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Research article

Blow-up phenomena for a class of metaparabolic equations with time dependent coefficient

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Abstract: This paper deals with the initial boundary value problem for a metaparabolic equations with time dependent coefficient. Under suitable conditions on initial data, a blow-up criterion which ensures that u cannot exist all time is given, and an upper bound for blow up time is derived. Moreover, we also obtain a lower bound for blow-up time if blow up does occur by means of a differential inequality technique.

Keywords: metaparabolic equations; blow up; upper bound; lower bound
Mathematics Subject Classification: 35K70, 35K61, 35B44, 35D40

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1. Introduction

In this paper, we investigate the initial-boundary value problem for the following metaparabolic equations

$$u_t - u_{xx} - u_{xxt} + u_{xxxx} = k(t)f(u_x)_x, \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.2}$$

$$u(0,t) = u(1,t) = 0, \quad u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad t \geq 0, \tag{1.3}$$

where $u_0(x)$ is the initial value function defined on $\Omega = (0, 1)$. The coefficient $k(t)$ is assumed non-positive or strictly negative depending of the different situation. The nonlinear smooth function $f(s)$

satisfies the following assumptions:

$$\begin{cases} (i)\ \ f(0)=0; f(s) \text{ is monotone and is convex for } s>0,\ \text{concave for } s<0; \\ (ii)\ \ |f(s)| \le \alpha|s|^p,\ \ \alpha>0,\ \ 1<p<+\infty,\ \ \forall\ s\in R; \\ (iii)\ \ (p+1)F(s) \le sf(s) \text{ for some } p>1,\ \ \forall\ s\in R,\ \ F(s)=\int_0^s f(\tau)d\tau. \end{cases} \tag{1.4}$$

The Eq.(1.1) is a typical higher-order metaparabolic equation [1, 2], which has extensive physical background and rich theoretical connotation. This type of equation can be regarded as the regularization of Sobolev-Galpern equation by adding a fourth-order term u_{xxxx}. The Sobolev-Galpern equation appear in the study of various problems of fluid mechanics, solid mechanics and heat conduction theory [3]-[5]. There have been many outstanding results about the qualitative theory for Sobolev-Galpern which include the existence, nonexistence, asymptotic behavior, regularities and other some special properties of solutions. We also refer the reader to see [6]-[9] and the papers cited therein. In (1.1), u is the concentration of one of the two phases, the fourth-order term u_{xxxx} denotes the capillarity-driven surface diffusion, and the nonlinear term $f(u_x)$ is an intrinsic chemical potential.

If we differentiate Eq.(1.1) with respect to x and take $v = u_x$, $k(t) = 1$, then it becomes the well-known viscous Cahn-Hilliard equation

$$v_t - v_{xxt} + v_{xxxx} = \varphi(v)_{xx},\ \ x\in\Omega,\ \ t>0, \tag{1.5}$$

where $\varphi(v) = f(v) + v$. The Eq.(1.5) appears in the dynamics of viscous first order phase transitions in cooling binary solutions such as such as glasses, alloys and polymer mixtures [10]-[12]. On the other hand, the Eq.(1.5) appear in the study of the regularization of nonclassical diffusion equations by adding a fourth-order term v_{xxxx}. During the past years, many authors have paid much attention to the viscous Cahn-Hilliard equation. Some numerical approaches and basic qualitative theory results of this type equation have been developed [13]-[17]. Liu and Yin [15] studied the Eq.(1.5) for $\varphi(v) = -v+\gamma_1 v^2+\gamma_2 v^3$ in R^3, they proved the existence, nonexistence of global classical solutions, and pointed out that the sign of γ_2 is crucial to the global existence of solutions. In [16], Grinfeld and Novick-Cohen studied a Morse decomposition of the stationary solutions of the one-dimensional viscous Cahn-Hilliard equation by explicit energy calculations. They also proved that a partial picture of the variation in the structure of the attractor (n=1) for the viscous Cahn-Hilliard equation as the mass constraint and homotopy parameter are varied. Zhao and Liu [17] considered the initial boundary problem for the viscous Cahn-Hilliard Eq.(1.5). In their paper, the optimal control under boundary condition was given and the existence of optimal solution to the equation was proved.

Let us mention that there is an abounding literature about the blow-up phenomena to nonlinear parabolic and hyperbolic equations. We refer the reader to the monographs [18, 19] which devoted to the second-order parabolic and pseudo-parabolic problems. For the fourth-order nonlinear parabolic equation, there are also some results about the initial boundary value and Cauchy problems, especially on global existence/nonexistence, uniqueness/nonuniqueness, asymptotic behavior and so on [20]-[26]. In [22], Liu studied the following metaparabolic equation

$$u_t - ku_{xxt} + A(u)_{xxxx} = f(x,t),\ \ 0\le x\le 1,\ \ 0\le t\le T<+\infty, \tag{1.6}$$

where $A(u) = \int_0^u a(s)ds$, $a_0 + a_1|s|^b \le a(s)$, $|a''(s)| \le a_2|s|^b(a_0, a_1, a_2$ and b are positive constants). He proved the existence of weak solutions by using the method of continuity.

In 2009, Khudaverdiyev and Farhadova [23] discussed the following fourth-order semilinear pseudo-parabolic equation

$$u_t - \alpha u_{xxt} + u_{xxxx} = f(t, x, u, u_x, u_{xx}, u_{xxx}), \;\; 0 \le x \le 1, \;\; 0 \le t \le T < +\infty, \tag{1.7}$$

where $\alpha > 0$ is a fixed number. They proved the existence in large theorem (i.e. true for sufficiently large values of T) for generalized solution by means of Schauder stronger fixed point principle.

In [24], Zhao and Xuan studied the following generalized BBM-Burgers equations

$$u_t - \alpha u_{xx} - \gamma u_{xxt} + \beta u_{xxxx} + f(u)_x = 0, \;\; x \in \mathbb{R}, \;\; t > 0. \tag{1.8}$$

They obtained the existence and convergence behavior of the global smooth solutions.

Recently, Philippin [25, 26] studied the following fourth-order parabolic equation

$$u_t - k_1(t)\triangle u + k_2(t)\triangle^2 u = k_3(t)u|u|^{p-1}, \;\; x \in \Omega, \;\; t > 0, \tag{1.9}$$

where k_i, $i = 1, 2, 3$ are positive constants or in general positive derivable functions of time t. Under appropriate assumptions on the data, he proved that the solutions u cannot exist for all time, and an upper bound is derived. Furthermore, a lower bound for the finite time blow-up were derived when blow up occurs.

The Eq.(1.1) is also closely connected with many equations [27]-[32]. For example, in the study of a weakly nonlinear analysis of elasto-plastic-microstructure models for longitudinal motion of an elasto-plastic bar [27], there arises the model equation

$$u_{tt} + \alpha u_{xxxx} = \beta(u_x^2)_x, \;\; x \in (0, 1), \;\; t > 0. \tag{1.10}$$

Moreover, because of this propagation of the wave in the medium with the dissipation effect, it is meaningful [28] to consider the following nonlinear wave equations with the viscous damping term

$$u_{tt} - 2bu_{xxt} + \alpha u_{xxxx} = \beta(u_x^n)_x, \;\; x \in (0, 1), \;\; t > 0. \tag{1.11}$$

In [29], Chen and Lu studied the initial boundary value problem to the fourth-order wave equation

$$u_{tt} - 2bu_{xxt} + \alpha u_{xxxx} = f(u_x)_x, \;\; x \in (0, 1), \;\; t > 0. \tag{1.12}$$

They proved the existence and uniqueness of the global generalized solution and global classical solution by the Galerkin method. Furthermore, Xu, Wang et al. [30] considered the initial boundary value problem and proved the global existence, nonexistence of solutions by adopting and modifying the so called concavity method under some conditions with low initial energy. Ali Khelghati and Khadijeh Baghaei [31] proved that the blow up for Eq.(1.12) occurs in finite time for arbitrary positive initial energy. When the strong damping term $-u_{xxt}$ of Eq.(1.12) is replaced by the weak damping term u_t, Yang [32] studied the asymptotic property of the solution and gave some sufficient conditions of the blow-up.

Motivated by the above researches, in the present work we main study the blow-up phenomena for the metaparabolic problem (1.1)-(1.3). As far as we known, there is little information on the bounds for blow-up time to the problem (1.1)-(1.3). Especially, the appearance of the dispersion term u_{xxt}

and nonlinear term $k(t)f(u_x)_x$ cause some difficulties such that we cannot apply the normal concavity and potential methods directly, we have to invent some new methods and skills to overcome these difficulties.

The rest of this paper is organized as follows: In Section 2, we obtain a sufficient condition which guarantees that blow-up occurs at some finite time T^* and an upper bound for T^* is given, where $(0, T^*)$ is the interval of existence of the solution u. In Section 3, we determine a lower bound for blow-up time T^* when blow up occurs to the initial boundary problem (1.1)-(1.3) under some certain conditions on the data.

2. Upper bound for blow-up time

In this section, we show that u cannot exist for all time and an upper bound for blow-up time T^* is derived. Throughout this paper, the following abbreviations are used for precise statement:

$$L^p(\Omega) = L^p,\ \ W^{m,p}(\Omega) = W^{m,p},\ \ H^m(\Omega) = W^{m,2}(\Omega) = H^m,\ \ H_0^m(\Omega) = H_0^m,$$

$$V = \left\{u \in H^2(\Omega)\Big|\ u(0,t) = u(1,t) = 0,\ u_{xx}(0,t) = u_{xx}(1,t) = 0\right\},$$

$$\|u\|_{L^p(\Omega)} = \|u\|_p,\ \ \|u\|_{W^{m,p}(\Omega)} = \|u\|_{W^{m,p}},\ \ \|u\|_{H_0^m(\Omega)} = \|u\|_{H_0^m}.$$

And the notation $(\cdot,\cdot)$ for the L^2-inner product will also be used for the notation of duality paring between dual spaces.

Firstly, we start with a local existence result for the problem (1.1)-(1.3) which can be obtained by Faedo-Galerkin methods.

Theorem 2.1. *Assume that (1.4) hold. Then for any $u_0 \in W^{1,p+1}(\Omega) \cap V$, there exists a number $T > 0$ such that the problem (1.1)-(1.3) has a unique local solution*

$$u \in L^\infty([0,T]; W^{1,p+1}(\Omega) \cap V), \quad u_t \in L^2([0,T]; H_0^1(\Omega)),$$

satisfying

$$(u_t, v) + (u_{xt}, v_x) + (u_x, v_x) + (u_{xx}, v_{xx}) = -k(t)(f(u_x), v_x),\ \ \text{for all } v \in W^{1,p+1}(\Omega) \cap V. \tag{2.1}$$

Next, we prove that the solution u cannot exist for all time and seek an upper bound for blow up of the problem (1.1)-(1.3).

Replacing v by u_t in the Eq. (2.1), we have

$$\int_\Omega \left[|u_t|^2 + |u_{xt}|^2\right] dx + \frac{d}{dt}\left[\frac{1}{2}\|u_x\|_2 + \frac{1}{2}\|u_{xx}\|_2\right] = -k(t)\frac{d}{dt}\int_\Omega F(u_x)dx. \tag{2.2}$$

We then define the following auxiliary functions:

$$\Phi(t) = \int_\Omega u^2 dx + \int_\Omega u_x^2 dx, \tag{2.3}$$

and

$$\Psi(t) = \int_\Omega \left\{k(t)F(u_x) + \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_x^2\right\} dx. \tag{2.4}$$

Theorem 2.2. *Let f satisfy the assumptions (1.4) and $u_0 \in W^{1,p+1}(\Omega) \cap V$. Assume that the data of the problem (1.1)-(1.3) satisfy the following conditions:*

$$k(t) < 0, \quad k'(t) \leq 0, \tag{2.5}$$

$$\Psi(0) = \int_\Omega \left\{ k(0)F(u_{0x}) + \frac{1}{2}u_{0xx}^2 + \frac{1}{2}u_{0x}^2 \right\} dx < 0. \tag{2.6}$$

Then, we conclude that the solutions u of problem (1.1)-(1.3) cannot exist for all time. Moreover, an upper bound for blow-up time T^* is given by

$$T^* \leq \frac{\Phi(0)}{(1-p^2)\Psi(0)}, \tag{2.7}$$

where $\Phi(0) = \int_\Omega u_0^2 dx + \int_\Omega u_{0x}^2 dx$.

Proof. First, differentiating (2.4) with respect to t and integrating by parts, then we have

$$\begin{aligned}\Psi'(t) &= \int_\Omega k'(t)F(u_x)dx + \int_\Omega k(t)f(u_x)u_{xt}dx + \int_\Omega [u_{xx}u_{xxt} + u_x u_{xt}]\, dx \\ &= \int_\Omega k'(t)F(u_x)dx + \int_\Omega [u_{xxxx} - u_{xx} - k(t)f(u_x)_x]\, u_t dx. \end{aligned} \tag{2.8}$$

From the lemma 2.2 of [33], we know that $F(s)$ is nonnegative for all $s \in R$ under the assumptions (1.4)(i). Hence, by the combination of (1.1), (2.5) and (2.8), we obtain

$$\begin{aligned}\Psi'(t) &= \int_\Omega k'(t)F(u_x)dx - \int_\Omega [u_t - u_{xxt}]\, u_t dx \\ &\leq - \int_\Omega \left[u_t^2 + u_{xt}^2\right] dx < 0. \end{aligned} \tag{2.9}$$

It then follows from (2.9) that $\Psi(t)$ is nonincreasing, so we have

$$\Psi(t) \leq \Psi(0) < 0. \tag{2.10}$$

For simplicity, we define function $G(t) = -\Psi(t)$ for all $t \in [0, \infty)$. By (2.9) and (2.10), we see

$$G'(t) = -\Psi'(t) \geq \int_\Omega \left[u_t^2 + u_{xt}^2\right] dx > 0, \text{ and } G(t) \geq G(0) > 0. \tag{2.11}$$

Differentiating $\Phi(t)$ with respect to t, we get from the definition $G(t)$ and (1.4)(iii) that

$$\begin{aligned}\Phi'(t) &= 2\int_\Omega uu_t dx + 2\int_\Omega u_x u_{xt} dx \\ &= 2\int_\Omega u\,[-u_{xxxx} + u_{xx} + k(t)f(u_x)_x]\, dx \\ &= -2\int_\Omega k(t)f(u_x)u_x dx - 2\int_\Omega u_{xx}^2 dx - 2\int_\Omega u_x^2 dx \end{aligned}$$

$$
\begin{aligned}
&\geq -2(p+1)\int_\Omega k(t)F(u_x)dx - 2\int_\Omega u_{xx}^2 dx - 2\int_\Omega u_x^2 dx\\
&= 2(p+1)\left[-\int_\Omega k(t)F(u_x)dx - \frac{1}{p+1}\int_\Omega u_{xx}^2 dx - \frac{1}{p+1}\int_\Omega u_x^2 dx\right]\\
&\geq 2(p+1)\left[-\int_\Omega k(t)F(u_x)dx - \frac{1}{2}\int_\Omega u_{xx}^2 dx - \frac{1}{2}\int_\Omega u_x^2 dx\right]\\
&= 2(p+1)G(t).
\end{aligned} \tag{2.12}
$$

Multiplying $\Phi(t)$ by $G'(t)$, then we have

$$
\begin{aligned}
\Phi G' &\geq \int_\Omega \left[u_t^2 + u_{xt}^2\right]dx \int_\Omega \left[u^2 + u_x^2\right]dx\\
&= \int_\Omega u^2 dx \int_\Omega u_t^2 dx + \int_\Omega u_x^2 dx \int_\Omega u_{xt}^2 dx\\
&\quad + \int_\Omega u^2 dx \int_\Omega u_{xt}^2 dx + \int_\Omega u_x^2 dx \int_\Omega u_t^2 dx.
\end{aligned} \tag{2.13}
$$

Using the Schwarz and Young inequalities, we get

$$
\int_\Omega u u_t dx \leq \left(\int_\Omega u^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega u_t^2 dx\right)^{\frac{1}{2}}, \tag{2.14}
$$

$$
\int_\Omega u_x u_{xt} dx \leq \left(\int_\Omega u_x^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega u_{xt}^2 dx\right)^{\frac{1}{2}}, \tag{2.15}
$$

and

$$
\int_\Omega u u_t dx \int_\Omega u_x u_{xt} dx \leq \frac{1}{2}\int_\Omega u^2 dx \int_\Omega u_{xt}^2 dx + \frac{1}{2}\int_\Omega u_x^2 dx \int_\Omega u_t^2 dx. \tag{2.16}
$$

Inserting (2.14)-(2.16) into (2.13), we have from (2.12) that

$$
\Phi G' \geq \int_\Omega \left[u_t^2 + u_{xt}^2\right]dx \int_\Omega \left[u^2 + u_x^2\right]dx \geq \frac{1}{4}\left(\Phi'(t)\right)^2 \geq \frac{p+1}{2}\Phi' G. \tag{2.17}
$$

The above inequality may be rewritten as

$$
\left(G\Phi^{-\frac{p+1}{2}}\right)' = \Phi^{-\frac{p+3}{2}}\left\{\Phi G' - \frac{p+1}{2}\Phi' G\right\} \geq 0. \tag{2.18}
$$

Integrating (2.18), we obtain

$$
G(t)\left(\Phi(t)\right)^{-\frac{p+1}{2}} \geq G(0)\left(\Phi(0)\right)^{-\frac{p+1}{2}} = M. \tag{2.19}
$$

By (2.12) and (2.19), we get

$$
\frac{1}{1-p^2}\left(\Phi^{\frac{1-p}{2}}\right)' = \frac{1}{2(p+1)}\Phi'\Phi^{-\frac{p+1}{2}} \geq G\Phi^{-\frac{p+1}{2}} \geq M. \tag{2.20}
$$

Integrating (2.20), we have

$$(\Phi(t))^{\frac{1-p}{2}} \leq (\Phi(0))^{\frac{1-p}{2}} - (p^2-1)Mt, \tag{2.21}$$

which implies that

$$\Phi(t) \geq \frac{1}{\left[(\Phi(0))^{\frac{1-p}{2}} - (p^2-1)Mt\right]^{\frac{2}{p-1}}}. \tag{2.22}$$

Clearly, the above inequality cannot hold for all $t > 0$. In fact, (2.22) leads to the upper bound $\frac{\Phi(0)}{(1-p^2)\Psi(0)}$ for T^*.

3. Lower bound for blow-up time

In this section, our aim is to determine a lower bound for blow-up time T^* when blow up occurs to the initial boundary problem (1.1)-(1.3) under some certain conditions.

Theorem 3.1. *Let u be a blow-up solution of problem (1.1)-(1.3) and f satisfy the assumptions* (1.4). *Furthermore assume that $u_0 \in W^{1,p+1}(\Omega) \cap V$ and $k(t) < 0$ satisfies the condition*

$$\frac{k'(t)}{k(t)} \leq \beta, \quad \forall \; t \geq 0, \tag{3.1}$$

for some constant $\beta \geq 0$. Then, we conclude that the auxiliary functions

$$\theta(t) = (-k(t))^{\frac{2}{p-1}} \left[\int_\Omega u_x^2 dx + \int_\Omega u_{xx}^2 dx\right], \tag{3.2}$$

becomes unbounded at finite time T^*. Moreover, an lower bound for blow up time T^* can be estimated by

$$T^* \geq \begin{cases} \dfrac{2\theta(0)^{1-p}}{(p-1)\alpha^2\gamma^{2p}}, & if\, \beta = 0, \\ \frac{1}{2\beta} \ln\left(\dfrac{4\beta\theta(0)^{1-p}}{(p-1)\alpha^2\gamma^{2p}} + 1\right), & if\, \beta > 0, \end{cases} \tag{3.3}$$

where γ is the optimal constant satisfying the inequality $\|u_x\|_{2p} \leq \gamma \|u_{xx}\|_2$.

Proof. To obtain the lower bound for the blow-up time, we construct a differential inequality for $\theta(t)$. Differentiating (3.2) with respect to t and integrating by parts, then we have

$$\begin{aligned} \theta'(t) = & \frac{2}{p-1} (-k(t))^{\frac{3-p}{p-1}} (-k'(t)) \left[\int_\Omega u_x^2 dx + \int_\Omega u_{xx}^2 dx\right] \\ & + 2(-k(t))^{\frac{2}{p-1}} \int_\Omega [u_{xx}u_{xxt} + u_x u_{xt}]\, dx \\ \leq & \frac{2\beta}{p-1}\theta(t) + 2(-k(t))^{\frac{2}{p-1}} \int_\Omega [u_{xxt} - u_t]\, u_{xx} dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{2\beta}{p-1}\theta(t) + 2\left(-k(t)\right)^{\frac{2}{p-1}} \int_\Omega \left[-k(t)f(u_x)_x - u_{xx} + u_{xxxx}\right] u_{xx} dx \\
&= \frac{2\beta}{p-1}\theta(t) - 2\left(-k(t)\right)^{\frac{p+1}{p-1}} \int_\Omega f(u_x)u_{xxx} dx \\
&- 2\left(-k(t)\right)^{\frac{2}{p-1}} \int_\Omega u_{xx}^2 dx - 2\left(-k(t)\right)^{\frac{2}{p-1}} \int_\Omega u_{xxx}^2 dx. \qquad (3.4)
\end{aligned}
$$

Considering the Young inequality and (1.4)(ii), we have

$$
\begin{aligned}
&- 2\left(-k(t)\right)^{\frac{p+1}{p-1}} \int_\Omega f(u_x)u_{xxx} dx \\
&\leq 2\left(-k(t)\right)^{\frac{p+1}{p-1}} \int_\Omega |f(u_x)u_{xxx}| dx \\
&\leq \frac{1}{2}\left(-k(t)\right)^{\frac{2p}{p-1}} \int_\Omega |f(u_x)|^2 dx + 2\left(-k(t)\right)^{\frac{2}{p-1}} \int_\Omega u_{xxx}^2 dx \\
&\leq \frac{\alpha^2}{2}\left(-k(t)\right)^{\frac{2p}{p-1}} \int_\Omega |u_x|^{2p} dx + 2\left(-k(t)\right)^{\frac{2}{p-1}} \int_\Omega u_{xxx}^2 dx. \qquad (3.5)
\end{aligned}
$$

From the Sobolev inequality, we have

$$
\int_\Omega |u_x|^{2p} dx \leq \gamma^{2p}\left(\int_\Omega u_{xx}^2 dx + \int_\Omega u_x^2 dx\right)^p. \qquad (3.6)
$$

Inserting (3.5),(3.6) into (3.4), we obtain

$$
\theta'(t) \leq \frac{2\beta}{p-1}\theta(t) + \frac{\alpha^2\gamma^{2p}}{2}\left(-k(t)\right)^{\frac{2p}{p-1}}\left(\int_\Omega u_{xx}^2 dx + \int_\Omega u_x^2 dx\right)^p \leq \frac{2\beta}{p-1}\theta(t) + \frac{\alpha^2\gamma^{2p}}{2}\theta(t)^p. \qquad (3.7)
$$

Next, we need to consider inequality (3.7) in two cases.

(1) $\beta = 0$. Integrating (3.7), we have

$$
\theta(t)^{1-p} \geq \theta(0)^{1-p} - \frac{(p-1)\alpha^2\gamma^{2p}}{2}t. \qquad (3.8)
$$

(2) $\beta > 0$. The inequality (3.7) is integrable and leads to

$$
\int_{\theta(0)}^{\theta(t)} \frac{d\eta}{\frac{2\beta}{p-1}\eta + \frac{\alpha^2\gamma^{2p}}{2}\eta^p} \leq t,
$$

or by means of $\eta = \varphi^{\frac{1}{p-1}}$,

$$
\frac{1}{p-1}\int_{\theta(0)^{p-1}}^{\theta(t)^{p-1}} \frac{d\varphi}{\varphi\left(\frac{2\beta}{p-1} + \frac{\alpha^2\gamma^{2p}}{2}\varphi\right)} \leq t. \qquad (3.9)
$$

From (3.9), we can get

$$
\theta(t)^{1-p} \geq e^{-2\beta t}\left[\theta(0)^{1-p} + \frac{(p-1)\alpha^2\gamma^{2p}}{4\beta}\right] - \frac{(p-1)\alpha^2\gamma^{2p}}{4\beta}. \qquad (3.10)
$$

Clearly the inequality (3.3) follows easily from (3.8) and (3.10). This completes the proof of theorem 3.1.

Remark 3.1. For the more general PDE

$$\frac{1}{k_1(t)}[u_t - u_{xxt}] - k_2(t)[u_{xx} - u_{xxxx}] = k_3(t)f(u_x)_x, \quad x \in \Omega, \quad t > 0, \tag{3.11}$$

where $k_1(t)$, $k_2(t)$ are positive functions of t and $k_3(t)$ are negative functions of t, we note that Theorem 2.1 and 3.1 may be adapted. In fact, when the time t is replaced by another new time variable

$$\tau(t) = \int_0^t k_1(\xi)k_2(\xi)d\xi, \tag{3.12}$$

then Eq.(3.11) reduced to the Eq.

$$u_\tau - u_{xx\tau} - u_{xx} + u_{xxxx} = k(\tau)f(u_x)_x, \quad x \in \Omega, \quad \tau > 0, \tag{3.13}$$

with

$$k(\tau) = \frac{k_3(t(\tau))}{k_2(t(t))}. \tag{3.14}$$

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Conflict of Interest

The authors declare that there is no conflicts of interest in this paper.

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