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Research article

Applications of the lichnerowicz Laplacian to stress energy tensors

Paul Bracken*

Department of Mathematics, University of Texas, TX 78539-2999 Edinburg, USA

* **Correspondence:** paul.bracken@utrgv.edu

Abstract: A generalization of the Laplacian for p-forms to arbitrary tensors due to Lichnerowicz will be applied to a 2-tensor which has physical applications. It is natural to associate a divergence-free symmetric 2-tensor to a critical point of a specific variational problem and it is this 2-tensor that is studied. Numerous results are obtained for the stress-energy tensor, such as its divergence and Laplacian. A remarkable integral formula involving a symmetric 2-tensor and a conformal vector field is obtained as well.

Keywords: Basis; tensor; connection; differential system; Laplacian; bundle; harmonic map
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1. Introduction

Variational problems arise in various areas of mathematics and physics. Suppose (M, g) is a Riemannian manifold with volume form dv_M, it is the case that functionals of the form

$$I(\varphi, g) = \int_M \sigma(\varphi, g)\, dv_M \tag{1.1}$$

very often occur [1, 2]. Here φ could be a mapping between Riemannian manifolds of a vector bundle valued differential form. Given a variational problem starting from (1.1), the stress-energy tensor S can be derived by considering variations of the metric on M. If I has a critical point with respect to variations of φ, then the stress-energy tensor is divergence free, and there are conservation laws.

To provide some motivation, let (M, g) and (N, h) be two smooth Riemannian manifolds which are connected, compact, orientable and without boundary, and $\varphi : (M, g) \to (N, h)$ a smooth map. The differential of φ which is $d\varphi$ can be thought of as a section of the bundle $T^*M \otimes \varphi^{-1}TN$, with norm $|d\varphi|$. If $\{x^i\}$ and $\{u^a\}$ constitute local coordinate systems around x and $\varphi(x)$, respectively, then in terms of coordinates, we can write

$$|d\varphi|^2 = g^{ij}\, h_{ab}(\varphi) \frac{\partial \varphi^a}{\partial x^i} \frac{\partial \varphi^b}{\partial x^j}, \tag{1.2}$$

where $(\partial\varphi^a/\partial x^i)$ is the local representation of $d\varphi$. Then the energy density of φ can be defined as $e(\varphi) = 1/2|d\varphi|^2$ and the energy density of the field is given by the positive functional $E(\varphi) = \int_M e(\varphi)\, dv_M$.

A large class of maps which come up in physics, especially in gravity, are called harmonic. A mapping $\varphi : M \to N$ is harmonic if and only if it is an extremal of the energy. Consequently, it is the case that a map φ is harmonic if and only if it satisifes the Euler-Lagrange equation

$$\tau(\varphi) = -d^* d\,\varphi = \mathrm{tr}\nabla\, d\varphi.$$

This defines the tension field of φ, and may be expressed in local coordinates on M and N as follows

$$\begin{aligned}\nabla_{\partial_i}(d\varphi) &= (\frac{\partial\varphi^a_j}{\partial x^i})\, dx^j \frac{\partial}{\partial u^a} + \varphi^a_j(\nabla_{\partial_i} dx^j)\frac{\partial}{\partial u^a} + \varphi^a_j\, dx^j\, \nabla_{\partial_i}\frac{\partial}{\partial u^a} \\ &= \varphi^a_{ij} - {}^M\Gamma^k_{ij}\varphi^a_k + {}^N\Gamma^b_{b\gamma}\varphi^\gamma_j.\end{aligned} \tag{1.3}$$

The tension field is the trace of (1.3),

$$\tau(\varphi)^a = g^{ij}(\nabla\, d\varphi)^a_{ij} = -\Delta\varphi^a + {}^N\Gamma^a_{b\gamma}\varphi^b_i\varphi^\gamma_j g^{ij}. \tag{1.4}$$

Thus (1.4) is a semilinear, elliptic, second-order system. If N is the space $\mathbb{R}$, a harmonic map is called a harmonic function [4, 7, 8].

2. Energy Functional and Critical Point

Now let us extend this idea to another object which may be defined on a manifold. Let M be a Riemannian manifold and E a Riemannian vector bundle over M, where each fiber carries a positive definite inner product denoted by $\langle\cdot,\cdot\rangle^E$. Let $\Omega^p(E)$ be the space of smooth p-forms which have values in E, where it is assumed throughout that $p \geq 1$. For $\omega \in \Omega^p(E)$, define the energy functional

$$I(\omega, g) = \int_M \langle\omega(e_{i_1},\dots,e_{i_p}), \omega(e_{i_1},\dots,e_{i_p})\rangle^E\, dv_m. \tag{2.1}$$

where $\{e_i\}$ is an orthonormal basis on M and repeated indices are summed for $1 \leq i_1,\dots,i_p \leq m$ and $m = \dim M$. With respect to a local coordinate system $\{x^i\}$ on M and local frame $\{s_a\}$ of E, the norm of ω, which is the integrand of (2.1) can be written

$$|\omega|^2 = \langle\omega(e_{i_1},\dots,e_{i_p}), \omega(e_{i_1},\dots,e_{i_p})\rangle = g^{i_1 j_1}\cdots g^{i_p j_p}\omega^a_{i_1\cdots i_p}\omega^b_{j_1\cdots j_p}h_{ab}. \tag{2.2}$$

Suppose M is compact, then vary the integral (2.1) with respect to metric g. If $g(u)$ is a smooth, one-parameter family of metrics such that $g(0) = g$, then the variation $\delta g = \partial g/\partial u|_{u=0}$ is a smooth symmetric tensor on M.

Theorem 1. For $\omega \in \Omega^p(E)$ and $p \geq 1$,

$$\frac{dI}{du}|_{u=0} = \int_M \langle S(\omega), \frac{\partial g}{\partial u}|_{u=0}\rangle\, dv_M, \tag{2.3}$$

where $S(\omega)$ is the symmetric two-tensor defined by

$$S(\omega) = \frac{1}{2}|\omega|^2 g - p\sum_{i_2,\cdots,i_p}\langle\omega(\cdot, e_{i_2},\dots,e_{i_p}), \omega(\cdot, e_{i_2},\dots,e_{i_p})\rangle^E. \tag{2.4}$$

where $\{e_i\}$ is an orthonormal basis on M.

Proof: Let $\{x^i\}$ be a local coordinate system on M, and $\{s_a\}$ a local frame for E,

$$\frac{dI(\omega)}{du}|_{u=0} = \int_M \frac{\partial|\omega|^2}{\partial g_{ij}}\delta g_{ij}\, dv_M + \int_M |\omega|^2 \frac{\partial(dv_M)}{\partial g_{ij}}\,\delta g_{ij}. \tag{2.5}$$

The volume form on M is given by

$$dv_M = (\det g)^{1/2}\, dx^1 \wedge \ldots \wedge dx^m,$$

where $\det g$ is the determinant of the metric tensor g_{ij} and therefore,

$$\frac{\partial}{\partial g_{ij}} dv_M = \frac{1}{2}(\det g)^{-1/2}\frac{\partial}{\partial g_{ij}}(\det g)\, dx^1 \wedge \ldots \wedge dx^m = \frac{1}{2} g^{ij}\, dv_M.$$

Differentiating the expression for the metric tensor $g^{ij} g_{jk} = \delta^i_k$, the following relation holds

$$\frac{\partial g^{i_s j_s}}{\partial g_{ij}} = -g^{i_s i} g^{j_s j}.$$

The first term in (2.5) is

$$\frac{\partial|\omega|^2}{\partial g_{ij}} = \frac{\partial}{\partial g_{ij}}(g^{i_1 j_1} \cdots g^{i_p j_p}\omega^a_{i_1\cdots i_p}\omega^b_{j_1\cdots j_p} h_{ab})$$
$$= -\sum_{s=1}^{p} g^{i_1 j_1} \cdots g^{i_s i} g^{j_s j} \cdots g^{i_p j_p}\omega^a_{i_1\cdots i_s\cdots i_p}\omega^b_{j_1\cdots j_s\cdots j_p} \cdot h_{ab}.$$

Therefore,

$$\frac{\partial|\omega|^2}{\partial g_{ij}} g_{ik} g_{jl} = -\sum_{s=1}^{p} g^{i_1 j_1} \cdots \delta^{i_s}_k \delta^{j_s}_l \cdots g^{i_p j_p}\omega^a_{i_1\cdots i_s\cdots i_p}\omega^b_{j_1\cdots j_s\cdots j_p} h_{ab}$$
$$= -p g^{i_2 j_2} \cdots g^{i_p j_p}\omega^a_{k i_2\cdots i_p}\omega^b_{l j_2\cdots j_p} \cdot h_{ab}.$$

□

Definition 1. Let M be an arbitrary not necessarily compact Riemannian manifold, and let E be a Riemannian vector bundle over M. Let $\omega \in \Omega^p(E)$ and define the stress-energy tensor of the form ω to be the following symmetric 2-tensor

$$S(\omega) = \frac{1}{2}|\omega|^2 g - p \sum_{j_1,\cdots,j_p} \langle \omega(\cdot, e_{j_2}, \cdots, e_{j_p}), \omega(\cdot, e_{j_2}, \cdots, e_{j_p})\rangle^E \tag{2.6}$$

at each point x where there is an orthonormal basis $\{e_i\}$.

Let the vector bundle E be endowed with a Riemannian connection denoted by ∇^E so

$$X\langle s, t\rangle^E = \langle \nabla^E_X s, t\rangle^E + \langle s, \nabla^E_X t\rangle^E, \tag{2.7}$$

Theorem 2. Let $\omega \in \Omega^p(E)$ and $S(\omega)$ be the stress-energy tensor associated with ω, then for all $x \in M$ and each $X \in T_x M$,

$$\operatorname{div} S(\omega)(X) = \langle \omega, d\omega \rfloor X\rangle + p\langle d^*\omega, \omega \rfloor X\rangle \tag{2.8}$$

where the contraction of a p-form with a vector field X is given by

$$(\omega\rfloor X)(X_1,\dots,X_{p-1})=\omega(X,X_1,\dots,X_{p-1}).$$

Proof: Let $\{e_i\}$ be an orthonormal basis at x and extend objects to a neighborhood of x. Suppose that $\nabla_{e_i}e_j=0$ at x for all i,j. The tensorial property allows one to evaluate at $X=e_k$ without loss of generality. Consequently,

$$\begin{aligned}\operatorname{div}S(X)&=\sum_j(\nabla_{e_j}S)(e_j,e_k)=\sum_j e_jS(e_j,e_k)=\sum_j e_j\{\frac{1}{2}|\omega|g(e_j,e_k)-p\langle\omega\rfloor e_j,\omega\rfloor e_k\rangle\}\\&=\langle\nabla_{e_k}\omega,\omega\rangle-p\sum_j\langle\nabla_{e_j}(\omega\rfloor e_j),\omega\rfloor e_k\rangle-p\sum_j\langle\omega\rfloor e_j,\nabla_{e_j}(\omega\rfloor e_k)\rangle.\end{aligned}\tag{2.9}$$

At x, it is the case that

$$d^*\omega(e_{j_1},\dots,e_{j_{p-1}})=-\nabla^E_{e_j}(\omega(e_j,e_{j_1},\dots,e_{j_{p-1}})),$$

and

$$d\omega(e_k,e_{i_1},\dots,e_{i_p})=\nabla^E_{e_k}(\omega(e_{i_1},\dots,e_{i_p}))+\sum_{k=1}^{p}(-1)^k\nabla^E_{e_{i_k}}(\omega(e_k,e_{i_1},\dots,\hat{e}_{i_k},\dots,e_{i_p})).$$

Solving for $\nabla^E_{e_k}$ and substituting it into the divergence, we get

$$\begin{gathered}\operatorname{div}S(X)=\langle d\omega\rfloor e_k,\omega\rangle\\-\sum_{i_1,\cdots,i_p}\sum_{k=1}^{p}(-1)^k\langle\nabla^E_{e_{i_k}}(\omega(e_k,e_{i_1},\dots,\hat{e}_{i_k},\dots,e_{i_p})),\omega(e_{i_1},\dots,e_{i_p})\rangle^E\\+p\langle d^*\omega,\omega\rfloor X\rangle-p\sum_j\langle\omega\rfloor e_j,\nabla_{e_j}(\omega\rfloor e_k)\rangle.\end{gathered}$$

The double sum in this can be simplified to the form

$$\begin{gathered}\sum_{i_1,\cdots,i_p}\sum_{k=1}^{p}(-1)^{k+1}\langle\nabla^E_{e_{i_k}}(\omega(e_k,e_{i_1},\dots,\hat{e}_{i_k},\dots,e_{i_p})),\omega(e_1,\dots,e_{i_p})\rangle\\=p\sum_{j=1}^{n}\langle\nabla_{e_j}(\omega\rfloor e_k),\omega\rfloor e_j\rangle.\end{gathered}$$

The claim follows as a result of these

$$\begin{aligned}\operatorname{div}S(\omega)&=\langle\omega,d\omega\rfloor e_k\rangle+p\langle d^*\omega,\omega\rfloor e_k\rangle+p\sum_j\langle\nabla_{e_j}(\omega\rfloor e_k),\omega\rfloor e_j\rangle_j\langle\omega\rfloor e_j,\nabla_{e_j}(\omega\rfloor e_k)\rangle\\&=\langle\omega,d\omega\rfloor e_k\rangle+p\langle d^*\omega,,\omega\rfloor e_k\rangle.\end{aligned}$$

□

The form $\omega\in\Omega^p(E)$ is called harmonic if and only if $d\omega=d^*\omega=0$. Substituting these derivatives into the right-hand side of (2.8), we prove the following result.

Corollary 1. If $\omega \in \Omega^p(E)$ is a harmonic form, then $\operatorname{div} S(\omega) = 0$. □

Clearly, S is symmetric and so given a symmetric, divergence-free 2-tensor S, conservation laws can be formulated by contracting with a Killing vector field. If X is a Killing vector field, then $S \rfloor X$ is also divergence free. However, the converse of Corollary 1 is not true. If $\operatorname{div} S(\omega) = 0$, it may not be concluded that ω is harmonic. However, when ω is a differential of a submersive mapping, there is equivalence.

Corollary 2. Let $\varphi : M \to N$ be a smooth mapping between Riemannian manifolds and let $\omega = d\varphi$ be the corresponding $\varphi^{-1}TN$- valued one-form on M. Then for each vector field X,

$$\operatorname{div} S(d\varphi)(X) = \langle \tau(\varphi),\ d\varphi(X) \rangle. \tag{2.10}$$

In (2.10), $\tau(\varphi)$ is called the tension field of φ and is defined as

$$\tau(\varphi) = -d^* d\varphi. \tag{2.11}$$

If φ is a submersive almost everywhere, then $\tau(\varphi) = 0$ if and only if $\operatorname{div} S(d\varphi) = 0$.

Proof: Substitute $\omega = d\varphi$ into (2.8) and use the identity $d(d\varphi) = 0$ to obtain,

$$\operatorname{div} S(d\varphi)(X) = \langle d\varphi, dd\varphi \rfloor X \rangle + \langle d^* d\varphi, d\varphi \rfloor X \rangle = -\langle \tau(\varphi), d\varphi \rfloor X \rangle. \tag{2.12}$$

Clearly, if $\tau(\varphi) = 0$, then the right side of (2.12) vanishes, $\operatorname{div} S(d\varphi)(X) = 0$, and conversely. □

The form ω is a critical form with respect to variations of the metric, so the conditions under which the stress-energy tensor vanishes should be studied.

Definition 2. The form $\omega \in \Omega^p(E)$ is conformal if the map $X \to \omega \rfloor X$ is conformal for each $x \in X$, or equivalently, ω is conformal if and only if there is a real λ such that

$$\langle \omega \rfloor X, \omega \rfloor Y \rangle = \lambda^2 \langle X, Y \rangle, \qquad X, Y \in T_x M. \tag{2.13}$$

To obtain an expression for λ^2, evaluate $S(\omega)$ in (2.13) on the vectors $X = Y = e_i \in T_x M$ and carry out the trace on both sides

$$m\lambda^2 = |\omega|^2. \tag{2.14}$$

If $\varphi : M \to N$ is a smooth map between Riemannian manifolds, then $d\varphi \in \Omega^1(\varphi^{-1}TN)$ is a conformal form if and only if the map is conformal as well.

Lemma 1. For $\omega \in \Omega^p(E)$ which is not identically zero, the stress-energy tensor (2.6) vanishes identically if and only if $m = 2p$ and ω is conformal.

Proof: Evaluating $S(\omega)$ on the pair (e_i, e_i) we obtain,

$$S(\omega)(e_i, e_i) = \frac{1}{2}|\omega|^2 - p\langle \omega \rfloor e_i, \omega \rfloor e_i \rangle.$$

Tracing on both sides gives

$$\operatorname{tr} S(\omega) = (\frac{m}{2} - p)|\omega|^2.$$

This vanishes for $\omega \neq 0$ when $m = 2p$. The definition of conformal form (2.13) gives equivalence. □

3. Weizenböck Formulas and Applications to the Stress-Energy Tensor

Now consider the Laplacian of the tensor $S(\omega)$. The Laplacian on p-forms was extended to arbitrary tensors on a Riemannian manifold by Lichnerowicz [5,6]. For a 2-tensor Q, the Laplacian on Q is given by

$$\begin{aligned}\Delta Q(X,Y) &= -\mathrm{tr}\nabla^2 Q(X,Y) + S(\mathrm{Ricci}^M(X),Y) + S(X,\mathrm{Ricci}^M(Y)) \\ &\quad - 2\sum_{i,j}\langle R^M(X,e_i)Y,e_j\rangle S(e_i,e_j). \end{aligned} \tag{3.1}$$

where $\{e_i\}$ is an orthonormal basis. The curvature tensor on M is represented by R^M with sign convention

$$R^M(X,Y)Z = -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z. \tag{3.2}$$

In (3.2), Ricci^M is the Ricci tensor defined by $\mathrm{Ricci}^M(X,Y) = \mathrm{tr}(Z \to R^M(X,Z)Y)$.

The following theorem given first by Lichnorowicz [5, 6] is very useful and is presented without proof.

Proposition 1. (Lichnerowicz) Let M be a Riemannian manifold with $\nabla\mathrm{Ricci}^M = 0$. Let Q be a 2-tensor on M, then the divergence commutes with the Laplacian

$$\mathrm{div}(\Delta Q) = \Delta(\mathrm{div} Q), \tag{3.3}$$

where the right-hand side is now the standard Laplacian on one-forms.

Define the symmetric 2-tensor $L(Q)$ to be

$$L(Q) = Q(\mathrm{Ricci}^M(X),Y) + Q(X,\mathrm{Ricci}^M(Y)) - 2\sum_{i,j}\langle R^M(X,e_i)Y,e_j\rangle\, Q(e_i,e_j). \tag{3.4}$$

Suppose $E \to M$ is a vector bundle endowed with a metric and Riemannian connection and associated curvature R^E. The Ricci operator on a p-form ω written as $\mathrm{Ricci}^p = \mathrm{Ricci}$ is defined to be

$$(\mathrm{Ricci}(\omega))(X_1,\dots,X_p) = \sum_{i,k}(R^p(e_i,X_k)\omega)(e_i,X_1,\dots,\hat{X}_k,\dots,X_p). \tag{3.5}$$

In (3.5), R^p is the canonical curvature and is defined to be [4]

$$\begin{aligned}(R^p(X,Y)\omega)(X_1,\dots,X_p) &= R^E(X,Y)(\omega(X_1,\dots,X_p)) \\ &\quad - \sum_k \omega(X_1,\dots,R^M(X,Y)X_k,\dots,X_p). \end{aligned} \tag{3.6}$$

The following theorem is due to Weizenböck [3].

Proposition 2. (Weitzenböck) Let $E \to M$ be a Riemannian vector bundle over a Riemannian manifold M. Then for any $\omega \in \Omega^p(E)$,

$$\begin{aligned} &(i) \qquad \Delta\omega = -\mathrm{tr}\nabla^2\omega + \mathrm{Ricci}(\omega). \\ &(ii) \qquad \tfrac{1}{2}\Delta|\omega|^2 = \langle\Delta\omega,\omega\rangle - |\nabla\omega|^2 - \langle\mathrm{Ricci}(\omega),\omega\rangle. \end{aligned} \tag{3.7}$$

Proof: Let us prove (ii) by using (i) for $-\mathrm{tr}\nabla^2\omega$,

$$\frac{1}{2}\Delta|\omega|^2 = -\frac{1}{2}\mathrm{tr}\nabla\, d|\omega|^2 = -\mathrm{tr}\nabla\langle\, \nabla\omega, \omega\rangle = -\mathrm{tr}\langle\nabla\nabla\omega, \omega\rangle - \langle\nabla\omega, \nabla\omega\rangle$$

$$= \langle\Delta\omega, \omega\rangle - \langle\mathrm{Ricci}(\omega), \omega\rangle - |\nabla\omega|^2.$$

□

It is worth noting a few important applications of this proposition. If $\omega \in \Omega^0(E)$, then (3.7) reduces to

$$\Delta\omega = -\mathrm{tr}\nabla d\omega, \qquad \frac{1}{2}\Delta|\omega|^2 = \langle\Delta\omega, \omega\rangle - |\nabla\omega|^2. \tag{3.8}$$

If $\omega \in \Omega^1(E)$ and $X \in \Gamma(TM)$, then

$$\Delta\omega(X) = -\mathrm{tr}\nabla^2\omega(X) - \sum_s R^E(e_s, X)\omega(e_s) + \sum_s \omega(R^M(e_s, X)e_s), \tag{3.9}$$

where $\{e_s\}$ is an orthonormal basis for TM and

$$\frac{1}{2}\Delta|\omega|^2 = \langle\Delta\omega, \omega\rangle - |\nabla\omega|^2 + \sum_{i,j}\langle R^E(e_i, e_j)\omega(e_i), \omega(e_j)\rangle - \sum_k \langle\omega(\mathrm{Ricci}(e_k)), \omega(e_k)\rangle. \tag{3.10}$$

Theorem 3. Let $X, Y \in \Gamma(TM)$, it holds that

$$(\mathrm{tr}\nabla^2\langle\omega\lrcorner\cdot, \omega\lrcorner\cdot\rangle)(X, Y) = \langle(\mathrm{tr}\nabla^2\omega)\lrcorner X, \omega\lrcorner Y\rangle + \langle\omega\lrcorner X, (\mathrm{tr}\nabla^2\omega)\lrcorner Y\rangle$$
$$+ 2\sum_i\langle(\nabla_{e_i}\omega)\lrcorner X, (\nabla_{e_i}\omega)\lrcorner Y\rangle. \tag{3.11}$$

Proof: Differentiating once gives,

$$\nabla_{e_j}\langle\omega\lrcorner\cdot, \omega\lrcorner\cdot\rangle(X, Y) = \langle\nabla_{e_j}\omega\lrcorner X, \omega\lrcorner Y\rangle + \langle\omega\lrcorner X, \nabla_{e_j}\omega\lrcorner Y\rangle.$$

Differentiating a second time gives

$$\nabla_{e_i}\nabla_{e_j}\langle\omega\lrcorner\cdot, \omega\lrcorner\cdot\rangle(X, Y) = \langle\nabla_{e_i}\nabla_{e_j}\omega\lrcorner X, \omega\lrcorner Y\rangle + \langle\nabla_{e_j}\omega\lrcorner X, \nabla_{e_j}\omega\lrcorner Y\rangle + \langle\nabla_{e_i}\omega\lrcorner X, \nabla_{e_j}\omega\lrcorner Y\rangle$$

$$+\langle\omega\lrcorner X, \nabla_{e_i}\nabla_{e_j}\omega\lrcorner Y\rangle.$$

Tracing on both sides of this equation yields (3.11). □

Theorem 4. Let $\omega \in \Omega^p(E)$ be a vector bundle valued p-form with stress-energy tensor $S(\omega)$. The Laplacian of $S(\omega)$ can be written in the following way,

$$\Delta S(\omega) = 2S(\langle\Delta\omega\lrcorner\cdot, \omega\lrcorner\cdot\rangle) - 2S(\sum_j\langle\nabla_{e_j}\omega\lrcorner\cdot, \nabla_{e_j}\omega\lrcorner\cdot\rangle) - 2S(\langle\mathrm{Ricci}(\omega)\lrcorner\cdot, \omega\lrcorner\cdot\rangle) + L(S(\omega)). \tag{3.12}$$

where Ricci^M is the Ricci tensor and the symmetric 2-tensor $L(S)$ was introduced in (3.4). Substituting Weitzenböck formula (3.7) (i) into (3.11), it follows that

$$\mathrm{tr}\nabla^2\langle\omega\lrcorner\cdot, \omega\lrcorner\cdot\rangle(X, Y) = \langle(\Delta\omega - \mathrm{Ricci}(\omega))\lrcorner X, \omega\lrcorner Y\rangle + \langle\omega\lrcorner X, (\Delta\omega - \mathrm{Ricci}(\omega))\lrcorner Y\rangle$$

$$-2\sum_i \langle \nabla_{e_i}\omega\lrcorner X, \nabla_{e_i}\omega\lrcorner Y\rangle$$

$$= \langle \Delta\omega\lrcorner X, \omega\lrcorner Y\rangle - \langle \mathrm{Ricci}(\omega)\lrcorner X, \omega\lrcorner Y\rangle + \langle \omega\lrcorner X, \Delta\omega\lrcorner Y\rangle - \langle \omega\lrcorner X, \mathrm{Ricci}(\omega)\lrcorner Y\rangle$$
$$-2\sum_i \langle \nabla_{e_i}\omega\lrcorner X, \nabla_{e_i}\omega\lrcorner Y\rangle. \tag{3.13}$$

Substituting (3) into (3.12), it follows by symmetry and linearity that,

$$\Delta S(\omega)(X,Y) = 2S(\langle \Delta\omega\lrcorner X, \omega\lrcorner Y\rangle) - 2S(\sum_i \langle \nabla_{e_i}\omega\lrcorner X, \nabla_{e_i}\omega\lrcorner Y\rangle)$$

$$-2S(\langle \mathrm{Ricci}(\omega)\lrcorner X, Y\rangle) + L(S(\omega))(X,Y).$$

This is the required result. □

For any 2-tensor $Q \in \Gamma(\otimes^2 T^*M)$, define the p-th stress-energy tensor associated to Q to be the 2-tensor

$$\mathcal{S}_p(Q) = \frac{1}{2}(\mathrm{tr}_g Q)g - p\,\mathrm{sym}\,Q, \tag{3.14}$$

where $\mathrm{tr}Q = \sum_i Q(e_i, e_i)$, and $\{e_i\}$ is orthonormal with respect to the metric g. Moreover, define

$$(\mathrm{sym}\,Q)(X,Y) = \frac{1}{2}(Q(X,Y) + Q(Y,X)). \tag{3.15}$$

Then sym Q in (3.15) is called the symmetrization of Q. For a p-form $\omega \in \Omega^p(E)$, this is simply

$$S(\omega) = \mathcal{S}_p(\langle \omega\lrcorner\cdot, \omega\lrcorner\cdot\rangle). \tag{3.16}$$

The previous result can be written in completely symmetric form in terms of $\mathcal{S}_p$.

Corollary 3. Let $\omega \in \Omega^p(E)$ be a vector-bundle valued p-form with associated stress-energy tensor $S(\omega)$. The Laplacian of $S(\omega)$ is given by

$$\Delta S(\omega) = 2\mathcal{S}_p(\langle \Delta\omega\lrcorner\cdot, \omega\lrcorner\cdot\rangle) - 2\mathcal{S}_p(\sum_i \langle \nabla_{e_i}\omega\lrcorner \cdot \nabla_{e_i}\omega\lrcorner\cdot\rangle) - 2\mathcal{S}_p(\langle \mathrm{Ricci}(\omega)\lrcorner\cdot, \omega\lrcorner\cdot\rangle) + L(S(\omega)). \tag{3.17}$$

□

It is also worth writing this out for the special case of a 1-form.

Corollary 4. Let $\omega \in \Omega^1(E)$ be a vector bundle valued 1-form, then (3.17) takes the form

$$\Delta S(\omega) = 2\mathcal{S}(\langle \Delta\omega, \omega\rangle) - 2\mathcal{S}(\sum_i \langle \nabla_{e_i}\omega, \nabla_{e_i}\omega\rangle) + 2\mathcal{S}(\sum_i \langle R^E(\omega)\lrcorner\cdot, \omega\lrcorner\cdot\rangle)$$

$$-2\mathcal{S}\Big(\langle \omega(\mathrm{Ricci}^M(\cdot)), \omega\rangle\Big) + L(S(\omega)). \tag{3.18}$$

Moreover, for $\omega \in \Omega^1(E)$, a closed vector bundle valued 1-form,

$$\Delta S(\omega) = 2\mathcal{S}(\langle \Delta\omega, \omega\rangle) - 2S(\nabla\omega) + 2\mathcal{S}(\sum_i \langle R^E(e_i, \cdot)\omega(e_i), \omega(\cdot)\rangle)$$

$$-2\mathcal{S}(\langle \omega(\mathrm{Ricci}^M(\cdot), \omega\rangle) + L(S(\omega)). \tag{3.19}$$

where $S(\nabla\omega)$ denotes the stress-energy tensor of the $T^*M \otimes E$ valued one-form $\nabla_X\omega$.

Proof: First (3.18) is a consequence of (3.17). For (3.19), since ω is closed, $d\omega = 0$, and it follows that $(\nabla_{e_i}\omega)(X) = (\nabla_X\omega)(e_i)$, consequently $\sum_i \langle \nabla_{e_i}\omega, \nabla_{e_i}\omega\rangle = \langle \nabla\omega, \nabla\omega\rangle$. □

Theorem 5. Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds and let $d\varphi \in \Omega^1(\varphi^{-1}TN)$ be its exterior derivative.

$$\Delta S(d\varphi) = -2\mathcal{S}(\langle \nabla\tau(\varphi), d\varphi\rangle) - 2S(\nabla d\varphi) - 2\mathcal{S}(\langle d\varphi(\mathrm{Ricci}^M(\cdot)), d\varphi(\cdot)\rangle)$$
$$- 2\mathcal{S}(\sum_i \langle R^N(d\varphi(e_i), d\varphi(\cdot))\, d\varphi(e_i), d\varphi(\cdot)\rangle) + L(S(d\varphi)), \tag{3.20}$$

where $\tau(\varphi) = -d^* d\varphi$ denotes the tension field of the map φ.

Proof: For the one-form ω, substitute $\omega = d\varphi$ into (3.19). Since it follows that $\Delta d\varphi = (d^*d + dd^*)d\varphi = -d\tau(\varphi) = -\nabla\tau(\varphi)$, and $R^N \circ d\varphi$ is the induced curvature in the bundle $E = \varphi^{-1}TN$. □

Theorem 6. Let $\varphi : (M^2, g) \to (N^2, h)$ be a harmonic map between surfaces where M^2 has Gaussian curvature c^M, then the folowing holds,

$$\Delta S(d\varphi) = -2S(\nabla d\varphi) + 2c^M S(d\varphi). \tag{3.21}$$

In particular, if M has constant curvature, then it follows that $\mathrm{div} S(\nabla d\varphi) = 0$.

Proof: If φ is harmonic, then $\nabla\tau(\varphi) = 0$, so the first term in (3.20) vanishes. Suppose $\sigma(Q) = \frac{1}{2}(\mathrm{tr}Q)g - \mathrm{sym}\, Q$ is a stress-energy tensor associated to the 2-tensor Q on M. Suppose z is an isothermal coordinate, then $\sigma(Q)$ can be expressed in diagonal form $\sigma(Q) = a\, dz \otimes dz + \bar{a}\, d\bar{z} \otimes d\bar{z}$ for some function $a = a(z, \bar{z})$. If $\sigma(Q)$ is divergence free, then $a_{\bar{z}}$ vanishes, so $\sigma(Q) = a\, dz \otimes dz$ defines a quadratic differential form on M^2. When φ is a smooth map, then locally $\varphi^* h$ can be diagonalized to the form $\varphi^* h = \lambda_1\, \omega_1 \otimes \omega_1 + \lambda_2 \omega_2 \otimes \omega_2$, where $\{\omega_1, \omega_2\}$ is an orthonormal basis of 1-forms. If M and N have Gaussian curvatures c^M and c^N, then letting $\{e_1, e_2\}$ be the basis dual to $\{\omega_1, \omega_2\}$, summing on $i = 1, 2$ we have $\langle R^N(d\varphi(e_i), d\varphi(\cdot))d\varphi(e_i), d\varphi(\cdot)\rangle = c^N \lambda_1 \lambda_2 g$, and consequently $\sigma(\langle R^N(d\varphi(e_i), d\varphi(\cdot))d\varphi(e_i), d\varphi(\cdot)\rangle) = 0$. Moreover, $L(S(\varphi)) = 4c^M S(d\varphi)$ and $\sigma(\langle d\varphi(\mathrm{Ricci}^M(\cdot)), d\varphi(\cdot)\rangle) = c^M S(d\varphi)$, so substituting these calculations into (3.20), equation (3.21) is the result.

Recall that if $\omega \in \Omega^p(E)$ is harmonic, then $\mathrm{div}\, S(\omega) = 0$. If we use $\omega = d\varphi \in \Omega^1(E)$, then $\mathrm{div} S(d\varphi) = 0$. Take the divergence of both sides of (3.21) and note that c^M is constant, then the result follows since divergence and Laplacian commute be Proposition 1. □

Now $\tau(\varphi)$ defines an energy functional which is called the biharmonic energy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2\, dv_M, \tag{3.22}$$

and has the associated stress-energy tensor $S_2(d\varphi)$,

$$S_2(d\varphi) = \frac{1}{2}|\tau(\varphi)|^2 g + 2\mathcal{S}(\langle \nabla\tau(\varphi), d\varphi\rangle). \tag{3.23}$$

Theorem 7. Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds and let $d\varphi \in \Omega^1(\varphi^{-1}TN)$ be the derivative.

(i) The Laplacian of $S(d\varphi)$ is given as

$$\Delta S(d\varphi) = \frac{1}{2}|\tau(\varphi)|^2 g - S_2(d\varphi) - 2S(\nabla d\varphi) - 2\mathcal{S}(\langle d\varphi(\mathrm{Ricci}^M(\cdot), d\varphi(\cdot)\rangle)$$

$$+\,2\mathcal{S}(\sum_i \langle R^N(d\varphi(e_i), d\varphi(\cdot))\, d\varphi(e_i), d\varphi(\cdot)\rangle^{TN}) + L(S\,(d\varphi)). \tag{3.24}$$

(ii) If $\varphi : M \to N$ is an isometric immersion,

$$\frac{1}{2}|\tau(\varphi)|^2 g - S_2(d\varphi) - 2S\,(\nabla d\varphi) - 2\mathcal{S}(\mathrm{Ricci}^M)$$
$$+\,2\mathcal{S}(\sum_i \langle R^N(d\varphi(e_i), d\varphi(\cdot))\, d\varphi(e_i), d\varphi(\cdot)\rangle) = 0. \tag{3.25}$$

(iii) If N is a space form of constant curvature $K^N = -1, 0, +1$,

$$\frac{1}{2}|\tau(\varphi)|^2 g - S_2(d\varphi) - 2\mathcal{S}(d\gamma) - 2\mathcal{S}(\mathrm{Ricci}^M) + 2(m-1)K^N S\,(d\varphi) = 0. \tag{3.26}$$

Proof: (i) Let $\omega = d\varphi$ then solve for the second term in (3.23) and substitute it into the previous result (3.21) which gives (3.24). (ii) If $\varphi : (M, g) \to (N, h)$ is an isometric immersion, then

$$S\,(d\varphi) = \frac{1}{2}(m-2)\,g, \tag{3.27}$$

hence the left-hand side of (3.24) as well as $L(S\,d\varphi))$ both vanish. (iii) Finally, if we map into a space form, the second fundamental form $\nabla d\varphi$ can be identified with the derivative of the associated Gauss map Γ and (3.26) follows. □

As an application of this theorem, suppose φ is a minimal immersion of a surface into Euclidean space, then $\tau(\varphi) = 0$, $S_2(d\varphi) = 0$ and $\sigma(\mathrm{Ricci}^M) = 0$. Then equation (3.26) implies that $S\,(d\gamma) = 0$.

If the divergence of both sides of (3.26) is worked out, since $\mathrm{div}\mathcal{S}(\mathrm{Ricci}^M) = 0$ as a consequence of the Bianchi identity, (3.26) implies that if $\varphi : M \to N$ is an isometric immersion into a space form, it follows that

$$\frac{1}{2}d|\tau(\varphi)|^2 - \mathrm{div}S_2(d\varphi) - 2\mathrm{div}\,S\,(d\gamma) = 0. \tag{3.28}$$

4. Further Results Including an Integral Theorem

Let us examine the influence of conformal vector fields to finish [3]. A monotonicity formula describes the growth properties of extremals of the functional (2.1) and may be used to establish their regularity in appropriate situations.

Let (M, g) be a Riemannian manifold of dimension m. A vector field X on M is called conformal if it satisfies

$$\mathcal{L}_X\, g = \xi g \tag{4.1}$$

for some function $\xi : M \to \mathbb{R}$ and $\mathcal{L}$ is the Lie derivative in the direction of X. Equivalently, X is conformal if and only if

$$\mathrm{sym}\, g(\nabla_i) = \frac{1}{m}(\mathrm{div}X)g, \tag{4.2}$$

in which case, we have $\xi = (2/m)\mathrm{div}X$.

Theorem 8. Let T be a symmetric 2-tensor, and let X be a conformal vector field, then

$$\mathrm{div}(T \lrcorner X) = (\mathrm{div}\,T)(X) + \frac{1}{m}(\mathrm{div}X)\cdot \mathrm{tr}T. \tag{4.3}$$

Proof: Let $\{e_i\}_1^m$ be a locally defined frame field. If X is a conformal vector field, the divergence can be calculated as follows,

$$\begin{aligned}\mathrm{div}(T \lrcorner X) &= \nabla_{e_i}(T \lrcorner X)(e_i) = (\nabla_{e_i} T)(X, e_i) + T(\nabla_{e_i} X, e_i) \\ &= (\mathrm{div} T)(X) + T(\langle \nabla_{e_i} X, e_j \rangle e_j, e_i).\end{aligned}$$

Since T is a symmetric 2-tensor, the definition of conformal implies that

$$\begin{aligned}\mathrm{div}(T \lrcorner X) &= (\mathrm{div} T)(X) + \frac{1}{2}(\langle \nabla_{e_i} X, e_j \rangle + \langle \nabla_{e_j} X, e_i \rangle) T(e_i, e_j) \\ &= (\mathrm{div} T)(X) + \frac{1}{m} \mathrm{div} X \, g(e_i, e_j) \, T(e_i, e_j) = (\mathrm{div} T) + \frac{1}{m} \mathrm{div} X \, \mathrm{tr} \, T,\end{aligned}$$

since we have $\mathrm{tr} T = g(e_i, e_j) \, T(e_i, e_j)$.

Integrate (4.3) over a compact region, U, to obtain the monotonicity formula. The divergence theorem permits us to write

$$\int_U \mathrm{div}(T \lrcorner X) \, dv_M = \int_{\partial U} T(X, \mathbf{n}) \, da_M. \tag{4.4}$$

where da_M is the volume form on ∂U.

Theorem 9. Let (M, g) be an oriented Riemannian manifold and let X be a conformal vector field on M. Suppose that $U \subset M$ is a compact region with a smooth boundary ∂U. Then for any symmetric 2-tensor T on M,

$$\int_{\partial U} T(X, \mathbf{n}) \, da_M = \int_U (\mathrm{div} T)(X) \, dv_M + \frac{1}{m} \int_M \mathrm{div} X \, \mathrm{tr} T \, dv_M, \tag{4.5}$$

where dv_M is the volume element on M and da_M is the volume element on ∂U, $\mathbf{n}$ the outward pointing normal on ∂U. □

This is a remarkable theorem as there are numerous applications of it, but only one will be presented here. Let $\varphi : B^m \to (N, h)$ be a harmonic map from the Euclidean m-ball of radius R and $S = e(\varphi) g - \varphi^* h$ the corresponding stress-energy tensor. Clearly, $\mathrm{div} S = 0$ and $\mathrm{tr} S = e \, \mathrm{tr} g - \mathrm{tr} \varphi^* h = e \, m - 2e = (m-2) \, e(\varphi)$. Take X to be the conformal vector field $X = r \partial_r$ where $r = |\mathbf{x}|$, $\mathbf{x} \in \mathbb{R}^m$. By direct calculation, $\mathrm{div} X = m$, hence substituting these facts into (4.5), the following result holds for $r < R$,

$$\int_{\partial B^m} S(r \partial_r, \partial_r) \, da = \int_{B^m} (m-2) \, e(\varphi) \, dv_M. \tag{4.6}$$

Substituting for S yields

$$(m-2) \int_{B^m} e(\varphi) \, dv_M = \int_{\partial B^m} r \{ e(\varphi) - h(\frac{\partial \varphi}{\partial r}, \frac{\partial \varphi}{\partial r}) \} \, da_M. \tag{4.7}$$

The following conclusion can be drawn from (4.7). If $m > 2$ and $\varphi|_{\partial B_r^m} = c$ for some $r < R$ and constant c, then φ must be constant, that is, for $e(\varphi)|_{\partial B^m} = \frac{1}{2} h(\partial_r \varphi, \partial_r \varphi)$ gives the upper bound

$$\int_{B^m} e(\varphi) \, dv_M \leq 0,$$

which in turn implies that $e(\varphi) \equiv 0$.

Conflict of Interest

The author declares no conflicts of interest in this paper.

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