Mathematics

Research article

The exact traveling wave solutions of a class of generalized Black-Scholes equation

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Abstract: In this paper, the traveling wave solutions of a class of generalized Black-Scholes equation are considered. By using the first integral method and the G'/G-expansion method, several exact traveling wave solutions of the equation are obtained.

1. Introduction

In financial mathematics and financial engineering, the classical Black-Scholes equation is a practical partial differential equation. In 1973, Black and Scholes derived the famous Black-Scholes Option Pricing Model [1]. In [2], Sunday O. Edeki etc successfully calculated the European option valuation using the Projected Differential Transformation Method. The results obtained converge faster to their associated exact solutions. In [3], the author studied the Black-Scholes equation in stochastic volatility model. In [4], the author considered to deal with the Black-Scholes equation in financial mathematics by the volatility of a variable and the abstract boundary conditions.

In this paper, we consider to obtain the traveling wave solutions of a class of generalized Black-Scholes equation by using the first integral method and the G'/G-expansion method. In 2002, Feng first proposed the first integral method [5]. This method has been widely used to solve the exact solutions of some partial differential equations. In [6], authors applied first integral method and functional variable method to obtain optical solitons from the governing nonlinear Schrödinger equation with spatio-temporal dispersion. In [7], the first integral method is applied for solving the system of nonlinear partial differential equations which are (2 + 1)-dimensional Broer-Kaup-Kupershmidt system and (3 + 1)-dimensional Burgers equations exactly. In [8], authors applied first integral method to construct travelling wave solutions of modified Zakharov-Kuznetsov equation and ZK-MEW equation. This

method can also be applied to other systems of nonlinear differential equations [9–12]. The advantage of the first integral method is that the calculation is concise. A more accurate traveling wave solution can be obtained by the first integral method. In 2008, Mingliang Wang et al introduced the G'/G-expansion method in detail [13]. In [14], the G'/G-expansion method is applied to address the resonant nonlinear Schrödinger equation with dual-power law nonlinearity. In [15], the author constructed the traveling wave solutions involving parameters for some nonlinear evolution equations in mathematical physics via the $(2 + 1)$-dimensional Painlevé integrable Burgers equations, the $(2 + 1)$-dimensional Nizhnik-Novikov-Vesselov equations, the $(2 + 1)$-dimensional Boiti-Leon-Pempinelli equations and the $(2 + 1)$-dimensional dispersive long wave equations by using the G'/G-expansion method. In [16], a generalized G'/G-expansion method is proposed to seek exact solutions of the mKdV equation with variable coefficients. The G'/G-expansion method has been proposed to construct more explicit travelling wave solutions to many nonlinear evolution equations [17–19]. The performance of this method is effective, simple, convenient and gives many new solutions.

The classical celebrated Black-Scholes option pricing model is as follows:

$$\frac{\partial f}{\partial \tau} + \frac{1}{2} S^2 \delta^2 \frac{\partial^2 f}{\partial S^2} + rS \frac{\partial f}{\partial S} - rf = 0.$$

We consider the class of generalized Black-Scholes equation is as follows:

$$v_t + \frac{1}{2} A^2 x^2 v_{xx} + Bxv_x - Cv + Dv^3 = 0, \tag{1}$$

where $v = v(t, x), A, B, C, D \neq 0$ are arbitrary constants, when $D = 0$, (1) changes to the classical Black-Scholes equation.

Using the wave transformation $v(t, x) = v(\xi)$, $\xi = \ln x - at$, and a is wave velocity, we have the following ordinary differential equation,

$$-\frac{1}{2} A^2 v'' + (\frac{1}{2} A^2 - B + a)v' + Cv - Dv^3 = 0. \tag{2}$$

Letting $w = v'$, (2) is equivalent to the autonomous system,

$$\begin{cases} v' = w, \\ \\ w' = \frac{A^2 - 2B + 2a}{A^2} w + \frac{2C}{A^2} v - \frac{2D}{A^2} v^3. \end{cases} \tag{3}$$

2. Traveling wave solutions of (1) by using the first integral method

In this section, we apply the first integral method to obtain the traveling wave solution to (1). We assume that $p(v, w) = \sum_{i=0}^{N} \alpha_i(v)w^i$ is an irreducible polynomial in $C[v, w]$, and $p(v, w) = \sum_{i=0}^{N} \alpha_i(v)w^i = 0$ is a first integral of (3), such that,

$$\frac{dP}{d\xi}\bigg|_{(3)} = 0.$$

Owing to Division Theorem, there exists a polynomial $g(v) + h(v)w$ in the complex domain $C(v, w)$, such that,

$$\frac{dP}{d\xi} = \frac{\partial P}{\partial v} \frac{dv}{d\xi} + \frac{\partial P}{\partial w} \frac{dw}{d\xi} = [g(v) + h(v)w] \sum_{i=0}^{N} \alpha_i(v)w^i = 0. \tag{4}$$

Here, we mainly consider (4) in two cases: $N = 1$ and $N = 2$.

2.1. N=1

At present,

$$P(v, w) = \alpha_0(v) + \alpha_1(v)w = 0.$$

From (4), we have

$$
\begin{aligned}
\frac{dP}{d\xi} &= \frac{d\alpha_0}{dv}w + \frac{d\alpha_1}{dv}w^2 + \alpha_1(v)(\frac{A^2 - 2B + 2a}{A^2}w + \frac{2C}{A^2}v - \frac{2D}{A^2}v^3) \\
&= g(v)\alpha_0(v) + [g(v)\alpha_1(v) + h(v)\alpha_0(v)]w + h(v)\alpha_1(v)w^2.
\end{aligned}
\tag{5}
$$

By observing the coefficients of $w^i (i = 0, 1)$ of the two sides of (5), obviously, we have

$$
\begin{cases}
\dfrac{d\alpha_1}{dv} = h(v)\alpha_1(v), \\[2mm]
\dfrac{d\alpha_0}{dv} = g(v)\alpha_1(v) + h(v)\alpha_0(v) - \dfrac{A^2 - 2B + 2a}{A^2}\alpha_1(v), \\[2mm]
\alpha_0(v)g(v) = \alpha_1(v)(\dfrac{2C}{A^2}v - \dfrac{2D}{A^2}v^3).
\end{cases}
\tag{6}
$$

Since $\alpha_i(v)(i = 0, 1)$ are polynomials, then from the first equation of (6), we deduce that $\alpha_1(v)$ is constant and $h(v) = 0$. For simplification, taking $\alpha_1(v) = 1$. In order to keep balancing the degree of $g(v)$, $\alpha_1(v)$ and $\alpha_0(v)$ in the second and the third equations of (6), one can conclude that deg $g(v) = 1$. Suppose that $g(v) = g_0v + d_0$, where g_0 and d_0 are arbitrary constants. By solving the above equations, one can obtain

$$
\begin{cases}
\alpha_1(v) = 1, \\[2mm]
\alpha_0(v) = \dfrac{1}{2}g_0v^2 + (d_0 - \dfrac{A^2 - 2B + 2a}{A^2})v + d_1,
\end{cases}
$$

where d_1 is an integration constant. Substituting $g(v)$, $\alpha_1(v)$ and $\alpha_0(v)$ into the third equation of (6) and setting all the coefficients of $v^i (i = 0, 1, 2, 3)$ to be zeros, then one can get

$$
\begin{cases}
\dfrac{1}{2}g_0^2 = -\dfrac{2D}{A^2}, \\[2mm]
\dfrac{3}{2}g_0d_0 - g_0\dfrac{A^2 - 2B + 2a}{A^2} = 0, \\[2mm]
g_0d_1 + d_0^2 - d_0\dfrac{A^2 - 2B + 2a}{A^2} = \dfrac{2C}{A^2}, \\[2mm]
d_0d_1 = 0.
\end{cases}
\tag{7}
$$

From the last equation of (7), we consider to solve (7) in two cases $d_0 = 0$ or $d_0 \neq 0$.

Case 1: $d_0 = 0$

By solving (7), we have

$$\begin{cases} g_0 = \pm 2\sqrt{\dfrac{-D}{A^2}}, \\[2ex] d_0 = 0, \\[2ex] d_1 = \pm\dfrac{C}{A^2}\sqrt{\dfrac{A^2}{-D}}, \end{cases}$$

and $D < 0, A^2 - 2B + 2a = 0$, such that,

$$\begin{cases} \alpha_0(v) = \pm\sqrt{\dfrac{-D}{A^2}}v^2 \pm \dfrac{C}{A^2}\sqrt{\dfrac{A^2}{-D}}, \\[2ex] \alpha_1(v) = 1. \end{cases}$$

So,

$$P(v, w) = \pm\sqrt{\frac{-D}{A^2}}v^2 \pm \frac{C}{A^2}\sqrt{\frac{A^2}{-D}} + w = 0.$$

One can obtain the following one order ordinary differential equation,

$$w = \frac{dv}{d\xi} = \mp\sqrt{\frac{-D}{A^2}}v^2 \mp \frac{C}{A^2}\sqrt{\frac{A^2}{-D}}. \tag{8}$$

Integrating (8) once with respect to ξ, we can get the following results.

I: $\frac{C}{D} > 0$, we have

$$v(\xi) = \frac{\sqrt{\frac{C}{D}}\left(1 + \xi_0 e^{\mp 2\sqrt{\frac{-D}{A^2}}\xi}\right)}{1 - \xi_0 e^{\mp 2\sqrt{\frac{-D}{A^2}}\xi}},$$

where ξ_0 is an integration constant.

The traveling wave solution to (1) can be got as follows:

$$v(t, x) = \frac{\sqrt{\frac{C}{D}}\left(1 + \xi_0 e^{\mp 2\sqrt{\frac{-D}{A^2}}(\ln x - at)}\right)}{1 - \xi_0 e^{\mp 2\sqrt{\frac{-D}{A^2}}(\ln x - at)}}.$$

II: $\frac{C}{D} < 0$, we have

$$v(\xi) = \sqrt{\frac{C}{-D}}\tan\left[\mp\sqrt{\frac{C}{A^2}}\xi + \xi_1\right],$$

where ξ_1 is an arbitrary integration constant.

The traveling wave solution to (1) can be got as follows:

$$v(t, x) = \sqrt{\frac{C}{-D}} \tan[\mp \sqrt{\frac{C}{A^2}}(\ln x - at) + \xi_1].$$

Case 2: $d_0 \neq 0$

From (7), we have

$$\begin{cases} g_0 = \pm 2\sqrt{\dfrac{-D}{A^2}}, \\[3mm] d_0 = \dfrac{2(A^2 - 2B + 2a)}{3A^2}, \\[3mm] d_1 = 0, \end{cases}$$

and $D < 0$, $6C + (A^2 - 2B + 2a) = 0$, such that,

$$\begin{cases} \alpha_0(v) = \pm\sqrt{\dfrac{-D}{A^2}}v^2 - \dfrac{A^2 - 2B + 2a}{3A^2}v, \\[3mm] \alpha_1(v) = 1. \end{cases}$$

So,

$$P(v, w) = \pm\sqrt{\frac{-D}{A^2}}v^2 - \frac{A^2 - 2B + 2a}{3A^2}v + w = 0.$$

One can obtain the following one order ordinary differential equation,

$$w = \frac{dv}{d\xi} = \mp\sqrt{\frac{-D}{A^2}}v^2 + \frac{A^2 - 2B + 2a}{3A^2}v. \tag{9}$$

Integrating equation (9) once with respect to ξ, we can get the following results,

$$v(\xi) = \frac{\frac{A^2 - 2B + 2a}{3A^2}\xi_2 e^{\frac{A^2 - 2B + 2a}{3A^2}\xi}}{1 \pm \sqrt{\frac{-D}{A^2}}\xi_2 e^{\frac{A^2 - 2B + 2a}{3A^2}\xi}},$$

where ξ_2 is an integration constant.

The exact traveling wave solution to (1) can be got as follows:

$$v(t, x) = \frac{\frac{A^2 - 2B + 2a}{3A^2}\xi_2 e^{\frac{A^2 - 2B + 2a}{3A^2}(\ln x - at)}}{1 \pm \sqrt{\frac{-D}{A^2}}\xi_2 e^{\frac{A^2 - 2B + 2a}{3A^2}(\ln x - at)}}.$$

2.2. N = 2

At present,

$$P(v, w) = \alpha_0(v) + \alpha_1(v)w + \alpha_2(v)w^2 = 0. \tag{10}$$

From (4), we have

$$\frac{dP}{d\xi} = \frac{d\alpha_0}{dv}w + \frac{d\alpha_1}{dv}w^2 + \alpha_1(v)(\frac{A^2 - 2B + 2a}{A^2}w + \frac{2C}{A^2}v - \frac{2D}{A^2}v^3)$$

$$+ \frac{d\alpha_2}{dv}w^3 + 2\alpha_2(v)w(\frac{A^2 - 2B + 2a}{A^2}w + \frac{2C}{A^2}v - \frac{2D}{A^2}v^3)$$

$$= g(v)\alpha_0(v) + [g(v)\alpha_1(v) + h(v)\alpha_0(v)]w + [g(v)\alpha_2(v) + h(v)\alpha_1(v)]w^2 + h(v)\alpha_2(v)w^3. \tag{11}$$

By observing the coefficients of $w^i (i = 0, 1, 2, 3)$ of the two sides of (11), obviously, we have

$$\begin{cases} \dfrac{d\alpha_2}{dv} = h(v)\alpha_2(v), \\ \dfrac{d\alpha_1}{dv} = g(v)\alpha_2(v) + h(v)\alpha_1(v) - 2\dfrac{A^2 - 2B + 2a}{A^2}\alpha_2(v), \\ \dfrac{d\alpha_0}{dv} = g(v)\alpha_1(v) + h(v)\alpha_0(v) - \dfrac{A^2 - 2B + 2a}{A^2}\alpha_1(v) - 2\alpha_2(v)(\dfrac{2C}{A^2}v - \dfrac{2D}{A^2}v^3), \\ \alpha_0(v)g(v) = \alpha_1(v)(\dfrac{2C}{A^2}v - \dfrac{2D}{A^2}v^3). \end{cases} \tag{12}$$

Similarly, from the first equation of (12), we deduce that $\alpha_2(v)$ is constant and $h(v) = 0$. From the second and the third equations of (12), we assume that deg $g(v) = k$, then deg $\alpha_1(v) = k + 1$ and deg $\alpha_0(v) = 2k + 2$. By the last equation of (12), we have the degree of the polynomial $\alpha_1(v)(\frac{2C}{A^2}v - \frac{2D}{A^2}v^3)$ is $k + 4$, and the degree of the polynomial $\alpha_0(v)g(v)$ is $3k + 2$. By balancing the degree of the last equation of (12), we have $k + 4 = 3k + 2$, obviously, $k = 1$. Specially, we conclude that deg $g(v) = 0$, then deg $\alpha_1(v) = 1$. The degree on both sides of the last equation of (12) is still true.

Case 1: deg $g(v) = 0$

Assume $g(v) = g_1$. From (12), we have

$$\begin{cases} \alpha_0(v) = \dfrac{D}{A^2}v^4 + (\dfrac{1}{2}g_1^2 - \dfrac{3}{2}g_1\dfrac{A^2 - 2B + 2a}{A^2} + \dfrac{(A^2 - 2B + 2a)^2}{A^4} - \dfrac{2C}{A^2})v^2 \\ \\ +d_2(g_1 - \dfrac{A^2 - 2B + 2a}{A^2})v + d_3, \\ \\ \alpha_1(v) = (g_1 - 2\dfrac{A^2 - 2B + 2a}{A^2})v + d_2, \\ \\ \alpha_2(v) = 1, \end{cases} \tag{13}$$

where d_2, d_3 are integration constants. From the last one of equation (12), we have

$$\begin{cases} g_1\dfrac{D}{A^2} = -\dfrac{2D}{A^2}(g_1 - 2\dfrac{A^2 - 2B + 2a}{A^2}), \\[2mm] -d_2\dfrac{2D}{A^2} = 0, \\[2mm] g_1(\dfrac{1}{2}g_1^2 - \dfrac{3}{2}g_1\dfrac{A^2 - 2B + 2a}{A^2} + \dfrac{(A^2 - 2B + 2a)^2}{A^4} - \dfrac{2C}{A^2}) = \dfrac{2C}{A^2}(g_1 - 2\dfrac{A^2 - 2B + 2a}{A^2}), \\[2mm] g_1d_2(g_1 - \dfrac{A^2 - 2B + 2a}{A^2}) = d_2\dfrac{2C}{A^2}, \\[2mm] g_1d_3 = 0. \end{cases} \tag{14}$$

Solving (14), we have

$$\begin{cases} g_1 = \dfrac{4(A^2 - 2B + 2a)}{3A^2}, \\[2mm] 9CA^2 + (A^2 - 2B + 2a)^2 = 0, \\[2mm] d_2 = 0, \\[2mm] d_3 = 0. \end{cases} \tag{15}$$

Then from (15), (13) and (10), one can obtain the following equation,

$$P(v, w) = \dfrac{D}{A^2}v^4 + \dfrac{(A^2 - 2B + 2a)^2}{9A^4}v^2 - \dfrac{2(A^2 - 2B + 2a)}{3A^2}vw + w^2 = 0.$$

Solving the above algebraic equation with respect to the variable w, we have

$$w = \dfrac{dv}{d\xi} = \pm\sqrt{\dfrac{-D}{A^2}v^2 + \dfrac{A^2 - 2B + 2a}{3A^2}}v. \tag{16}$$

Integrating (16) once with respect to ξ, then we have

$$v(\xi) = \dfrac{\dfrac{A^2 - 2B + 2a}{3A^2}\xi_3 e^{\frac{A^2 - 2B + 2a}{3A^2}\xi}}{1 \mp \sqrt{\dfrac{-D}{A^2}}\xi_3 e^{\frac{A^2 - 2B + 2a}{3A^2}\xi}},$$

where ξ_3 is an integration constant.

The exact traveling wave solution to (1) can be got as follows:

$$v(t, x) = \dfrac{\dfrac{A^2 - 2B + 2a}{3A^2}\xi_3 e^{\frac{A^2 - 2B + 2a}{3A^2}(\ln x - at)}}{1 \mp \sqrt{\dfrac{-D}{A^2}}\xi_3 e^{\frac{A^2 - 2B + 2a}{3A^2}(\ln x - at)}}.$$

Case 2: deg $g(v) = 1$

Assume $g(v) = g_2 v + d_4$. From (12), we have

$$
\begin{cases}
\alpha_0(v) = \dfrac{1}{4}(\dfrac{1}{2}g_2^2 + \dfrac{4D}{A^2})v^4 + \dfrac{1}{3}(\dfrac{3}{2}g_2 d_4 - \dfrac{5}{2}g_2\dfrac{A^2 - 2B + 2a}{A^2})v^3 + (d_4 d_5 - d_5\dfrac{A^2 - 2B + 2a}{A^2})v \\
+\dfrac{1}{2}(g_2 d_5 + d_4^2 - 3d_4\dfrac{A^2 - 2B + 2a}{A^2} + 2\dfrac{(A^2 - 2B + 2a)^2}{A^4} - \dfrac{4C}{A^2})v^2 + d_6, \\
\alpha_1(v) = \dfrac{1}{2}g_2 v^2 + (d_4 - 2\dfrac{A^2 - 2B + 2a}{A^2})v + d_5, \\
\alpha_2(v) = 1,
\end{cases}
$$

where d_5, d_6 are integration constants.

From the last equation of (12), we have

$$
\begin{cases}
-g_2\dfrac{D}{A^2} = \dfrac{1}{8}g_2^3 + g_2\dfrac{D}{A^2}, \\[2mm]
-\dfrac{2D}{A^2}(d_4 - 2\dfrac{A^2 - 2B + 2a}{A^2}) = \dfrac{5}{8}g_2^2 d_4 - \dfrac{5}{6}g_2^2\dfrac{A^2 - 2B + 2a}{A^2} + d_4\dfrac{D}{A^2}, \\[2mm]
g_2\dfrac{C}{A^2} - d_5\dfrac{2D}{A^2} = g_2 d_4^2 - \dfrac{7}{3}g_2 d_4\dfrac{A^2 - 2B + 2a}{A^2} + \dfrac{1}{2}g_2^2 d_5 + g_2\dfrac{(A^2 - 2B + 2a)^2}{A^4} - g_2\dfrac{2C}{A^2}, \\[2mm]
\dfrac{-4C(A^2 - 2B + 2a)}{3A^4} = \dfrac{3}{2}g_2 d_4 d_5 + \dfrac{1}{2}d_4^3 - \dfrac{3(A^2 - 2B + 2a)}{2A^2}d_4^2 \\[2mm]
+d_4(\dfrac{(A^2 - 2B + 2a)^2}{A^4} - \dfrac{2C}{A^2}) - g_2 d_5\dfrac{A^2 - 2B + 2a}{A^2}, \\[2mm]
d_5\dfrac{2C}{A^2} = g_2 d_6 + d_4^2 d_5 - d_4 d_5\dfrac{A^2 - 2B + 2a}{A^2}, \\[2mm]
d_4 d_6 = 0.
\end{cases} \tag{17}
$$

From the last equation of (17), we need to discuss (17) in two cases $d_4 = 0$ or $d_4 \neq 0$.

Case I: $d_4 = 0$

Solving (17), we have

$$
\begin{cases}
g_2 = \pm 4\sqrt{\dfrac{-D}{A^2}}, \\[2mm]
d_4 = 0, \\[2mm]
d_5 = \mp 2\sqrt{\dfrac{-D}{A^2}}\dfrac{C}{D}, \\[2mm]
d_6 = -\dfrac{C^2}{A^2 D},
\end{cases}
$$

and $D < 0, A^2 - 2B + 2a = 0$, such that,

$$\begin{cases} \alpha_0(v) = \dfrac{-D}{A^2}v^4 + \dfrac{2C}{A^2}v^2 - \dfrac{C^2}{A^2D}, \\[4mm] \alpha_1(v) = \pm 2\sqrt{\dfrac{-D}{A^2}}v^2 \mp 2\sqrt{\dfrac{-D}{A^2}}\dfrac{C}{D}, \\[4mm] \alpha_2(v) = 1. \end{cases}$$

So,

$$P(v,w) = \dfrac{-D}{A^2}v^4 + \dfrac{2C}{A^2}v^2 - \dfrac{C^2}{A^2D} + (\pm 2\sqrt{\dfrac{-D}{A^2}}v^2 \mp 2\sqrt{\dfrac{-D}{A^2}}\dfrac{C}{D})w + w^2 = 0.$$

Solving the above algebraic equation with respect to the variable w, we have

$$w = \dfrac{dv}{d\xi} = \mp\sqrt{\dfrac{-D}{A^2}}v^2 \pm \sqrt{\dfrac{-D}{A^2}}\dfrac{C}{D}. \tag{18}$$

Integrating (18) once with respect to ξ, we can get the following results.

I: $\dfrac{C}{D} > 0$.

$$v(\xi) = \dfrac{\sqrt{\dfrac{C}{D}}(1 + \xi_4 e^{\mp 2\sqrt{\frac{-C}{A^2}}\xi})}{1 - \xi_4 e^{\mp 2\sqrt{\frac{-C}{A^2}}\xi}},$$

where ξ_4 is an integration constant.

The traveling wave solution to (1) can be got as follows:

$$v(t,x) = \dfrac{\sqrt{\dfrac{C}{D}}(1 + \xi_4 e^{\mp 2\sqrt{\frac{-C}{A^2}}(\ln x - at)})}{1 - \xi_4 e^{\mp 2\sqrt{\frac{-C}{A^2}}(\ln x - at)}}.$$

II: $\dfrac{C}{D} < 0$.

$$v(\xi) = \sqrt{\dfrac{C}{-D}}\tan[\mp\sqrt{\dfrac{C}{A^2}}\xi + \xi_5],$$

where ξ_5 is an integration constant.

The traveling wave solution to (1) can be got as follows:

$$v(t,x) = \sqrt{\dfrac{C}{-D}}\tan[\mp\sqrt{\dfrac{C}{A^2}}(\ln x - at) + \xi_5].$$

Case II: $d_4 \neq 0$

By solving (17), we have

$$
\begin{cases}
g_2 = \pm 4 \sqrt{\dfrac{-D}{A^2}}, \\[3mm]
d_4 = \dfrac{4(A^2 - 2B + 2a)}{3A^2}, \\[3mm]
C = -\dfrac{(A^2 - 2B + 2a)^2}{9A^2}, \\[2mm]
d_5 = 0, \\[1mm]
d_6 = 0.
\end{cases}
\tag{19}
$$

Substituting (19) into (13), we have

$$
\begin{cases}
\alpha_0(v) = -\dfrac{D}{A^2} v^4 \mp \dfrac{2}{3} \sqrt{\dfrac{-D}{A^2}} \dfrac{A^2 - 2B + 2a}{A^2} v^3 + \dfrac{(A^2 - 2B + 2a)^2}{9A^4} v^2, \\[4mm]
\alpha_1(v) = \pm 2 \sqrt{\dfrac{-D}{A^2}} v^2 - \dfrac{2(A^2 - 2B + 2a)}{3A^2} v, \\[4mm]
\alpha_2(v) = 1.
\end{cases}
$$

So,

$$
P(v, w) = -\frac{D}{A^2} v^4 \mp \frac{2}{3} \sqrt{\frac{-D}{A^2}} \frac{A^2 - 2B + 2a}{A^2} v^3 + \frac{(A^2 - 2B + 2a)^2}{9A^4} v^2
$$
$$
+ \left(\pm 2 \sqrt{\frac{-D}{A^2}} v^2 - \frac{2(A^2 - 2B + 2a)}{3A^2} v \right) w + w^2 = 0.
$$

Solving the above algebraic equation with respect to the variable w, we have

$$
w = \frac{dv}{d\xi} = \mp \sqrt{\frac{-D}{A^2}} v^2 + \frac{A^2 - 2B + 2a}{3A^2} v.
\tag{20}
$$

Integrating (20) once with respect to ξ, then we have

$$
v(\xi) = \frac{\dfrac{A^2 - 2B + 2a}{3A^2} \xi_6 e^{\frac{A^2 - 2B + 2a}{3A^2} \xi}}{1 \pm \sqrt{\dfrac{-D}{A^2}} \xi_6 e^{\frac{A^2 - 2B + 2a}{3A^2} \xi}},
$$

where ξ_6 is an integration constant.

The traveling wave solution to (1) can be got as follows:

$$
v(t, x) = \frac{\dfrac{A^2 - 2B + 2a}{3A^2} \xi_6 e^{\frac{A^2 - 2B + 2a}{3A^2} (\ln x - at)}}{1 \pm \sqrt{\dfrac{-D}{A^2}} \xi_6 e^{\frac{A^2 - 2B + 2a}{3A^2} (\ln x - at)}}.
$$

3. Traveling wave solutions of (1) by the G'/G-expansion method

In this section, we get the traveling wave solutions to (1) using the G'/G-expansion method. Introducing the solution $v(\xi)$ of equation (2) in a form of finite series:

$$v(\xi) = \sum_{l=0}^{N} a_l \left(\frac{G'(\xi)}{G(\xi)}\right)^l, \tag{21}$$

where a_l are real constants with $a_N \neq 0$ and N is a positive integer that needs to be determined. The function $G(\xi)$ is the solution of the auxiliary linear ODE

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{22}$$

where λ and μ are real constants to be determined. By balancing v'' with v^3 in the equation (2), we have $N + 2 = 3N$, obviously, $N = 1$. Therefore, (21) is written as

$$v(\xi) = a_0 + a_1 \left(\frac{G'(\xi)}{G(\xi)}\right). \tag{23}$$

Correspondingly,

$$v'(\xi) = -a_1 \lambda \frac{G'}{G} - a_1 \left(\frac{G'}{G}\right)^2 - a_1 \mu, \tag{24}$$

$$v''(\xi) = (a_1 \lambda^2 + 2a_1 \mu)\frac{G'}{G} + 3a_1 \lambda \left(\frac{G'}{G}\right)^2 + 2a_1 \left(\frac{G'}{G}\right)^3 + a_\lambda \mu. \tag{25}$$

Substituting (23), (24), (25) and v^3 into (2), we have

$$[-\frac{1}{2}A^2(a_1\lambda^2 + 2a_1\mu) - (\frac{1}{2}A^2 - B + a)a_1\lambda + Ca_1 - 3Da_0^2 a_1]\frac{G'}{G}$$
$$- [\frac{3}{2}A^2 a_1 \lambda + (\frac{1}{2}A^2 - B + a)a_1 + 3Da_0 a_1^2](\frac{G'}{G})^2$$
$$- (A^2 a_1 + Da_1^3)(\frac{G'}{G})^3 - [\frac{1}{2}A^2 a_1 \lambda \mu + (\frac{1}{2}A^2 - B + a)a_1\mu - Ca_0 + Da_0^3] = 0.$$

Setting all the coefficients of $(\frac{G'}{G})^i (i = 0, 1, 2, 3)$ to be zeros, we yield a set of algebraic equations for a_0, a_1, λ and μ as follows:

$$
\begin{cases}
-\dfrac{1}{2}A^2(a_1\lambda^2 + 2a_1\mu) - (\dfrac{1}{2}A^2 - B + a)a_1\lambda + Ca_1 - 3Da_0^2 a_1 = 0, \\[2mm]
\dfrac{3}{2}A^2 a_1 \lambda + (\dfrac{1}{2}A^2 - B + a)a_1 + 3Da_0 a_1^2 = 0, \\[2mm]
A^2 a_1 + Da_1^3 = 0, \\[2mm]
\dfrac{1}{2}A^2 a_1 \lambda \mu + (\dfrac{1}{2}A^2 - B + a)a_1\mu - Ca_0 + Da_0^3.
\end{cases}
$$

Solving the above algebraic equations, we have

$$
\begin{cases}
a_0 = \pm\left(\dfrac{\sqrt{\frac{A^2}{-D}}(A^2 - 2B + 2a)}{3A^2} + \dfrac{\sqrt{\frac{A^2}{-D}}(A^2 - 2B + 2a)^3}{27A^4C}\right), \\[4mm]
a_1 = \pm\sqrt{\dfrac{A^2}{-D}}, \\[4mm]
\lambda = -\left(\dfrac{16(\frac{1}{2}A^2 - B + a)^3}{27A^4C} + \dfrac{A^2 - 2B + 2a}{A^2}\right), \\[4mm]
\mu = \dfrac{(A^2 - 2B + 2a)^6}{729A^8C^2} + \dfrac{(A^2 - 2B + 2a)^4}{27A^6C} + \dfrac{5(A^2 - 2B + 2a)^2}{12A^4} + \dfrac{C}{A^2}.
\end{cases}
\tag{26}
$$

From equations (23) and (26), the solution of (2) can be written as the following equation,

$$
V(\xi) = \mp\left(\frac{\sqrt{\frac{A^2}{-D}}(A^2 - 2B + 2a)}{3A^2} + \frac{\sqrt{\frac{A^2}{-D}}(A^2 - 2B + 2a)^3}{27A^4C}\right) \pm \sqrt{\frac{A^2}{-D}}\left(\frac{G'}{G}\right).
\tag{27}
$$

By solving the second order linear ODE (22), then (27) gives three types traveling wave solutions.

Case 1: $\Delta = \lambda^2 - 4\mu = -\frac{2(A^2 - 2B + 2a)^2}{3A^4} - \frac{4C}{A^2} > 0$, we obtain

$$
\frac{G'}{G} = -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4u}}{2}\frac{C_1 e^{\frac{\sqrt{\lambda^2-4u}}{2}\xi} - C_2 e^{-\frac{\sqrt{\lambda^2-4u}}{2}\xi}}{C_1 e^{\frac{\sqrt{\lambda^2-4u}}{2}\xi} + C_2 e^{-\frac{\sqrt{\lambda^2-4u}}{2}\xi}},
$$

where C_1, C_2 are integration constants. The traveling wave solution to (1) can be got as follows:

$$
v(t, x) = \mp\left(\frac{\sqrt{\frac{A^2}{-D}}(A^2 - 2B + 2a)}{3A^2} + \frac{\sqrt{\frac{A^2}{-D}}(A^2 - 2B + 2a)^3}{27A^4C}\right) \pm \sqrt{\frac{A^2}{-D}}\Bigg\{\frac{(A^2 - 2B + 2a)^3}{27A^4C} + \frac{A^2 - 2B + 2a}{2A^2} +
$$

$$
\frac{1}{2}\sqrt{-\frac{2(A^2 - 2B + 2a)^2}{3A^4} - \frac{4C}{A^2}}\left[\frac{C_1 e^{H_1(\ln x - at)} - C_2 e^{-H_1(\ln x - at)}}{C_1 e^{H_1(\ln x - at)} + C_2 e^{-H_1(\ln x - at)}}\right]\Bigg\}.
$$

where $H_1 = \frac{1}{2}\sqrt{-\frac{2(A^2 - 2B + 2a)^2}{3A^4} - \frac{4C}{A^2}}$. Specially, when $C_1 = C_2$, we obtain traveling wave solution of (1) as follows:

$$
v(t, x) = \mp\left(\frac{\sqrt{\frac{A^2}{-D}}(A^2 - 2B + 2a)}{3A^2} + \frac{\sqrt{\frac{A^2}{-D}}(A^2 - 2B + 2a)^3}{27A^4C}\right) \pm \sqrt{\frac{A^2}{-D}}\Bigg\{\frac{(A^2 - 2B + 2a)^3}{27A^4C} + \frac{A^2 - 2B + 2a}{2A^2} +
$$

$$
\frac{1}{2}\sqrt{-\frac{2(A^2 - 2B + 2a)^2}{3A^4} - \frac{4C}{A^2}}\tanh\left\{\frac{1}{2}\sqrt{-\frac{2(A^2 - 2B + 2a)^2}{3A^4} - \frac{4C}{A^2}}(\ln x - at)\right\}.
$$

Case 2: $\Delta = \lambda^2 - 4\mu = -\frac{2(A^2-2B+2a)^2}{3A^4} - \frac{4C}{A^2} < 0$, we obtain

$$\frac{G'}{G} = -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4u}}{2} \frac{C_2 \cos \frac{\sqrt{\lambda^2-4u}}{2}\xi - C_1 \sin \frac{\sqrt{\lambda^2-4u}}{2}\xi}{C_1 \cos \frac{\sqrt{\lambda^2-4u}}{2}\xi + C_2 \sin \frac{\sqrt{\lambda^2-4u}}{2}\xi},$$

where C_1, C_2 are integration constants. The traveling wave solution to (1) can be got as follows:

$$v(t, x) = \mp \left(\frac{\sqrt{\frac{A^2}{-D}}(A^2 - 2B + 2a)}{3A^2} + \frac{\sqrt{\frac{A^2}{-D}}(A^2 - 2B + 2a)^3}{27A^4C} \right) \pm \sqrt{\frac{A^2}{-D}} \{ \frac{(A^2 - 2B + 2a)^3}{27A^4C}$$

$$+ \frac{A^2 - 2B + 2a}{2A^2} + \frac{1}{2}\sqrt{-\frac{2(A^2 - 2B + 2a)^2}{3A^4} - \frac{4C}{A^2}}[\frac{C_2 \cos H_2(\ln x - at) - C_1 \sin H_2(\ln x - at)}{C_1 \cos H_2(\ln x - at) + C_2 \sin H_2(\ln x - at)}]\}.$$

where $H_2 = \frac{1}{2}\sqrt{-\frac{2(A^2-2B+2a)^2}{3A^4} - \frac{4C}{A^2}}$.

Case 3: $\Delta = \lambda^2 - 4\mu = -\frac{2(A^2-2B+2a)^2}{3A^4} - \frac{4C}{A^2} = 0$, we obtain

$$\frac{G'}{G} = -\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2\xi},$$

where C_1, C_2 are integration constants. The traveling wave solution to (1) can be got as follows:

$$v(t, x) = \mp \left(\frac{\sqrt{\frac{A^2}{-D}}(A^2 - 2B + 2a)}{3A^2} + \frac{\sqrt{\frac{A^2}{-D}}(A^2 - 2B + 2a)^3}{27A^4C} \right)$$

$$\pm \sqrt{\frac{A^2}{-D}} \{ \frac{(A^2 - 2B + 2a)^3}{27A^4C} + \frac{A^2 - 2B + 2a}{2A^2} + \frac{C_2}{C_1 + C_2(\ln x - at)} \}.$$

4. Conclusion

We have obtained several traveling wave solutions of a class of generalized Black- Scholes equation under certain parametric conditions by using the first integral method and the G'/G-expansion method. From the above investigation, we see that the two methods are effective for obtaining the exact traveling wave solutions of the class of generalized Black-Scholes equation. The first integral method is based on the ring theory of commutative algebra, and supposes that (2) has rational first integrals of the variables v and v'. The G'/G-expansion method is to obtain a special class of solutions of (2) based on the balance principle. When $D = 0$, the class of generalized Black-Scholes equation is changed to the classical Black-Scholes equation, that is, the classical Black-Scholes equation is a special case of (2). The results obtained in this paper can provide some useful reference and help for the relevant financial research.

Conflict of Interest

All authors declare no conflicts of interest in this paper.

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