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Research Article

Asymptotic stability of degenerate stationary solution to a system of viscous conservation laws in half line

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Abstract: In this paper, we study a system of viscous conservation laws given by a form of a symmetric parabolic system. We consider the system in the one-dimensional half space and show existence of a degenerate stationary solution which exists in the case that one characteristic speed is equal to zero. Then we show the uniform a priori estimate of the perturbation which gives the asymptotic stability of the degenerate stationary solution. The main aim of the present paper is to show the a priori estimate without assuming the negativity of non-zero characteristics. The key to proof is to utilize the Hardy inequality in the estimate of low order terms.

Keywords: stationary waves; boundary layer solutions; compressible viscous gases; energy method; center manifold theory
Mathematics Subject Classification: 35B35, 35B40

1. Introduction

We consider a large-time behavior of solutions to a system of viscous conservation laws

$$u_t + f(u)_x = (Bu_x)_x \tag{1.1}$$

over a one-dimensional half line $\mathbb{R}_+ := (0, \infty)$. Here m is a positive integer; $u = u(t, x) \in \mathbb{R}^m$ is an unknown m-vector function; $f(u) \in \mathbb{R}^m$ is a flux function which is a smooth given function of u; B is a viscosity matrix which is an $m \times m$ symmetric and positive constant matrix.

We prescribe an initial condition for (1.1) as

$$u(0, x) = u_0(x) \quad (x \in \mathbb{R}_+), \tag{1.2}$$

where $u_0(x)$ is an initial data satisfying $u_0(x) \to 0$ as $x \to \infty$. We also put a Dirichlet boundary condition

$$u(t,0) = u_{\mathrm{b}} \quad (t > 0), \tag{1.3}$$

where $u_{\mathrm{b}} \in \mathbb{R}^m$ is a constant.

Related to the system (1.1), existence and asymptotic stability of a boundary layer solution, which is a smooth stationary solution connecting a boundary data and a spatial asymptotic data, for model systems of compressible viscous gases are proved in the papers [1, 4, 8, 9, 10]. These results are generalized in the papers [7, 14] for a quasi-linear symmetric system of hyperbolic equations and parabolic equations under the stability condition discussed in [3, 11, 13]. Especially, in order to prove asymptotic stability of a degenerate boundary layer solution, which exists if the corresponding inviscid system has a characteristic field with speed zero, it is assumed in [7] that the speed of the non-zero characteristics are negative. In the paper [6], the simplified system (1.1) with the flux function $f(u)$ given by the following form and satisfying the following assumption [A1] of symmetricity is considered:

$$f(u) = Au + \frac{1}{2}F(u,u), \tag{1.4}$$

where $A = (a_1, \ldots, a_m)$ $(a_j \in \mathbb{R}^m)$ is a constant $m \times m$ matrix; $F(\cdot,\cdot)$ is a bilinear map on $\mathbb{R}^m$ of the form

$$F(u,v) = \sum_{i,j=1}^{m} f_{ij} u_j v_i = \begin{pmatrix} \langle F_1 u, v\rangle \\ \vdots \\ \langle F_m u, v\rangle \end{pmatrix} \in \mathbb{R}^m$$

for $u = {}^t(u_1, \ldots, u_m), v = {}^t(v_1, \ldots, v_m) \in \mathbb{R}^m$, where $f_{ij} = {}^t(f_{ij}^1, \ldots, f_{ij}^m) \in \mathbb{R}^m$ $(i, j = 1, \ldots, m)$ are constant vectors and $F_k = (f_{ij}^k)_{ij}$ $(k = 1, \ldots, m)$ are constant $m \times m$ matrices.

Assumption [A1]. (i) The matrix A is symmetric.

(ii) The bilinear map $F(\cdot,\cdot)$ is symmetric in the sense of $f_{ij}^k = f_{ji}^k = f_{kj}^i$.

From Assumption [A1], we see that $f_{ij} = f_{ji}$ and $F(u,v) = F(v,u)$, so that F_k is symmetric.

In the paper [6], the simplified system (1.1) with non-positive characteristics is considered and the convergence rate of solutions toward the degenerate boundary layer solution is obtained provided that the initial perturbation belongs to the weighted L^2 space. The important property of the system in [6] is a negativity of non-zero characteristics which enable us to obtain the weighted L^2 estimate.

The aim of the present paper is to show asymptotic stability of the degenerate boundary layer solution for (1.1) without assuming that the initial perturbation belongs to weighted spaces. Namely we show the uniform a priori estimate (3.4) under the assumption [A4]-(i) which means that the characteristic speed of the system is non-positive. The key to proof is to utilize a weight function defined in (3.9). In the case if the viscosity effect is strong enough, we can also show the estimate (3.4) without assuming the negativity of non-zero characteristics. This case corresponds to the assumption [A4]-(ii). To study this problem, we prescribe the following assumption.

Assumption [A2]. The matrix A has a simple zero-eigenvalue.

Note that Assumption [A2] corresponds to analysis on the transonic flow for the model system of compressible viscous gases studied in the papers [1, 2, 9, 12].

Notations. For vectors $u, v \in \mathbb{R}^m$, $|u|$ denotes the Euclidean norm of u; $\langle u, v\rangle$ denotes the Euclidean inner product of u and v. For real matrices A and B of which eigenvalues are real number, we use a notation $A \sim B$ if the numbers of positive eigenvalues, negative eigenvalues and zero eigenvalues of A coincide with those of B. For $p \in [1, \infty]$, L^p denotes a standard Lebesgue space over $\mathbb{R}_+$ equipped with a norm $\| \cdot \|_{L^p}$.

2. Existence of stationary solution

In this section, we summarized the existence result of the degenerate boundary layer solution studied in [6, 7]. Let $\tilde{u} = \tilde{u}(x)$ be a boundary layer solution, which is a smooth solution to a system of equations

$$f(\tilde{u})_x = (B\tilde{u}_x)_x \quad (x \in \mathbb{R}_+), \tag{2.1}$$

which is rewritten to

$$\tilde{u}_x = B^{-1}A\tilde{u} + \frac{1}{2}B^{-1}F(\tilde{u}, \tilde{u}) \tag{2.2}$$

by integrating (2.1) over (x, ∞) with using $\tilde{u}_x(x) \to 0$ as $x \to \infty$. We prescribe boundary conditions for $\tilde{u}$ as

$$\tilde{u}(0) = u_{\mathrm{b}}, \tag{2.3}$$

$$\tilde{u}(x) \to 0 \quad (x \to \infty). \tag{2.4}$$

To solve the above stationary problem, we introduce a following lemma proved in [6, 7].

Lemma 2.1 ([6, 7]). *Let B be a symmetric and positive definite matrix and A be a symmetric matrix.*

(i) *The matrix $B^{-1}A$ is diagonalizable.*

(ii) *There exists an orthogonal matrix Q such that $P := B^{-1/2}Q$ diagonalizes the matrix $B^{-1}A$ and satisfies ${}^tP = P^{-1}B^{-1}$.*

(iii) *$B^{-1}A \sim A$.*

Here $B^{1/2}$ is a symmetric and positive definite matrix satisfying $(B^{1/2})^2 = B$ and $B^{-1/2}$ is an inverse matrix of $B^{1/2}$.

We give a brief outline of proof of the solvability theorem to the stationary problem (2.2)–(2.4) by following the argument in [6, 7]. Due to Lemma 2.1, there exists a matrix P of the form

$$P = (r, P_*), \quad P_* : \ m \times (m-1) \text{ matrix},$$

which diagonalizes $B^{-1}A$, that is,

$$P^{-1}B^{-1}AP = \begin{pmatrix} 0 & {}^t\mathbf{0} \\ \mathbf{0} & \Lambda \end{pmatrix}, \tag{2.5}$$

where $\mathbf{0}$ is a column zero-vector and Λ is a diagonal $(m-1) \times (m-1)$ matrix satisfying $\det \Lambda \neq 0$. Note that the column vector r is an eigenvector of $B^{-1}A$ corresponding to the zero-eigenvalue, that is, $B^{-1}Ar = 0$ and hence $Ar = 0$, which yields that r is also an eigenvector of A corresponding to the zero-eigenvalue. We employ a new unknown function $\tilde{w}(x) := P^{-1}\tilde{u}(x)$ and deduce the system (2.2) to that for $\tilde{w}$ as

$$\tilde{w}_{1x} = g_1(\tilde{w}), \tag{2.6a}$$

$$\tilde{w}_{*x} = \Lambda\tilde{w}_* + g_*(\tilde{w}), \tag{2.6b}$$

where

$$\tilde{w} = \begin{pmatrix}\tilde{w}_1\\ \tilde{w}_*\end{pmatrix}, \quad \tilde{w}_1 = \tilde{w}_1(x) \in \mathbb{R}, \quad \tilde{w}_* = \tilde{w}_*(x) \in \mathbb{R}^{m-1},$$

$$g(\tilde{w}) = \begin{pmatrix}g_1(\tilde{w})\\ g_*(\tilde{w})\end{pmatrix} := \frac{1}{2}P^{-1}B^{-1}F(P\tilde{w}, P\tilde{w}), \quad g_1(\tilde{w}) \in \mathbb{R}, \quad g_*(\tilde{w}) \in \mathbb{R}^{m-1}.$$

Let $z = z(x) \in \mathbb{R}$ be a solution to (2.6a) restricted on the local center manifold $\tilde{w}_* = \Phi^{\mathrm{c}}(\tilde{w}_1)$. Namely, $z(x)$ satisfies

$$z_x = g_1(z, \Phi^{\mathrm{c}}(z)) = \kappa z^2 + O(|z|^3), \tag{2.7}$$

where κ is a constant given by

$$\kappa := \frac{1}{2}\langle r, F(r, r)\rangle. \tag{2.8}$$

Here the second equality in (2.7) is obtained by using $P^{-1}B^{-1} = {}^tP$ and $P\tilde{w} = r\tilde{w}_1 + P_*\tilde{w}_*$ as well as a bilinearity of F. To solve the equation (2.7), we put the following assumption.

Assumption [A3]. Let r be an eigenvector of the matrix A corresponding to the zero-eigenvalue. Then it is assumed that the constant κ defined by (2.8) is not equal to zero.

Note that we assume $\kappa < 0$ without loss of generality. We also note that the assumption [A3] is equivalent to the genuine nonlinearity of the zero-characteristic field.

If the boundary data u_{b} belongs to a certain region $\mathscr{M} \subset \mathbb{R}^m$, which is a one side of a neighborhood of the equilibrium divided by a local stable manifold, and if the boundary strength $\delta = |u_{\mathrm{b}}|$ is sufficiently small, then the equation (2.7) has a solution z satisfying

$$\begin{gathered} z(x) > 0, \quad z_x(x) < 0, \quad z(x) \sim \frac{\delta}{1+\delta x}, \\ |\partial_x^k z(x)| \le C\frac{\delta^{k+1}}{(1+\delta x)^{k+1}} \quad (k = 0, 1, \dots). \end{gathered} \tag{2.9}$$

By virtue of the center manifold theory, the solution $\tilde{w}$ to (2.6) is given by using z as

$$\tilde{w}_1(x) = z(x) + O(\delta e^{-cx}),$$

$$\tilde{w}_*(x) = \Phi^{\mathrm{c}}(z(x)) + O(\delta e^{-cx}),$$

which gives the existence of the solution $\tilde{u}$.

Theorem 2.2 ([6, 7])**.** *Assume that Assumptions* [A1], [A2] *and* [A3] *hold. Then there exists a certain region $\mathscr{M} \subset \mathbb{R}^m$ such that if $u_{\mathrm{b}} \in \mathscr{M}$ holds and $\delta = |u_{\mathrm{b}}|$ is sufficiently small, then the problem* (2.2)–(2.4) *has a unique smooth solution $\tilde{u}(x)$ satisfying*

$$\tilde{u}(x) = rz(x) + O(z(x)^2 + \delta e^{-cx}), \tag{2.10}$$

$$\tilde{u}_x(x) = \kappa r z(x)^2 + O(z(x)^3 + \delta e^{-cx}). \tag{2.11}$$

3. Energy estimates

In this section, we show the uniform a priori estimate of a perturbation

$$\varphi(t,x) := u(t,x) - \tilde{u}(x)$$

which gives the existence of a solution globally in time. From (1.1) and (2.1), the equation for φ is given by

$$\varphi_t + D_u f(u)\varphi_x = B\varphi_{xx} - (D_u f(u) - D_u f(\tilde{u}))\tilde{u}_x \quad (x \in \mathbb{R}_+,\ t > 0), \tag{3.1}$$

where $D_u f(u) = A + (\langle f_{ij}, u\rangle)_{ij}$. The initial condition and the boundary condition are prescribed as

$$\varphi(0,x) = \varphi_0(x) := u_0(x) - \tilde{u}(x) \quad (x \in \mathbb{R}_+), \tag{3.2}$$

$$\varphi(t,0) = 0 \quad (t > 0). \tag{3.3}$$

To obtain the uniform a priori estimate, we prescribe the following assumption.

Assumption [A4]. It is assumed that either of the following two conditions is satisfied:

(i) The matrix A is non-positive definite, that is, the diagonal matrix Λ is negative definite, or

(ii) The viscosity effect is strong enough to satisfy

$$\langle B\phi, \phi\rangle > C_1 \tilde{F}|\phi|^2, \quad \tilde{F} := \max_{i,j,k} |f_{ij}^k|$$

for an arbitrary $\phi \in \mathbb{R}^m$, where C_1 is a positive constant in (3.19).

Notice that Assumption [A4]-(i) corresponds to analysis on the transonic flow for the outflow problem of compressible viscous gases. Assumption [A4]-(ii) corresponds to the condition that the Reynolds number is sufficiently small for the model system of compressible viscous gases. The a priori estimate for φ is summarized in the following theorem.

Theorem 3.1. *Assume that Assumptions* [A1], [A2], [A3] *and* [A4] *hold. Let $\tilde{u}$ be a degenerate boundary layer solution obtained in Theorem* 2.2 *and let $\varphi \in C^0([0,T]; L^2)$ be a solution to* (3.1)–(3.3) *for a certain $T > 0$. Then there exists a positive constant ε_0 such that if $\|\varphi_0\|_{L^2} + \delta \le \varepsilon_0$, then φ satisfies the following uniform estimate*

$$\|\varphi(t)\|_{L^2}^2 + \int_0^t \|\varphi_x(\tau)\|_{L^2}^2\, d\tau \le C\|\varphi_0\|_{L^2}^2 \quad (0 \le t \le T). \tag{3.4}$$

Remark. By combining the existence of the solution locally in time with the a priori estimate (3.4), we can construct a solution $\varphi \in C^0([0,\infty); L^2)$ globally in time. Moreover, if we construct the solution in H^1 framework, we can show the asymptotic stability $\|\varphi(t)\|_{L^\infty} \to 0$ as $t \to \infty$. In this paper, we only give the derivation of the basic estimate (3.4) in L^2 framework.

To obtain the estimate (3.4), it is convenient to define the weighted L^2 norm

$$|\varphi|_\alpha := \Big(\int_{\mathbb{R}_+} z^{-\alpha}|\varphi|^2\, dx \Big)^{1/2},$$

where $z(x) > 0$ is a solution to (2.7) satisfying (2.9).

Proof of Theorem 3.1. We employ the energy form $\mathscr{E}$ and the energy flux $\mathscr{F}$ defined by

$$\mathscr{E} = \frac{1}{2}|\varphi|^2, \quad \mathscr{F} = \frac{1}{2}\mathbf{A}[\varphi,\varphi] + \frac{1}{2}\mathbf{F}[\tilde{u},\varphi,\varphi] + \frac{1}{3}\mathbf{F}[\varphi,\varphi,\varphi],$$
$$\mathbf{A}[u,v] := \langle Au, v\rangle = \sum_{i,j} a_{ij}u_i v_j, \quad \mathbf{F}[u,v,w] := \langle u, F(v,w)\rangle = \sum_{i,j,k} f_{ij}^k u_i v_j w_k,$$

where $u = {}^t(u_1,\dots,u_m)$, $v = {}^t(v_1,\dots,v_m)$ and $w = {}^t(w_1,\dots,w_m)$. Note that $\mathbf{A}[u,v]$ and $\mathbf{F}[u,v,w]$ are multi-linear forms. Then we see that $\mathscr{E}$ and $\mathscr{F}$ satisfy

$$\mathscr{E}_t + \mathscr{F}_x + \mathscr{G} + \langle B\varphi_x, \varphi_x\rangle = (\langle B\varphi_x, \varphi\rangle)_x, \tag{3.5}$$
$$\mathscr{G} := \langle \tilde{u}_x, f(u) - f(\tilde{u}) - D_u f(\tilde{u})\varphi\rangle = \frac{1}{2}\mathbf{F}[\tilde{u}_x,\varphi,\varphi].$$

We firstly show the proof of (3.4) under the assumption [A4]-(i) by following the idea in [7]. We change the variable φ to ψ defined by $\psi(t,x) := P^{-1}\varphi(t,x)$ where P is a diagonalization matrix of $B^{-1}A$ satisfying (2.5). Then we see

$$\varphi = P\psi = r\psi_1 + P_*\psi_*, \tag{3.6}$$
$$\psi = \begin{pmatrix}\psi_1\\ \psi_*\end{pmatrix}, \quad \psi_1(t,x) \in \mathbb{R}, \quad \psi_*(t,x) \in \mathbb{R}^{m-1}.$$

By using ${}^tP = P^{-1}B^{-1}$, we see

$$\mathbf{A}[\varphi,\varphi] = \langle AP\psi, P\psi\rangle = \langle {}^tPAP\psi, \psi\rangle = \langle P^{-1}B^{-1}AP\psi, \psi\rangle = \langle \Lambda\psi_*, \psi_*\rangle.$$

Also, (2.10), (2.11), (3.6) and multi-linearity of $\mathbf{F}$ yield

$$\mathscr{F} = \frac{1}{2}\langle \Lambda\psi_*, \psi_*\rangle + O(|\varphi|^3 + \delta e^{-cx}|\varphi|^2) \le -c_1|\psi_*|^2 + O(|\varphi|^3 + \delta e^{-cx}|\varphi|^2), \tag{3.7}$$
$$\mathscr{G} = \kappa^2 z^2 \psi_1^2 + z^2\mathscr{G}' + O(z^3|\varphi|^2 + \delta e^{-cx}|\varphi|^2), \tag{3.8}$$

where c_1 is a positive constant. Here $\mathscr{G}'$ is a quadratic form consisting of ψ_1 and ψ_* satisfying

$$|\mathscr{G}'| \le C_1(|\psi_1||\psi_*| + |\psi_*|^2),$$

where C_1 is a positive constant. To obtain the estimate (3.4), we employ the weighted energy method with using a weight function

$$W(x) = \frac{\omega}{\omega - \kappa z(x)}, \quad \omega := \frac{2c_1\kappa^4}{9(3C_1^2 + 2C_1\kappa^2)}. \tag{3.9}$$

If δ is small enough to satisfy $|z(x)| \le \omega/2$, we see that W satisfies

$$\frac{2}{3} \le W(x) \le 2, \quad W_x(x) = \frac{1}{\omega}\kappa^2 W^2 z^2 + O(z^3) > 0. \tag{3.10}$$

Multiplying (3.5) by W, we get

$$(W\mathscr{E})_t + (W\mathscr{F})_x - W_x\mathscr{F} + W\mathscr{G} + W\langle B\varphi_x, \varphi_x\rangle = (W\langle B\varphi_x, \varphi\rangle)_x - W_x\langle B\varphi_x, \varphi\rangle. \tag{3.11}$$

We estimate $-W_x\mathscr{F} + W\mathscr{G}$ in (3.11). From (3.7) and (3.10), we have

$$-W_x\mathscr{F} \geq \frac{4\kappa^2 c_1}{9\omega} z^2|\psi_*|^2 + O(z^2|\varphi|^3 + \delta e^{-cx}|\varphi|^2). \tag{3.12}$$

Also, (3.8) and (3.10) give

$$\begin{aligned} W\mathscr{G} &\geq \frac{2\kappa^2}{3} z^2\psi_1^2 - 2C_1 z^2(|\psi_1||\psi_*| + |\psi_*|^2) + O(z^3|\psi|^2 + \delta e^{-cx}|\varphi|^2) \\ &\geq \frac{\kappa^2}{3} z^2\psi_1^2 - \frac{3C_1^2 + 2C_1\kappa^2}{\kappa^2} z^2|\psi_*|^2 + O(z^3|\psi|^2 + \delta e^{-cx}|\varphi|^2), \end{aligned} \tag{3.13}$$

where we have used $2C_1|\psi_1||\psi_*| \leq \frac{\kappa^2}{3}\psi_1^2 + \frac{3C_1^2}{\kappa^2}|\psi_*|^2$. Therefore, (3.12) and (3.13) yield

$$-W_x\mathscr{F} + W\mathscr{G} \geq cz^2|\varphi|^2 + O(z^3|\psi|^2 + z^2|\psi|^3 + \delta e^{-cx}|\varphi|^2). \tag{3.14}$$

The last term in the right-hand side of (3.11) is estimated as

$$|W_x\langle B\varphi_x, \varphi\rangle| \leq C\delta(z^2|\varphi|^2 + |\varphi_x|^2). \tag{3.15}$$

Integrating (3.11) over $[0, T] \times \mathbb{R}_+$, substituting (3.14) and (3.15) in the resultant equality and letting δ suitable small, we have

$$\|\varphi\|_{L^2}^2 + \int_0^t (|\varphi|_{-2}^2 + \|\varphi_x\|_{L^2}^2)\, d\tau \leq C\|\varphi_0\|_{L^2}^2 + C\int_0^t \int_{\mathbb{R}_+} (z^2|\varphi|^3 + \delta e^{-cx}|\varphi|^2)\, dx\, d\tau. \tag{3.16}$$

To estimate the remainder terms in the right-hand side, we compute

$$\int_{\mathbb{R}_+} z^2|\varphi|^3\, dx \leq \|\varphi\|_{L^\infty}|\varphi|_{-2}^2 \leq C\|\varphi\|_{L^2}(|\varphi|_{-2}^2 + \|\varphi_x\|_{L^2}^2), \tag{3.17}$$

where we have used the Sobolev inequality $\|\varphi\|_{L^\infty} \leq C\|\varphi\|_{L^2}^{1/2}\|\varphi_x\|_{L^2}^{1/2}$ and $|\varphi|_{-2} \leq C\|\varphi\|_{L^2}$. Also, due to the Poincaré type inequality, we see

$$\int_{\mathbb{R}_+} e^{-cx}|\varphi|^2\, dx \leq C\|\varphi_x\|_{L^2}^2. \tag{3.18}$$

Substituting (3.17) and (3.18) in (3.16), we get

$$\|\varphi\|_{L^2}^2 + \int_0^t (|\varphi|_{-2}^2 + \|\varphi_x\|_{L^2}^2)\, d\tau \leq C\|\varphi_0\|_{L^2}^2 + C\sup_{0\leq t\leq T}\|\varphi\|_{L^2}\int_0^t (|\varphi|_{-2}^2 + \|\varphi_x\|_{L^2}^2)\, d\tau$$

which yields the desired estimate (3.4) provided that $\|\varphi_0\|_{L^2}$ is sufficiently small.

Next, we show the proof of (3.4) under Assumption [A4]-(ii). By using

$$|\mathscr{G}| \leq \frac{1}{2}|\tilde{u}_x||F(\varphi, \varphi)| \leq C\tilde{F}z^2|\varphi|^2 \leq C\tilde{F}\frac{1}{x^2}|\varphi|^2$$

and the Hardy inequality, we see

$$\int_{\mathbb{R}_+} |\mathscr{G}|\, dx \leq C\tilde{F}\int_{\mathbb{R}_+} \frac{1}{x^2}|\varphi|^2\, dx \leq C_1\tilde{F}\|\varphi_x\|_{L^2}^2, \tag{3.19}$$

where C_1 is a positive constant. Thus, integrating (3.5) and substituting the above inequality with using Assumption [A4]-(ii), we obtain the desired estimate (3.4). Consequently, we complete the proof. □

Notice that the computation in the present paper is also applicable to the model system of compressible and viscous gas which is given by a hyperbolic-parabolic system. We also note that the condition [A4] is assumed because of the technical reason. It is open problem that we can remove this condition or not.

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Conflict of Interest

The author declare no conflicts of interest in this paper.

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