\Gamma(2-r)}.
\end{aligned}$$

From the above inequalities, we obtain :

$$\begin{aligned}
\|Fx\| \leq & c_1\Big(\Big(\frac{T^q\theta_1}{\Gamma(q+1)} + |\xi_1\lambda_1|\frac{T^q\theta_1}{\Gamma(q+1)} + |\xi_2\lambda_2||1+\lambda_1|\frac{T^q\theta_1}{\Gamma(q)} \\
& + |\xi_2\mu_2||1+\lambda_1|T^2L_2 + |\mu_1\xi_1|TL_1\Big)|x| \\
& + \Big(\frac{T^q\theta_2}{\Gamma(q+1)} + |\xi_1\lambda_1|\frac{T^q\theta_2}{\Gamma(q+1)} + |\xi_2\lambda_2||1+\lambda_1|\frac{T^q\theta_2}{\Gamma(q)}\Big)|{}^cD^rx| \\
& + \frac{T^qM}{\Gamma(q+1)} + |\xi_1\lambda_1|\frac{T^qM}{\Gamma(q+1)} + |\xi_2\lambda_2||1+\lambda_1|\frac{T^qM}{\Gamma(q)} \\
& + |\xi_2\mu_2||1+\lambda_1|T^2M_2 + |\mu_1\xi_1|TM_1\Big) \\
& + c_2\Big(\Big(\frac{T^{q-r}\theta_1}{\Gamma(q-r+1)} + \frac{T^{q-r}|\lambda_2\xi_2||\lambda_1-1|\theta_1}{\Gamma(2-r)\Gamma(q)} + \frac{T^{2-r}|\mu_2\xi_2||\lambda_1-1|L_2}{\Gamma(2-r)}\Big)|x|
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{T^{q-r}\theta_2}{\Gamma(q-r+1)}+\frac{T^{q-r}|\lambda_2\xi_2||\lambda_1-1|\theta_2}{\Gamma(2-r)\Gamma(q)}\Big)|^cD^rx|\\
&+\frac{T^{q-r}M}{\Gamma(q-r+1)}+\frac{T^{q-r}|\lambda_2\xi_2||\lambda_1-1|M}{\Gamma(2-r)\Gamma(q)}+\frac{T^{2-r}|\mu_2\xi_2||\lambda_1-1|M_2}{\Gamma(2-r)}\Big)\\
&\leq\Big(\frac{c_1T^q\theta_1}{\Gamma(q+1)}+\frac{c_1|\xi_1\lambda_1|T^q\theta_1}{\Gamma(q+1)}+c_1|\xi_2\lambda_2||1+\lambda_1|\frac{T^q\theta_1}{\Gamma(q)}+c_1|\xi_2\mu_2||1+\lambda_1|T^2L_2\\
&+c_1|\mu_1\xi_1|TL_1+\frac{T^{q-r}\theta_1c_2}{\Gamma(q-r+1)}+\frac{T^{q-r}|\lambda_2\xi_2||\lambda_1-1|\theta_1c_2}{\Gamma(2-r)\Gamma(q)}\\
&+\frac{T^{2-r}|\mu_2\xi_2||\lambda_1-1|L_2c_2}{\Gamma(2-r)}\Big)|x|\\
&+\Big(\frac{T^q\theta_2c_1}{\Gamma(q+1)}+\frac{|\xi_1\lambda_1|T^q\theta_2c_1}{\Gamma(q+1)}+\frac{|\xi_2\lambda_2||1+\lambda_1|T^q\theta_2c_1}{\Gamma(q)}+\frac{T^{q-r}\theta_2c_2}{\Gamma(q-r+1)}\\
&+\frac{T^{q-r}|\lambda_2\xi_2||\lambda_1-1|\theta_2c_2}{\Gamma(2-r)\Gamma(q)}\Big)|^cD^rx|+\frac{MT^qc_1}{\Gamma(q+1)}+\frac{|\xi_1\lambda_1|T^qMc_1}{\Gamma(q+1)}\\
&+\frac{|\xi_2\lambda_2||1+\lambda_1|T^qMc_1}{\Gamma(q)}+|\xi_2\mu_2||1+\lambda_1|T^2M_2c_1+|\xi_1\mu_1|TM_1c_1\\
&+\frac{T^{q-r}Mc_2}{\Gamma(q-r+1)}+\frac{T^{q-r}|\lambda_2\xi_2||\lambda_1-1|Mc_2}{\Gamma(2-r)\Gamma(q)}+\frac{T^{2-r}|\mu_2\xi_2||\lambda_1-1|M_2c_2}{\Gamma(2-r)}\\
\leq\ & L(c_1|x|+c_2|^cD^rx|)+\gamma\\
\leq\ & LR+\gamma\\
\leq\ & R.
\end{aligned}
$$

Now, for any $x, y \in X$ and for each $t \in [0, T]$, we obtain

$$
|(F_1x)(t)-(F_1y)(t)| \leq \frac{T^q}{\Gamma(q+1)}\Big(\theta_1|x-y|+\theta_2|^cD^rx-^cD^ry|\Big),
$$

$$
\begin{aligned}
|(F_2x)(t)-(F_2y)(t)| \leq\ & \Big(\frac{T^{q-r}\theta_1}{\Gamma(q-r-1)}+\frac{T^{q-r}|\lambda_2\xi_2||\lambda_1-1|\theta_1}{\Gamma(2-r)\Gamma(q)}\\
&+\frac{T^{2-r}|\mu_2\xi_2||\lambda_1-1|L_2}{\Gamma(2-r)}\Big)|x-y|\\
&+\Big(\frac{T^{q-r}\theta_2}{\Gamma(q-r-1)}+\frac{T^{q-r}|\lambda_2\xi_2||\lambda_1-1|\theta_2}{\Gamma(2-r)\Gamma(q)}\Big)|^cD^rx-^cD^ry|.
\end{aligned}
$$

We obtain :

$$
\begin{aligned}
|(Fx)(t)-(Fy)(t)| \leq\ & \Big(\frac{T^q\theta_1}{\Gamma(q+1)}+|\xi_1\lambda_1|\frac{T^q\theta_1}{\Gamma(q+1)}+|\xi_2\lambda_2||1+\lambda_1|\frac{T^q\theta_1}{\Gamma(q)}\\
&+|\xi_2\mu_2||1+\lambda_1|T^2L_2+|\mu_1\xi_1|TL_1\Big)|x-y|\\
&+\Big(\frac{T^q\theta_2}{\Gamma(q+1)}+\frac{|\xi_1\lambda_1|T^q\theta_2}{\Gamma(q+1)}+|\xi_2\lambda_2||1+\lambda_1|\frac{T^q\theta_2}{\Gamma(q)}\Big)|^cD^rx-^cD^ry|.
\end{aligned}
$$

Similary, we have :

$$
\begin{aligned}
|({}^cD^rFx)(t)-({}^cD^rFy)(t)| &= |(I^{q-r}Nx)(t)\\
&\quad -\frac{t^{1-r}}{\Gamma(2-r)}\frac{\lambda_2}{(\lambda_2-1)\Gamma(q-1)}\int_0^T(T-s)^{q-2}N(x)(s)ds\\
&\quad -\frac{t^{1-r}}{\Gamma(2-r)}\frac{\mu_2}{(\lambda_2-1)}\int_0^T h(s,x(s))ds-(I^{q-r}Ny)(t)\\
&\quad +\frac{t^{1-r}}{\Gamma(2-r)}\frac{\lambda_2}{(\lambda_2-1)\Gamma(q-1)}\int_0^T(T-s)^{q-2}N(y)(s)ds\\
&\quad +\frac{t^{1-r}}{\Gamma(2-r)}\frac{\mu_2}{(\lambda_2-1)}\int_0^T h(s,y(s))ds|\\
&\leq \Big(\frac{T^{q-r}\theta_1}{\Gamma(q-r+1)}+\frac{T^{q-r}|\lambda_2\xi_2||\lambda_1-1|\theta_1}{\Gamma(2-r)\Gamma(q)}\\
&\quad +\frac{T^{2-r}|\mu_2\xi_2||\lambda_1-1|L_2}{\Gamma(2-r)}\Big)|x-y|\\
&\quad +\Big(\frac{T^{q-r}\theta_2}{\Gamma(q-r+1)}+\frac{T^{q-r}|\lambda_2\xi_2||\lambda_1-1|\theta_2}{\Gamma(2-r)\Gamma(q)}\Big)|{}^cD^rx-{}^cD^ry|.
\end{aligned}
$$

From the above inequalities, we obtain

$$
\begin{aligned}
\|(Fx)(t)-(Fy)(t)\| &\leq \Big(\frac{T^q\theta_1c_1}{\Gamma(q+1)}+\frac{|\xi_1\lambda_1|T^q\theta_1c_1}{\Gamma(q+1)}+|\xi_2\lambda_2||1+\lambda_1|\frac{T^q\theta_1c_1}{\Gamma(q)}\\
&\quad +|\xi_2\mu_2||1+\lambda_1|T^2L_2c_1+|\mu_1\xi_1|TL_1c_1+\frac{T^{q-r}\theta_1c_2}{\Gamma(q-r+1)}\\
&\quad +\frac{T^{q-r}|\lambda_2\xi_2||\lambda_1-1|\theta_1c_2}{\Gamma(2-r)\Gamma(q)}+\frac{T^{2-r}|\mu_2\xi_2||\lambda_1-1|L_2c_2}{\Gamma(2-r)}\Big)|x-y|\\
&\quad +\Big(\frac{T^q\theta_2c_1}{\Gamma(q+1)}+\frac{|\xi_1\lambda_1|T^q\theta_2c_1}{\Gamma(q+1)}+|\xi_2\lambda_2||1+\lambda_1|\frac{T^q\theta_2c_1}{\Gamma(q)}\\
&\quad +\frac{T^{q-r}\theta_2c_2}{\Gamma(q-r+1)}+T^{q-r}|\lambda_2\xi_2|\frac{|\lambda_1-1|\theta_2c_2}{\Gamma(2-r)\Gamma(q)}\Big)|{}^cD^rx-{}^cD^ry|\\
&\leq L(c_1|x-y|+c_2|{}^cD^rx-{}^cD^rx|)\\
&\leq L\|x-y\|.
\end{aligned}
$$

Which implies that F is a contraction mapping. By means of the Banach contraction mapping principle, F has a unique fixed point which is a unique solution of the boundary value problem (1.1). □

Now, we state a known result due to Schauder which is needed to prove the existence of at least one solution of (1.1).

Theorem 3.3. *Let $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and $g,h:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous functions. Assume that*

$$| f(t,x,y) | \leq m_1(t) + d_1|x|^{\rho_1} + d_1'|y|^{\rho_1'},$$

$$| g(t,x) | \leq m_2(t) + d_2|x|^{\rho_2}, \quad | h(t,x) | \leq m_3(t) + d_3|x|^{\rho_3},$$

for each $t \in [0,T]$ *and* $x, y \in \mathbb{R}$ *with* $m_1 \in L^\infty([0,T],\mathbb{R}^+), m_2, m_3 \in L^1([0,T],\mathbb{R}^+)$ *and* $d_i, d_1' \geq 0, 0 \leq \rho_i, \rho_1' < 1, i = 1,2,3$. *Then problem* (1.1) *has at least one solution on* $[0,T]$.

Proof. Schauder's Fixed point theorem is used to prove that F defined by Eq. (3.3) has a fixed point. The proof will be given in several steps.

Step 1: F maps the bounded sets into the bounded sets in X.
Denote $\|m_1\| = \sup_{t\in[0,T]} | m_1(t) |$, let $B_R = \{x \in X, \quad \|x\| \leq R\}$ and $R > 0$ is a positive number. It is clear that B_R is a closed, bounded and convex subset of the Banach space X. For any $x \in B_R$, we have:

$$|(F_1x)(t)| \leq \frac{\|m_1\|T^q}{\Gamma(q+1)} + \frac{(d_1r^{\rho_1} + d_1'r^{\rho_1'})T^q}{\Gamma(q+1)},$$

$$\begin{aligned}|(F_2x)(t)| \leq & \Big(|\xi_1\lambda_1|\frac{T^q}{\Gamma(q+1)} + |\xi_2\lambda_2||1+\lambda_1|\frac{T^q}{\Gamma(q)}\Big)\|m_1\| + |\xi_2\mu_2||1+\lambda_1|T\|m_3\| \\ & +|\xi_1\mu_1|\|m_2\| + \Big(|\xi_1\lambda_1|\frac{T^q}{\Gamma(q+1)} + |\xi_2\mu_2||1+\lambda_1|\frac{T^q}{\Gamma(q)}\Big)(d_1r^{\rho_1} + d_1'r^{\rho_1'}) \\ & +|\xi_1\mu_1|Td_2r^{\rho_2} + |\xi_2\mu_2||1+\lambda_1|T^2d_3r^{\rho_3}.\end{aligned}$$

So, we have

$$\begin{aligned}|(Fx)(t)| \leq & \Big(\frac{T^q}{\Gamma(q+1)} + \frac{|\xi_1\lambda_1|T^q}{\Gamma(q+1)} + \frac{\xi_2\lambda_2||1+\lambda_1|T^q}{\Gamma(q)}\Big)\|m_1\| \\ & +|\xi_1\mu_1|\|m_2\| + |\xi_2\mu_2||1+\lambda_1|T\|m_3\| \\ & +\Big(\frac{T^q}{\Gamma(q+1)} + \frac{|\xi_1\lambda_1|T^q}{\Gamma(q+1)} + \frac{|\xi_2\lambda_2||1+\lambda_1|T^q}{\Gamma(q)}\Big)(d_1r^{\rho_1} + d_1'r'^{\rho_1}) \\ & +|\xi_1\mu_1|Td_2r^{\rho_2} + |\xi_2\mu_2|\,|1+\lambda_1|\,T^2d_3r^{\rho_3}.\end{aligned}$$

Then from Eq. (3.4), we have

$$|({}^cD^rFx)(t)| \leq \frac{T^{q-r}\|m_1\|}{\Gamma(q-r+1)} + \frac{T^{q-r}}{\Gamma(q-r+1)}(d_1r^{\rho_1} + d_1'r'^{\rho_1'}) + |k|\frac{T^{1-r}}{\Gamma(2-r)},$$

where

$$
\begin{aligned}
|k| \leq & \frac{|\lambda_2\xi_2||\lambda_1-1|T^{q-1}}{\Gamma(q)}\|m_1\|+|\mu_2\xi_2||\lambda_2-1|\|m_3\| \\
& +\frac{|\xi_2\lambda_2||\lambda_1-1|T^{q-1}}{\Gamma(q)}(d_1r^{\rho_1}+d_1'r^{\rho_1'})+|\xi_2\mu_2||\lambda_2-1|Td_3|r|^{\rho_3}.
\end{aligned}
$$

So, we have:

$$
\begin{aligned}
|({}^cD^rFx)(t)| \leq & \Big(\frac{T^{q-r}}{\Gamma(q-r+1)}+\frac{|\lambda_2\xi_2||\lambda_1-1|T^{q-r}}{\Gamma(2-r)\Gamma(q)}\Big)\|m_1\| \\
& +\frac{|\mu_2\xi_2||\lambda_1-1|T^{1-r}}{\Gamma(2-r)}\|m_3\| \\
& +\Big(\frac{T^{q-r}}{\Gamma(q-r+1)}+\frac{|\lambda_2\xi_2||\lambda_1-1|T^{q-r}}{\Gamma(q)\Gamma(2-r)}\Big)(d_1r^{\rho_1}+d_1'r^{\rho_1'}) \\
& +\frac{T^{2-r}}{\Gamma(2-r)}|\mu_2\xi_2||\lambda_1-1|d_3r^{\rho_3}.
\end{aligned}
$$

From above inequalities, we obtain

$$
\begin{aligned}
\|(Fx)(t)\| \leq & \Big(\frac{c_1T^q}{\Gamma(q+1)}+\frac{|\xi_1\lambda_1|c_1T^q}{\Gamma(q+1)}+\frac{\xi_2\lambda_2||1+\lambda_1|c_1T^q}{\Gamma(q)}\Big)\|m_1\| \\
& +c_1|\xi_1\mu_1|\|m_2\|+c_1|\xi_2\mu_2||1+\lambda_1|T\|m_3\| \\
& +\Big(\frac{c_1T^q}{\Gamma(q+1)}+\frac{|\xi_1\lambda_1|c_1T^q}{\Gamma(q+1)}+\frac{|\xi_2\lambda_2||1+\lambda_1|c_1T^q}{\Gamma(q)}\Big)(d_1r^{\rho_1}+d_1'r'^{\rho_1}) \\
& +c_1|\xi_1\mu_1|Td_2r^{\rho_2}+c_1\,|\xi_2\mu_2|\,|1+\lambda_1|\,T^2d_3r^{\rho_3} \\
& +\Big(\frac{c_2T^{q-r}}{\Gamma(q-r+1)}+\frac{|\lambda_2\xi_2||\lambda_1-1|c_2T^{q-r}}{\Gamma(2-r)\Gamma(q)}\Big)\|m_1\| \\
& +\frac{|\mu_2\xi_2||\lambda_1-1|c_2T^{1-r}}{\Gamma(2-r)}\|m_3\| \\
& +\Big(\frac{c_2T^{q-r}}{\Gamma(q-r+1)}+\frac{|\lambda_2\xi_2||\lambda_1-1|c_2T^{q-r}}{\Gamma(q)\Gamma(2-r)}\Big)(d_1r^{\rho_1}+d_1'r^{\rho_1'}) \\
& +\frac{c_2T^{2-r}}{\Gamma(2-r)}|\mu_2\xi_2||\lambda_1-1|d_3r^{\rho_3}.
\end{aligned}
$$

Denote:

$$
\begin{aligned}
L = & \Big(\frac{c_1T^q}{\Gamma(q+1)}+\frac{|\xi_1\lambda_1|c_1T^q}{\Gamma(q+1)}+\frac{\xi_2\lambda_2||1+\lambda_1|c_1T^q}{\Gamma(q)}\Big)\|m_1\| \\
& +c_1|\xi_2\mu_2||1+\lambda_1|T\|m_3\|+c_1|\xi_1\mu_1|\|m_2\| \\
& +\Big(\frac{c_2T^{q-r}}{\Gamma(q-r+1)}+\frac{|\lambda_2\xi_2||\lambda_1-1|c_2T^{q-r}}{\Gamma(2-r)\Gamma(q)}\Big)\|m_1\| \\
& +\frac{|\mu_2\xi_2||\lambda_1-1|c_2T^{1-r}}{\Gamma(2-r)}\|m_3\|,
\end{aligned}
$$

$$M_1 = \Big(\frac{c_1T^q}{\Gamma(q+1)} + \frac{|\xi_1\lambda_1|c_1T^q}{\Gamma(q+1)} + \frac{|\xi_2\lambda_2||1+\lambda_1|c_1T^q}{\Gamma(q)} + \frac{c_2T^{q-r}}{\Gamma(q-r+1)} + \frac{|\lambda_2\xi_2||\lambda_1-1|c_2T^{q-r}}{\Gamma(q)\Gamma(2-r)}\Big)(d_1r^{\rho_1} + d_1'r'^{\rho_1}),$$

$$M_2 = c_1|\xi_1\mu_1|T, \quad M_3 = c_1|\xi_2\mu_2||1+\lambda_1|T^2 + c_2|\mu_2\xi_2||\lambda_1-1|\frac{T^{2-r}}{\Gamma(2-r)}.$$

Now let R be a positive number such that:

$$R \geq \max\Big(5L, (5M_1d_1)^{\frac{1}{1-\rho_1}}, (5M_1d_1')^{\frac{1}{1-\rho_1'}}, (5M_2d_2)^{\frac{1}{1-\rho_2}}, (5M_3d_3)^{\frac{1}{1-\rho_3}}\Big).$$

Then it is obvious that for any $x \in B_R$,

$$\|Fx\| \leq L + M_1(d_1r^{\rho_1} + d_1'r^{\rho_1'}) + M_2d_2r^{\rho_2} + M_3d_3r^{\rho_3} \leq \frac{R}{5} + \frac{R}{5} + \frac{R}{5} + \frac{R}{5} + \frac{R}{5} = R.$$

This implies that $F : B_R \to B_R$.

Step 2: F is continuous.
Suppose that $\{x_n\}_{n=1}^{\infty} \subset X$ and $x_n(t)$ converges to $x(t)$ uniformly on $[0,T]$ as $n \to \infty$; that is, $\lim_{n\to\infty}\|x_n - x\| = 0$.

So we have

$$\lim_{n\to\infty}\|x_n - x\|_\infty = 0 \quad \text{and} \quad \lim_{n\to\infty}\|{}^cD^rx_n - {}^cD^rx\|_\infty = 0,$$

which implies that

$$\lim_{n\to\infty}x_n(t) = x(t) \quad \text{and} \quad \lim_{n\to\infty}{}^cD^rx_n(t) = {}^cD^rx(t), \quad t \in [0,T],$$

therefore

$$\lim_{n\to\infty}f(t, x_n(t), {}^cD^rx_n(t)) = f(t, x(t), {}^cD^rx(t))$$
$$\lim_{n\to\infty}g(t, x_n(t)) = g(t, x(t)), \quad \lim_{n\to\infty}h(t, x_n(t)) = h(t, x(t)), \quad t \in [0,T],$$

which gives

$$\begin{aligned}|(Fx_n)(t) - (Fx)(t)| \leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s, x_n, {}^cD^rx_n) - f(s, x, {}^cD^rx)|ds \\ & + |\xi_1\lambda_1|\int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}|f(s, x_n, {}^cD^rx_n) - f(s, x, {}^cD^rx)|ds \\ & + |\xi_2\lambda_2||1+\lambda_1|T\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}|f(s, x_n, {}^cD^rx_n) - f(s, x, {}^cD^rx)|ds \\ & + |\xi_2\mu_2||1+\lambda_2|T\int_0^T |h(s, x_n) - h(s, x)|ds\end{aligned}$$

$$+|\mu_1\xi_1|\int_0^T |g(s,x_n)-g(s,x)|ds$$

and

$$\begin{aligned}|({}^cD^rFx_n)(t)-({}^cD^rFx)(t)| &\leq \int_0^t \frac{(t-s)^{q-r-1}}{\Gamma(q-r)}\left(|f(s,x_n,{}^cD^rx_n)-f(s,x,{}^cD^rx)|\right)ds\\ &+\frac{T^{1-r}}{\Gamma(2-r)}|\xi_2\lambda_2||\lambda_1-1|\int_0^T\frac{(T-s)^{q-2}}{\Gamma(q-1)}\\ &\times(|f(s,x_n,{}^cD^rx_n)-f(s,x,{}^cD^rx)|)ds\\ &+\frac{T^{1-r}}{\Gamma(2-r)}|\xi_2\mu_2||\lambda_1-1|\int_0^T|h(s,x_n)-h(s,x)|ds.\end{aligned}$$

Finely, we have

$$\|(Fx_n)(t)-(Fx)(t)\| = c_1\|(Fx_n)(t)-(Fx)(t)\|_\infty + c_2\|({}^cD^rFx_n)(t)-({}^cD^rFx)(t)\|_\infty \underset{n\to\infty}{\longrightarrow} 0,$$

which means that F is continuous.

Step 3: $F(B_R)$ is equicontinuous with B_R defined as in **Step 2**.
Since f is continuous, we can assume, without any loss of generality, that $|f(t,x(t),{}^cD^rx(t))| \leq N_1$ and $|h(t,x(t))| \leq N_2$ for any $x \in B_R$ and $t \in [0,T]$.

Now let, $0 \leq t_1 \leq t_2 \leq T$. Then we have

$$\begin{aligned}|(F_1x)(t_2)-(F_1x)(t_1)| &= \Big|\int_0^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}f(s,x(s),{}^cD^rx(s))ds\\ &-\int_0^{t_1}\frac{(t_1-s)^{q-1}}{\Gamma(q)}f(s,x(s),{}^cD^rx(s))ds\Big|\\ &\leq \int_0^{t_1}\frac{|(t_2-s)^{q-1}-(t_1-s)^{q-1}|}{\Gamma(q)}|f(s,x(s),{}^cD^rx(s))|ds\\ &+\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}|f(s,x(s),{}^cD^rx(s))|ds\\ &\leq \frac{|t_2^q-(t_2-t_1)^q-t_1^q|}{\Gamma(q+1)}N_1+\frac{(t_2-t_1)^q}{\Gamma(q+1)}N_1\\ &\leq \frac{2N_1(t_2-t_1)^q}{\Gamma(q+1)}+\frac{N_1|t_2^q-t_1^q|}{\Gamma(q+1)},\end{aligned}$$

$$|(F_2x)(t_2)-(F_2x)(t_1)| \leq \Big(|\xi_2\lambda_2||1-\lambda_1|\frac{T^{q-1}}{\Gamma(q)}N_1+|\xi_2\mu_2||1-\lambda_2|TN_2\Big)|t_2-t_1|.$$

So, we have:

$$
\begin{aligned}
|(Fx)(t_2)-(Fx)(t_1)| \;\le\; & \frac{2N_1(t_2-t_1)^q}{\Gamma(q+1)}+\frac{N_1|t_2^q-t_1^q|}{\Gamma(q+1)}+\Big(|\xi_2\lambda_2||1-\lambda_1|\frac{T^{q-1}}{\Gamma(q)}N_1\\
&+|\xi_2\mu_2||1-\lambda_2|TN_2\Big)|t_2-t_1|,
\end{aligned}
$$

we find that

$$
\begin{aligned}
\Big|({}^cD^rFx)(t_2)-({}^cD^rFx)(t_1)\Big| \;=\; & \Big|\frac{1}{\Gamma(q-r)}\int_0^{t_1}((t_2-s)^{q-r-1}\\
&-(t_1-s)^{q-r-1})f(s,x(s),{}^cD^rx(s))ds\\
&+\frac{1}{\Gamma(q-r)}\int_{t_1}^{t_2}(t_2-s)^{q-r-1}f(s,x(s),{}^cD^rx(s))ds\\
&-\Big(\frac{\lambda_2}{(\lambda_2-1)\Gamma(q-1)}\int_0^T(T-s)^{q-2}f(s,x(s),{}^cD^rx(s))ds\\
&+\frac{\mu_2}{(\lambda_2-1)}\int_0^T h(s,x(s))ds\Big)\frac{(t_2^{1-r}-t_1^{1-r})}{\Gamma(2-r)}\Big|\\
\le\; & \frac{N_1|t_2^{q-r}-t_1^{q-r}|}{\Gamma(q-r+1)}+\frac{2N_1(t_2-t_1)^{q-r}}{\Gamma(q-r+1)}\\
&+\Big(|\lambda_2\xi_2||\lambda_1-1|\frac{T^{q-1}}{\Gamma(q)}N_1\\
&+|\mu_1\xi_2||\lambda_1-1|TN_2\Big)\frac{|t_2^{1-r}-t_1^{1-r}|}{\Gamma(2-r)}.
\end{aligned}
$$

Hence we have (since $q>1, q-r>0$ and $1-r\ge 0$)

$$\|(Fx)(t_2)-(Fx)(t_1)\|\longrightarrow 0 \quad as \quad t_2\longrightarrow t_1$$

and the limit is independent of $x\in B_R$. As a consequence of step 1 to 3 together with the Arzela-Ascoli theorem implies that $F(B_R)$ is relatively compact in X. From Theorem 2.4 the problem (1.1) has at least one solution and the proof is completed. □

Now, we prove the existence of solution of (1.1) by applying Alternative of Leray-Schauder fixed point theorem.

Theorem 3.4. *Let $f:[0,T]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ is continuous function and that*

(H_1) *There exists positive functions* $a_i(t), b_i(t), d_i(t)\in C([0,T],\mathbb{R})$ *such that*

$$|f(t,x,y)|\le a_1(t)+a_2(t)|x|+a_3(t)|y|,$$
$$|g(t,x)|\le b_1(t)+b_2(t)|x|,\quad |h(t,x)|\le d_1(t)+d_2(t)|x|,\quad \forall t\in[0,T].$$

(H_2) *Suppose that A and ρ positive constants such that,* $0<A<\infty$ *and* $0<\rho<1$.

Then the problem (1.1) *has at least one solution.*

Proof. It is trivially that $F : X \longrightarrow X$.
We have shown in Theorem 3.3 that F is continuous.

Firstly, Let $\bar{B}$ be a uniformly bounded subset of X and let $R > 0$ be such that $\|x\| \leq R$ for all $x \in \bar{B}$. We prove that $F : \bar{B} \to \bar{B}$. For any $x \in \bar{B}$, we have

$$\begin{aligned}
|(Fx)(t)| &\leq \Big\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} ds + |\xi_1\lambda_1| \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} ds \\
&\quad + |\xi_2\lambda_2||1+\lambda_1|T \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} ds \Big\} |f(s, x(s), {}^c D^r x(s))| \\
&\quad + |\xi_2\mu_2||1+\lambda_1|T \int_0^T |h(s, x(s))|\, ds + |\mu_1\xi_1| \int_0^T |g(s, x(s))|\, ds \\
&\leq \Big\{ \frac{T^q}{\Gamma(q+1)} + \frac{|\xi_1\lambda_1|T^q}{\Gamma(q+1)} + \frac{|\xi_1\lambda_1||1+\lambda_1|T^q}{\Gamma(q)} \Big\} M_1 + |\xi_2\mu_2||1+\lambda_1|TM_2 + |\mu_1\xi_1|M_3.
\end{aligned}$$

So, we have

$$\begin{aligned}
|({}^c D^r Fx)(t)| &\leq \Big\{ \frac{1}{\Gamma(q-r)} \int_0^t (t-s)^{q-r-1} ds \\
&\quad + \frac{T^{1-r}}{\Gamma(2-r)} |\xi_2\lambda_2||\lambda_1 - 1| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} ds \Big\} |f(s, x(s), {}^c D^r x(s))| \\
&\quad + |\xi_2\mu_2||\lambda_1 - 1| \int_0^T |h(s, x(s))|\, ds \\
&\leq \Big\{ \frac{T^{q-r}}{\Gamma(q-r+1)} + \frac{|\xi_2\lambda_2||\lambda_1 - 1|T^{q-r}}{\Gamma(q)} \Big\} M_1 + \frac{|\mu_2\xi_2||\lambda_1 - 1|T^{1-r}}{\Gamma(2-r)} M_2.
\end{aligned}$$

Finely, we have

$$\begin{aligned}
|(Fx)(t)| &\leq \Big\{ \frac{c_1 T^q}{\Gamma(q+1)} + \frac{c_1|\xi_1\lambda_1|T^q}{\Gamma(q+1)} + \frac{c_1|\xi_1\lambda_1||1+\lambda_1|T^q}{\Gamma(q)} + \frac{c_2 T^{q-r}}{\Gamma(q-r+1)} \\
&\quad + \frac{c_2|\xi_2\lambda_2||\lambda_1 - 1|T^{q-r}}{\Gamma(q)} \Big\} M_1 + \Big\{ c_1|\xi_2\mu_2||1+\lambda_1|T + \frac{c_2|\mu_2\xi_2||\lambda_1 - 1|T^{1-r}}{\Gamma(2-r)} \Big\} M_2 \\
&\quad + c_1|\mu_1\xi_1|M_3 \\
&\leq K_1 M_1 + K_2 M_2 + K_3 M_3.
\end{aligned}$$

Where:

$$M_1 = \max_{(s,z_1,z_2)\in[0,T]\times\mathbb{R}^2} |f(s, z_1, z_2)|, M_2 = \max_{(s,z_1)\in[0,T]\times\mathbb{R}} \int_0^T |h(s, z_1)|,$$

$$M_3 = \max_{(s,z_1)\in[0,T]\times\mathbb{R}} \int_0^T |g(s, z_1)|$$

.

$$K_1 = \frac{c_1 T^q}{\Gamma(q+1)} + \frac{c_1|\xi_1\lambda_1|T^q}{\Gamma(q+1)} + \frac{c_1|\xi_1\lambda_1||1+\lambda_1|T^q}{\Gamma(q)} + \frac{c_2 T^{q-r}}{\Gamma(q-r+1)} + \frac{c_2|\xi_2\lambda_2||\lambda_1-1|T^{q-r}}{\Gamma(q)}$$

$$K_2 = c_1|\xi_2\mu_2||1+\lambda_1|T + \frac{c_2|\mu_2\xi_2||\lambda_1-1|T^{1-r}}{\Gamma(2-r)}, \quad K_3 = c_1|\mu_1\xi_1|.$$

Hence Fu is uniformly bounded.

Secondly, we prove the compactness of the operator F, we define $|f(t,x(t),^c D^r x(t))| \le N_1, |h(t,x(t))| \le N_2$. For any $t_1, t_2 \in [0,T]$ are such that $t_1 \le t_2$, we have the following facts:

$$\begin{aligned} |(Fx)(t_2)-(Fx)(t_1)| \le & \frac{2(t_2-t_1)^q + |t_2^q - t_1^q|}{\Gamma(q+1)} N_1 \\ & + \Big(|\xi_2\lambda_2||\lambda_1-1|\frac{T^{q-1}}{\Gamma(q)}N_1 + |\xi_2\mu_2||\lambda_1-1|TN_2\Big)|t_2-t_1|. \end{aligned}$$

So, we have

$$\begin{aligned} |(^c D^r Fx)(t_2)-(^c D^r Fx)(t_1)| \le & \frac{2|t_2-t_1|^{q-r} + (t_2^{q-r}-t_1^{q-r})}{\Gamma(q-r+1)} N_1 \\ & + \Big(|\xi_2\lambda_2||\lambda_1-1|\frac{T^{q-1}}{\Gamma(q)}N_1 \\ & + |\xi_2\mu_2||\lambda_1-1|TN_2\Big)\frac{\left|t_2^{1-r}-t_1^{1-r}\right|}{\Gamma(2-r)}. \end{aligned}$$

Hence

$$\begin{aligned} \|(Fx)(t_2)-(Fx)(t_1)\| \le & \frac{2c_1(t_2-t_1)^q + c_1|t_2^q - t_1^q|}{\Gamma(q+1)} N_1 \\ & + \Big(|\xi_2\lambda_2||\lambda_1-1|\frac{T^{q-1}}{\Gamma(q)}c_1N_1 + |\xi_2\mu_2||\lambda_1-1|Tc_1N_2\Big)|t_2-t_1| \\ & + \frac{2c_2|t_2-t_1|^{q-r} + c_2(t_2^{q-r}-t_1^{q-r})}{\Gamma(q-r+1)} N_1 + \Big(|\xi_2\lambda_2||\lambda_1-1|\frac{T^{q-1}}{\Gamma(q)}c_2N_1 \\ & + |\xi_2\mu_2||\lambda_1-1|Tc_2N_2\Big)\frac{\left|t_2^{1-r}-t_1^{1-r}\right|}{\Gamma(2-r)} \longrightarrow_{t_2\to t_1} 0 \end{aligned}$$

and the limit is independent of $x \in \bar{B}$. Therefor the operator F is equicontinuous. By the Arzela-Ascoli theorem, the operator F is completely continuous.

Thirdly, the result will follow from the Leray-Schauder nonlinear alternative (Theorem 2.5) once we have proved the boundeness of the set of all solutions to equations $x = \lambda Fx$ for $\lambda \in (0,1)$.

Let $U = \{x \in X : \|x\| < R\}$ where $R = \dfrac{A}{1-\rho}$. Then

$$\begin{aligned}
|(Fx)(t)| \leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} a_1(s)ds + |\xi_1\lambda_1| \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} a_1(s)ds \\
& + |\xi_2\lambda_2[\lambda_1(T-t)+T]| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q)} a_1(s)ds \\
& + |\xi_2\mu_2[\lambda_2(T-t)+T]| \int_0^T d_1(s)ds + |\mu_1\xi_1| \int_0^T b_1(s)ds \\
& + \Big\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} a_2(s)ds + |\xi_1\lambda_1| \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} a_2(s)ds \\
& + |\xi_2\lambda_2[\lambda_1(T-t)+T]| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q)} a_2(s)ds \\
& + |\xi_2\mu_2[\lambda_2(T-t)+T]| \int_0^T d_2(s)|x|ds + |\mu_1\xi_1| \int_0^T b_2(s)ds \Big\} |x| \\
& + \Big\{ \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} a_3(s)|^cD^r x|ds + |\xi_1\lambda_1| \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} a_3(s)ds \\
& + |\xi_2\lambda_2[\lambda_1(T-t)+T]| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q)} a_3(s)ds \Big\} |^cD^r x| \\
\leq & A_1 + A_2|x| + A_3|^cD^r x|.
\end{aligned}$$

Where

$$\begin{aligned}
A_1 = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} a_1(s)ds + |\xi_1\lambda_1| \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} a_1(s)ds \\
& + |\xi_2\lambda_2[\lambda_1(T-t)+T]| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q)} a_1(s)ds \\
& + |\xi_2\mu_2[\lambda_2(T-t)+T]| \int_0^T d_1(s)ds + |\mu_1\xi_1| \int_0^T b_1(s)ds
\end{aligned}$$

$$\begin{aligned}
A_2 = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} a_2(s)ds + |\xi_1\lambda_1| \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} a_2(s)ds \\
& + |\xi_2\lambda_2[\lambda_1(T-t)+T]| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q)} a_2(s)ds \\
& + |\xi_2\mu_2[\lambda_2(T-t)+T]| \int_0^T d_2(s)|x|ds + |\mu_1\xi_1| \int_0^T b_2(s)ds
\end{aligned}$$

$$\begin{aligned}
A_3 = & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} a_3(s)|^cD^r x|ds + |\xi_1\lambda_1| \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} a_3(s)ds \\
& + |\xi_2\lambda_2[\lambda_1(T-t)+T]| \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q)} a_3(s)ds
\end{aligned}$$

By the definition of the Caputo fractional derivative with $0 < r \leq 1$,

$$\begin{aligned}
|^{c}D^{r}(Fx)(t)| \leq\ & \frac{1}{\Gamma(q-r)}\int_0^t (t-s)^{q-r-1}a_1(s)ds \\
&+\frac{T^{1-r}|\lambda_2|}{\Gamma(2-r)\Gamma(q-1)|\lambda_2-1|}\int_0^T (T-s)^{q-2}a_1(s)ds \\
&+\frac{T^{1-r}|\mu_2|}{\Gamma(2-r)|\lambda_2-1|}\int_0^T d_1(s)ds \\
&+\Big\{\frac{1}{\Gamma(q-r)}\int_0^t (t-s)^{q-r-1}a_2(s)ds \\
&+\frac{T^{1-r}|\lambda_2|}{\Gamma(2-r)\Gamma(q-1)|\lambda_2-1|}\int_0^T (T-s)^{q-2}a_2(s)ds \\
&+\frac{T^{1-r}|\mu_2|}{\Gamma(2-r)|\lambda_2-1|}\int_0^T d_2(s)ds\Big\}|x| \\
&+\Big\{\frac{1}{\Gamma(q-r)}\int_0^t (t-s)^{q-r-1}a_3(s)ds \\
&+\frac{T^{1-r}|\lambda_2|}{\Gamma(2-r)\Gamma(q-1)|\lambda_2-1|}\int_0^T (T-s)^{q-2}a_3(s)ds\Big\}|^{c}D^{r}x| \\
\leq\ & A'_1 + A'_2|x| + A'_3|^{c}D^{r}x|.
\end{aligned}$$

Where

$$\begin{aligned}
A'_1 =\ & \frac{1}{\Gamma(q-r)}\int_0^t (t-s)^{q-r-1}a_1(s)ds \\
&+\frac{T^{1-r}|\lambda_2|}{\Gamma(2-r)\Gamma(q-1)|\lambda_2-1|}\int_0^T (T-s)^{q-2}a_1(s)ds \\
&+\frac{T^{1-r}|\mu_2|}{\Gamma(2-r)|\lambda_2-1|}\int_0^T d_1(s)ds
\end{aligned}$$

$$\begin{aligned}
A'_2 =\ & \frac{1}{\Gamma(q-r)}\int_0^t (t-s)^{q-r-1}a_2(s)ds \\
&+\frac{T^{1-r}|\lambda_2|}{\Gamma(2-r)\Gamma(q-1)|\lambda_2-1|}\int_0^T (T-s)^{q-2}a_2(s)ds \\
&+\frac{T^{1-r}|\mu_2|}{\Gamma(2-r)|\lambda_2-1|}\int_0^T d_2(s)ds
\end{aligned}$$

$$\begin{aligned}
A'_3 =\ & \frac{1}{\Gamma(q-r)}\int_0^t (t-s)^{q-r-1}a_3(s)ds \\
&+\frac{T^{1-r}|\lambda_2|}{\Gamma(2-r)\Gamma(q-1)|\lambda_2-1|}\int_0^T (T-s)^{q-2}a_3(s)ds
\end{aligned}$$

Therefore, we can obtain that

$$\begin{aligned}
\|Fx(t)\| &\leq c_1(A_1 + A_2|x(t)| + A_3|^cD^r x(t)|) + c_2(A_1' + A_2'|x(t)| + A_3'|^cD^r x(t)|)\\
&= \underbrace{(c_1A_1 + c_2A_1')}_{=A} + (c_1A_2 + c_2A_2')|x(t)| + (c_1A_3 + c_2A_3')|^cD^r x(t)|\\
&\leq A + \rho(c_1 \max_{t\in I} |x(t)| + c_2 \max_{t\in I} |^cD^r x(t)|)\\
&\leq A + \rho\|x\|.
\end{aligned}$$

Suppose there exists a $x \in \partial U$ and a $\lambda \in (0,1)$ such that $x = \lambda Fx$, then for this x and λ we have

$$R = \|x\| = \lambda\|Fx\| < A + \rho\|x\| = R,$$

which is a contradiction. By Theorem 2.5, there exists a fixed point $x \in \bar{U}$ of F. This fixed point is a solution of (1.1) and the proof is complete.

□

4. Examples

Example 4.1. Consider the following boundary value problem :

$$\begin{cases}
{}^cD^{\frac{3}{2}}x(t) = \dfrac{1}{(t+4)^2}\tan^{-1}(x) + \dfrac{|^cD^{\frac{3}{4}}x(t)|}{(t+3)^2(1+|^cD^{\frac{3}{4}}x(t)|)}, \quad t \in [0,1]\\
x(0) + \dfrac{1}{2}x(1) = \dfrac{1}{3}\displaystyle\int_0^1 \dfrac{3|x|}{4+|x|}dx,\\
x'(0) + \dfrac{1}{3}x(1) = \dfrac{3}{2}\displaystyle\int_0^1 \dfrac{2|x|}{3+|x|}dx.
\end{cases} \tag{4.1}$$

Here,

$$q = \frac{3}{2}, r = \frac{3}{4}, \lambda_1 = \frac{-1}{2}, \lambda_2 = \frac{-1}{3}, \mu_1 = \frac{1}{3}, \mu_2 = \frac{3}{2}, c_1 = \frac{1}{2}, c_2 = \frac{1}{3}, \xi_1 = \frac{-2}{3},$$

$$\xi_2 = \frac{1}{2}, T = 1 \quad \text{and} \quad |f(t,x_1,y_1) - f(t,x_2,y_2)| \leq \frac{1}{16}|x_1 - x_2| + \frac{1}{9}|y_1 - y_2|,$$

$$| g(t,x) - g(t,y) | \leq \frac{3}{4}|x-y|, | h(t,x) - h(t,y) | \leq \frac{2}{3}|x-y|, \theta_1 = \frac{1}{16}, \theta_2 = \frac{1}{9},$$

$$L_1 = \frac{3}{4}, L_2 = \frac{2}{3}.$$

Furthermore,

$$\theta_1 \leq 0.14971, \quad \theta_2 \leq 0.54173, \quad 1 > L > 0,6666.$$

Thus, by Theorem 3.2 the boundary value problem (4.1) has a unique solution on $[0,1]$.

Example 4.2. Consider the following boundary value problem :

$$\begin{cases} {}^cD^{\frac{5}{3}}x(t) = (4t^2 - 9t)e^{-x^3(t)} + \frac{1}{4}|x(t)|^{\frac{1}{2}} + \frac{1}{2}\left(\frac{|{}^cD^{\frac{3}{4}}x(t)|}{1+\cos^2 x(t)}\right)^{\frac{1}{3}}, \quad t \in [0,1] \\ x(0) + \frac{1}{2}x(1) = \frac{1}{5}\int_0^1 [(s-1)e^{-x^2(s)} + \frac{1}{3}|x(s)|^{\frac{1}{2}}]ds, \\ x'(0) + \frac{1}{5}x'(1) = \frac{1}{3}\int_0^1 [(s^3-2s)e^{-x^2(s)} + \frac{1}{9}|x(s)|^{\frac{1}{2}}]ds. \end{cases} \tag{4.2}$$

In this case, we have

$$f(t,x,y) = (4t^2-9t)e^{-x^3} + \frac{1}{4}|x|^{\frac{1}{2}} + \frac{1}{2}\left(\frac{|y|}{1+\cos^2 x}\right)^{\frac{1}{3}}$$

and $q = \frac{5}{3}, \quad r = \frac{3}{4}, \quad T = 1, \quad \lambda_1 = \frac{-1}{2}, \quad \lambda_2 = \frac{-1}{5}, \quad \mu_1 = \frac{1}{5}, \quad \mu_2 = \frac{1}{3}$

$$g(t,x) = (t-1)e^{-x^2} + \frac{1}{3}|x(s)|^{\frac{1}{2}}, \quad h(t,x) = (t^3-2t)e^{-x^2} + \frac{1}{9}|x|^{\frac{1}{2}},$$

since

$$|f(t,x,y)| \le |4t^2 - 9t| + \frac{1}{4}|x|^{\frac{1}{2}} + \frac{1}{2}|y|^{\frac{1}{3}},$$

$$|g(t,x)| \le |t-1| + \frac{1}{3}|x|^{\frac{1}{2}}, \quad |h(t,x)| \le |t^3-2t| + \frac{1}{9}|x|^{\frac{1}{2}}.$$

Let

$$d_1 = \frac{1}{4}, \quad d_2 = \frac{1}{2}, \quad \rho_1 = \frac{1}{2}, \quad \rho_2 = \frac{1}{3}, d_3 = \frac{1}{3}, \quad d_4 = \frac{1}{9}, \quad \rho_3 = \rho_4 = \frac{1}{2}$$

and

$$m(t) = |4t^2 - 9t| \in L^\infty(0,1), \quad m_1(t) = |t-1| \in L^1(0,1), \quad m_2(t) = |t^3 - 2t| \in L^1(0,1)$$

Now it is easy to verify that all conditions of Theorem 3.3 are satisfied. Therfore, the fractional boundary value problem (4.2) has at least one solution on [0, 1].

Conflict of interest

The authors declare no conflict of interest.

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