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Research article

Dynamics analysis of stochastic tuberculosis model transmission with immune response

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Abstract: In this paper we extend the tuberculosis epidemic model from a deterministic framework to a deterministic model with immunue response and after to stochastic one. We formulate it as a stochastic differential equation. We, then, etablish the stabilities of different equilibria, and give conditions for extinction and persistence of the desease.

Keywords: nonlinear epidemic model; Lyapounov function; stochastic asymptotic stability; *Itô* 's formula
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1. Introduction

We consider the following model of tuberculosis transmission:

$$\begin{cases} \dot{S} = \Lambda - \beta\frac{SI}{N} - \mu S \\ \dot{E} = \beta(1-p)\frac{SI}{N} + r_2 I - (\mu + k(1-r_1))E \\ \dot{I} = \beta p\frac{SI}{N} + k(1-r_1)E - (\mu + d + \delta + r_2)I \end{cases} \tag{1}$$

where S(t),E(t) and I(t) denote the numbers of susceptible, exposed and infected individuals at time t, respectively ,with the following parameters:
Λ is the recruitment into the population; β, the probability that a susceptible individual will be infected by infectious ; μ is the probability that an individual in the population died from reasons not related to the disease; d is the probability that an infectious individual dies because of the disease.An individual leaves his region to another for a new treatment with the probability δ, thus this individual goes missing of model. New infected individual may develop the disease directly with probability p. To account

for treatment, we define r_1E as the fraction of population receiving effective chemoprophylaxis and r_2 as the rate of effective per capita therapy. We assume that chemoprophylaxis of latently infected individuals E reduces their reactivation rate r_1 and that the initiation of of therapeutics immediately removes individuals from active status I and places them into state E,the time before latently infected individuals who does not received effective chemoprophylaxis become infectious is assumed to satisfy an exponential distribution, with time $\frac{1}{k}$. Thus, individuals leave the class E to I at rate $k(1 - r_1)$. Also, after receiving a therapeutic treatment, individuals leave the class I to E at rate r_2 I.
In [6], we have study dynamical and stochastic models of tuberculosis desease. In this paper we extend the model (1) by introducing, in two times,the effects of immune response and also of environmental fluctuations. This paper is organized as follows.In sections 2 and 3 we introduce tthe dynamical model with immune response and notations and give conditions of stability of these different equilibria. in section 4, we give we study stability of stochastic tuberculosis model.

2. Dynamical model with immune response

The transmission of tuberculosis is mainly by air, but occasionally by the oral or digestive route. This is mainly the case of pulmonary tuberculosis where the individual gets the disease by inhalation of particles (nuclei) that are in the air. Thus, the fact of the presence of these particles in the body triggers a network of immune cells, antibodies and other components of the immune response. The effectors are these organs can activate or inhibit an activity.The immune system of an organism provides an extraordinary defense against foreign attacks.Once it recongnize matter as non-self, it actives multiple chemical and physiological processes to control and eliminate the pathogen.
The immune reaction is represented by the term P representing the immune effectors and is subject to the following constraints:
1-The immune system responds to the presence of parasites by producing more immune effectors,
2-Immune effectors reduce the number of parasites. Thus the model of tuberculosis, defined below, is the SEI model augmented by the part that expresses the immune response. The new model for transmitting TB from one human to another will have two components: the first components are (S,E,I), writting with taking account of cholerae bacillus:

$$\begin{cases} \dot{S} = \Lambda - \beta\dfrac{SI}{N} - \beta_1\dfrac{BS}{K+B} - \mu S \\ \dot{E} = (1-p)\Big(\beta\dfrac{SI}{N} + \beta_1\dfrac{BS}{K+B}\Big) + r_2 I - (\mu + k(1-r_1))E \\ \dot{I} = p\Big(\beta\dfrac{SI}{N} + \beta_1\dfrac{BS}{K+B}\Big) + k(1-r_1)E - (\mu + d + \delta + r_2)I \end{cases} \tag{2}$$

And second components are B and P. Where B is the amount of bacilli of Koch and P the rate of effectors of immunity..
We assume that the bacillus population is suitable for logistic growth with a carrying capacity equal to K. Then,the model on immune response can be writting as follows:

$$\begin{cases} \dot{B} = rB(1 - \frac{B}{K}) - \varepsilon BP \\ \dot{P} = \alpha B - \gamma P \end{cases} \tag{3}$$

where

3. Mathematical analysis

Proposition 3.1. *Let (S(t),E(t),I(t),B(t),P(t)) be the solution of system* (2)–(3) *with initial conditions (S0),E(0),I(0),B(0), P(0)) and the compact set:*

$$\Delta_\theta = \{(S, E, I, B, P) \in \mathbb{R}^5_+, 0 \leq (S + E + I) \leq \frac{\Lambda}{\mu} + \theta; \theta > 0; B \leq K, P \leq \frac{\alpha K}{\gamma}\} \tag{4}$$

Then, under the flow described by system (2)–(3) *,* Δ *is positively set that attracts all solutions of* R^5_+

Proof: Let be $W(t) = (W_1(t), W_2(t))$ with $W_1(t) = S(t) + E(t) + I(t)$ and $W_2(t) = P$
Its time derivative satisfies:

$$\frac{dW(t)}{dt} = (\Lambda - \mu W_1(t) - (d + \delta)I; \alpha B - \gamma P) \tag{5}$$

One has $B \leq K$ which gives following inequalities:

$$\begin{cases} \frac{dW_1(t)}{dt} = (\Lambda - \mu W_1(t) - (d + \delta)I \leq \Lambda - \mu W_1(t) \leq 0 \qquad for\ W_1(t) \geq \frac{\Lambda}{\mu} \\ \frac{dW_2(t)}{dt} \leq \alpha K - \gamma P \qquad for\ W_2(t) \geq \frac{\alpha K}{\gamma} \end{cases} \tag{6}$$

which implies that Δ_θ is positively invariant set.
Solving this differential equation one has:

$$0 \leq (W_1(t), W_2(t)) \leq \{\frac{\Lambda}{\mu} + W_1(0)e^{-\mu t}, \frac{\alpha K}{\gamma} + W_2(0)e^{-\gamma t}\}$$

where W(0) is the initial condition of W(t). Then one can conclude that Δ_θ is an attractive set. $\theta \geq 0$.

3.1. Mathematical Analysis of immune response system

System (3) have an extinction equilibrium $E_0 = (0, 0)$, an immune response free equilibrium $E_1 = (K, 0)$ and an unique infection equilibrium $E_2 = (\frac{rK\gamma}{r\gamma+\alpha K}, \frac{\alpha r K}{r\gamma+\alpha K})$

3.1.1. Stability of extinction equilibrium

The jacobian matrix of immune response model at E_0 is:

$$J = \begin{pmatrix} r & 0 \\ \alpha & -\gamma \end{pmatrix}$$

and has its trace $trace(J) = r - \gamma < 0$ and its determinant $det(J) = -r\gamma > 0$,this means that there is always an eigenvalue which is positive.hence equilibrium E_0 is unstable.

3.1.2. Stability of immune response free equilibrim

System (3) has the following jacobian at immune response free equilibrium:

$$J = \begin{pmatrix} -r & K\varepsilon \\ \alpha & -\gamma \end{pmatrix}$$

We see that $Trace(J) = -r - \gamma < 0$ and $det(J) = r\gamma - K\alpha\varepsilon > 0$ for $\frac{K\alpha\varepsilon}{r\gamma} < 1$.In this case, by Routh-Hurwitz all eingenvalues are negative or have negative real parts. We can deduce that:

$$\hat{\mathcal{R}}_0 = \frac{K\alpha\varepsilon}{r\gamma}$$

and when $\hat{\mathcal{R}}_0 \leq 1$, this equilibrium is locally stable.

3.1.3. Stability of infection equilibrium

The jacobian matrix of immune response model (3) at E_2 is:

$$J = \begin{pmatrix} r - \frac{r(2\gamma r+\varepsilon\alpha K)}{¢r\gamma+\alpha K} & -\frac{r\varepsilon\gamma K}{¢r\gamma+\alpha K} \\ \alpha & -\gamma \end{pmatrix}$$

This jacobian has its $Trace(J(E_2) = r - \frac{r(2\gamma r+\varepsilon\alpha K)}{¢r\gamma+\alpha K} - \gamma < 0$ and its determinant $det(J(E_2)) = \frac{r^2\gamma^2-\gamma^2 rK+2\varepsilon\gamma}{r\gamma+\alpha Kr} > 0$. From Routh Hurwitz criterion all eingenvalues are negative or have negative real parts. Hence this equilibrium is always stable, for $\hat{\mathcal{R}}_0 \geq 1$

3.2. Equilibria and basic reproduction number

system (2)–(3) two equilibria points:
the desease free equilibrium $(\frac{\Lambda}{\mu}, 0, 0, K, 0)$ and the endemic equilibrium

$$\begin{aligned} S^* &= \frac{D(1-p) + pQ\Lambda}{\mu(1-p)D + pQ} \\ I^* &= \frac{(1-p)(\Lambda - \mu\bar{S})}{r_2 - (\mu + k(1-r_1))A} \\ E^* &= \frac{p(\mu + d + \delta + 2r_2)\bar{I}}{p\mu + k(1-r_1)} \\ B^* &= \frac{Kr}{r + \varepsilon K} \\ P^* &= \frac{\alpha Kr}{\gamma(r + \varepsilon K)} \end{aligned}$$

3.2.1. Local stability of the desease free equilibrium

The stability of the desease free equilibrium will be investigated using the next generator operator [11] Let be X=(E,I,S), system (2)-(3) an be writting as follows: $\frac{dX}{dt} = \mathcal{F} - \mathcal{V}$, where :

$$\mathcal{F} = \begin{pmatrix} (1-p)\left(\beta\frac{SI}{N} + \beta_1\frac{BS}{K+B}\right) \\ p\left(\beta\frac{SI}{N} + \beta_1\frac{BS}{K+B}\right) \\ 0 \end{pmatrix} \; et \; \mathcal{V} = \begin{pmatrix} -r_2 I + (\mu + k(1-r_1)E \\ -k(1-r_1)E + (\mu + d + \delta + r_2)I \\ -\Lambda + \beta\frac{SI}{N} + \beta_1\frac{BS}{K+B} + \mu S \end{pmatrix}$$

Jacobian matrices $\mathcal{F}$ and $\mathcal{V}$ on X_0 are respectively:

$$\mathcal{DF}(X_0) = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} \; et \; \mathcal{DV}(X_0) = \begin{pmatrix} V & 0 \\ J_1 & J_2 \end{pmatrix}$$

o

$$F = \begin{pmatrix} 0 & 0 \\ \beta(1-p) & \beta p \end{pmatrix} \; et \; V = \begin{pmatrix} \mu + k(1-r_1) & -k(1-r_1) \\ -r_2 & \mu + d + \delta + r_2 \end{pmatrix}$$

$$FV^{-1} \begin{pmatrix} 0 & 0 \\ \dfrac{\beta(1-p)k(1-r_1) + \beta p(\mu + d + \delta + r_2)}{(\mu + d + \delta)(\mu + k(1-r_1)) + \mu r_2} & \dfrac{\beta[\mu p + k(1-r1)]}{(\mu + d + \delta)(\mu + k(1-r_1)) + \mu r_2} \end{pmatrix}$$

is the next generation matrix of system (2)-(3). The radius of FV^{-1} is

$$\rho(FV^{-1}) = \frac{\beta(\mu p + k(1-r1))}{(\mu + d + \delta)(\mu + k(1-r_1)) + \mu r_2}$$

Hence , the basic reproductive number of system (2)-(3) is:

$$\mathcal{R}_0 = \frac{\beta(\mu p + k(1-r1))}{(\mu + d + \delta)(\mu + k(1-r_1)) + \mu r_2}$$

The following result is etablished (from theorem 2 of [11])

Lemma 3.2. [*11*] *The desease free equilibrium of system* (2)-(3) *is locally asymptotically stable whenever* $\mathcal{R}_0 < 1$ *, and unstable if* $\mathcal{R}_0 > 1$

It means that in this case, tuberculosis can be eliminated from community.

3.2.2. Stability of endemic equilibrium

Theorem 3.3. *If* $\mathcal{R}_0 > 1$ *et* $\hat{\mathcal{R}}_0 > 1$ *endemic equilibrium is globally asymptotically stable in* Δ_θ

Preuve : Consider the Lyapounov function V:$\Delta \rightarrow \mathbb{R}$ defined as:

$$V(S,E,I,B,P) = W_1[S - S\,ln\frac{S^*}{S^*}] + W_2[I - I^* ln\frac{I}{I^*}] + W_3[B - B^* ln\frac{B}{B^*}] + W_4[P - P^* ln\frac{P}{P^*}]$$

where W_1,W_2, W_3 and W_4 are positive constants to be choosen latter.
Set: $V_1(S;E,I,B,P) = W_1[S - S^* ln\frac{S}{S^*}] + W_2[I - I^* ln\frac{I}{I^*}]$
$V_2(S;E,I,B,P) = W_3[B - B^* ln\frac{B}{B^*}] + W_4[P - P^* ln\frac{P}{P^*}]$
one has:

$$\frac{dV_1}{dt} = W_1\frac{S-S^*}{S}(\Lambda - \beta\frac{SI}{N} - \beta_1\frac{BS}{K+B} - \mu S) + W_2\frac{I-I^*}{I}(\beta p\frac{SI}{N} + \beta_1\frac{BS}{K+B} + k(1-r_1)E - (\mu + d + \delta + r_2)I)$$

$$= W_1\frac{S-S^*}{S}(\beta\frac{S^*I^*}{N} + \beta_1\frac{B^*S^*}{K+B^*} - \beta_1\frac{BS}{K+B} + \mu S^* - \beta\frac{SI}{N} - \mu S) + W_2\frac{I-I^*}{I}(\beta p\frac{SI}{N} + p\beta_1\frac{BS}{K+B} + k(1-r_1)E$$
$$-(\mu + d + \delta + r_2)I - \beta p\frac{S^*I^*}{N} - k(1-r_1)E^* + (\mu + d + \delta + r_2)I^*) - p\beta_1\frac{B^*S^*}{K+B^*}$$
$$= W_1\frac{S-S^*}{S}[\beta(\frac{S^*I^*}{N} - \frac{SI}{N}) + \mu(S^* - S) + \beta_1(\frac{B^*S^*}{K+B^*} - \frac{BS}{K+B})] + W_2\frac{I-I^*}{I}[\beta p(\frac{SI}{N} - \frac{S^*I^*}{N})$$

$$+k(1-r_1)(E-E^*)-(\mu+d+\delta+r_2)(I-I^*)-p\beta_1(\frac{B^*S^*}{K+B^*}-\frac{BS}{K+B})]$$

$$=W_1\frac{S-S^*}{S}[\beta(\frac{S^*I^*}{N}-\frac{S^*I}{N}+\frac{S^*I}{N}-\frac{SI}{N})+\mu(S^*-S)+\beta_1(\frac{B^*S^*}{K+B^*}-\frac{B^*S}{K+B^*}$$

$$+\frac{B^*S}{K+B^*}-\frac{BS}{K+B})]+W_2\frac{I-I^*}{I}[\beta p(\frac{SI}{N}-\frac{SI^*}{N}+\frac{SI^*}{N}-\frac{S^*I^*}{N})$$

$$+\beta_1 p(-\frac{B^*S^*}{K+B^*}-\frac{B^*S}{K+B^*}+\frac{B^*S}{K+B^*}+\frac{BS}{K+B})]+k(1-r_1)(E-E^*)-(\mu+d+\delta+r_2)(I-I^*)]$$

$$\leq W_1\frac{S-S^*}{S}[\beta[\frac{S^*}{N}(I^*-I)+\frac{I}{N}(S^*-S)]+\mu(S^*-S)+\frac{B^*}{K+B^*}(S^*-S)]$$

$$+W_2\frac{I-I^*}{I}[\beta p[\frac{S}{N}(I-I^*)+\frac{I^*}{N}(S-S^*)]+k(1-r_1)(E-E^*)-(\mu+d+\delta+r_2)(I-I^*)$$

$$+\beta_1 p\frac{B^*}{K+B^*}(S-S^*)]\leq -W_1\beta\frac{I}{N}\frac{(S-S^*)^2}{S}-W_1\beta\frac{S^*}{N}\frac{(S-S^*)}{S}(I-I^*)-W_1(\mu+$$

$$\frac{B^*S^*}{K+B^*})\frac{(S-S^*)^2}{S}]+W_2\beta p\frac{S}{N}\frac{(I-I^*)^2}{I}+W_2\beta p\frac{I^*}{IN}(S-S^*)(I-I^*)+W_2(\frac{1}{I}k(1-r_1)(E-E^*)$$

$$+\beta_1 p\frac{B^*}{K+B^*}(S-S^*))(I-I^*)-W_2\frac{(\mu+d+\delta+r_2)}{I}(I-I^*)^2]$$

$$=-W_1\beta(\frac{I}{N}+\mu)\frac{(S-S^*)^2}{S}-W_2\frac{(\mu+d+\delta+r_2)}{I}(I-I^*)^2+(W_2\beta p\frac{I^*}{IN}$$

$$-W_1\beta\frac{S^*}{SN})(S-S^*)(I-I^*)+\frac{W_2}{I}k(1-r_1)(E-E^*)(I-I^*)$$

which gives the following inequality:

$$\leq -W_1\beta(\frac{I}{N}+\mu)\frac{(S-S^*)^2}{S}-W_1(\mu+\frac{B^*S^*}{K+B^*})\frac{(S-S^*)^2}{S}-W_2\frac{(\mu+d+\delta+r_2)}{I}(I-I^*)^2$$

$$+\frac{\beta}{INS}(W_2-W_1)[S^*(S-S^*)(I-I^*)^2+I^*(S-S^*)^2(I-I^*)]+\frac{W_2}{I}k(1-r_1)(E-E^*)(I-I^*)$$

For $W_1=W_2$ and take W_2 and for the fact that $\frac{W_2}{I}k(1-r_1)$ will be very small we deduce that:

$$\frac{dV_1}{dt}\leq 0$$

In the same way one has:

$$dV_2=-W_3\frac{(B-B^*)^2}{B}(rk(B+B^*)-r-\varepsilon P))-W_3\frac{B-B^*}{B}(\varepsilon B^*(P-P^*)$$

$$-W_4\gamma\frac{(P-P^*)^2}{P}+W_4\alpha\frac{P-P^*}{P}(B-B^*)$$

$$=-W_3\frac{(B-B^*)^2}{B}(rk(B+B^*)-r-\varepsilon p))-W_4\gamma\frac{(P-P^*)^2}{P}$$

$$-(\varepsilon PB^*W_3-\alpha BW_4)\frac{P-P^*}{BP}(B-B^*)\leq 0$$

and: $\frac{dV}{dt}=0$ for $S=S^*$, $I=I^*$, $B=B^*$ and $P=P^*$
Hence by the LaSalle's principe [8], endemic equilibrium is globally asymptotically stable in Δ_θ.

4. Stochastic model

Some authors take stochastic perturbations into account when they investigate the epidemic system [6, 12, 13, 14].We assume that the perturbation is of white noise type, that is $\beta \to \beta + \frac{\sigma_1 dW_1(t)}{dt}$, $\beta_1 \to \beta_1 + \frac{\sigma_2 dW_2}{dt}$, then we get the following stochastic system:

$$\begin{cases} \dot{S} = \Lambda - \beta\frac{SI}{N} - \beta_1\frac{BS}{K+B} - \mu S - \sigma_1 SI\frac{dW_1}{dt} - \sigma_2\frac{BS}{K+B}\frac{dW_2}{dt} \\ \dot{E} = (1-p)\left(\beta\frac{SI}{N} + \beta_1\frac{BS}{K+B}\right) + r_2 I - (\mu + k(1-r_1))E + (1-p)\sigma_1 SI\frac{dW_1}{dt} \\ +(1-p)\sigma_2\frac{BS}{K+B}\frac{dW_2}{dt} \\ \dot{I} = p\left(\beta\frac{SI}{N} + \beta_1\frac{BS}{K+B}\right) + k(1-r_1)E - (\mu + d + \delta + r_2)I + p\sigma_1 SI\frac{dW_1}{dt} + p\sigma_2\frac{BS}{K+B}\frac{dW_2}{dt} \\ \dot{B} = rB(1-\frac{B}{K}) - \varepsilon BP \\ \dot{P} = \alpha B - \gamma P \end{cases} \tag{7}$$

where $w_1(t)$ and $w_2(t)$ are standard one dimensional Brownian motion, $\sigma_i > 0$, i=1..3 are the intensity of the white noise.

Through this paper,unless otherwise specified , we let $(\Omega, F, \{F\}_{i\geq 0}, P)$ be a complete space with filtration $\{F_i\}$ satisfying the usual conditions (i.e. it is right continuous and increasing while F_0 contains all null sets).

The following Itô 's formula will be used in the sequel of this paper.

Lemma 4.1. *[14] Assume that X(t)∈ R^+ is an Itô's process of the form*

$$dx(t) = f(x,t)dt + \phi(x,t)dB(t) \tag{8}$$

where f:$R^n \times [0,+\infty) \to R^n$ and $\phi(x,t) : R^n \times [0,+\infty) \to R^n$ are measurable functions.

Given V (x,t) is a Lyapounov function, we define the operator LV by:

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x,t) + \frac{1}{2}trace[\phi^T V_{xx}(x,t)\phi(x,t)].$$

Where $V_t(x,t) = \frac{dV(x,t)}{¢dt}$; $V_x(x,t) = \frac{dV(x,t)}{dx_1}, \ldots, \frac{dV(x,t)}{dx_n})$, $V_{xx}(x,t) = (\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j})_{n\times n}$

Then the general Itô 's formula is given by:

$$dV(x,t) = LV(x,t) + V_x(x,t)G(x,t)dW(t)$$

For the sequel we need the following definitions :

Definition 4.2. *The solution of system* (7) *is stochastically ultimately bounded a.s. if for any $\epsilon \in (0,1)$, there exists a positive constant $\varrho = \varrho(\epsilon)$ such that for any initial value (S(0), E(0), I(0)) $\in R^3_+$, the solution of system* (7) *has the property :*

$$\limsup_{t\to\infty} P\{|X(t)| \geq \varrho\} < \epsilon \tag{9}$$

Definition 4.3. *The trivial solution x(t)=0 of* (7) *is said to be stable in probability if for all $\varepsilon > 0$,*

$$\lim_{x_0\to 0} \mathbb{P}(\sup_{t\geq 0} |x(t, x_0)| \geq \varepsilon) = 0$$

Definition 4.4. *The trivial solution x(t)=0 of* (7) *is said to be asymptotically stable if it is stable in probability and moreover*

$$\lim_{x_0\to 0}\mathbb{P}(\lim_{t\to\infty} x(t,x_0)=0)=1$$

Definition 4.5. *The trivial solution x(t)=0 of* (7) *is said to be globally asymptotically stable if it is stable in probability and moreover*

$$\mathbb{P}(\lim_{t\to\infty} x(t,x_0)=0)=1$$

Definition 4.6. *The trivial solution x(t)=0 of* (7) *is said to be almost surely exponentially stable if for all* $x_0\in\mathbb{R}^n$*,*

$$\lim_{t\to\infty}\frac{1}{t}\ln|x(t,x_0)|<0 \qquad a.s.$$

Definition 4.7. *The trivial solution x(t)=0 of* (7) *is said to be exponentially p stable if there is a pair of positive constants* C_1 *and* C_2 *such that for all* $x_0\in\mathbb{R}^n$, $\mathbb{E}(|x(t,x_0)|^p\leq C_1|x(t,x_0)|^p e^{-C_2 t}$ *on* $t\geq 0$

4.1. Existence and uniqueness of positive solutions

Lemma 4.8. *For any given* $(S(0),E(0),I(0),B(0),P(0))\in\mathbb{R}^5_+$ *, there is a unique solution* $(S(t),E(t);I(t),B(t),P(t))\in\Delta_\theta$*, on* $t\geq 0$ *and will remain in* R^5_+ *with probability one.*

Proof Since the coefficients of model (7) satisfy the local Lipchitz condition, then there exists a unique local solution on $[0,\tau_\varepsilon)$, where τ_ε is the explosion time. Proposition (3.1) shows us that $0\leq S(t)+E(t)+I(t)\leq\frac{\Lambda}{\mu}$,$B\leq K$ et $P(t)<\frac{\alpha K}{\gamma}$ for $t\in[0,\tau_\varepsilon)$
We, now, want to show that this solution is global, i.e. $\tau_\varepsilon=+\infty$ a.s. Let $n_0>0$ be sufficiently large for for any $(S(0),E(0),I(0),B(0),P(0)))$ remaining in the interval $[\frac{1}{n_0},n_0]$. For each integer $n>n_0$, we define the stopping time:
$\tau_n=\inf\{t\in[0,\tau_\varepsilon);S(t)\notin(\frac{1}{n},n),E(t)\notin(\frac{1}{n},n),I(t)\notin(\frac{1}{n},n),B(t)\notin(\frac{1}{n},n)\,or P(t)\notin(\frac{1}{n},n)\}$
By reduction to absurdity , we suppose that $\tau_\varepsilon=+\infty$ is false, there is a pair of constant $T>0$ and for any $\varepsilon\in(0,1)$ such that $P\{\tau_\infty\leq T\}>\varepsilon$. Consequently , there is an integer $n_1\geq n_0$ such that

$$P\{\tau_n\leq T\}\geq\varepsilon, n\geq n_1 \tag{10}$$

Define C^3 $V:\mathbb{R}^4_+\to\mathbb{R}$, lake this:

$$V(S,I,B,R)=(S-\ln S)+(E-\ln E)+(I-\ln I)+(B-\ln B)+(P-\ln P)$$

for $(S(t),E(t),I(t),B(t),P(t))\in\Delta_\theta$. One has:

$$\begin{aligned}
LV=&(1-\frac{1}{S})(\Lambda-\beta SI-\beta_1 S\frac{B}{K+B}-\mu S-\sigma_1 SIdW_1-\sigma_2 S\frac{B}{K+B}dW_2)\\
&+(1-\frac{1}{E})((1-p)(\beta SI+\beta_e S\frac{B}{K+B})+r_2 I-(\mu+k(1-r_2)E+(1-p)\sigma_1 SIdW_1\\
&+(1-p)\sigma_2 S\frac{B}{K+B}dW_2)+(1-\frac{1}{I})(p(\beta SI+\beta_e S\frac{B}{K+B})+k(1-r_2)E-(\mu+\delta+d+r_2)I\\
&+p\sigma_1 SIdW_1+p\sigma_2 S\frac{B}{K+B}dW_2)+(1-\frac{1}{B})(rB(1-\frac{B}{K}-\varepsilon BP)+(1-\frac{1}{P})(\alpha B-\gamma P)\\
&+\frac{1}{2}(\sigma_1^2 I^2+\sigma_2^2\frac{B^2}{(B+K)^2}+(1-p)^2\sigma_1^2\frac{S^2I^2}{E^2}+(1-p)^2\sigma_2^2 p^2\frac{S^2B^2}{E^2(B+K)^2})+p^2\sigma_1^2 S^2
\end{aligned}$$

$$+\sigma_2^2 p^2 \frac{S^2B^2}{I^2(B+K)^2})$$

$$LV = \Lambda + 3\mu + \delta + \gamma + \beta_e \frac{B}{K+B} + (\beta_h + \xi)I - \mu(S + I + R) - \beta_h S - \beta_e \frac{B}{K+B}\frac{S}{I} - \delta B$$
$$- \frac{1}{B}\xi I + \frac{1}{2}(\sigma_1^2 I^2 + \sigma_2^2 \frac{B^2}{(B+K)^2} + (1-p)^2\sigma_1^2 \frac{S^2I^2}{E^2}$$
$$+ (1-p)^2\sigma_2^2 p^2 \frac{S^2B^2}{E^2(B+K)^2}) + p^2\sigma_1^2 S^2 + \sigma_2^2 p^2 \frac{S^2B^2}{I^2(B+K)^2})$$

from (7) we have:

$$LV \le \Lambda + 3\mu + k(1-r_1) + r_2 + \delta + \gamma + \beta_e + (\beta_h + \xi)\frac{\lambda}{\mu} + \sigma_1^2(\frac{\Lambda}{\mu})^2 + p\sigma_2^2 = C$$

Therefore, we obtain:

$$dV \le Cdt - ((1-p)(\frac{SI}{E} - p)\sigma_1 dW_1(t) - (1-(1-p)\frac{S}{E} - p\frac{S}{I})\sigma_2 \frac{B}{K+B}dW_2(t) \tag{11}$$

By integrating both sides of (18) from 0 to $\tau_k \wedge T$ yields that:

$$\int_0^{t\wedge T} dV(S(t), E(t), I(t), B(t), P(t)) \le \int_0^{t\wedge T} Cdt - (1-p)\int_0^{t\wedge T} \sigma_1(\frac{SI}{E} - p)\sigma_1 dW_1 dW_1(t)$$
$$- \int_0^{t\wedge T}(1-(1-p)\frac{S}{E} - p\frac{S}{I})\sigma_2\frac{B}{K+B}dW_2(t)$$

where $\tau_n \wedge T = min\{\tau_n, T\}$. Whence taking the expectative of the above inequality leads to

$$EV(S(\tau_n \wedge T), E(\tau_n \wedge T), I(\tau_n \wedge T), B(\tau_n \wedge T), P(\tau_n \wedge T))) \le V(S(0), E(0), I(0), B(0), P(0)) + CT \tag{12}$$

Set $\Omega_n = \{\tau_n \le T\}$ for $n > n_1$ by inequality (18),we have $P(\Omega_n) \ge \varepsilon$. Note that every $\omega \in \Omega_n$, there exists at least one of $S(\tau_n, \omega)$, $I(\tau_n, \omega)$,$B(\tau_n, \omega)$ and $P(\tau_n, \omega)$ equals either n or $\frac{1}{n}$, hence

$$V(S(\tau_n, \omega), E(\tau_n, \omega), I(\tau_n \omega)) \ge (n - 1 - \ln n) \wedge (\frac{1}{n} - 1 - \ln\frac{1}{n})$$

as consequence from (22) one has:

$$V(S(0), E(0), I(0), B(0), P(0)) + CT \ge E[1_{\Omega_n(\omega)} V(S\tau_n, \omega), E(\tau_n, \omega), I(\tau_n, \omega), B(\tau_n, \omega),$$
$$P(\tau_n, \omega)] \ge \varepsilon(n - 1 - lnn) \wedge (\frac{1}{n} - 1 - ln\frac{1}{n})$$

where 1_{Ω_n} is indicator function of Ω_n. Let $n \to +\infty$ leads to the following contradiction:

$$+\infty > V(S(0), E(O), I(0), B(0), P(0)) + CT = +\infty \tag{13}$$

So we must have $\tau_\infty = \infty$. Therefore , the solution (S(t),E(t),I(t),B(t),P(t)) of model will not explode at a finite time with probability one. This completes the proof of lemma (4.8).

Theorem 4.9. *The solutions of System* (7) *are stochastically ultimately bounded for any initial value* $(S(0), I(0), B(0), R(0)) \in \Delta_\theta$

Proof From lemma (4.1) we know that the solution $(S(t), E(t), I(t), B(t), P(t))$ will remains in $\mathbb{R}^5_+$ for all $t \geq 0$ with probability 1. defines functions:

$$V_1 = e^t S^\theta; \;\; V_2 = e^t E^\theta \; et \; V_3 = e^t I^\theta \; pour 0 < \theta < 1.$$

By Ito's formula, one has :

$$\begin{aligned} dV_1 &= LV_1 dt + \theta e^t S^\theta(\sigma_1 S I dW_1 + \sigma_2 \tfrac{BS}{K+B} dW_2) \\ dV_2 &= LV_2 dt + (1-p)\theta e^t E^\theta(\sigma_1 S I dW_1 + \sigma_2 \tfrac{BS}{K+B} dW_2) \\ dV_3 &= LV_3 dt + p\theta e^t I^\theta(\sigma_1 S I dW_1 + \sigma_2 \tfrac{BS}{K+B} dW_2) \end{aligned} \tag{14}$$

where

$$\begin{aligned} LV_1 &= e^t S^\theta[1 + \theta(\tfrac{\Delta}{S} - \beta\tfrac{I}{N} - \beta_1 \tfrac{B}{K+B} - \mu) + \tfrac{\theta(\theta-1)}{2}(\sigma_1^2 I^2 + \sigma_2^2(\tfrac{B}{K+B})^2)e^{-t}] \\ LV_2 &= e^t E^\theta[1 + \theta(\beta(1-p)(\tfrac{SI}{NE} + \beta_1 S\tfrac{B}{(K+B)E}) + r_2\tfrac{I}{E} - (\mu + k(1-r_1)) \\ &\quad + \tfrac{\theta(\theta-1)}{2}(\sigma_1^2 \tfrac{S^2 I^2}{E^2} + \sigma_2^2(\tfrac{SB}{(K+B)E})^2)e^{-t}] \\ LV_3 &= e^t I^\theta[1 + \theta(\beta p \tfrac{S}{N} + \beta_1 p S \tfrac{B}{(K+B)I} + k(1-r_1)\tfrac{E}{I} - (\mu + d + \delta + r_2)) \\ &\quad + \tfrac{\theta(\theta-1)}{2}(\sigma_1^2 S^2 + \sigma_2^2(\tfrac{SB}{(K+B)I})^2)e^{-t}] \end{aligned} \tag{15}$$

Thus, there exists C_1, C_2 and C_3 such that:

$$LV_1 < C_1 e^t, LV_2 < C_2 e^t \; et \; LV_3 < C_3 e^t$$

It follows that:

$$\begin{aligned} e^t E(S^\theta(t)) - E(S^\theta(0)) &\leq C_1 e^t \\ e^t E(E^\theta(t)) - E(E^\theta(0)) &\leq C_2 e^t, \;\; et \\ e^t E(I^\theta(t)) - E(I^\theta(0)) &\leq C_3 e^t \end{aligned}$$

We get now:

$$\begin{aligned} &\limsup_{t\to\infty} ES^\theta(t) \leq C_1 < \infty \\ &\limsup_{t\to\infty} E(E)^\theta(t) \leq C_2 < \infty \\ &\limsup_{t\to\infty} EI^\theta(t) \leq C_3 < \infty \end{aligned} \tag{16}$$

for $X(t) = (S(t), E(t), I(t)) \in \mathbb{R}^3_+$, note that

$$\begin{aligned} &|X(t)|^\theta = (S^2(t) + E^2(t) + I^2(t))^{\frac{\theta}{2}} \leq 3^{\frac{\theta}{2}} max\{S^\theta(t), E^\theta(t), I^\theta(t)\} \\ &\leq 3^{\frac{\theta}{2}}(S^\theta(t) + E^\theta(t) + I^\theta(t)) \end{aligned} \tag{17}$$

consequently:

$$\limsup_{t\to\infty} E|X(t)| \leq 3^{\frac{\theta}{2}}(C_1 + C_2 + C_3)$$

as result, there exists a positive δ_1 sutch that

$$\limsup_{t\to\infty} E|\sqrt{X(t)}| < \delta_1 \tag{18}$$

now for $\varepsilon > 0$, let $\delta = \frac{\delta_1^2}{\varepsilon^2}$, by Chebychev'inequality,

$$P\{|X(t)|\} \leq \frac{E|\sqrt{X(t)}|}{\sqrt{\delta}} = \varepsilon \tag{19}$$

wich gives the desired assertion.

5. Moment exponential stability

In this section we study the p^{th} moment exponentially stability of the desease free equilibrium::

Theorem 5.1. *Set* $p \geq 2$*, if* $\mathcal{R}_0 < 1$*, the disease free equilibrium is* p^{th} *moment exponentially stable in* Δ_θ

The proof of this theorem needs the two next results:

Theorem 5.2. *(Afanas'ev et Komanowski, [2]) Suppose that there exists a function V(t,x)* $\in C^{1,2}(R_+, R^n)$*, satisfying the following inequalities:*

$$K_1|x| \leq V(t,x) \leq K_2|x|^p$$

and

$$LV(t,x) \leq -K_3|x|^p, t \geq 0$$

where p, K_1*,*K_2 *and* K_3 *are positive constants.Then the equilibrium of* (7) *is pth moment exponentially stable.When p=2, it is usually said to be exponentially stable in mean square and the disease free equilibrium is globally asymptotically stable.*

Lemma 5.3. *If* $p \geq 2$ *and ε,x,y* > 0*. then*

$$x^{p-1}y \leq \frac{(p-1)\varepsilon}{p}x^p + \frac{1}{p}\varepsilon^{1-p}y^p$$

and

$$x^{p-2}y^2 \leq \frac{(p-2)\varepsilon}{p}x^p + \frac{2}{p}\varepsilon^{(2-p)/2}y^p$$

Proof of theorem (5.1)**:** Set $p \geq 2$ and $(S(0), I(0), B(0), P(0)) \in \Delta_\theta$, from lemma (4.1) , the solution of the system remains in Δ_θ. Let be the following Lyapounov function

$$V = c_1(\frac{\Lambda}{\mu} - S)^p + \frac{1}{p}I^p + c_2B^p$$

One gets by Itô's formula

$$\begin{aligned}
LV &= -c_1p(\frac{\Lambda}{\mu} - S)^{p-1}(\Lambda - \beta_1 S\frac{B}{K+B} - \beta\frac{SI}{N} - \mu S) + \frac{1}{2}c_1p(p-1)(\frac{\Lambda}{\mu} - S)^{p-2}[\sigma_1^2S^2I^2 \\
&+ \sigma_2^2S^2\frac{B^2}{(K+B)^2}] + I^{p-1}(\beta_1pS\frac{B}{K+B} + \beta p\frac{SI}{N} + k(1-r_1)E - (\mu + d + \delta + r_2)I] \\
&+ \frac{1}{2}(p-1)[\sigma_1^2I^pS^2 + \sigma_2^2I^{p-2}S^2\frac{B^2}{(K+B)^2}] + c_2pB^{p-1}(rB(1-\frac{B}{K}) - \varepsilon Bp] \\
&= -c_1p\mu(\frac{\Lambda}{\mu} - S)^p + c_1p(\frac{\Lambda}{\mu} - S)^{p-1}(\beta_1S\frac{B}{K+B} + \beta SI) \\
&+ \frac{1}{2}C_1p(p-1)(\frac{\Lambda}{\mu} - S)^{p-2}[\sigma_1^2S^2I^2 + \sigma_2^2S^2\frac{B^2}{(K+B)^2}] + I^{p-1}[\beta_1pS\frac{B}{K+B} + \beta p\frac{SI}{N} \\
&- (\mu + d + \delta + r_2)I] + \frac{1}{2}(p-1)[\sigma_1^2I^pS^2 + \sigma_2^2I^{p-2}S^2\frac{B^2}{(K+B)^2}] + c_2pB^p(r - r\frac{B^2}{K} - \varepsilon P)
\end{aligned}$$

of (4) gives us $S \leq \frac{\Lambda}{\mu}$ and the fact that $\frac{B}{K+B} < 1$, we obtain:

$$
\begin{aligned}
LV \leq & -c_1 p\mu(\frac{\Lambda}{\mu} - S)^p + c_1 p(\frac{\Lambda}{\mu} - S)^{p-1}(\beta\frac{\Lambda}{\mu} + \beta_1\frac{\Lambda}{\mu}I) + c_1\mu p(\frac{\Lambda}{\mu} - S)^{p-1}S \\
& - \frac{1}{2}c_1 p(p-1)(\frac{\Lambda}{\mu} - S)^{p-2}[\sigma_1^2\frac{\Lambda}{\mu}^2 I^2 + \sigma_2\frac{\Lambda}{\mu}^2] + I^{p-1}(\beta_1 p\frac{\Lambda}{\mu} + \beta p\frac{\Lambda}{\mu}I - (\mu + r + \delta + r_2)I) \\
& + \frac{1}{2}(p-1)[\sigma_1^2\frac{\Lambda}{\mu}^2 I^P + \frac{1}{2}(p-1)\sigma_2^2\frac{\Lambda}{\mu}^2] \\
\leq & -c_1 p\mu(\frac{\Lambda}{\mu} - S)^p + c_1 p\beta\frac{\Lambda}{\mu}(\frac{\Lambda}{\mu} - S)^{p-1} + c_1 p\beta_1\frac{\Lambda}{\mu}(\frac{\Lambda}{\mu} - S)^{p-1}I - \frac{1}{2}c_1 p(p-1)(\frac{\Lambda}{\mu} - S)^{p-2}[\sigma_1^2\frac{\Lambda}{\mu}^2 I^2 \\
& + \sigma_2\frac{\Lambda}{\mu}^2](\beta\frac{\Lambda}{\mu} - (\mu + d + \delta + r_2))I^p + \beta_1\frac{\Lambda}{\mu}I^{p-1} + \frac{1}{2}(p-1)\{\sigma_1\frac{\Lambda}{\mu}^2 I^P + \sigma_2\frac{\Lambda}{\mu}^2\} + c_2 prB^{p-1}I \\
& - c_2 p\delta B^p
\end{aligned}
$$

and by application of lemma (4.1), one gets now :

$$
\begin{aligned}
LV \leq & -c_1(p\mu - (p-1)\varepsilon(\beta\frac{\Lambda}{\mu} + \frac{1}{2}(p-1)(p-2)\sigma^2(\frac{\Lambda}{\mu})^2))(\frac{\Lambda}{\mu} - S)^p + pc_1\beta_1\frac{\Lambda}{\mu}(\frac{\Lambda}{\mu} - S)^{p-1} \\
& - ((\mu + d + \delta + r_2) - (\beta\frac{\Lambda}{\mu}(1 + c_1\varepsilon^{1-p}) + c_1(p-1)\sigma^2\varepsilon^{(2-p)/2} + c_2 r\varepsilon^{1-p} \\
& + \frac{1}{2}(p-1)\sigma^2\frac{\Lambda}{\mu}^2) + c_3(d + \delta + r_2)\varepsilon^{1-p})I^p - (c_2 p\delta - c_2 r\varepsilon)B^p \\
\leq & -c_1(p\mu - (p-1)\varepsilon(\beta\frac{\Lambda}{\mu} + \frac{1}{2}(p-1)(p-2)\sigma_1^2(\frac{\Lambda}{\mu})^2))(\frac{\Lambda}{\mu} - S)^p \\
& - ((\mu + d + \delta + r_2) - (\beta\frac{\Lambda}{\mu}(1 + c_1\varepsilon^{1-p}) + c_1(p-1)\sigma_2^2\varepsilon^{(2-p)/2} + (c_2 r + c_3\gamma)\varepsilon^{1-p} \\
& + \frac{1}{2}(p-1)\sigma^2\frac{\Lambda}{\mu}^2))I^p - c_2(p\delta - r\varepsilon)B^p
\end{aligned}
$$

We choose ε sufficiently small such that the coefficients of $(\frac{\Lambda}{\mu} - S)^p$ and B^p be negative.
We ,also, can choose c_1, c_2 and c_3 positive such that the coefficient of I^p be negative. according to theorem (5.1), the proof is complete.

5.1. Almost sure exponential stability of tuberculosis model with immune response

In this subsection, we investigate stochastic stability of the desease free equilibrium, $E_0 = (\frac{\Lambda}{\mu}, 0, 0, K, 0)$. the following result gives the suffucient condition for almost surely exponential stability.

Theorem 5.4. *If $\mathcal{R}_0 \leq 1$ and $\beta^2 \leq 2\sigma_1^2\mu$ then the disease free equilibrium is almost surely exponential stable in Δ_θ*

Proof: Define

$$V = \ln((\frac{\Lambda}{\mu} - S) + E + I + B + P)$$

Using Itô 's formula:

$$LV = \frac{1}{\frac{\Lambda}{\mu} - S + E + I + B + P}(-dS + dE + dI + dB + dP) + \frac{1}{(\frac{\Lambda}{\mu} - S + E + I + B + P)^2}(dSdS + dEdE + dIdI + dBdB + dPdP) + \frac{1}{(\frac{\Lambda}{\mu} - S + E + I + B + P)^2}(dSdI + dSdB + dSdP + dIdB + dIdP + dBdP)$$

one has:

$$\begin{aligned} LV &= \frac{1}{(\frac{\Lambda}{\mu} - S) + E + I + B + P}(-\Lambda + 2(\beta_1 S\frac{B}{K+B} + \beta SI) + \mu S - \mu(E+I) - (d+\delta)I \\ &\quad + (r+\alpha)B - r\frac{B^2}{K} - \varepsilon BP - \gamma P] - \frac{1}{\frac{\Lambda}{\mu} - S + E + I + B + R)^2}(2\sigma_1^2 S^2 I^2 + \sigma_2^2 S^2 \frac{B^2}{(K+B)^2}) \\ &\leq \frac{1}{\frac{\Lambda}{\mu} - S + E + I + B + P}[-\Lambda + \beta SI + \mu S - \mu(E+I)] - \frac{2\sigma_1^2 S^2 I^2}{(\frac{\Lambda}{\mu} - S + E + I + B + P)^2} \\ &\quad - \frac{1}{\frac{\Lambda}{\mu} - S + E + I + B + P}(2\beta_1 S\frac{B}{K+B} + (r+\alpha)B) \end{aligned}$$

define $U = \frac{2\sigma_1 SI}{\frac{\Lambda}{\mu} - S + E + I + B + P}$,

$$2\beta_1 U - 2\sigma_1^2 U^2 - \mu = -2\sigma^2(U - \frac{\beta_1}{2\sigma_1})^2 + (\beta_1^2 - 2\sigma_1^2\mu)/2\sigma_1^2$$

one has:

$$\begin{aligned} dV &\leq [-2\sigma^2(U - \frac{\beta_1}{2\sigma_1})^2 + (\beta_1^2 - 2\sigma_1^2\mu)/2\sigma_1^2]dt + 2\sigma_1 U dW(t) \\ &\leq (\beta_1^2 - 2\sigma_1^2\mu)/2\sigma_1^2)dt + 2\sigma_1 U dW_1(t) \end{aligned}$$

By integrating from 0 to t, we cheek:

$$\ln(\frac{\Lambda}{\mu} - S + E + I + B + P) \leq \ln(\frac{\Lambda}{\mu} - S(0) + E(0) + I(0) + B(0) + P(0)) + (\beta_1^2 - 2\sigma_1^2\mu)/2\sigma_1^2)t + G(t) \quad (20)$$

with G(t)a martingale defined by :$G(t) = \sigma_1 \int_0^t Z dW_1(t)$, and in vertue of lemma (5.3) the solution of model(7) remains in Δ, it exists a positive constatnt C sutch that

$$< G, G >_t = \sigma_1^2 \int_0^t Z^2 ds \leq Ct$$

finally by th strong law of large numbers for local martingales, we have:

$$\limsup_{t\to+\infty} \ln(\frac{\Lambda}{\mu} - S + E + I + B + P) \leq (\beta_1^2 - 2\sigma_1^2\mu)/2\sigma_1^2) \leq 0 \qquad a.s.$$

5.2. Almost sure convergence

5.2.1. extinction

The following result gives conditions for extinction of tuberculosis desease, it means that the desease dies out with probability 1.:

Theorem 5.5. *If* $\left[(\beta+\beta_1) - (\mu+d+\delta) - \frac{1}{2}\sigma_1^2(\frac{\Lambda}{\mu})^2\right] \leq 0$ *et* $\mathcal{R}_0 < 1$*, then I(t) converge almost surely exponentially to 0.*
Mathematicaly, we have to show that:

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \leq 0$$

Proof: One has

$$\dot{E} + \dot{I} = \left(\beta\frac{SI}{N} + \beta_1\frac{BS}{K+B}\right) - \mu E - (\mu+d+\delta)I + \sigma_1 SIdW_1 + \sigma_2\frac{BS}{K+B}dW_2$$

which gives

$$\dot{I} \leq \left(\beta\frac{SI}{N} + \beta_1\frac{BS}{K+B}\right) - \mu E - (\mu+d+\delta)I + \sigma_1 SIdW_1 + \sigma_2\frac{BS}{K+B}dW_2$$

Let be a Lyapounov function $V(I(t)) = \ln I(t)$, By Itô's calculus:

$$dV(I(t)) = \frac{1}{I}dI(t) - \frac{1}{2I^2}(dI(t))^2$$

and then

$$dV(I(t)) \leq \left(\beta\frac{S}{N} + \beta_1\frac{BS}{I(K+B)}\right) - (\mu+d+\delta) - \frac{1}{2}[\sigma_1^2 S^2 + \sigma_2^2\frac{B^2S^2}{I^2(K+B)^2}] + \sigma_1 SIdW_1 + \sigma_2\frac{BS}{K+B}dW_2$$

$$\leq (\beta+\beta_1) - (\mu+d+\delta) - \frac{1}{2}\sigma_1^2(\frac{\Lambda}{\mu})^2 + \sigma_1 SIdW_1 + \sigma_2\frac{BS}{K+B}dW_2$$

gives the following equation:

$$\ln I(t) = \ln I_0 + (\beta+\beta_1) - (\mu+d+\delta) - \frac{1}{2}\sigma_1^2(\frac{\Lambda}{\mu})^2 dt + \int_0^T \sigma_1 SIdW_1(t) + \int_0^t \sigma_2\frac{BS}{K+B}dW_2$$

$$\ln I(t) \leq \ln I_0 + \left[\beta+\beta_1) - (\mu+d+\delta) - \frac{1}{2}\sigma_1^2(\frac{\Lambda}{\mu})^2\right]t + G(t) \tag{21}$$

wher G(t) is a martingale defined by:

$$G(t) = \int_0^T \sigma_1(\frac{\Lambda}{\mu})^2 dW_1(t) + \int_0^t \sigma_2 dW_2$$

THis calculus implies that:

$$< G, G >_t \leq (\sigma_1^2 (\frac{\Lambda}{\mu})^2 + \sigma_2^2) t.$$

And by the strong law of large numbers for local martingales [12, 14] we have:

$$\limsup_{t\to\infty} \frac{G(t)}{t} = 0 \qquad almost \qquad surely.$$

and then:

$$\limsup_{t\to\infty} \frac{I(t)}{t} \leq \left[\beta + \beta_1) - (\mu + d + \delta) - \frac{1}{2}\sigma_1^2 (\frac{\Lambda}{\mu})^2\right] \leq 0 \qquad a.s. \tag{22}$$

this completes the proof.

6. Persistance

Definition 6.1. *System* (7) *is said to be persistent in the mean, if*

$$\lim_{t\to+\infty} \inf \frac{1}{t} \int_0^t I(s)ds > 0$$

Theorem 6.2. *If* $\frac{\beta\Lambda}{\mu(\mu+d+\delta+r_2)} > 1$ *then (7) is persistent in the mean , moreover we have:*

$$\lim_{t\to+\infty} \inf \frac{1}{t} \int_0^t I(s)ds > \frac{(\mu + d + \delta + r_2)(\beta \frac{\Lambda}{\mu(\mu+d+\delta+r_2)} - 1)}{(\beta \frac{\Lambda}{\mu} + (\mu + d + \delta))}$$

$$\lim_{t\to+\infty} \inf \frac{1}{t} \int_0^t E(s)ds \geq \frac{r_2}{\mu + k(1 + r_1)} \frac{(\mu + d + \delta + r_2)(\beta \frac{\Lambda}{\mu(\mu+d+\delta+r_2)} - 1)}{(\beta \frac{\Lambda}{\mu} + (\mu + d + \delta))}$$

$$\lim_{t\to+\infty} \inf \frac{1}{t} \int_0^t (\frac{\Lambda}{\mu} - S(s))ds > \mu \frac{(\mu + d + \delta + r_2)(\beta \frac{\Lambda}{\mu(\mu+d+\delta+r_2)} - 1)}{(\beta \frac{\Lambda}{\mu} + (\mu + d + \delta))}$$

The proof is based on the following lemma:

Lemma 6.3. *[1] Let* $g \in C([0, \infty) \times \Omega, [0, \infty))$ *and* $G \in C([0, \infty) \times \Omega, [0, \infty))$*. If there exists positive constants* λ_0 *and* λ *such that:*

$$\ln g(t) \geq \lambda_0 t - \lambda \int_0^t g(s)ds + G(t) \qquad a.s.$$

for all $t \geq 0$*, and* $\lim_{t\to+\infty} \frac{G(t)}{t} = 0$ *a.s., then*

$$\lim_{t\to+\infty} \inf \frac{1}{t} \int_0^t g(t)dt \geq \frac{\lambda_0}{\lambda} \qquad a.s.$$

Proof of theorem (6.2): Consider the following Lyapounov function:

$$V(S,E,I) = \alpha_1(S+E+I) + \alpha_2 S + \ln I$$

where α_1 and α_2 are defined below. By Ito's formula we have:

$$\begin{aligned}
dV &= \alpha_1[\Lambda - \mu(S+E+I) - (d+\delta)I] + \alpha_2[\Lambda - \beta\frac{SI}{N} - \beta_1\frac{BS}{K+B} - \mu S - \sigma_1 SIdW_1 \\
&- \sigma_2\frac{BS}{K+B}dW_2] + p\left(\beta\frac{S}{N} + \beta_1\frac{BS}{I(K+B)}\right) + k(1-r_1)\frac{E}{I} - (\mu+d+\delta+r_2) + p\sigma_1 S\,dW_1 \\
&+ p\sigma_2\frac{BS}{I(K+B)}dW_2 + (2p^2-2p+2)(\sigma_1^2 S^2 I^2 + \frac{S^2B^2}{(K+B)^2}) \\
&\geq \alpha_1[(\Lambda - \mu(S+E+I) - (d+\delta)I] + \alpha_2[(\Lambda - \mu S) - \beta_1 S\frac{B}{K+B} - \beta\frac{\Lambda}{\mu}I)dt - \sigma_1 SIdW_1 \\
&- \sigma_2 S\frac{B}{K+B}dW_2] + p(\beta_1 S\frac{B}{I(K+B)} + \beta\frac{\Lambda}{\mu} - (\mu+d+\delta+r_2) + \sigma_1^2 S^2)dt + p\sigma_1 S\,dW_1 \\
&+ p\sigma_2 S\frac{B}{I(K+B)}dW_2 \\
&\geq p(\mu+d+\delta+r_2)(\beta\frac{\Lambda}{\mu(\mu+d+\delta+r_2)} - 1) + [(\alpha_1+\alpha_2)\mu - \beta](\frac{\Lambda}{\mu} - S) \\
&- (\alpha_2 - \frac{\mu}{\Lambda})\beta_1 S\frac{B}{K+B} + (\alpha_1 - \beta_h\frac{\Lambda}{\mu})I + (1-\alpha_2 I)S\sigma_1 S\,dW_1 + (\frac{1}{I} - \alpha_2)\sigma_2 S\frac{B}{K+B}dW_2
\end{aligned}$$

with $\alpha_2 = \frac{\mu}{\Lambda}$ and $\beta = (\alpha_1+\alpha_2)\mu$ one has:

$$\begin{aligned}
dV &\geq (\mu+d+\delta+r_2)(\beta\frac{\Lambda}{\mu(\mu+d+\delta+r_2)} - 1)dt - (\beta(\frac{\Lambda}{\mu} + (\mu+d+\delta))Idt \\
&+ (1-\alpha_2 I)S\sigma_1 S\,dW_1 + (\frac{1}{I} - \alpha_2)\sigma_2 S\frac{B}{K+B}dW_2
\end{aligned}$$

and integrating both sides, one obtains:

$$\begin{aligned}
V(S,E,I) &\geq V(S_0,E_0,I_0) + (\mu+d+\delta+r_2)(\beta\frac{\Lambda}{\mu(\mu+d+\delta+r_2)} - 1)t - (\beta(\frac{\Lambda}{\mu} + (\mu+d+\delta))\int_0^t Idt \\
&+(1-\alpha_2 I)\sigma_1\int_0^t S\,dW_1 + (\frac{1}{I} - \alpha_2)\sigma_2\int_0^t S\frac{B}{K+B}dW_2
\end{aligned}$$

hence

$$\ln I \geq (\mu+d+\delta+r_2)(\beta\frac{\Lambda}{\mu(\mu+d+\delta+r_2)} - 1)t - (\beta\frac{\Lambda}{\mu} + (\mu+d+\delta))\int_0^t Idt + D(t)$$

with

$$D(t) = V(S_0,E_0,I_0) - (\alpha_1+\alpha_2)S - \alpha_1 E - \alpha_1 I + (1-\alpha_2 I)\sigma_1\int_0^t S\,dW_1 + (\frac{1}{I} - \alpha_2)\sigma_2\int_0^t S\frac{B}{K+B}dW_2$$

by the the strong low of martingale, one deduces that:

$$\lim_{t\to+\infty} \frac{D(t)}{t} = 0$$

By the lemma (4.8) one has:

$$I(t) \geq \frac{(\mu + d + \delta + r_2)(\beta\frac{\Lambda}{\mu(\mu+d+\delta+r_2)} - 1)}{(\beta\frac{\Lambda}{\mu} + (\mu + d + \delta))}$$

From (4)

$$\dot{E} \geq r_2 I - (k(1 - r_1) + \mu)E$$

wich gives

$$\begin{aligned}\lim_{t\to\infty}\inf E(t) &\geq \frac{r_2}{\mu + k(1 - r_1)} \lim_{t\to\infty}\inf(I(t)) + \lim_{t\to\infty}\inf \frac{E_0 - E}{(\mu + k(1 - r_1)t}\\ &\geq \frac{r_2}{\mu + k(1 + r_1)} \frac{(\mu + d + \delta + r_2)(\beta\frac{\Lambda}{\mu(\mu+d+\delta+r_2)} - 1)}{(\beta\frac{\Lambda}{\mu} + (\mu + d + \delta))}\end{aligned}$$

For the last equality, we take account of the following relation::

$$\begin{aligned}dN = d(S + E + I) &= (\mu(\frac{\Lambda}{\mu} - S) - \mu E - (\mu + d + \delta)I)dt\\ &\geq (\mu(\frac{\Lambda}{\mu} - S) - \mu E - \mu I)dt\end{aligned}$$

hence

$$\begin{aligned}\lim_{t\to\infty}\inf(\frac{\Lambda}{\mu} - S) &\geq \lim_{t\to\infty} \frac{N - N_0}{\mu t} + \mu \lim_{t\to\infty}\inf I(t) + \mu \lim_{t\to\infty}\inf E(t)\\ &\geq \mu \frac{(\mu + d + \delta + r_2)(\beta\frac{\Lambda}{\mu(\mu+d+\delta+r_2)} - 1)}{(\beta\frac{\Lambda}{\mu} + (\mu + d + \delta))}\end{aligned}$$

Conflict of interest

The authors declare no conflict of interest.

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