Research article

Approximation of solutions of multi-dimensional linear stochastic differential equations defined by weakly dependent random variables

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Abstract: It is well-known that under suitable conditions there exists a unique solution of a d-dimensional linear stochastic differential equation. The explicit expression of the solution, however, is not given in general. Hence, numerical methods to obtain approximate solutions are useful for such stochastic differential equations. In this paper, we consider stochastic difference equations corresponding to linear stochastic differential equations. The difference equations are constructed by weakly dependent random variables, and this formulation is raised by the view points of time series. We show a convergence theorem on the stochastic difference equations.

Keywords: Difference equation; weakly dependent random variables; Euler-Maruyama scheme; strong invariance principle; linear stochastic differential equation
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1. Introduction

Stochastic differential equations have been used to describe stochastic models in many areas. Under suitable conditions, a stochastic differential equation has a unique solution, and the representation of the solution is given by a stochastic integral. To express the explicit solutions of stochastic differential equations, numerical methods on stochastic differential equations have been well established. For example, we refer [1] and [3].

In the theory of mathematical finance, Black-Scholes type stochastic differential equation is an important example. Let $\{W(t)\}$ be a standard one-dimensional Wiener process. For fixed constants $\mu \in \mathbb{R}$ and $\sigma > 0$, Black-Scholes model is given by the following stochastic differential equation:

$$dX(t) = X(t)(\nu dt + \sigma dW(t)), \ 0 \le t \le T, \tag{1.1}$$

where T is a maturity. Fixing $n \in \mathbb{N}$, we observe the stochastic differential equation at times $\{t_k = kT/n, k = 1, 2, \ldots, n\}$. To simplify the problem, we consider the case where $\nu = 0$. By the Euler-Maruyama scheme, we can consider an approximation of the solution of (1.1) such that

$$X(t_k) - X(t_{k-1}) \simeq \sigma X(t_{k-1})\{W(t_k) - W(t_{k-1})\}, \ k = 1, 2, \ldots, n,$$

where $\simeq$ means *nearly equal to*. We rewrite the approximation above such that

$$\sigma\{W(t_k) - W(t_{k-1})\} \simeq \frac{X(t_k) - X(t_{k-1})}{X(t_{k-1})}.$$

In mathematical finance, we regard the difference $W(t_k) - W(t_{k-1})$ as a rate of returns. For the Black-Scholes model, the difference is given by a sequence of independent and identically distributed random variables. If we consider a stochastic difference equation corresponding to the stochastic differential equation (1.1), then time series analysis for the difference can be investigated. From the view of time series analysis, it is natural to consider the model based on dependent data. For the model, the difference should be given by a sequence of dependent random variables.

In [7], Yoshihara studied a stochastic differential equation such that

$$X(t_k) - X(t_{k-1}) = (\nu + \sigma\xi_{k-1})X(t_{k-1}),$$

where $\{\xi_k\}$ is a strictly stationary stochastic sequence. This difference equation corresponds to a Black-Scholes type stochastic differential equation. Under suitable conditions on $\{\xi_k\}$, Yoshihara showed an almost sure convergence theorem for the solution of the difference equation above by using results related to strong Wiener approximations of partial sums of some dependent random variables. Yoshihara's result was extended to multi-dimensional cases in [4], and application to finance models were also considered in [5] and [6]. Following the previous studies, we consider strong approximation of linear stochastic differential equations by weakly dependent random variables in this paper.

Let $\{\xi_k\} = \{(\xi_{k,1}, \ldots, \xi_{k,d})\}$ be a strictly stationary sequence of d-dimensional centered random vectors defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and satisfies the strong mixing condition

$$\alpha(n) = \sup_{A\in\mathbf{M}_{-\infty}^0, B\in\mathbf{M}_n^\infty} |P(AB) - P(A)P(B)| \to 0 \quad (n \to \infty), \tag{1.2}$$

where $\mathbf{M}_a^b (a < b)$ denotes the σ-algebra generated by $\{\xi_a, \ldots, \xi_b\}$. Under some conditions on $\{\xi_k\}$, which shall be mentioned below, there exists a $d \times d$ matrix $\mathbf{\Gamma} = (\gamma_{q,q'})_{q,q'=1,\ldots,d}$ such that

$$\begin{cases} \gamma_{q,q} &= E\xi_{1,q}^2 + 2\sum_{i=2}^\infty E\xi_{1,q}\xi_{i,q}, \\ \gamma_{q,q'} &= E\xi_{1,q}\xi_{1,q'} + \sum_{i=2}^\infty (E\xi_{1,q}\xi_{i,q'} + \xi_{1,q'}\xi_{i,q}). \end{cases} \tag{1.3}$$

We write

$$\mathbf{R} = (r_{q,q'})_{q,q'=1,\ldots,d} = (E\xi_{1,q}\xi_{1,q'})_{q,q'=1,\ldots,d}. \tag{1.4}$$

We remark that if $\{\xi_k\}$ is a sequence of independent and identically distributed $\mathbb{R}^d$-valued random variables, then Γ equals to $\mathbf{R}$. In the paper, we always assume that $d \times d$ matrix Γ and $\mathbf{R}$ are always positive definite. Let $\{\mathbf{W}(t), t \geq 0\} = \{(W_1(t), \ldots, W_d(t)), t \geq 0\}$ be a d-dimensional Wiener process with covariance matrix Γ, i.e.,

$$E\mathbf{W}(t)\mathbf{W}(t)^\mathsf{T} = t\Gamma \quad \text{for all } t \geq 0,$$

where $\mathbf{W}^\mathsf{T}$ is the transpose of the matrix $\mathbf{W}$ and set $\{\mathcal{F}_t, t \geq 0\}$ as a filtration generated by the Wiener process.

For a stationary sequence $\{\xi_k\}$, the following strong invariance principle was shown by Liu and Lin [2]. We remark that their result holds under more general conditions:

Proposition 1.1. *Let $\{\xi_k\}$ be a stationary strong mixing sequence of centered d-dimensional random vectors. Assume that*

 (i) $E[\xi_k] = \mathbf{0}$,
 (ii) ξ_k *has the third moment,*
(iii) *for some $\tau > 0$ there exists positive constant c such that $\alpha(n) \leq cn^{-3(1+\tau)}$.*

Then, on a richer probability space we can redefine the sequence $\{\xi_k\}$ together with a d-dimensional Brownian motion $\{\mathbf{W}(t)\}$ whose covariance matrix is Γ such that

$$\left| \sum_{k \leq t} \xi_k - \mathbf{W}(t) \right| = O(t^{1/4}) \tag{1.5}$$

almost surely, where $O(t)$ is the Landau notation, that is, $O(t)$ is any $\mathbb{R}$-valued sequence for which $\limsup_{t \to \infty}(O(t)/t) < \infty$.

Afterward, for the strong approximation (1.5) of Proposition 1.1 we use a notation such that

$$\sum_{k \leq t} \xi_k \Rightarrow \mathbf{W}(t) \quad \text{a.s.}$$

We denote by $A_n \to A$ a.s. the almost surely convergence of random variables.

2. Setting and result

Let $A(t), B_1(t), \ldots, B_d(t)$ be $p \times p$-matrix functions and $a(t), b_1(t), \ldots, b_d(t)$ be p-dimensional vector functions. We assume that all components of the above matrix and vector functions have continuous bounded derivatives on $[0, \infty)$. On $(\Omega, \mathcal{F}, P)$, we denote a d-dimensional Wiener process with covariance matrix Γ by $\{\tilde{\mathbf{W}}(t), t \geq 0\} = \{(\tilde{W}_1(t), \ldots, \tilde{W}_d(t)), t \geq 0\}$. It is known that the d-dimensional linear stochastic differential equation

$$
\begin{aligned}
dX(t) &= \{A(t)X(t) + a(t)\}dt + \sum_{q=1}^{d} \{B_q(t)X(t) + b_q(t)\}d\tilde{W}_q(t), \\
X(0) &= \boldsymbol{x} \in \mathbb{R}^p
\end{aligned}
\tag{2.1}
$$

has a unique solution $X(t)$, which is given by

$$\Phi(t)\left\{x + \int_0^t \Phi(s)^{-1}\left[a(s) - \sum_{q=1}^d B_q(s)b_q(s)\right]ds + \sum_{q=1}^d \int_o^t \Phi(s)^{-1}b_q(s)d\tilde{W}_q(s)\right\},\tag{2.2}$$

where $\Phi(t)$ denotes the solution of the matrix stochastic differential equation such that

$$
\begin{aligned}
d\Phi(t) &= A(t)\Phi(t)dt + \sum_{q=1}^d B_q(t)\Phi(t)d\tilde{W}_q(t),\\
\Phi(0) &= I,
\end{aligned}
\tag{2.3}
$$

where I denote the identity matrix.

We remark that unlike the scalar homogeneous linear equation, we cannot solve (2.3) explicitly for its fundamental solution $\Phi(t)$, even when all of the matrices are constant. If $A, B_1, \ldots, B_d$ are constant matrices and commutative, then we obtain the following explicit expression for the fundamental matrix solution:

$$\tilde{\Phi}(t) = \exp\left\{\left(A - \frac{1}{2}\mathbf{R}\sum_{q=1}^d B_q^2\right)t - \sum_{q=1}^d B_q\tilde{W}_q(t)\right\}.$$

See Section 2.2 in [1] for more detail. Hence, approximation of solution of stochastic differential equation is a critical tool.

We fix $T > 0$ arbitrarily. For an arbitrary positive integer n, let $m = [n^{1/6}]$ and $l = [n/m]$, where $[x]$ is a floor function. Using them, we define time point for a sufficiently large n such that

$$
\begin{cases}
s_{i,j} &= \left(\dfrac{i}{l} + \dfrac{j}{lm}\right)T, \ i = 0, \ldots, l-1, \ j = 1, \ldots m,\\
s_{0,0} &= 0.
\end{cases}
\tag{2.4}
$$

We identify $s_{k,0}$ with $s_{k-1,m}$ for $k = 1, \ldots, l$.

Corresponding to (2.1), we consider the following difference equation:

$$
\begin{aligned}
\Delta X^{(n)}(s_{i,j}) &= X^{(n)}(s_{i,j}) - X^{(n)}(s_{i,j-1})\\
&:= \left\{A(s_{i,j-1})X^{(n)}(s_{i,j-1}) + a(s_{i,j-1})\right\}\frac{T}{lm}\\
&\quad + \sum_{q=1}^d \left\{B_q(s_{i,j-1})X^{(n)}(s_{i,j-1}) + b_q(s_{i,j-1})\right\}\xi_{im+j,q}\sqrt{\frac{T}{lm}},\\
X^{(n)}(0) &= x \in \mathbb{R}^p.
\end{aligned}
\tag{2.5}
$$

In addition, corresponding to (2.3), we also consider the following difference equation:

$$
\begin{aligned}
\Delta\Phi^{(n)}(s_{i,j}) &= \Phi^{(n)}(s_{i,j}) - \Phi^{(n)}(s_{i,j-1})\\
&:= \left\{A(s_{i,j-1})\frac{T}{lm} + \sum_{q=1}^d B_q(s_{i,j-1})\xi_{im+j,q}\sqrt{\frac{T}{lm}}\right\}\Phi^{(n)}(s_{i,j-1}),\\
\Phi^{(n)}(0) &= I.
\end{aligned}
\tag{2.6}
$$

For the stochastic difference equation (2.6), we obtain a convergence result by using (i) of Theorem 1 and Theorem 2 in [4].

Proposition 2.1. *We assume that a strictly stationary strong mixing sequence $\{\xi\}$ satisfies that (i) $E[|\xi_i|^{6+\delta_1}] < \infty$ with $\delta_1 > 0$ and (ii) $\alpha(n) \le cn^{-3(1+\tau)}$ with some $\tau(\delta) = \tau > 0$. Then, we have the following:*

(1) The solution of (2.6) (we denote by $\Phi^{(n)}(T)$) is given by

$$\prod_{i=0}^{l-1} \prod_{j=1}^{m} \exp\left\{ I + A(s_{i,j-1})\frac{T}{lm} + \sum_{q=1}^{d} B_q(s_{i,j-1})\xi_{im+j,q}\sqrt{\frac{T}{lm}} \right\}.$$

(2) On a richer probability space we can redefine the sequence $\{\xi_k\}$ together with a d-dimensional Brownian motion $\{\mathbf{W}(t)\}$ whose covariance matrix is $\mathbf{\Gamma}$ such that $\Phi^{(n)}(T)$ converges almost surely to a matrix $\Phi(T)$ which corresponds to the solution of the stochastic differential equation (2.3).

Using the solutions $\Phi^{(n)}(T)$ and $\Phi(T)$, we obtain the solution of the stochastic difference (2.5) as follows:

Theorem 2.1. *Under the same assumptions as those in Proposition 2.1, we obtain the following:*
(1) The solution of (2.5) (we denote by $X^{(n)}(T)$) is given by

$$\Phi^{(n)}(T)\left[x + \sum_{i=0}^{l-1}\sum_{j=1}^{m} \Phi^{(n)}(s_{i,j})^{-1}\left\{ a(s_{i,j}) + \sum_{q,q'=1}^{d} B_q(s_{i,j-1})b_{q'}(s_{i,j-1})\xi_{im+j,q}\xi_{im+j,q'} \right\}\frac{T}{lm} \right.$$
$$\left. + \sum_{i=0}^{l-1}\sum_{j=1}^{m}\sum_{q=1}^{d} \Phi^{(n)}(s_{i,j-1})^{-1}b_q(s_{i,j-1})\xi_{im+j,q}\sqrt{\frac{T}{lm}} \right].$$

(2) On a richer probability space we can redefine the sequence $\{\xi_k\}$ together with a p-dimensional Brownian motion $\{\mathbf{W}(t)\}$ whose covariance matrix is $\mathbf{\Gamma}$ such that $X^{(n)}(T)$ converges almost surely to a matrix $X(T)$ which corresponds to the solution of the stochastic differential equation (2.2).

3. Proof

Assertion (1) is shown by constructing $X^{(n)}$ in assertion (2). We consider the following difference equation:

$$\begin{aligned} \Delta Z^{(n)}(s_{i,j}) &= Z^{(n)}(s_{i,j}) - Z^{(n)}(s_{i,j-1}) \\ &:= \Phi^{(n)}(s_{i,j-1})^{-1}\left\{ a(s_{i,j-1}) - \sum_{q,q'=1}^{d} B_q(s_{i,j-1})b_{q'}(s_{i,j-1})\xi_{im+j,q}\xi_{im+j,q'} \right\}\frac{T}{lm} \\ &\quad + \Phi^{(n)}(s_{i,j-1})^{-1}\sum_{q=1}^{d} b_q(s_{i,j-1})\xi_{im+j,q}\sqrt{\frac{T}{lm}}, \\ Z(0) &= x. \end{aligned}$$

Then for each $(s_{i,j})_{0\le i\le l-1, 1\le j\le m}$ we have that

$$\begin{aligned} \Delta(\Phi^{(n)}Z^{(n)}) &= \Delta\Phi^{(n)}\,Z^{(n)} + \Phi^{(n)}\,\Delta Z^{(n)} + \Delta\Phi^{(n)}\,\Delta Z^{(n)} \\ &=: U_1^{(n)} + U_2^{(n)} + U_3^{(n)}. \end{aligned} \tag{3.1}$$

Firstly, we put that $X(t) := \Phi(t)Z(t)$. Then, Theorem 1 in [4] implies that

$$
\sum_{i=0}^{l-1} \sum_{j=1}^{m} U_1^{(n)} = \sum_{i=0}^{l-1} \sum_{j=1}^{m} \left\{ A(s_{i,j-1}) \frac{T}{lm} + \sum_{q=1}^{d} B_q(s_{i,j-1}) \xi_{im+j,q} \sqrt{\frac{T}{lm}} \right\} \Phi^{(n)}(s_{i,j-1}) Z^{(n)}(s_{i,j-1})
$$

$$
\Rightarrow \int_0^T A(s)X(s)ds + \sum_{q=1}^{d} \int_0^T B_q(s)X(s)dW_q(s) \quad \text{a.s.} \tag{3.2}
$$

Secondly, we consider that

$$
\sum_{i=0}^{l-1} \sum_{j=1}^{m} U_2^{(n)} = \sum_{i=0}^{l-1} \sum_{j=1}^{m} a(s_{i,j}) \frac{T}{lm} - \sum_{i=0}^{l-1} \sum_{j=1}^{m} \sum_{q,q'=1}^{d} B_q(s_{i,j-1}) b_{q'}(s_{i,j-1}) \xi_{im+j,q} \xi_{im+j,q'} \frac{T}{lm}
$$

$$
+ \sum_{i=0}^{l-1} \sum_{j=1}^{m} \sum_{q=1}^{d} b_q(s_{i,j-1}) \xi_{im+j,q} \sqrt{\frac{T}{lm}}
$$

$$
=: V_1^{(n)} - V_2^{(n)} + V_3^{(n)}. \tag{3.3}
$$

Since $a(t)$ has continuous bounded derivatives on $[0, \infty)$, it is obvious that

$$
V_1^{(n)} \to \int_0^T a(s)ds. \tag{3.4}
$$

For $V_2^{(n)}$, we use the law of large numbers for $\{\xi_k\}$. Then, we obtain that

$$
V_2^{(n)} \to \sum_{q,q'=1}^{d} r_{q,q'} \int_0^T B_q(s) b_{q'}(s)ds \quad \text{a.s.} \tag{3.5}
$$

Further, using Theorem 1 in [4] again, we have that

$$
V_3^{(n)} \Rightarrow \sum_{q=1}^{d} \int_0^T b_q(s)dW_q(s) \quad \text{a.s.} \tag{3.6}
$$

Finally, we consider that

$$
\sum_{i=0}^{l-1} \sum_{j=1}^{m} U_3^{(n)} = \sum_{i=0}^{l-1} \sum_{j=1}^{m} \left(A(s_{i,j-1}) \frac{T}{lm} + \sum_{q=1}^{d} B_q(s_{i,j-1}) \xi_{im+j,q} \sqrt{\frac{T}{lm}} \right)
$$

$$
\left[\left\{ a(s_{i,j-1}) - \sum_{q,q'=1}^{d} B_q(s_{i,j-1}) b_{q'}(s_{i,j-1}) \xi_{im+j,q} \xi_{im+j,q'} \right\} \frac{T}{lm} + \sum_{q=1}^{d} b_q(s_{i,j-1}) \xi_{im+j,q} \sqrt{\frac{T}{lm}} \right].
$$

$$
\tag{3.7}
$$

Since under the condition of the theorem on $\{\xi_k\}$, there exists a covariant matrix $\mathbf{\Gamma}$. We thus obtain that the right hand side of (3.7) equals almost surely to

$$
\frac{T}{lm} \sum_{i=0}^{l-1} \sum_{j=1}^{m} \sum_{q,q'=1}^{d} B_q(s_{i,j-1}) b_{q'}(s_{i,j-1}) \xi_{m(i-1)+j,q} \xi_{m(i-1)+j,q'} + O((lm)^{-3/2}). \tag{3.8}
$$

To estimate the first term of the right hand side of (3.8), we consider the term of random variables. Then we obtain that

$$\frac{1}{lm} \sum_{i=0}^{l-1} \sum_{j=1}^{m} \xi_{m(i-1)+j,q} \xi_{m(i-1)+j,q'} \to r_{q,q'} \quad \text{a.s.} \tag{3.9}$$

In addition, all components of B_q and $b_q (1 \le q \le p)$ have continuous bounded derivatives. Thus, there exists some $c > 0$ such that for any $1 \le j \le m$

$$\|B_q(s_{i,j-1})b_{q'}(s_{i,j-1}) - B_q(s_{i,0})b_{q'}(s_{i,0})\|$$
$$\le \sum_{k=1}^{j-1} \|B_q(s_{i,k})b_{q'}(s_{i,k}) - B_q(s_{i,0})b_{q'}(s_{i,0})\|$$
$$\le \frac{c(j-1)}{lm} \le \frac{c}{l},$$

which implies that

$$\frac{T}{lm} \sum_{i=0}^{l-1} \sum_{j=1}^{m} B_q(s_{i,j-1})b_{q'}(s_{i,j-1})\xi_{m(i-1)+j,q} \xi_{m(i-1)+j,q'} \to r_{q,q'} \int_0^T B_q(s)b_{q'}(s)ds \quad \text{a.s.} \tag{3.10}$$

Hence, by (3.9) and (3.10) we obtain that

$$\sum_{i=0}^{l-1} \sum_{j=1}^{m} \Delta\Phi^{(n)}(s_{i,j})\Delta Z^{(n)}(s_{i,j}) \to \sum_{q,q'=1}^{p} r_{q,q'} \int_0^T B_q(s)b_{q'}(s)ds \quad \text{a.s.} \tag{3.11}$$

Combining (3.1), (3.2), (3.3)-(3.6), (3.7) and (3.11), we obtain that

$$\sum_{i=0}^{l-1} \sum_{j=1}^{m} \Delta(\Phi^{(n)}(s_{i,j})Z^{(n)}(s_{i,j}))$$
$$\Rightarrow \int_0^T A(s)X(s)ds + \sum_{q=1}^{p} B_q(s)X(s)dW_q(s) + \int_0^T a(s)ds + \sum_{q=1}^{p} \int_0^T b_q(s)dW_q(s) \quad \text{a.s.}$$
$$= \int_0^T (A(s)X(s) + a(s))ds + \sum_{q=1}^{p} \int_0^T (B_q(s)X(s) + b_q(s))dW_q(s) \quad \text{a.s.}$$

On the other hand, the solution $X^{(n)}(T)$ of the difference equation

$$\begin{aligned} \Delta X^{(n)}(s_{i,j}) &= X^{(n)}(s_{i,j}) - X^{(n)}(s_{i,j-1}) \\ &:= \Delta(\Phi^{(n)}(s_{i,j})Z^{(n)}(s_{i,j})), \\ X^{(n)}(0) &= x \end{aligned}$$

satisfies that

$$X^{(n)}(T) = x + \sum_{i=0}^{l-1} \sum_{j=1}^{m} \Delta X^{(n)}(s_{i,j}).$$

Thus, we obtain that

$$X^{(n)}(T) \Rightarrow \boldsymbol{x} + \int_0^T (A(s)X(s) + a(s))ds + \sum_{q=1}^p \int_0^T (B_q(s)X(s) + b_q)dW_q(s) \quad \text{a.s,}$$

which shows assertion (2) of Theorem 2.1. □

Acknowledgments

Conflict of Interest

All authors declare no conflicts of interest in this paper.

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